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YANG-MILLS THEORY AND GRAVITATION: A COMPARISON*

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Introduction

The purpose of theoretical physics is to construct mathematical models of physical phenomena and, on the basis of such models, to explain what is observed, suggest new experiments and predict their outcome. This ideal activity is supplemented and supported by research, done by mathematicians and physicists, on the properties of the theoretical constructions themselves. One considers questions such as: Are the equations of a theory consistent? Are their solutions stable? Can the Cauchy problem be correctly formulated? In what space of functions? Answers to such questions have no direct bearing on the predictive power of a theory, but they may throw light on the range of its applicability or need for modifications. Successful physical theories are often studied in order to construct, by analogy, models of phenomena outside their scope. For example, in the 19th century, there was a trend to reduce all of physics to classical mechanics, whereas now quantum electrodynamics is the theory relative to which all others are evaluated.

The quantum-mechanical description of charged particles led to an important change in the original interpretation, due to Weyl [1], of gauge transformations as maps inducing conformal changes of the metric tensor in space-time. The idea that the electromagnetic field is a 'compensating' or 'gauge' field [7] associated with the circle group U(1) was generalized, by Yang and Mills [10], by the introduction of a gauge field corresponding to the 'isotopic' group SU(2). Soon after, it became clear that essentially any Lie group can be so 'gauged' and that Einstein's theory of gravitation fits - though not quite - into the scheme (cf. the Annotated Bibliography for references and further remarks on the history of the development of the notion of a gauge field).

Present-day physics is dominated by the striking successes of quantum electrodynamics and the current trends in the description of fundamental interactions

* The actual lectures given by the author at the 1981 Scheveningen Conference contained, besides the material reproduced here, an introduction to the geometrical aspects of gauge theories, based on articles published elsewhere [52,61].

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(chromodynamics and the Weinberg-Salam theory). As a result of this, the theory of gravitation is sometimes required to conform to the principles and fashions prevalent in elementary particle physics. In my opinion, one should rather regard Einstein's theory of general relativity in its own right, as a very successful, classical, relativistic theory of gravitation. Its structure is worth studying and comparing to that of theories of the Yang-Mills type, but not necessarily with an intention of formulating all gauge theories according to one pattern. If a unified picture is desired, it may be achieved not so much by replacing Einstein's equations by those arising from a Lagrangian quadratic in the field strengths, as by combining gravitation with Yang-Mills fields in a suitably generalized theory of the Kaluza-Klein type [12,18,22-25,27,44,57].

Superficial Observations

Consider the following three classical, relativistic field theories:

(i) Maxwell's electrodynamics,

(ii) Yang-Mills theory based on SU(2),

(iii) Einstein's theory of gravitation.

They share some fundamental properties: on the mathematical side, each of the theories is based on an infinitesimal connection defined on a suitable principal bundle over space-time; they all exhibit 'large' groups of gauge transformations. From the point of view of physics, the similarities between (i) and (iii) are obvious: the Coulomb law is analogous to Newton's. In fact, electromagnetism and gravitation seem to be the only two long range forces existing in nature. Free Yang-Mills equations also have Coulomb-like solutions, but their physical relevance is probably restricted by the phenomenon of confinement and/or the Higgs-Kibble mechanism of mass generation through a spontaneous breakdown of symmetry.

A superficial analysis indicates analogies between (i) and (ii), as well as between (ii) and (iii), but not so much between (i) and (iii). Since the Maxwell and Yang-Mills Lagrangians are both quadratic in the field strengths, they yield equations of a similar form. On the other hand, Yang-Mills and Einstein equations exhibit non-linearities which, in both cases, may be traced back to the non-Abelian character of the corresponding structure groups. These non-linearities induce a self-interaction of the corresponding particles.

In the rest of the paper, the analogies and differences among the three theories (i) - (iii) will be considered and some unexpected formal similarities between gravitation and electromagnetism stressed.

A Dictionary

Much of the language of theoretical physics is sufficiently imprecise to allow vivid disputes between authors who attribute different meanings to the words they use. This is the way it has to be: the exact significance of the notions used in physics becomes clear only in the final stages of formation of the theories in which they occur. It is now being felt that classical gauge theories have reached the point when their fundamental notions can be given a precise meaning, i.e. translated into clearly defined mathematical terms. Such a dictionary has been initiated by Wu and Yang [26] and I supplement it here with a few entries.

