Published as Chapter 34 in
On Einstein's Path
Essays in honor of Engelbert Schucking
A. Harvey, editor, Springer, New York 1999

ON COMPLEX STRUCTURES IN PHYSICS

Andrzej Trautman

Instytut Fizyki Teoretycznej UW
Hoża 69, 00681 Warszawa, Poland
E-mail: amt@fuw.edu.pl

Abstract

Complex numbers enter fundamental physics in at least two rather distinct ways. They are needed in quantum theories to make linear differential operators into Hermitian observables. Complex structures appear also, through Hodge duality, in vector and spinor spaces associated with space-time. This paper reviews some of these notions. Charge conjugation in multidimensional geometries and the appearance of Cauchy-Riemann structures in Lorentz manifolds with a congruence of null geodesics without shear are presented in considerable detail.

1 Introduction

In 1960, Ivor Robinson and I studied a class of solutions of Einstein's equations on a Lorentzian manifold, foliated by a shearfree, non-rotating and diverging congruence of null geodesics. We were surprised to find that, in the coordinate system we were using, the vanishing of the Ricci tensor implied that two components of the metric tensor satisfied Cauchy-Riemann equations [16]. Since that time I have been interested in the question how and why complex numbers and structures appear in physical problems.

Complex numbers have been used in physics so much and for so long that one is taking them, most of the time, for granted. Their origin is in 'pure' mathematics: they appeared, in the XVIth century, in connection with solving polynomial equations. They were used in solutions of physical problems, such as reflection and diffraction of waves, early in the XIX century. Complex roots of polynomial equations often appear in physical problems in connection with linear differential equations with constant coefficients. An early, ingenious use of complex analytic functions was made by Arnold Sommerfeld in his rigorous solution of the problem of diffraction of waves on a half-plane [20][1]. Roger Penrose, in his twistor theory, put forward convincing arguments in favor of the relevance of holomorphic structures for fundamental physics [11, 13]. His methods and ideas have been successfully used in a variety of mathematical and physical problems. Complex numbers and analytic functions permeate now all of quantum physics.

Complex numbers are introduced in physical theories in several ways; it is not obvious that 'all the square roots of -1 are the same'. There is the $\sqrt{-1}$ of quantum mechanics that is 'universal' in the sense that it appears irrespective of the details of the model under consideration. Chiral (Weyl, reduced) spinors in Minkowski space-time are also complex, but this property reflects the signature of its metric tensor. Dirac spinors can be restricted to be real, provided one uses a metric of signature $(3,1)$; but to write the wave equation of an electron, in any signature, one has to introduce complex numbers because electromagnetism is a gauge theory with U_1 as the structure group. The 'electromagnetic' $\sqrt{-1}$ seems to have a quantum-theoretical origin: in the classical theory of a particle of charge e the potential A_μ appears in expressions such as $p_\mu + eA_\mu$; in quantum theory this becomes

$$\mathrm{i}\partial_\mu + eA_\mu = \mathrm{i}(\partial_\mu - \mathrm{i}eA_\mu).$$

and the i next to eA_μ reflects the nature of the Lie algebra of U_1.

In this article, I present some thoughts on the origin of the appearance of complex numbers in physics, emphasizing the geometric, rather than the analytic, aspects of the problem. After recalling the notion of complex structures in real vector spaces (Section 2), I show, on a simple example, how such structures may be considered to appear in quantum mechanics (Section

[1] I am grateful to Jürgen Ehlers for this reference.

3). For some signatures of the metric tensor, Clifford-Hodge-Kähler duality introduces complex structures in spaces of spinors and multivectors (Section 4). Charge conjugation is also closely related to the appearance of complex numbers in quantum theory; its generalization to higher dimensions is described in Section 5. In a final section, influenced by my collaboration with W. Kopczyński, P. Nurowski and J. Tafel, I describe the geometry underlying shearfree congruences of null geodesics, its relation to Cauchy-Riemann structures in three dimensions, and the close analogy between optical geometries in Lorentzian manifolds and Hermitian geometries in proper Riemannian manifolds.

In 1961, Engelbert Schucking and I spent some time together at Syracuse University in Peter Bergmann's group that included also Dick Arnowitt, Asim Barut, Art Komar, Ted Newman, Roger Penrose, Ivor Robinson, Ralph Schiller and Mel Schwartz. Since that time, I have had the pleasure to see Engelbert on various occasions and to talk with him on many issues of science and life. These discussions included also the topics touched upon in this text. On one of my visits to New York, Engelbert presented me with a copy of [2], an excellent account of the history of number systems. Several times, my family and I enjoyed his very kind hospitality at Washington Square. This article is dedicated to Engelbert as a token of my friendship and respect.

2 Definitions and notation

Recall that a *complex structure* in a real vector space W is a linear automorphism J of W such that $J^2 = -\mathrm{id}_W$; if W is finite-dimensional, then its dimension is even. A real vector space W with a complex structure J can be made into a complex vector space in two ways, by defining, for every $w \in W$, either $\mathrm{i}w = J(w)$ or $\mathrm{i}w = -J(w)$. The automorphism J extends, in an obvious way, to an automorphism $J_{\mathbb{C}}$ of the *complexification* of W, i.e. to the complex vector space $\mathbb{C} \otimes W$. This space can be decomposed into the direct sum,

$$(1) \qquad \mathbb{C} \otimes W = W_+ \oplus W_-, \ \text{ where } \ W_\pm = \{w \in \mathbb{C} \otimes W : J_{\mathbb{C}}(w) = \pm \mathrm{i} w\}.$$

Considered as a complex vector space, W is isomorphic to W_+ or W_-, depending on whether the multiplication by i in W is defined as $\mathrm{i}w = J(w)$ or $\mathrm{i}w = -J(w)$, respectively. Note that if W is of real dimension $2n$, then its complex dimension is n. Every complex vector space of dimension n can be 'realified', i.e. considered as a real vector space of real dimension $2n$. Such a realification has a natural complex structure.

