Pin structures and the modified Dirac operator

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Abstract

General theorems on pin structures on products of manifolds and on homogeneous (pseudo-)Riemannian spaces are given and used to find explicitly all such structures on odd-dimensional real projective quadrics, which are known to be non-orientable [3]. It is shown that the product of two manifolds has a pin structure if, and only if, both are pin and at least one of them is orientable. This general result is illustrated by the example of the product of two real projective planes. It is shown how the Dirac operator should be modified to make it equivariant with respect to the twisted adjoint action of the Pin group. A simple formula is derived for the spectrum of the Dirac operator on the product of two pin manifolds, one of which is orientable, in terms of the eigenvalues of the Dirac operators on the factor spaces.

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1 Introduction

This paper is a continuation of our work on spin structures on symmetric spaces [2] and on the modified Dirac operator on pin manifolds [9]. It is based, in part, on the lectures given, in September 1994, by two of the authors at the Erwin Schrödinger Institute in Vienna [5,10]. A brief review of the spinor representations of Clifford algebras and Pin groups is followed by a description of how to construct the representation of the Clifford algebra $\mathrm{Cl}(h_1 \oplus h_2) = \mathrm{Cl}(h_1) \otimes^{\mathrm{gr}} \mathrm{Cl}(h_2)$ from the representations of $\mathrm{Cl}(h_i)$, $i = 1, 2$. We extend the results of [3] to pin structures on non-orientable homogeneous (pseudo-)Riemannian spaces and illustrate them by constructing such structures on the quadrics $Q_{k,l}$ for $k+l$ odd. A theorem is given on the relation between the existence of a pin structure on a manifold and on its universal covering space.

We consider products of (s)pin manifolds and describe the relations between the pin structures and the spectrum of the Dirac operator on the product and on the factors.

We use the notation traditional in differential geometry. All manifolds and maps under consideration are smooth. If V is a finite-dimensional vector space, then V^* denotes its dual and the value $f(v)$ of the 1-form $f \in V^*$ on $v \in V$ is often denoted by $\langle v, f\rangle$. If $h : V \to W$ is a homomorphism of vector spaces, then its *transpose* ${}^t h : W^* \to V^*$ is defined by $\langle v, {}^t h(f)\rangle = \langle h(v), f\rangle$ for every $v \in V$ and $f \in W^*$. Let V be a real m-dimensional vector space with an isomorphism $h : V \to V^*$ which is symmetric, $h = {}^t h$, and such that the quadratic form $V \to \mathbb{R}$, given by $v \mapsto \langle v, h(v)\rangle$, is of signature (k, l), $k+l = m$. One says that the pair (V, h) is a *quadratic space* of dimension m and signature (k, l). The orthogonal group $\mathrm{O}(h)$ consists of all automorphisms of (V, h). A *Riemannian space* is defined as a connected manifold M with a metric tensor, i.e. a symmetric isomorphism $g : TM \to T^*M$ of vector bundles over M; for every $x \in M$ the pair (T_xM, g_x) is a quadratic space; the quadratic space (V, h) is a *local model* of the Riemannian space (M, g) if the spaces (V, h) and (T_xM, g_x) are isometric; an isometry $p : V \to T_xM$ is then said to be an *orthonormal frame* at x. We say that M is a *proper* Riemannian space if the quadratic form associated with h is definite; since we deal often with the case when the quadratic form is indefinite, this terminology is more convenient than the traditional one of pseudo-Riemannian spaces. For every Riemannian space M with a local model (V, h) there is the principal $\mathrm{O}(h)$-bundle $P \to M$ of all orthonormal frames on M. The group $\mathrm{O}(h)$ acts on P on the right by composition of isometries; the symbol of composition of maps is often omitted; e.g. if $p \in P$ and $a \in \mathrm{O}(h)$, then we write pa instead of $p \circ a$; a similar notation is used for the action of structure groups on other principal bundles. If P is a principal G-bundle over M and $f : G \to H$ is a homomorphism of groups, then the principal H-bundle over M, associated with P by f, is denoted by

$P \times_f H$.

2 Clifford algebras and their representations

In this section, we give a brief description of the properties of real Clifford algebras and their complex representations, relevant to our work. Details and proofs can be found in the literature; see, e.g. [1,6,7] and the references given there.