A classical gauge theory is any physical theory which includes among its dynamical variables a connection on a principal G-bundle P over space-time M. The structure group G is a Lie group; physicists often call it the 'gauge group', but this is misleading as the same name is used (more appropriately) for a group of automorphisms of the bundle. In the physicist's language 'to gauge a group G' means 'to consider (sometimes: to construct) a connection on a bundle over space-time with structure group G'. A connection form ω on $\pi: P \to M$ describes a 'gauge configuration' and a local section $s: U \to P$, $U \subset M$, $\pi \circ s = id$, defines a 'gauge'. The pull-back $A = s^*\omega$ is the 'potential of the gauge configuration in the gauge s'. Similarly, if $\Omega = d\omega + \frac{1}{2}[\omega,\omega]$ is the curvature two-form on P, then $F = s^*\Omega$ is the 'field strength in the gauge s'.

Let M be an oriented Riemannian space (conformal geometry suffices if M is four-dimensional) and let $*$ denote the Hodge (duality) isomorphism of the vector structure of the Grassmann algebra over M. This isomorphism lifts to horizontal forms on P. Let $k: \mathcal{g} \times \mathcal{g} \to R$ be a scalar product on the Lie algebra $\mathcal{g}$ of G, invariant under the adjoint action of G in $\mathcal{g}$. If (e_i) is a linear basis in $\mathcal{g}$, $k_{ij} = k(e_i, e_j)$ and $\Omega = \Omega^i e_i$, then

$$k_{ij} *\Omega^i \wedge \Omega^j \tag{1}$$

is a G-invariant, horizontal form of degree n = dim M. The pull-back of (1) with a section $s: M \to P$ does not depend on s; upon integration over M it gives the classical action from which field equations are derived by variation. A gauge theory is said to be of the 'Yang-Mills type' if its action contains a term derived from (1). If G is semi-simple and compact, then k may be taken as a multiple of its Killing-Cartan form: this is the case of the 'Yang-Mills theory'. For example, Maxwell's electrodynamics is a theory of the Yang-Mills type, but not a Yang-Mills theory in the strict sense. Einstein's general relativity is not a theory of the Yang-Mills type (see, however, [28,38,45-47] for different views on this problem).

Some Analogies and Differences

It is convenient to summarize the analogies and differences between gauge theories of the Yang-Mills type and gravitation, some of which have already been mentioned, in a table where the following notation is used:

$\theta = (\theta^\mu)$ is the canonical (soldering) $\mathbb{R}^n$-valued 1-form on the bundle $LM \to M$ of linear frames of an n-dimensional manifold M, μ and other Greek indices run from 1 to n;

$\omega = (\omega^\mu{}_\nu)$ is the 1-form of a linear connection;

$\Gamma = (\Gamma^\mu{}_\nu)$ are its coefficients, obtained by pull-back of ω by a (local) section $s: M \to LM$;

$e^\mu = s^*\theta^\mu$ is the μ-th element of the coframe field on M, dual to the frame field $s = (s_\mu)$, i.e. $\langle s_\mu, e^\nu\rangle = \delta^\nu_\mu$;

D denotes the covariant exterior derivative; if ϕ is a V-valued field of k-forms of type ρ, defined by a homomorphism $\rho: \mathcal{g} \to L(V)$ of Lie algebras, then $D\phi =$ $= d\phi + \rho(A) \wedge \rho$ [49];

$R = (R^\mu{}_\nu)$ is the curvature two-form, referred to the frame s, $R = s^*\Omega$, where $\Omega^\mu{}_\nu = d\omega^\mu{}_\nu + \omega^\mu{}_\rho \wedge \omega^\rho{}_\nu$;

$Q = (Q^\mu)$ is the torsion two-form, referred to the frame s, $Q = s^*\Theta$, where $\Theta^\mu = d\theta^\mu + \omega^\mu{}_\nu \wedge \theta^\nu$;

g is the metric tensor and $g_{\mu\nu} = g(s_\mu, s_\nu)$;

$\eta_{\mu\nu}$ is the Hodge dual of $e_\mu \wedge e_\nu$, where $e_\mu = g_{\mu\nu}e^\nu$;

$\eta_{\mu\nu\rho}$ is the Hodge dual of $e_\mu \wedge e_\nu \wedge e_\rho$;

$T_\mu = T_{\mu\nu}e^\nu$ is the $\mathbb{R}^n$-valued 1-form of energy.momentum of the sources of the gravitational field; similarly, t_μ corresponds to the 'pseudotensor' of energy-momentum of the gravitational field itself;

j is the $\mathcal{g}$-valued 1-form of the current corresponding to the sources of the gauge field;

ϕ is a (generalized) Higgs field, i.e. a V-valued field of k-forms of type ρ.