Assume now that the real vector space W has a (generalized) scalar product, i.e. a map $g : W \times W \to \mathbb{R}$ which is bilinear, symmetric and non-degenerate. The scalar product g extends to a $\mathbb{C}$-bilinear scalar product $g_{\mathbb{C}}$ on $\mathbb{C} \otimes W$. If the complex structure J in W is orthogonal with respect to g, i.e. if $g(J(w_1), J(w_2)) = g(w_1, w_2)$ for every $w_1, w_2 \in W$, then the vector spaces W_+ and W_- are both totally null (isotropic) with respect to $g_{\mathbb{C}}$. The vector space W, considered as a complex vector space such that $\mathrm{i}w = J(w)$ has a Hermitian scalar product $h : W \times W \to \mathbb{C}$ defined by

$$h(w_1, w_2) = g(w_1, w_2) + \mathrm{i}g(J(w_1), w_2), \quad w_1, w_2 \in W, \tag{2}$$

so that $h(w_1, \mathrm{i}w_2) = \mathrm{i}h(w_1, w_2)$, $\overline{h(w_1, w_2)} = h(w_2, w_1)$ and $h(w, w) = g(w, w)$.

Consider now a *complex, finite-dimensional vector space* S. Its (complex) dual S^* consists of all $\mathbb{C}$-linear maps $s' : S \to \mathbb{C}$; it is often convenient to denote here the value of s' on $s \in S$ by $\langle s', s\rangle$. If $f : S_1 \to S_2$ is a $\mathbb{C}$-linear map of complex vector spaces, then the dual (transposed) map $f^* : S_2^* \to S_1^*$ is defined by $\langle f^*(s'), s\rangle = \langle s', f(s)\rangle$ for every $s \in S_1$ and $s' \in S_2^*$. The spaces S^{**} and S can be identified. A map $h : S_1 \to S_2$ is said to be *antilinear* (semi-linear) if it is $\mathbb{R}$-linear and $h(\mathrm{i}s) = -\mathrm{i}h(s)$ for every $s \in S_1$. The *complex conjugate* $\bar{S}$ of a complex vector space S is the complex vector space of all antilinear maps of S^* into $\mathbb{C}$; there is a canonical antilinear isomorphism $S \to \bar{S}$, $s \mapsto \bar{s}$, given by $\langle \bar{s}, s'\rangle = \overline{\langle s', s\rangle}$. With every linear map $f : S_1 \to S_2$ there is associated the linear map $\bar{f} : \overline{S_1} \to \overline{S_2}$ defined by $\bar{f}(\bar{s}) = \overline{f(s)}$ for $s \in S_1$; the map $f \mapsto \bar{f}$ is antilinear. If $g : S_2 \to S_3$ is another linear map, then $(g \circ f)^* = f^* \circ g^*$ and $\overline{g \circ f} = \bar{g} \circ \bar{f}$. One often writes gf instead of $g \circ f$.

All manifolds and maps are assumed to be smooth. Einstein's summation convention over repeated indices is used. If L is a vector bundle over a manifold M, then $\Gamma(L)$ denotes the module of sections of $L \to M$. The zero bundle is denoted by 0. The tangent and cotangent bundles of M are denoted by TM and T^*M, respectively. The contraction of a vector (field) v with a p-form ω is the $(p-1)$-form $v \lrcorner \omega$ given by its value on the vectors $v_2, \ldots, v_p$, $(v \lrcorner \omega)(v_2, \ldots, v_p) = \omega(v, v_2, \ldots, v_p)$. If g is a scalar product on a vector space V and $v \in V$, then $g(v) \in V^*$ is defined by $v' \lrcorner g(v) = g(v', v)$ for every $v' \in V$. The exterior differential of a form ω is denoted by $\mathrm{d}\omega$.

3 A complex structure defined by differentiation

The usual argument for complex numbers in quantum mechanics, in a simplified form, runs as follows: differential operators such as $\partial/\partial x$, because of their relation to translations, are needed to represent components of momentum; to make them (formally) self-adjoint, one has to multiply by i. One can reformulate this argument into a statement about the appearance of a complex structure in the vector space of wave functions, initially considered as a real vector (Hilbert) space. The key observation is that the Laplacian on a compact, proper Riemannian manifold is a *negative* operator.

To illustrate this argument on a simple example, and make it explicit, consider the infinite-dimensional real Hilbert space $L^2_{\mathbb{R}}(\mathbb{S}_1)$ of square-integrable functions on the circle $\mathbb{S}_1$. Let x be a coordinate on the circle, $0 \leqslant x \leqslant 2\pi$. The scalar product of two functions $\varphi, \psi : \mathbb{S}_1 \to \mathbb{R}$, is given by

$$g(\varphi, \psi) = \int_0^{2\pi} \varphi(x)\psi(x)\,\mathrm{d}x$$

so that $g(\varphi, \varphi) \geqslant 0$. Let W be the vector subspace of $L^2_{\mathbb{R}}(\mathbb{S}_1)$ containing all functions orthogonal to the constants on the circle,

$$W = \{\varphi \in L^2_{\mathbb{R}}(\mathbb{S}_1) : \int_0^{2\pi} \varphi(x)\,\mathrm{d}x = 0\}.$$

Smooth functions in W constitute a dense subspace of that space; for every two such functions φ and ψ one has

$$g(\varphi', \psi) = -g(\varphi, \psi'),$$

where $\varphi'(x) = \mathrm{d}\varphi(x)/\,\mathrm{d}x$. The operator $\mathrm{d}^2/\,\mathrm{d}x^2$ is (formally) self-adjoint and negative on W: if φ is smooth and $\varphi \neq 0$, then

$$g(\varphi, \varphi'') = -g(\varphi', \varphi') < 0.$$

The set of eigenfunctions of $\mathrm{d}^2/\,\mathrm{d}x^2$,

$$\{\cos kx,\ \sin kx\}, \quad k = 1, 2, \ldots,$$

is a basis in W. The operator $-\,\mathrm{d}^2/\,\mathrm{d}x^2$ has only positive eigenvalues; as such it has a unique positive square root X, i.e. a (formally) self-adjoint operator with positive eigenvalues such that $X^2 = -\,\mathrm{d}^2/\,\mathrm{d}x^2$. The operator X in W, which may be characterized by its action on the basis vectors,

$$X(\cos kx) = k\cos kx, \quad X(\sin kx) = k\sin kx,$$

is *invertible* and *commutes* with the operator $\mathrm{d}/\,\mathrm{d}x$. Therefore, the linear operator

