A Simple Proof of the Robinson Theorem

by

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1. — Notation

Standard terminology and notation of algebra and differential geometry is used in this paper. The exterior algebra of an \( n \)-dimensional real vector space \( V \) is denoted by

\[
\Lambda^* V = \bigoplus_{k=0}^{n} \Lambda^k V^*,
\]

where \( \Lambda^0 V^* = \mathbb{R} \) and \( \Lambda^1 V^* = V^* \) is the dual of \( V \). If \( u \in V \), then

\[
i(u) : \Lambda^* V \to \Lambda^* V
\]

is the (anti) derivation of degree \(-1\) defined by

\[
i(u)\alpha = \langle u, \alpha \rangle \quad \text{for any } \alpha \in V^*
\]

and

\[
i(u)(\beta \land \gamma) = (i(u)\beta) \land \gamma + (-1)^k \beta \land i(u)\gamma
\]

for any \( \beta \in \Lambda^k V^* \). Sometimes one writes \( u \_\alpha \) instead of \( i(u)\alpha \). If

\[
A : V \to V
\]

is a linear map, then

\[
\overline{A} : \Lambda^* V \to \Lambda^* V
\]

denotes the derivation of degree 0 defined by

\[
\langle u, \overline{A}\alpha \rangle = \langle Au, \alpha \rangle \quad \text{for any } u \in V \quad \text{and } \alpha \in V^*
\]

and

\[
\overline{A}(\beta \land \gamma) = (\overline{A}\beta) \land \gamma + \beta \land \overline{A}\gamma, \quad \beta, \gamma \in \Lambda^* V.
\]
It is easy to check that

\[(1) \quad [i(u), A] = i(Au)\]

for any \(u \in V\) and \(A \in \mathcal{L}(V)\). If \(u \in V\) and \(\alpha \in V^*\) then the map \(A\) defined by \(Av = \alpha(v)u\) is written as \(A = u \otimes \alpha\).

A scalar product in \(V\) is defined as a bilinear symmetric map \(g : V \times V \to \mathbb{R}\) which is non-degenerate, but the quadratic form \(u \to g(u, u)\) needs not be positive-definite. The same letter \(g\) will be used to denote the isomorphism of \(V\) onto \(V^*\) defined by

\[\langle v, g(u) \rangle = g(u, v), \quad u, v \in V.\]

A linear map \(A : V \to V\) is symmetric with respect to \(g\) if, for any \(u, v \in V\),

\[g(Au, v) = g(u, Av).\]

If \(A\) is symmetric, then \(g(Au) = \overline{A}g(u)\).

Let \((e_\mu)\), \(\mu = 1, \ldots, n\), be a linear frame (basis) in \(V\) and let \((e^\nu)\) denote its dual:

\[\langle e_\mu, e^\nu \rangle = \delta_\mu^\nu.\]

The \(n\)-form

\[(2) \quad e = e^1 \wedge e^2 \wedge \ldots \wedge e^n\]

spans \(\Lambda^n V^*\) and

\[(3) \quad \overline{A}e = eTrA.\]

Assume now that \(V\) has a preferred orientation and consider a frame which agrees with the orientation and is unimodular, i.e.:

\[|\det(g_{\mu\nu})| = 1,\]

where

\[g_{\mu\nu} = g(e_\mu, e_\nu).\]

The \(n\)-form (2) is now called an (oriented) volume element. The Hodge dual is an isomorphism of the vector space \(\Lambda V^*\) on itself,

\[\ast : \Lambda V^* \to \Lambda V^*,\]

defined as follows. Let \(\alpha \in \Lambda^k V^*\) and \(u_{k+1}, \ldots, u_n \in V\), then \(\ast \alpha \in \Lambda^{n-k} V^*\) is given by

\[(4) \quad \ast \alpha(u_{k+1}, \ldots, u_n) e = \alpha \wedge g(u_{k+1}) \wedge \ldots \wedge g(u_n).\]
One has

\[ i(u) \ast \alpha = \ast (\alpha \wedge g(u)) \]

and, if \( A \in \mathcal{L}(V) \) is symmetric,

\[ \overline{A} \ast \ast \overline{A} = (TrA)^\ast. \]