2.1 Definitions

The *Clifford algebra* $\mathrm{Cl}(h)$ of a quadratic space (V, h) is an associative real algebra with a unit element 1, containing $\mathbb{R} \oplus V$ as a vector subspace. The algebra is $\mathbb{Z}_2$-graded by the *main automorphism* α,

$$\mathrm{Cl}(h) = \mathrm{Cl}^0(h) \oplus \mathrm{Cl}^1(h), \quad a = a_0 + a_1,$$

where $a_\varepsilon \in \mathrm{Cl}^\varepsilon(h)$ and $\alpha(a_\varepsilon) = (-1)^\varepsilon a_\varepsilon$ for $\varepsilon = 0$ or 1. If $a \in \mathrm{Cl}^\varepsilon(h)$, then we write $\varepsilon = \deg a$.

The Clifford algebra is characterized by its *universal property*: if $f : V \to A$ is a *Clifford map*, i.e. a linear map into an algebra A with unit element 1_A and such that $f(v)^2 = \langle v, h(v)\rangle 1_A$ for every $v \in V$, then there is a homomorphism of algebras with units $\tilde{f} : \mathrm{Cl}(h) \to A$ such that $\tilde{f}|V = f$. In particular, the inclusion map $V \to \mathrm{Cl}(h)$ is Clifford.

Lemma 1 *Let (V, h) and (V_0, h_0) be quadratic spaces of dimensions m and one, respectively. Assume that h_0 is a negative form; there then exists $e_{m+1} \in V_0$ such that $\langle e_{m+1}, h_0(e_{m+1})\rangle = -1$. The Clifford map*

$$V \to \mathrm{Cl}^0(h \oplus h_0), \quad v \mapsto v e_{m+1},$$

extends to an isomorphism of algebras,

$$\iota : \mathrm{Cl}(h) \to \mathrm{Cl}^0(h \oplus h_0).$$

2.2 Pin and Spin groups

An element $u \in V$ is said to be a *unit vector* if either $u^2 = 1$ or $u^2 = -1$. The group $\mathrm{Pin}(h)$ is defined as the subset of $\mathrm{Cl}(h)$ consisting of products of all finite sequences of unit vectors; the group multiplication is induced by the Clifford product and $\mathrm{Spin}(h) = \mathrm{Pin}(h) \cap \mathrm{Cl}^0(h)$. For every $a \in \mathrm{Pin}(h)$ the map $\rho(a) : V \to V$, given by

$$\rho(a)v = \alpha(a)va^{-1} \tag{1}$$

is orthogonal,

$${}^t\rho(a) \circ h \circ \rho(a) = h, \tag{2}$$

and defines the *twisted adjoint representation* ρ of $\mathrm{Pin}(h)$ in V. The two exact sequences

$$1 \to \mathbb{Z}_2 \to \mathrm{Pin}(h) \xrightarrow{\rho} \mathrm{O}(h) \to 1 \;\text{ and }\; 1 \to \mathbb{Z}_2 \to \mathrm{Pin}(-h) \xrightarrow{\rho} \mathrm{O}(h) \to 1,$$

give two inequivalent (central) extensions of the orthogonal group $\mathrm{O}(h)$ by $\mathbb{Z}_2$. If the dimension m of V is *even*, then one can use the *adjoint representation* Ad such that $\mathrm{Ad}(a)v = ava^{-1}$ to form two inequivalent extensions of $\mathrm{O}(h)$ by $\mathbb{Z}_2$, namely

$$1 \to \mathbb{Z}_2 \to \mathrm{Pin}(h) \xrightarrow{\mathrm{Ad}} \mathrm{O}(h) \to 1 \;\text{ and }\; 1 \to \mathbb{Z}_2 \to \mathrm{Pin}(-h) \xrightarrow{\mathrm{Ad}} \mathrm{O}(h) \to 1.$$

If $V = \mathbb{R}^{k+l}$ and one wants to specify the signature (k, l) of h, then one writes $\mathrm{Cl}(k, l)$, $\mathrm{O}(k, l)$, $\mathrm{Pin}(k, l)$ and $\mathrm{Spin}(k, l)$ instead of $\mathrm{Cl}(h)$, $\mathrm{O}(h)$, $\mathrm{Pin}(h)$ and $\mathrm{Spin}(h)$, respectively. Since the groups $\mathrm{Spin}(h)$ and $\mathrm{Spin}(-h)$ are isomorphic, one writes $\mathrm{Spin}(m)$ instead of $\mathrm{Spin}(m, 0) = \mathrm{Spin}(0, m)$.