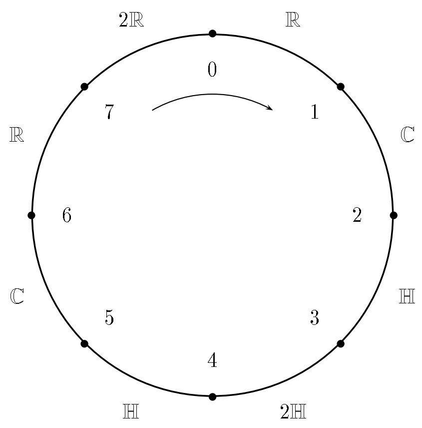
$$J = X^{-1}\circ\frac{\mathrm{d}}{\mathrm{d}x} \quad \text{satisfies} \quad J^2 = -\,\mathrm{id}_W$$

and defines a complex structure on W. Introducing the complex vector spaces $W_\pm$, as in the previous section, one obtains

$$X = \mp\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}x} \quad \text{on} \quad W_\pm.$$

4 Complex structures associated with pseudo-Euclidean vector spaces

Let V be a real, m-dimensional vector space with a scalar product g of signature (k,l), $k+l=m$. The Clifford algebra associated with the pair (V,g) is denoted by $\mathsf{Cl}_{k,l}$. The algebra is generated by V; by declaring the elements of V to be *odd*, one defines a $\mathbb{Z}_2$-grading of $\mathsf{Cl}_{k,l}$: one writes $\mathsf{Cl}^0_{k,l}\to\mathsf{Cl}_{k,l}$ to emphasize this grading and exhibit the even subalgebra $\mathsf{Cl}^0_{k,l}$. The *degree* $\deg a$ of an even (resp., odd) element $a\in\mathsf{Cl}_{k,l}$ is 0 (resp., 1). Recall that if $\mathcal{A}$ and $\mathcal{B}$ are $\mathbb{Z}_2$-graded algebras, then multiplication in their *graded product* $\mathcal{A}\otimes_{\mathrm{gr}}\mathcal{B}$ is defined, for homogeneous elements $a'\in\mathcal{A}$ and $b\in\mathcal{B}$, by $(a\otimes b)(a'\otimes b') = (-1)^{\deg b \deg a'}aa'\otimes bb'$. For every $k,l\in\mathbb{N}$ the algebras $\mathsf{Cl}_{k,l}$ and $\mathsf{Cl}^0_{k,l+1}$ are isomorphic. Denote by $\mathbb{R}(N)$ the algebra of real N by N matrices. For every algebra $\mathcal{A}$ over $\mathbb{R}$, put $2\mathcal{A}=\mathcal{A}\oplus\mathcal{A}$ and $\mathcal{A}(N)=\mathcal{A}\otimes\mathbb{R}(N)$. Every Clifford algebra $\mathsf{Cl}_{k,l}$ is isomorphic to one of the following algebras: $\mathbb{R}(2^p)$, $\mathbb{C}(2^p)$, $\mathbb{H}(2^p)$, $2\mathbb{R}(2^p)$, $2\mathbb{H}(2^p)$, $p\in\mathbb{N}$. Recall the *Chevalley theorem*: $\mathsf{Cl}_{k,l}\otimes_{\mathrm{gr}}\mathsf{Cl}_{k',l'} = \mathsf{Cl}_{k+k',l+l'}$ and the isomorphisms: $\mathsf{Cl}_{k+4,l}=\mathsf{Cl}_{k,l+4}$, $\mathsf{Cl}_{k+1,l+1}=\mathsf{Cl}_{k,l}\otimes\mathbb{R}(2)$. Two Clifford algebras, $\mathsf{Cl}_{k,l}$ and $\mathsf{Cl}_{k',l'}$, are said to be of the same *type* if $k+l'\equiv k'+l \bmod 8$. When grading is taken into account, there are eight types of Clifford algebras; with respect to graded tensor multiplication the set of these eight types forms a group (the *Brauer-Wall group* of $\mathbb{R}$) isomorphic to $\mathbb{Z}_8$; for this reason, the algebras are conveniently represented on the *spinorial clock* [1]:



The spinorial clock can be used to find the structure of real Clifford algebras. To determine $\mathsf{Cl}^0_{k,l} \to \mathsf{Cl}_{k,l}$, compute the corresponding *hour* $h \in \{0, \ldots, 7\}$, $l - k = h + 8r$, $r \in \mathbb{Z}$. Read off the sequence $\mathcal{A}^0_h \xrightarrow{h} \mathcal{A}_h$ from the clock. If $\dim_{\mathbb{R}} \mathcal{A}_h = 2^{\nu_h}$, then $\mathsf{Cl}_{k,l} = \mathcal{A}_h(2^{\frac{1}{2}(k+l-\nu_h)})$, etc. The algebra $\mathcal{A}_h \otimes_{\mathrm{gr}} \mathcal{A}_{h'}$ is of the same type as $\mathcal{A}_{h+h' \bmod 8}$.

As a vector space, the algebra $\mathsf{Cl}_{k,l}$ is isomorphic, in a natural way, to the vector space $\bigwedge V$ underlying the exterior algebra of V. This isomorphism

$$\kappa : \mathsf{Cl}_{k,l} \to \bigwedge V$$

is characterized by $\kappa(1) = 1$ and

$$\kappa(va) = v \wedge \kappa(a) + g(v) \lrcorner \kappa(a) \quad \text{for every } v \in V \subset \mathsf{Cl}_{k,l} \text{ and } a \in \mathsf{Cl}_{k,l}.$$

It respects the $\mathbb{Z}_2$-grading of the vector spaces in question.