Let \( M \) be an \( n \)-dimensional smooth oriented manifold with a metric tensor \( g \). The algebraic notions and constructions described above are extended, in a natural manner, to smooth fields on \( M \). For example, if \( \Gamma(M) = \bigoplus \Gamma^k(M) \) is the Cartan algebra of differential forms on \( M \) and \( u \) is a vector field, then \( i(u) : \Gamma(M) \to \Gamma(M) \) is a derivation of degree \(-1\). The exterior derivative

\[ d : \Gamma(M) \to \Gamma(M) \]

is a derivation of degree \(+1\). If \( u \) and \( v \) are vector fields, then

\[ \mathcal{L}(u) = d \circ i(u) + i(u) \circ d \]

is a derivation of degree \(0\) (the Lie derivative with respect to \( u \)); we have:

\[ [\mathcal{L}(u), d] = 0 \]

and

\[ [\mathcal{L}(u), i(v)] = i([u, v]), \]

where \([u, v]\) is the usual bracket of vector fields,

\[ \mathcal{L}([u, v]) = [\mathcal{L}(u), \mathcal{L}(v)]. \]

If \( A : TM \to TM \) is an endomorphism of the tangent bundle \( TM \), then \( \overline{A} \) denotes the corresponding derivation of the Cartan algebra; there are obvious extensions of formulae (1) - (6) to fields on \( M \). If \( u \) and \( v \) are vector fields and \( g(u) \) denotes the 1-form corresponding to \( u \) under the isomorphism \( g : TM \to T^*M \), then the map

\[ v \mapsto \mathcal{L}(u)(g(v)) - g([u, v]) \]

defines a tensor field (the Lie derivative of \( g \) with respect to \( u \)),

\[ \mathcal{L}_u g : TM \to T^*M, \]

given by:

\[ (\mathcal{L}_u g)(v) = \mathcal{L}(u)(g(v)) - g([u, v]). \]

This tensor field is symmetric, \( \langle w, (\mathcal{L}_u g)(v) \rangle = \langle v, (\mathcal{L}_u g)(w) \rangle \), and it vanishes if and only if \( u \) generates a group of isometries of \( g \). The composed map
(12) \[ A_u = g^{-1} \circ \mathcal{L}_u g : TM \to TM \]
occurs in the following lemma.

**Lemma 1.** Let \[ \ast : \Gamma(M) \to \Gamma(M) \]
be the Hodge dual acting on differential forms. If \( u \) is a vector field on \( M \), then

\[ [\mathcal{L}(u), \ast] = (\mathcal{A}_u - (1/2) \text{Tr} A_u \text{id}) \ast. \]

Moreover, according to (1), there holds

\[ [i(u), \mathcal{A}_u] = i(A_u u), \]

where

\[ g(A_u u) = (\mathcal{L}_u g)(u) \]

and

\[ \text{Tr} A_u = 2 \text{div} u. \]

It is also clear that \([\mathcal{L}(u), \ast]\) anticommutes with \( \ast \) and

\[ \text{id} \mid \Gamma^k(M) = k \text{id} \mid \Gamma^k(M). \]

2. — Spacetime and the Maxwell Equations

A spacetime is a (space and time) orientable four-dimensional manifold \( M \) with a metric tensor \( g \) of signature \(-2\). The Hodge dual acting on 2-forms is invariant under conformal changes of \( g \). It depends only on the conformal geometry of \( M \). As a result of this, Maxwell's equations in empty space,

\[ (15) \quad dF = 0, \quad d\ast F = 0 \]

where \( F \in \Gamma^2(M) \), are conformally invariant.

Let \( k \) be a complete, nowhere vanishing vector field on \( M \). There then exists a smooth map

\[ \varphi : \mathbb{R} \times M \to M, \quad \varphi(t, p) = \varphi_t(p), \]

such that

\[ \varphi_t \circ \varphi_s = \varphi_{t+s}, \quad \varphi_0 = \text{id}_M \]

and
\[
\frac{d}{dt} (f \circ \varphi_t) = \mathcal{L}(k) (f \circ \varphi_t)
\]
for any smooth function \( f \). One says that \((\varphi_t)\) is the flow generated by \( k \). Assume that there exists a hypersurface \( S \subset M \) transversal to \( k \) and that the restriction \( \psi \) of \( \varphi \) to \( \mathbb{R} \times S \) is a diffeomorphism of \( \mathbb{R} \times S \) onto \( M \). A system of (local) coordinates \((x', y', z')\) on \( S \) can be used to define coordinates \((t, x, y, z)\) in (a suitable region of) \( M \) by putting (cf. fig. 1)

\[
t = pr_1 \circ \psi^{-1}
\]

and

\[
x = x' \circ pr_2 \circ \psi^{-1}, \text{ etc.}
\]

![Fig. 1](image)

It follows from the definition that \( k = \partial / \partial t \) and

\[
\langle k, dt \rangle = 1, \quad \langle k, dx \rangle = 0, \quad \text{etc.}
\]

\[
p = \varphi_t(p_0), \quad x(p) = x'(p_0), \quad \text{etc.}
\]

The following lemmas are straightforward.