Assume V to be oriented and let $(e_1, \ldots, e_m)$ be an orthonormal frame in V, of the preferred orientation. The square of the *volume element*, $\mathrm{vol}(h) = e_1 \ldots e_m$, is either 1 or -1, depending on the signature of h. Putting $\mathrm{i} = \sqrt{-1}$, it is convenient to define $i(h) \in \{1, \mathrm{i}\}$ so that $\mathrm{vol}(h)^2 = i(h)^2$. Clearly, $u\,\mathrm{vol}(h) = (-1)^{m+1}\,\mathrm{vol}(h)u$ for every $u \in V$ and $\rho(\mathrm{vol}(h)) = -\,\mathrm{id}_V$.

2.3 Spinor representations

The following Lemma summarizes facts about representations of Clifford algebras [1,7] relevant to our work and introduces a notation and terminology used in theoretical physics [9].

Lemma 2 *(i) If the dimension m of V is* even, *$m = 2\nu$, then the algebra* $\mathrm{Cl}(h)$ *is central simple; as such it has only one, up to equivalence, irreducible and faithful representation*

$$\gamma : \mathrm{Cl}(h) \to \mathrm{End}\, S$$

in a complex, 2^ν-dimensional space of 'Dirac' spinors. On restriction to the even subalgebra $\mathrm{Cl}^0(h)$ *this representation decomposes into the sum $\gamma_+ \oplus \gamma_-$ of two irreducible, $2^{\nu-1}$-dimensional 'Weyl' representations.*

(ii) If the dimension m of V is odd, *$m = 2\nu - 1$, then the algebra $\mathrm{Cl}^0(h)$ is central simple; its (unique up to equivalence) irreducible and faithful representation extends to two 'Pauli' representations γ_+ and γ_- of the full algebra $\mathrm{Cl}(h)$ in a complex $2^{\nu-1}$-dimensional space of Pauli spinors. These representations are related by $\gamma_+ = \gamma_- \circ \alpha$; they are complex-inequivalent and irreducible, but not necessarily faithful. Their direct sum, $\gamma = \gamma_+ \oplus \gamma_-$, is a faithful representation of $\mathrm{Cl}(h)$ in the 2^ν-dimensional space S of 'Cartan' spinors. The latter representation can be identified with the restriction to $\mathrm{Cl}(h)$ of the Dirac representation*

$$\gamma' : \mathrm{Cl}(h \oplus h_0) \to \mathrm{End}\, S. \tag{3}$$

The commutant of the Cartan representation is generated by $\gamma(\mathrm{vol}(h))$.

We use indiscriminately the name of *spinor representation* for any one of the representations of the *type* described above, also when they are restricted to $\mathrm{Pin}(h)$ or $\mathrm{Spin}(h)$.

If the representation γ is as in Lemma 2, then there exists a *Dirac intertwiner* defined to be an isomorphism $\Gamma : S \to S$, intertwining the representations γ and $\gamma \circ \alpha$,

$$\gamma \circ \alpha(a) = \Gamma \gamma(a) \Gamma^{-1} \quad \text{for every} \quad a \in \mathrm{Cl}(h),$$

and such that $\Gamma^2 = \mathrm{id}_S$. Referring to Lemma 1 we see that such an intertwiner can be used to extend the representation $\gamma : \mathrm{Cl}(h) \to \mathrm{End}\, S$ to a representation $\gamma' : \mathrm{Cl}(h \oplus h_0) \to \mathrm{End}\, S$ by putting

$$\gamma'(v) = \mathrm{i}\, \gamma(v) \Gamma \quad \text{for} \quad v \in V \quad \text{and} \quad \gamma'(e_{m+1}) = -\,\mathrm{i}\, \Gamma \tag{4}$$

so that $\gamma = \gamma' \circ \iota$.

Lemma 3 *(i) If the dimension of V is even and $\gamma : \mathrm{Cl}(h) \to \mathrm{End}\, S$ is a Dirac representation, then the Dirac intertwiner Γ equals either $i(h)\gamma(\mathrm{vol}(h))$ or $-i(h)\gamma(\mathrm{vol}(h))$.*

(ii) If the dimension of V is odd and $\gamma : \mathrm{Cl}(h) \to \mathrm{End}\, S$ is a Cartan representation, then the Dirac intertwiner Γ can be any element of the set

$$\{\mathrm{i}\, \gamma'(e_{m+1})(\cosh z + i(h)\gamma(\mathrm{vol}(h)) \sinh z) : z \in \mathbb{C}\},$$

where γ' is the extension defined in (4). If Γ and Γ' are two such intertwiners, then there is $\Omega \in \mathrm{GL}(S)$, belonging to the commutant of the representation γ and such that

$$\Gamma' = \Omega \Gamma \Omega^{-1}. \tag{5}$$

The following two Propositions describe the construction of spinor representations of the algebra $\mathrm{Cl}(h_1 \oplus h_2)$ from suitably 'twisted' tensor products of the representations of the algebras $\mathrm{Cl}(h_i)$, $i=1,2$, and also the corresponding Dirac intertwiners.