An orthonormal frame (e_μ) in $V \subset \mathsf{Cl}_{k,l}$ satisfies $e_\mu e_\nu + e_\nu e_\mu = 0$ for $\mu \neq \nu$, $e_\mu^2 = 1$ for $\mu = 1, \ldots, k$, and $= -1$ for $\mu = k+1, \ldots, m$. The square of the *volume element* $\eta = e_1 \ldots e_m$ is $\eta^2 = (-1)^{\frac{1}{2}(l-k)(l-k+1)}$. Hodge duality, as defined by Kähler, is given by

$$\star\kappa(a) = \kappa(\eta a), \quad \text{where} \quad a \in \mathsf{Cl}_{k,l}.$$

Whenever $\eta^2 = -1$, there is a natural complex structure in the real vector space $\bigwedge V$. Therefore,
(i) if $V = \mathbb{R}^{2n}$ and $l - k \equiv 2 \bmod 4$, then $\star$ defines a complex structure in $\bigwedge^n \mathbb{R}^{2n}$;
(ii) if $V = \mathbb{R}^{2n+1}$ and $l - k \equiv 1 \bmod 4$, then $\star$ defines a complex structure in $\bigwedge^n \mathbb{R}^{2n+1} \oplus \bigwedge^{n+1} \mathbb{R}^{2n+1}$.

If $m = k + l$ is even, $m = 2n$, then the algebra $\mathsf{Cl}_{k,l}$ is central simple and has one, up to equivalence, representation

$$\gamma : \mathsf{Cl}_{k,l} \to \mathsf{End}\, S \tag{3}$$

in a complex, 2^n-dimensional space of Dirac spinors. In particular, the contragredient representation and the complex conjugate representation are each equivalent to γ. In terms of the Dirac matrices (endomorphisms of S), $\gamma_\mu = \gamma(e_\mu)$, this equivalence may be expressed by the equations

$$\gamma_\mu^* = B\gamma_\mu B^{-1}, \quad B : S \to S^*, \tag{4}$$

and

$$\overline{\gamma_\mu} = C\gamma_\mu C^{-1}, \quad C : S \to \bar{S}. \tag{5}$$

The intertwining isomorphisms B and C are defined up to multiplication by non-zero complex numbers. The matrix

$$\gamma_{2n+1} = \gamma_1 \dots \gamma_{2n}$$

anticommutes with γ_μ for $\mu = 1, \dots, 2n$. The representation γ, restricted to $\mathsf{Cl}^0_{k,l}$, decomposes into the direct sum $\gamma_+ \oplus \gamma_-$ of representations,

$$\gamma_\pm : \mathsf{Cl}^0_{k,l} \to \mathsf{End}\, S_\pm,$$

in the spaces S_+ and S_- of Weyl spinors of opposite *chirality*,

$$S_\pm = \{\varphi \in S : \gamma_{2n+1}\varphi = \pm\iota\varphi\},$$

where

$$\iota = 1 \text{ for } \eta^2 = 1 \text{ and } \iota = \mathrm{i} \text{ for } \eta^2 = -1. \tag{6}$$

The relevant properties of B, C and γ_{2n+1} can be summarized in (see, e.g., [1])

Proposition 1. *If $k + l = 2n$, then*

$$B^* = (-1)^{\frac{1}{2}n(n-1)} B, \tag{7}$$

$$\gamma^*_{2n+1} = (-1)^n B \gamma_{2n+1} B^{-1} \tag{8}$$

$$\overline{\gamma_{2n+1}} = C \gamma_{2n+1} C^{-1}. \tag{9}$$

One can normalize the intertwining isomorphisms defined by (4) *and* (5) *so that*

$$\bar{C} C = (-1)^{\frac{1}{8}(l-k)(l-k+2)} \tag{10}$$

and

$$\bar{B} C = \bar{C}^* B^*. \tag{11}$$

The proof of (11) is based on the observation that $\bar{B}C(\bar{C}^* B^*)^{-1}$ is in the commutant of the irreducible representation γ.

From the spinorial clock one obtains, for every $p \in \mathbb{N}$, the isomorphism

$$\mathsf{Cl}_{3+p,1+p} = \mathbb{R}(2^{2+p}).$$

Since in this case $\eta^2 = -1$, there is a complex structure $J = \gamma(\eta)$ in the real space of Dirac-Majorana spinors $W = \mathbb{R}^{2^{2+p}}$. Defining, as in (1), the complex, 2^{1+p}-dimensional spaces W_+ and W_-, one sees that they can be identified with the two spaces of Weyl spinors of opposite chirality.

5 Charge conjugation

Charge conjugation is intrinsically connected with the equivalence of the representations γ and $\bar{\gamma}$. The notion of charge conjugation, defined originally by physicists for spinors associated with Minkowski space, admits a generalization to higher dimensions [1]. In view of some controversy surrounding this generalization [8], I present it here, in considerable detail, for the case of an even-dimensional, flat space-time with a metric of signature $(2n-1, 1)$.

Consider first the general case of $k + l$ even, $k + l = 2n$; given a representation (3) of the algebra $\mathsf{Cl}_{k,l}$ in a complex vector space S of Dirac spinors, one defines *charge conjugation* to be the antilinear map $S \to S$,

$$\varphi \mapsto \varphi_c = C^{-1} \bar{\varphi}. \tag{12}$$

If φ is a Weyl spinor, $\gamma_{2n+1}\varphi = \pm\iota\varphi$, then φ_c is also such a spinor and its chirality is the same as (resp., opposite to) that of φ if $\eta^2 = 1$ (resp., if $\eta^2 = -1$). If $\bar{C}C = \mathrm{id}_S$, then the map $\varphi \mapsto \varphi_c$ is involutive, $(\varphi_c)_c = \varphi$, and there is the real vector space

$$S_{\mathbb{R}} = \{\varphi \in S : \varphi_c = \varphi\}$$

of Dirac-Majorana spinors. Charge conjugation is *not*, however, restricted to that case. If γ_μ are the Dirac matrices corresponding to a representation of $\mathsf{Cl}_{k,l}$, then the matrices $\mathrm{i}\gamma_\mu$ correspond to a representation of $\mathsf{Cl}_{l,k}$.