**LEMMA 2.** Let \( \alpha \in \Gamma(M) \) and \( B \) be an endomorphism of the tangent bundle of \( M \). If

\[
\alpha | S = 0 \quad \text{and} \quad \mathcal{L}(k)\alpha = B\alpha
\]

then

\[
\alpha = 0.
\]
LEMMA 3. If $i(k)\alpha = 0$ and $dt \wedge \alpha = 0$, then $\alpha = 0$.

LEMMA 4. If

$L(k)\alpha = 0$, \hspace{1em} i(k)\alpha \big| S = 0 \hspace{1em} and \hspace{1em} (dt \wedge d\alpha) \big| S = 0$

then

$i(k)\alpha = 0 \hspace{1em} and \hspace{1em} d\alpha = 0$.

Indeed,

$L(k)(i(k)\alpha) = i(k) \hspace{1em} L(k)\alpha = 0$

implies $i(k)\alpha = 0$ (Lemma 2). Moreover,

$i(k)d\alpha = L(k)\alpha - d(i(k)\alpha) = 0,$

$L(k)(dt \wedge d\alpha) = dL(k)t \wedge d\alpha + dt \wedge dL(k)\alpha = 0$;

therefore, again by Lemma 2, $dt \wedge d\alpha = 0$. Also, $i(k)d\alpha = L(k)\alpha - d(i(k)\alpha) = 0$

so that $d\alpha = 0$ holds by Lemma 3. \hspace{1em} (Q.E.D.)

3. Null Elements and the Robinson Theorem

A vector field $k$ on $M$ is null if $g(k, k) = 0$. It is, moreover, geodesic if

(16) \hspace{1em} (L_k g)(k) \wedge g(k) = 0.

In this case the lines of the flow generated by $k$ are null geodesics.

The form $\alpha \in \Gamma(M)$ is null if there exists a nowhere vanishing vector field $k$ such that

$i(k)\alpha = 0 \hspace{1em} and \hspace{1em} i(k) \star \alpha = 0$.

If $\alpha \neq 0$ then the vector field $k$ is necessarily null (use (5) to prove this).

THEOREM 1. Let $k$ be null and geodesic. If

$L(k)\alpha = 0 \hspace{1em} and \hspace{1em} i(k) \star \alpha \big| S = 0$

then $i(k) \star \alpha = 0$.

Proof. Since

$i(k) \star \alpha = \star (\alpha \wedge g(k))$,

the theorem is equivalent to the following: if $k$ is null geodesic, $L(k)\alpha = 0$ and
$\alpha \land g(k) \mid S = 0$ then $\alpha \land g(k) = 0$. Now,

$$\mathcal{L}(k)(\alpha \land g(k)) = (\mathcal{L}(k)\alpha) \land g(k) + \alpha \land (\mathcal{L}_k g)(k) = \alpha \land (\mathcal{L}_k g)(k).$$

Since $(\mathcal{L}_k g)(k)$ is parallel to $g(k)$, the right-hand side of the last equation is proportional to $\alpha \land g(k)$ and Theorem 1 follows from Lemma 2. (Q.E.D.)

**LEMA 5.** Let $F \in \Gamma^2(M)$ be non-zero and null, $i(k)F = 0 = i(k) \ast F$, $k \neq 0$ and let $B$ be a traceless, symmetric endomorphism of $TM$. Condition $BF = 0$ is equivalent to the existence of a vector field $u$ such that

$$B = u \otimes g(k) + k \otimes g(u) - \frac{1}{2} g(u, k) \text{id}. \quad (17)$$

A proof of the lemma is obtained by constructing a frame $(e_\mu)$ such that $g = g_{\mu\nu} e^\mu \otimes e^\nu = e^3 \otimes e^4 + e^4 \otimes e^3 - e^1 \otimes e^1 - e^2 \otimes e^2$ and $F = f e^1 \land e^3$, $\ast F = f e^2 \land e^3$. One writes $B = B^\mu_\nu e^\mu \otimes e^\nu$ where $B_{\mu\nu} = B_{\nu\mu} = g_{\mu\rho} B^\rho_\nu$ and $TrB = B^{\rho}_{\rho} = 0$. It follows from (6) that $B$ anticommutes with $\ast$ so that $B \ast F = 0$. The rest is a computation. (Q.E.D.)