Proposition 1 *Consider two quadratic spaces (V_i, h_i) $i=1,2$, and the spinor representations $\gamma_i : \mathrm{Cl}(h_i) \to \mathrm{End}\, S_i$. Assume that γ_1 has a Dirac intertwiner Γ_1. Then*

(i) The map $V_1 \times V_2 \to \mathrm{End}\, S_1 \otimes \mathrm{End}\, S_2$ given by

$$(v_1, v_2) \mapsto \gamma_1(v_1) \otimes \mathrm{id}_{S_2} + \Gamma_1 \otimes \gamma_2(v_2) \tag{6}$$

is a Clifford map and thus extends to a representation γ of $\mathrm{Cl}(h_1 \oplus h_2)$ in $S_1 \otimes S_2$. If Γ_2 is a Dirac intertwiner for γ_2, then

$$\Gamma = \Gamma_1 \otimes \Gamma_2. \tag{7}$$

is a Dirac intertwiner for γ.

(ii) Let the dimension of V_1 be even *and let γ_1 be the Dirac representation. If γ_2 is a Dirac (resp., Cartan, Pauli) representation, then γ is a Dirac (resp., Cartan, Pauli) representation.*

(iii) Let the dimension of V_1 be odd *and let γ_1 be the Cartan representation. If γ_2 is a Dirac (resp., Pauli) representation, then γ is a Cartan (resp., Dirac) representation. If γ_2 is the Pauli representation, then the Dirac intertwiner for γ is given by $\Gamma = \mathrm{i}\, i(h_1)\gamma_1(\mathrm{vol}(h_1))\Gamma_1 \otimes \mathrm{id}_{S_2}$. If γ_2 is the Cartan representation, then γ decomposes into the direct sum of two (equivalent) Dirac representations.*

PROOF. Since the proofs of the different cases are similar, we give it only for the last, least obvious case. The Cartan representations γ_1 and γ_2 being faithful, so is the representation γ. Let m_i be the dimension of V_i, $i=1,2$. The dimension of the carrier space $S_1 \otimes S_2$ is now $2^{1+(m_1+m_2)/2}$, i.e. it is twice the dimension of the space of Dirac spinors associated with $\mathrm{Cl}(h_1 \oplus h_1)$. One checks readily that $\mathrm{id}_{S_1} \otimes \gamma_2(\mathrm{vol}(h_2))$ generates the commutant of the representation γ and that the endomorphisms $\frac{1}{2}(\mathrm{id}_{S_1 \otimes S_2} + i(h_2)\mathrm{id}_{S_1} \otimes \gamma_2(\mathrm{vol}(h_2)))$ and $\frac{1}{2}(\mathrm{id}_{S_1 \otimes S_2} - i(h_2)\mathrm{id}_{S_1} \otimes \gamma_2(\mathrm{vol}(h_2)))$ are projections on two complementary and irreducible, with respect to γ, subspaces of $S_1 \otimes S_2$. $\square$

In order to cover the case when both γ_1 and γ_2 are Pauli representations, it is convenient to use the physicists' *Pauli matrices*,

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \quad \text{and} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Proposition 2 *Let the quadratic spaces* (V_i, h_i), $i = 1, 2$, *be both of* odd *dimension. Consider the Pauli representations,*

$$\gamma_i : \mathrm{Cl}(h_i) \to \mathrm{End}\, S_i, \quad i = 1, 2.$$

The Clifford map

$$V_1 \times V_2 \to \mathrm{End}\, \mathbb{C}^2 \otimes \mathrm{End}\, S_1 \otimes \mathrm{End}\, S_2,$$

given by

$$(v_1, v_2) \mapsto \sigma_1 \otimes \gamma_1(v_1) \otimes \mathrm{id}_{S_2} + \sigma_2 \otimes \mathrm{id}_{S_1} \otimes \gamma_2(v_2), \tag{8}$$

extends to the Dirac representation γ *of* $\mathrm{Cl}(h_1 \oplus h_2)$ *in* $\mathbb{C}^2 \otimes S_1 \otimes S_2$. *Its Dirac intertwiners are* $\mathit{\Gamma} = \pm\sigma_3 \otimes \mathrm{id}_{S_1} \otimes \mathrm{id}_{S_2}$.