Assume now that the signature is Lorentzian, $k = 2n-1$ and $l = 1$. In view of the previous remark, the case of signature $(1, 2n-1)$ can be easily reduced to the one under consideration. The properties of the intertwiner C, described in Prop. 1, are now expressed by the equation

$$\bar{C}C = (-1)^{\frac{1}{2}(n-1)(n-2)}\,\mathrm{id}_S\,. \tag{13}$$

The Dirac equation for a particle of mass m and electric charge e can be written as

$$\gamma^\mu(\partial_\mu - \mathrm{i}eA_\mu)\psi = m\psi, \tag{14}$$

where $\psi : \mathbb{R}^{2n} \to S$ is the wave function of the particle and A_μ, $\mu = 1, \ldots, 2n$, are the (real) components of the vector potential of the electromagnetic field. For a free particle ($A_\mu = 0$) one can consider a solution of (14) equal to a constant spinor times $\exp \mathrm{i}p_\mu x^\mu$; the Dirac equation then implies that the momentum vector (p_μ) is time-like: $p_{2n}^2 = p_1^2 + \cdots + p_{2n-1}^2 + m^2$. The *charge conjugate wave function* $\psi_c : \mathbb{R}^{2n} \to S$ is defined by $\psi_c(x) = \psi(x)_c$ for every $x \in \mathbb{R}^{2n}$.

Proposition 2. *If $\psi : \mathbb{R}^{2n} \to S$ is a wave function, then*
(i) *the vector field of current defined by*

$$j^\mu(\psi) = \mathrm{i}^{n+1}\langle B\gamma_{2n+1}\psi_c, \gamma^\mu\psi\rangle, \quad \mu = 1, \ldots, 2n, \tag{15}$$

is real *and invariant with respect to the replacement of ψ by ψ_c,*

$$j^\mu(\psi_c) = j^\mu(\psi); \tag{16}$$

(ii) *if ψ is a solution of the Dirac equation* (14)*, then the current is conserved,*

$$\partial_\mu j^\mu(\psi) = 0, \tag{17}$$

and the charge conjugate wave function satisfies the Dirac equation for a particle of charge $-e$,

$$\gamma^\mu(\partial_\mu + \mathrm{i}eA_\mu)\psi_c = m\psi_c. \tag{18}$$

The proof of part (i) the Proposition consists of simple, algebraic transformations, making use of equations (4), (5), (12), (13) and Prop. 1. Complex conjugating both sides of (14), multiplying the resulting equation by C^{-1} on the left and using (5) and (12), one obtains that ψ_c satisfies (18); it is then easy to check that (17) holds.

These simple observations are valid irrespective of whether the algebra $\mathsf{Cl}_{2n-1,1}$ is real ($\bar{C}C = \mathrm{id}_S$; $n \equiv 1$ or $2 \bmod 4$) or quaternionic ($\bar{C}C = -\,\mathrm{id}_S$; $n \equiv 0$ or $3 \bmod 4$). Charge conjugation is not related to the existence of Majorana spinors: even if the algebra $\mathsf{Cl}_{2n-1,1}$ is *real*, one has to use *complex* spinors to write the Dirac equation for a charged particle interacting with an electromagnetic field. The invariance of the current under the replacement of ψ by ψ_c, expressed by (16), reflects the classical (or rather: first-quantized) nature of the Dirac equation under consideration here. Upon second quantization, the wave function is replaced by an *anticommuting*, spinor-valued field Ψ; anticommutativity of Ψ and Ψ_c provides a change of sign, so that (16) is replaced by $j^\mu(\Psi_c) = -j^\mu(\Psi)$.[2]

As an example, consider the case of dimension 8: one has $\mathsf{Cl}_{7,1} = \mathbb{H}(8)$ and the space of Dirac spinors is complex 16-dimensional. Let

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

be the Pauli matrices. One can take, in this case, a representation such that

$$\begin{aligned}
\gamma_1 &= \sigma_x \otimes I \otimes I \otimes I, & \gamma_2 &= \sigma_y \otimes \sigma_y \otimes I \otimes I, \\
\gamma_3 &= \sigma_y \otimes \sigma_x \otimes \sigma_y \otimes I, & \gamma_4 &= \sigma_y \otimes \sigma_x \otimes \sigma_x \otimes \sigma_y, \\
\gamma_5 &= \sigma_y \otimes \sigma_x \otimes \sigma_z \otimes \sigma_y, & \gamma_6 &= \sigma_y \otimes \sigma_z \otimes I \otimes \sigma_y, \\
\gamma_7 &= \sigma_y \otimes \sigma_z \otimes \sigma_y \otimes \sigma_x, & \gamma_8 &= \mathrm{i}\sigma_y \otimes \sigma_z \otimes \sigma_y \otimes \sigma_z,
\end{aligned}$$

and

$$C = \sigma_x \otimes \sigma_z \otimes \sigma_y \otimes \sigma_z.$$

Note that the algebra $\mathsf{Cl}^0_{7,1} \to \mathsf{Cl}_{7,1}$ is of the same type as the algebra $\mathsf{Cl}^0_{1,3} \to \mathsf{Cl}_{1,3}$.

[2] I thank Engelbert Schucking for a discussion on this aspect of charge conjugation. See also Appendix I in [22].

6 CR structures associated with integrable optical geometries

In this section, intended to 'explain' the appearance of Cauchy-Riemann structures in the process of solving Einstein's equations for special Lorentz metrics, I restrict myself to four-dimensional Riemannian manifolds.

Consider first a Lorentz manifold, i.e. a Riemannian manifold M with a metric tensor field g of signature $(3, 1)$. Assume that M is space and time oriented and that there is given on M a bundle $K \subset TM$ of null lines; the flow generated by $k \in \Gamma(K)$ has null curves (rays) as trajectories. Since the fibres of K are null, the bundle

$$K^{\perp} = \{u \in TM : g(k, u) = 0 \text{ for every } k \in \Gamma(K)\}$$

contains K and there is the exact sequence of homomorphisms of vector bundles,

$$0 \to K \to K^{\perp} \to K^{\perp}/K \to 0.$$

The fibres of $K^{\perp}/K$ are 2-dimensional and have a positive-definite scalar product induced by g: they are the *screen spaces* of the 'optical' geometry of rays [3, 10, 13, 18]. Space and time orientation of M, together with the conformal structure of the screen spaces, induce a complex structure J in the fibres of $K^{\perp}/K$. There is a natural extension $J_{\mathbb{C}}$ of J to the complexified bundle $\mathbb{C} \otimes (K^{\perp}/K)$; the latter bundle can be identified with $(\mathbb{C} \otimes K^{\perp})/(\mathbb{C} \otimes K)$. For every $n \in \mathbb{C} \otimes K^{\perp}$, let $n + K \in (\mathbb{C} \otimes K^{\perp})/(\mathbb{C} \otimes K)$ denote the coset space containing n. The vector bundle

$$N = \{n \in \mathbb{C} \otimes K^{\perp} : J_{\mathbb{C}}(n + K) = \mathrm{i}n + K\} \tag{19}$$

is a subbundle of $\mathbb{C} \otimes TM$; its fibres are complex, totally null planes, $N^{\perp} = N$, and

$$N \cap \bar{N} = \mathbb{C} \otimes K, \quad N + \bar{N} = \mathbb{C} \otimes K^{\perp}. \tag{20}$$

A totally null, complex plane bundle N can be also considered in other possible signatures (namely, $(4, 0)$ and $(2, 2)$) of g on a 4-manifold.