The most important difference between theories of the Yang-Mills type and gravitation is that the underlying bundle of the latter - the bundle of linear frames - is 'concrete', has more structure than 'abstract' bundles occurring in other gauge theories. The additional structure is completely characterized by the soldering form which, upon differentiation, leads to torsion. In Einstein's theory torsion is assumed to vanish. This condition has no counterpart in theories of the Yang-Mills type.

The role played by the metric tensor in Einstein's theory is somewhat analogous to that of a Higgs field in a Yang-Mills theory. In both cases the additional structure 'breaks down the symmetry' by restricting the principal bundle to a subgroup H of its structure group G. If $\phi: P \to V$ is a V-valued map, equivariant under the action of G in P and in V defined by a representation $\rho: G \to GL(V)$, and such that the values

Table

Yang-Mills		Gravitation
A		Γ
F		R
$DF = 0$	Bianchi identity	$DR = 0$
—	torsion	Q
Higgs field ϕ		metric tensor g
$D\phi = 0$	compatibility	$Dg = 0$
$k_{ij} *F^i \wedge F^j$	field Lagrangian	$\eta^{\nu}{}_{\mu} \wedge R^{\mu}{}_{\nu}$
$D{*F} = 4\pi{*j}$	field equations	$\frac{1}{2}\eta_{\mu\nu\rho} \wedge R^{\nu\rho} = -8\pi{*T_\mu}$
$d{*F} = 4\pi{*j} - [A,{*F}]$	field equations in Gauss's form	$dU_\mu = 4\pi{*}(T_\mu + t_\mu)$, where $U_\mu = \frac{1}{4}\eta_{\mu\nu\rho} \wedge \Gamma^{\nu\rho}$
$d({*j} - \frac{1}{4\pi}[A,{*F}]) = 0$	conservation law	$d{*}(T_\mu + t_\mu) = 0$
$d{*A} = 0$	gauge fixing condition	$d{*e} = 0$
	Boundary conditions at spatial infinity for static configurations	
$\phi = O(1)$		g = Minkowski tensor + $O(1/r)$
$A = O(1/r)$		$\Gamma = O(1/r^2)$
$F = O(1/r^2)$		$R = O(1/r^3)$
$\frac{1}{4\pi}\oint {*F}$	total conserved quantity	$\frac{1}{4\pi}\oint U_\mu$

of ϕ lie in an orbit $W \subset V$ of G, then H is the isotropy group of some point of $W \simeq G/H$. In general, there are many orbits in V corresponding to the same H: they are all said to belong to the same stratum. For example, in a standard SO(3) Yang-Mills-Higgs theory, under the assumption of spherical symmetry and $\phi \neq 0$, the normalized field $\phi/||\phi||$ breaks the symmetry down to $H = SO(2)$. The radial Higgs equation selects, for each radius r, an orbit containing $\phi(r) \in \mathbb{R}^3$. All these orbits are diffeomorphic to $S_2 \simeq SO(3)/SO(2)$: they belong to the same stratum, without being isometric [48]. The situation is rather different in the theory of gravitation, where $G = GL(n,R)$ and H is an orthogonal group. According to the 'theorem on inertia' of quadratic forms, each stratum in $L^2_s(\mathbb{R}^n,\mathbb{R})$ consists of a single orbit, viz. the set of all quadratic forms with a given signature. As a result of this, there is no 'radial equation' and potential for the metric tensor; the symmetry breaking in the theory of gravitation is more of kinematic than dynamic nature.