**THEOREM 2.** If $F \in \Gamma^2(M)$ is non-zero and null, $i(k)F = 0$,

$$0 = i(k) \ast F, \quad k \neq 0,$$

and

$$\mathcal{L}(k)F = 0, \quad \mathcal{L}(k) \ast F = 0,$$

then there exists a vector field $u$ such that

$$A_k = \frac{1}{4} (TrA_k) \text{id} = u \otimes g(k) + k \otimes g(u) - \frac{1}{2} g(u, k) \text{id}. \quad (18)$$

**Proof.** It follows from the assumptions of the theorem that $[\mathcal{L}(k), \ast]F = 0$. Lemmas 1 and 5 complete the proof. (Q.E.D.)

**REMARK.** Condition (18) is equivalent to the following: there exist a function $a$ and a vector field $u$ such that

$$\mathcal{L}_k g = 2ag + g(u) \otimes g(k) + g(k) \otimes g(u). \quad (19)$$

Clearly, if (19) is satisfied, then $(\mathcal{L}_k g)(k) \land g(k) = 0$ so that $k$ is geodesic. It has been shown elsewhere [5] that the flow generated by $k$ subject to (19) preserves the distribution of subspaces orthogonal to $k$, together with their (degenerate)
conformal structure induced by $g$. For this reason, the flow, and $k$ itself, is said to be null, geodesic and shearfree ([2]; cf. also [1], [3], [6]).

**THEOREM 3.** Consider a null geodesic and shearfree, non-zero vector field $k$ and a hypersurface $S$ transversal to $k$ and such that the flow generated by $k$ determines a diffeomorphism of $\mathbb{R} \times S$ onto $M$. If $F \in \Gamma^2(M)$ satisfies the following initial conditions

\begin{align}
(20) & \quad i(k)F|_S = 0, \quad i(k) \star F|_S = 0, \\
(21) & \quad dt \wedge dF|_S = 0, \quad dt \wedge d \star F|_S = 0,
\end{align}

and is invariant by the flow,

$$\mathcal{L}(k)F = 0,$$

then $F$ is a null solution of Maxwell's equations,

$$dF = 0 \quad \text{and} \quad d \star F = 0.$$

**Proof.** It follows from Lemma 4 that $i(k)F = 0$ and $dF = 0$. Theorem 1 yields $i(k) \star F = 0$ so that $F$ is null. Since $\mathcal{L}(k) \star F = [\mathcal{L}(k), \star]F = 0$ by Lemmas 1 and 5, Lemma 4 can be applied to $\alpha = \star F$ to get $d \star F = 0$. (Q.E.D.)

**REMARK.** The initial data (21) contain derivatives of $F$ only in directions tangential to $S$. There always are non-zero initial data satisfying (20 - 21). This can be seen from the following argument [4]: let $v$ be a unit vector field on $S$, tangent to $S$ and orthogonal to $k$. Then $\star (g(k) \wedge g(v)) = g(k) \wedge g(w)$ where $w$ has unit length and is orthogonal to both $k$ and $v$; it may be chosen to be tangent to $S$. Put $F|_S = g(k) \wedge (ag(v) + bg(w))$ where $a$ and $b$ are functions on $S$. Conditions (20 - 21) reduce to two first order linear differential equations for $a$ and $b$ which may be solved.

**COROLLARY (The Robinson Theorem).** With any null, geodesic and shearfree vector field $k$ there is associated a non-trivial null solution of Maxwell's equations.

**References**


Abstract. It is shown that if a 2-form $F$ in a 4-dimensional conformal spacetime is invariant by the action of the flow generated by a null, geodesic and shearfree vector field $k$ and satisfies the initial conditions: $k \perp F = 0 = k \perp * F$ and $dt \wedge dF = 0 = dt \wedge d * F$ on a hypersurface $t = \text{const.}$ transversal to $k$, then $F$ is a null Maxwell field. The proof depends on a useful formula for the commutator of the Lie derivative with the Hodge $*$ operator.

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