If g is a *proper Riemannian* metric tensor, then

$$N \cap \bar{N} = 0 \quad \text{so that} \quad \mathbb{C} \otimes TM = N \oplus \bar{N} \tag{21}$$

and one can define an orthogonal *almost complex* structure J on M by putting

$$J(n + \bar{n}) = \mathrm{i}(n - \bar{n}) \quad \text{for every} \quad n \in N. \tag{22}$$

If g is *neutral* (i.e. of signature $(2,2)$), then there are two possibilities: either (21) holds and there is an orthogonal, almost complex structure on M or

$$N = \bar{N} \quad \text{so that} \quad N = \mathbb{C} \otimes K, \tag{23}$$

where $K = K^\perp$ is now a real, totally null, *plane* subbundle of TM.

In every one of the above cases, the complex, totally null, plane bundle N can be characterized, at least locally, by (the direction of) a complex, decomposable 2-form F such that

$$n \in N \quad \text{iff} \quad n \in \mathbb{C} \otimes TM \quad \text{and} \quad n \lrcorner F = 0. \tag{24}$$

If n_1 and $n_2 \in \Gamma(N)$ are linearly independent, then one can take $F = g(n_1) \wedge g(n_2)$. If F corresponds, in the sense of (24), to N, then $\star F$ corresponds to $N^\perp$; since $N^\perp = N$, the forms F and $\star F$ are parallel. Using the notation of (6) one has $\star F = \pm \iota F$. In signature $(3,1)$ one has $F \wedge \bar{F} = 0$ since F and $\bar{F}$ have $g(k)$ as a common factor; in the other two signatures, if (21) holds and $F \neq 0$, then $F \wedge \bar{F} \neq 0$.

There is also a convenient, spinorial description of the bundles N. Assume, for simplicity, that there is a spin structure Q on M; spinor and tensor fields can be then represented by equivariant maps from Q to suitable representation spaces; for example, a spinor field is given by a map $\varphi : Q \to S$ such that $\varphi(qa) = \gamma(a^{-1})\varphi(q)$ for $q \in Q$ and a in the spin structure group of the bundle $Q \to M$. Given a totally null plane bundle N on a Riemannian 4-manifold, there is a (locally defined) Weyl spinor field φ on M such that

$$N = \{n \in \mathbb{C} \otimes TM : \gamma(n)\varphi = 0\}. \tag{25}$$

The chiralities of φ and F coincide: if $\gamma_5 \varphi = \iota \varphi$, then for the corresponding 2-form F one has $\star F = \iota F$. The isomorphisms κ and B of Section 4, together with the representation γ, induce, in dimension 4, an isomorphism of $S_+ \otimes_{\mathrm{sym}} S_+$ onto the complex space $\bigwedge^2_+ \mathbb{C}^4$ of 2-forms F which are self-dual in the sense that $\star F = \iota F$; there is a similar isomorphism for spinors and 2-forms of the opposite chirality; these isomorphisms establish a correspondence between the descriptions of N by means of 2-forms and spinors [13]. In the Lorentzian case, the product $\varphi \otimes \varphi_c$ corresponds to $k \in \Gamma(K)$; in the proper Riemannian and neutral cases, if $\langle B\varphi_c, \varphi\rangle \neq 0$, then $\varphi \otimes \varphi_c / \langle B\varphi_c, \varphi\rangle$ corresponds to J; in the neutral case, if $\langle B\varphi_c, \varphi\rangle = 0$, then φ is (proportional to) a Weyl-Majorana spinor.

In the Lorentzian case, the real part of the 2-form F can be interpreted as representing a 'null' electromagnetic field $(\mathbf{E}, \mathbf{B})$, i.e. a field such that $(\mathbf{E}+\mathrm{i}\mathbf{B})^2 = 0$. In the 1950s, Ivor Robinson considered solutions of Maxwell's equations

$$\mathrm{d}F = 0 \tag{26}$$

for such a null field and has shown that the trajectories of the flow generated by $k \in \Gamma(K)$ constitute a congruence of null geodesics without shear [15]. He conjectured also that, given any such smooth congruence on a Lorentzian manifold, one can find a non-zero solution F of (26) such that $\star F = \mathrm{i}F$ and $k \lrcorner F = 0$. In 1985, Jacek Tafel [21] pointed out that this need not be true, because, to find such a solution, one has to solve a linear, partial differential equation of the first order, $\Lambda f = a$, of the type considered by Hans Lewy [7] and shown by that author not to have solutions, even locally, for some smooth, but non-analytic, functions a; see also [9]. Soon afterwards, it became clear [17] that the structure underlying shearfree congruences of null geodesics on Lorentzian manifolds is that of Cauchy-Riemann manifolds, earlier introduced into physics by Penrose, in his theory of twistors associated with Minkowski space, and its generalization to curved manifolds [11]-[13].