An essential difference between the two types of theories occurs in connection with the asymptotic behaviour (at large distances) of their static fields; this is indicated in the Table. A gauge transformation of the potential, $A \to A'$,

$$A' = S^{-1} A S + S^{-1} dS ,$$

where

$$S: M \to G ,$$

is compatible with the asymptotic behaviour of a time-independent A, if

$$S = a(\theta,\phi)(I + \beta(\theta,\phi)/r + \dots) ,$$

where θ, ϕ are coordinates on S_2, and $a: S_2 \to G$. Under such a transformation, the field strenghts change as follows,

$$F' = a^{-1} F a + O(1/r^3) .$$

Therefore, the total non-Abelian charge

$$\frac{1}{4\pi} \oint {*F}$$

is ill-defined [34]. By contrast, in the theory of gravitation, one has $\Gamma = O(1/r^2)$ for static configurations. To preserve this asymptotic behaviour, in the generic case, one has to restrict $a = (a^\mu{}_\nu)$ to be a constant matrix. This allows one to define unambiguously the total mass for such configurations. Indeed, the Von Freud 'superpotential' U transforms as follows,

$$U'_\mu = U_\nu a^\nu{}_\mu + O(1/r^3) , \qquad \text{where } a \in SO(1,3) .$$

The structure of the group of gauge transformations also reflects the similarities and differences among gauge theories [55]. A gauge transformation is an automorphism of the principal bundle $\pi: P \to M$ preserving the absolute elements of the gauge theory. A gauge transformation is said to be pure if it is vertical (based), i.e. if it induces the identity map on M. For any gauge theory one can construct the (horizontally) exact sequences of group homomorphisms,

$$\begin{array}{ccccccccc} I & \to & \mathcal{G}_0 & \to & \mathcal{G} & \to & \mathcal{G}/\mathcal{G}_0 & \to & I \\ & & \downarrow & & \downarrow & & \downarrow & & \\ I & \to & \mathrm{Aut}_0\, P & \to & \mathrm{Aut}\, P & \to & \mathrm{Diff}\, M & & \end{array}$$

where $\mathcal{G}$ (resp. $\mathcal{G}_0$) is the group of all gauge (resp. all pure gauge) transformations and Aut P (resp. Aut_0 P) is the group of all (resp. all vertical) automorphisms of P. In general relativistic theories of gravitation, the soldering form on P = LM is an absolute element and it reduces $\mathcal{G}$ to Diff M and $\mathcal{G}_0$ to the identity. By contrast, in a theory of the Yang-Mills type over Minkowski space, both $\mathcal{G}_0$ and $\mathcal{G}$ are 'large' groups, but $\mathcal{G}/\mathcal{G}_0$ is 'small', i.e. a Lie group [61].

Plane Gravitational Waves Are Abelian

Another aspect of Einstein's theory of gravitation, which makes it resemble electrodynamics rather than non-Abelian Yang-Mills theories, is associated with the nature of its plane waves.

In any theory of the Yang-Mills type, the potential

$$A = (a(u)x + b(u)y + c(u))du , \qquad (2)$$

where $u = t - z$ and $a,b,c\colon \mathbb{R} \to \mathfrak{g}$, represents in Minkowski space a solution of the source-free equation $D{*}F = 0$. The corresponding field strength

$$F = (adx + bdy) \wedge du$$

is invariant under translations in the (x,y)-plane, but the potential - and therefore the entire gauge configuration - is not, in general. For example, for $G = SO(3)$ and $[a,b] \neq 0$ the potential (2) is not invariant under any translation in that plane. On the other hand, if the functions a, b, and c span an Abelian Lie subalgebra of $\mathfrak{g}$, then (2) is invariant under translations in the (x,y)-plane and c can be eliminated by a gauge transformation.

The connection form Γ of plane gravitational waves, referred to a suitable orthonormal frame, can also be written in the form (2). In this case, however, the functions $a,b,c\colon R \to so(1,3)$ span a two-dimensional, Abelian subalgebra n of $so(1,3)$, corresponding to the nilipotent part of its Iwasawa decomposition. Therefore, c can be eliminated and the solution has a 5-dimensional group of isometrics isomorphic to the group of symmetries of a plane electromagnetic wave propagating in one direction. Incidentally, the restriction to n of the polarizational degrees of freedom is a result of the vanishing of torsion. There does not seem to exist an analogous, natural restriction on a and b in the non-Abelian Yang-Mills theory.

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The notion of 'gauge transformations' is introduced here, for the first time, in connection with an attempt to unify gravitation and electromagnetism. Gauge transformations act on the metric of space-time.

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Kaluza shows that the Einstein-Maxwell equations can be geometrically interpreted in a five-dimensional Riemannian space whose metric is independent on one of the coordinates, say x^5. Gauge transformations are reduced to coordinate changes, $\bar{x}^i = x^i$ $(i = 1,2,3,4)$ and $\bar{x}^5 = x^5 + f(x^1,x^2,x^3,x^4)$.

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