Proposition 3. *Let M be a Riemannian 4-manifold with a metric tensor g that is either proper Riemannian or Lorentzian or neutral. Let $N \to M$ be a totally null, complex, plane subbundle of $\mathbb{C} \otimes TM$ and let F be a 2-form such that* (24) *holds. Then*

(i) *equation* (26) *implies the* complex integrability condition:

$$[\Gamma(N), \Gamma(N)] \subset \Gamma(N); \tag{27}$$

(ii) *if $N \cap \bar{N} = 0$, then* (27) *is equivalent to the integrability of the almost complex structure J defined by* (22)*; if g is proper Riemannian (resp., neutral), then* (2) *defines a proper Hermitian (resp., Hermitian of signature (1,1)) tensor field h on M;*

(iii) *if g is Lorentzian, then* (27) *is equivalent to the statement that the trajectories of the flow generated by every $k \in \Gamma(K)$, K as in* (20)*, constitute a congruence of* null geodesics without shear*; moreover, if the congruence is regular in the sense that the quotient set $M' = M/K$ is a 3-manifold and* *the map* $\pi : M \to M'$ *is a submersion, then* N *projects to a complex line bundle* $H \to M'$, $H \subset \mathbb{C} \otimes TM'$, *defining a* CR-structure *on* M'; *the form* F *satisfying* (24) *and* (26) *descends to a complex 2-form* F' *on* M' *such that*

$$\text{(28)} \qquad \mathrm{d}F' = 0, \quad Z \lrcorner F' = 0 \ \textit{for every} \ Z \in H \quad \textit{and} \quad F = \pi^* F';$$

(iv) *if* g *is neutral and* (23) *holds, then* (27) *reduces to the* real integrability condition,

$$[\Gamma(K), \Gamma(K)] \subset \Gamma(K);$$

the leaves of the foliation defined by K *are 2-dimensional, totally null and totally geodesic submanifolds of* M.

The proof of Prop. 3 is straightforward; most of it can can be found in [4, 10, 13, 18, 24, 25]. There are interesting results and problems connected with the analogy between a shearfree congruence of null geodesics on a Lorentz manifold and the Hermitian geometry in the proper Riemannian case; one of them consists in the proof of the Goldberg-Sachs theorem in signatures $(4,0)$ and $(2,2)$ [14].

It is worth noting that, in the Lorentzian case, the complex structure in the fibres of $K^\perp/K$ is determined, in a natural manner, by giving only a space and time orientation of M and the bundle of null lines K; in the proper Riemannian case, the (almost) complex structure has to be introduced explicitly, by giving either J or N. It is for this reason that the appearance of the Cauchy-Riemann equations in [16] had been somewhat unexpected.

Recall that the (abstract) Cauchy-Riemann structure on a 3-manifold M', given by the complex line bundle $H \to M'$, $H \subset \mathbb{C} \otimes TM'$, $H \cap \bar{H} = 0$, can be conveniently locally described also as follows: let Z be a non-zero section of H and let λ be a non-zero, real 1-form on M such that $Z \lrcorner \lambda = 0$. One can find a complex 1-form μ such that $\lambda \wedge \mu \wedge \bar{\mu} \neq 0$, $Z \lrcorner \mu = 0$ and $Z \lrcorner \bar{\mu} \neq 0$. These forms are defined up to transformations

$$\text{(29)} \qquad \lambda \mapsto a\lambda, \quad \mu \mapsto b\mu + c\lambda,$$

where a is a real function and b, c are complex functions on M' such that $a, b \neq 0$. The direction of the 2-form $\lambda \wedge \mu$ is invariant with respect to the changes (29) and characterizes the CR structure. In the terminology of [5], such a form is a section of the *canonical bundle* of the CR 3-manifold, defined as

$$\{\omega \in \mathbb{C} \otimes {\textstyle\bigwedge}^2 T^* M' : Z \lrcorner \omega = 0 \ \text{ for every } \ Z \in H\}.$$

To alleviate the language, I shall use, from now on, the expression 'a CR space' instead of 'a three-dimensional manifold with a CR structure'.

Consider the fibration $\pi : M \to M'$. Let P and ξ be a real function and a real one-form on M, respectively, such that $P^2\pi^*(\mu \wedge \bar{\mu} \wedge \lambda) \wedge \xi$ vanishes nowhere on M. The symmetric tensor field on M,

$$g = P^2\pi^*(\mu \otimes_{\text{sym}} \bar{\mu}) + \pi^*\lambda \otimes_{\text{sym}} \xi, \tag{30}$$

is the most general Lorentz metric admitting the fibres of π as null geodesics constituting a congruence without shear [17]. One then has $\pi^*\lambda \wedge g(k) = 0$ for $k \in \Gamma(K)$ and $K^\perp = \ker \pi^*\lambda$.

Let $f : M' \to \mathbb{C}$ be a smooth function. If the *Cauchy-Riemann equation*

$$Z \lrcorner \, \mathrm{d}f = 0 \tag{31}$$

has two independent (local) solutions z and w, then the 2-form $\mathrm{d}z \wedge \mathrm{d}w$ is a non-zero section of the canonical bundle; using the freedom implied by (29), one can choose μ to coincide with $\mathrm{d}z$. The map

$$(z, w) : M' \to \mathbb{C}^2 \tag{32}$$

is a (local) embedding of M' in $\mathbb{C}^2$ and the CR structure is then said to be (locally) *embeddable*. Lewandowski, Nurowski and Tafel have shown that the CR space defined by a shearfree congruence of null geodesics on an Einstein-Lorentz manifold is so embeddable [6]. One can then introduce local coordinates (u, x, y) on M' such that $x + \mathrm{i}y = z$ and represent the form λ and the vector field Z as

$$\lambda = \mathrm{d}u + \bar{L}\,\mathrm{d}z + L\,\mathrm{d}\bar{z}, \quad Z = \frac{\partial}{\partial \bar{z}} - L\frac{\partial}{\partial u}.$$

If $L = 0$, then the CR structure is trivial in the sense that M' is foliated by a family of complex 1-manifolds of equation $u =$const.; the corresponding bundle $K^\perp$ is integrable, $\lambda \wedge \mathrm{d}\lambda = 0$, and (31) reduces to the classical Cauchy-Riemann equation: this is the special case of a *'hypersurface orthogonal'* congruence of shearfree null geodesics considered in [16].

According to part (iii) of Prop. 3 the general problem of finding a solution of Maxwell's equations (26) adapted to a shearfree congruence of null geodesics defined by K, i.e. such that $\star F = \mathrm{i}F$ and $k \lrcorner F = 0$, reduces to the following: given a CR space M', find a closed section F' of its canonical bundle. If M' is embeddable, then such sections exist and are of the form

$$F' = f(z, w)\,\mathrm{d}z \wedge \mathrm{d}w$$

where z and w are as in (32) and f is an analytic function of its arguments.

It is now known that there are CR spaces that are non-embeddable, but have one solution of (31) [19]; by the results of [21], extended to higher dimensions in [5], such CR spaces do not admit closed, non-zero sections of their canonical bundle.[3] Therefore, Lorentzian manifolds constructed on the basis of these CR spaces as in (30) do not admit any non-zero solutions F of Maxwell equations such that $k \lrcorner F = 0$, $g(k) \wedge F = 0$, where $g(k) = \lambda$. There are examples of non-embeddable 7-dimensional CR manifolds that have non-zero, closed, sections of their canonical bundle, but it is not clear whether there are such examples in dimensions 3 and 5. In connection with this, I formulate the following

Conjecture. *A CR 3-space admits locally a closed, non-zero section of its canonical bundle if, and only if, it is locally embeddable.*

The conjecture can be formulated as a problem of elementary vector calculus: given a complex vector field $\mathbf{F}$ on $\mathbb{R}^3$ such that $\mathbf{F} \times \overline{\mathbf{F}} \neq 0$ and $\operatorname{div} \mathbf{F} = 0$, show that there exist two complex functions z and w such that $\mathbf{F} = \operatorname{grad} z \times \operatorname{grad} w$.

Since it is known that real analytic CR spaces are locally embeddable, the proof of the conjecture—if it is true—should concern the smooth case.

Acknowledgments

Work on this article was supported in part by the Polish Committee on Scientific Research (KBN) under grant no. 2 P03B 017 12 and by the Foundation for Polish-German cooperation with funds provided by the Federal Republic of Germany. The paper has been completed in June 1997, during the Workshop on *Spaces of geodesics and complex structures in general relativity* at the Erwin Schrödinger International Institute for Mathematical Physics in Vienna. I have benefited there from discussions with C. Denson Hill, Paweł Nurowski, Roger Penrose and Helmuth Urbantke.

[3]The significance, in this context, of the examples found by Rosay and of the results of Jacobowitz has been explained at the Workshop in Vienna by C. Denson Hill.

References

[1] Budinich, P. and Trautman, A., *The spinorial chessboard*, Springer-Verlag, Berlin 1988.

[2] Ebbinghaus, H.-D., Hermes, H., Hirzebruch, F., Koecher, M., Mainzer, K., Prestel, A. and Remmert, R., *Zahlen*, Springer-Verlag, Berlin 1983.

[3] Ehlers, J. and Kundt, W., Exact solutions of the gravitational field equations, in: *Gravitation*, edited by L. Witten, Wiley, New York 1962.

[4] Hughston, L. P. and Mason, L. J., A generalised Kerr–Robinson theorem, *Class. Quantum Grav.* **5** (1988) 275–285.

[5] Jacobowitz, H., The canonical bundle and realizable CR hypersurfaces, *Pacific J. Math.* **127** (1987) 91–101.

[6] Lewandowski, J., Nurowski, P., and Tafel, J., Einstein's equations and realizability of CR manifolds, *Class. Quantum Grav.* **7** (1990) L241 - L246.

[7] Lewy, H., An example of a smooth partial differential equation without solution, *Ann. of Math.* **66** (1957) 155–158.

[8] Lounesto, P., Counter-examples in Clifford algebras, *Adv. Appl. Clifford Algebras* **6** (1996) 69–104.

[9] Nirenberg, L., On a question of Hans Lewy, *Russian Math. Surveys* **29** (1974) 251–262.

[10] Nurowski, P., Optical geometries and related structures, *J. Geom. Phys.* **18** (1996) 335–348.

[11] Penrose, R., The complex geometry of the natural world, in: *Proc. Int. Congress Math.*, pp. 189–194, Helsinki 1978.

[12] Penrose, R., Physical space-time and nonrealizable CR-structures, *Proc. Symp. Pure Math.* **39**, Part I (1983) 401–422.

[13] Penrose, R. and Rindler, W., *Spinors and space-time*, vols 1 and 2, Cambridge University Press, Cambridge 1984 and 1986.

[14] Przanowski, M. and Broda, B., Locally Kähler gravitational instantons, *Acta Phys. Polon.* **B14** (1983) 637–661.

[15] Robinson, I., Null electromagnetic fields, *J. Math. Phys.* **2** (1961) 290–291.

[16] Robinson, I. and Trautman, A., Spherical gravitational waves, *Phys. Rev. Lett.* **4** (1960) 431–432.

[17] Robinson, I. and Trautman, A., Cauchy–Riemann structures in optical geometry in: *Proc. of the Fourth Marcel Grossmann Meeting on General Relativity*, pp. 317–324, edited by R. Ruffini, Elsevier, 1986.

[18] Robinson, I. and Trautman, A., Optical geometry, in: *New Theories in Physics*, Procs. of the XI Warsaw Symposium on Elementary Particle Physics, edited by Z. Ajduk *et al.*, World Scientific, Singapore 1989.

[19] Rosay, J.-P., New examples of non-locally embeddable CR structures, *Ann. Inst. Fourier Grenoble* **39** (1989) 811–823.

[20] Sommerfeld, A., Mathematische Theorie der Diffraction, *Math. Annalen* **47** (1896) 317–374.

[21] Tafel, J., On the Robinson theorem and shearfree geodesic null congruences, *Lett. Math. Phys.* **10** (1985) 33–39.

[22] Thirring, W., *Principles of quantum electrodynamics*, Academic Press, New York 1958.

[23] Trautman, A., Optical structures in relativistic theories, in: *Élie Cartan et les mathémathiques d'aujourd'hui*, *Astérisque*, numéro hors série, (1985) 401–420.

[24] Trautman, A., Geometric aspects of spinors, in: *Clifford algebras and their applications in mathematical physics*, edited by R. Delanghe, F. Brackx and H. Serras, Kluwer Academic Publishers, Dordrecht 1993.

[25] Trautman, A., Clifford and the 'square root' ideas, *Contemporary Mathematics*, **203** (1997) 3–24.