2.4 The Clifford evaluation map

The tensor product of a spinor representation γ in S and of the representation contragredient to ρ defines a representation σ of $\mathrm{Pin}(h)$ in the vector space $\mathrm{Hom}(V, S)$: if $a \in \mathrm{Pin}(h)$ and $\mathit{\Phi} \in \mathrm{Hom}(V, S)$, then

$$\sigma(a)\mathit{\Phi} = \gamma(a) \circ \mathit{\Phi} \circ \rho(a^{-1}). \tag{9}$$

Identifying $\mathrm{Hom}(V, S)$ with $V^* \otimes S$, we define the *Clifford evaluation map*

$$\tilde{\gamma} : \mathrm{Hom}(V, S) \to S \quad \text{by} \quad \tilde{\gamma}(v^* \otimes \varphi) = \gamma(h^{-1}(v^*))\varphi, \tag{10}$$

where $v^* \in V^*$ and $\varphi \in S$. Using (1) and (2) one shows that

$$\tilde{\gamma} \circ \sigma(a) = (\gamma \circ \alpha)(a) \circ \tilde{\gamma} \tag{11}$$

for every $a \in \mathrm{Pin}(h)$.

3 Pin and spin structures

3.1 Definitions

Let (V, h) be the local model of a Riemannian manifold M and let $\pi : P \to M$ be the bundle of all orthonormal frames on M. A $\mathrm{Pin}(h)$-*structure* on M is a principal $\mathrm{Pin}(h)$-bundle $\varpi : Q \to M$, together with a morphism $\chi : Q \to P$ of principal bundles over M associated with the epimorphism $\rho : \mathrm{Pin}(h) \to \mathrm{O}(h)$. The morphism condition means that $\varpi = \pi \circ \chi$ and there is the commutative diagram

$$\begin{array}{ccc} Q \times \mathrm{Pin}(h) & \xrightarrow{\chi\times\rho} & P \times \mathrm{O}(h) \\ \downarrow & & \downarrow \\ Q & \xrightarrow{\chi} & P \end{array}$$

where the vertical arrows denote the action maps.

The expression $\mathrm{Pin}(k,l)$-*structure* is used when one wants the signature of h to appear explicitly. For brevity, we shall describe a $\mathrm{Pin}(h)$-structure by the sequence

$$\mathrm{Pin}(h) \to Q \xrightarrow{\chi} P \xrightarrow{\pi} M. \tag{12}$$

If M is orientable and admits a $\mathrm{Pin}(h)$-structure, then it has a spin structure. In an abbreviated style, similar to that of (12), it may be described by the sequence of maps

$$\mathrm{Spin}(h) \to SQ \to SP \to M, \tag{13}$$

where SP is now an $\mathrm{SO}(h)$-bundle; if the quadratic form associated with h is definite, then SP is one of the two connected components of P.

3.2 Existence of pin structures

Let $TM = T^+M \oplus T^-M$ be the decomposition of the tangent bundle of M into the Whitney sum of two vector bundles such that the metric tensor restricted to T^+M (resp., T^-M) is positive- (resp., negative-) definite. Denoting by w_i^+ (resp., by w_i^-) the ith Stiefel–Whitney class of T^+M (resp., of T^-M), one can formulate

Theorem (Karoubi) *A Riemannian space admits a* $\mathrm{Pin}(h)$*-structure* (12) *if, and only if,*

$$w_2^+ + w_2^- + w_1^-(w_1^+ + w_1^-) = 0. \tag{14}$$

A proof of the Theorem is in [6]. Introducing the Stiefel-Whitney classes w_i of TM, one can write (14) as

$$w_2 + (w_1^-)^2 = 0. \tag{15}$$

In particular, if M is proper Riemannian, then the condition for M to have a $\mathrm{Pin}(m,0)$-structure is $w_2 = 0$, whereas the corresponding condition for a $\mathrm{Pin}(0,m)$-structure is $w_2 + w_1^2 = 0$. The conditions $w_1^\pm = 0$ and $w_1 = 0$ are equivalent to the orientability of $T^\pm M$ and TM, respectively.

3.3 Homogeneous pin manifolds

The following theorem is an extension of Theorem 1 in [3] to the case of a homogeneous manifold that need not be orientable.

Theorem 1 *Let (V, h) be the local model of a Riemannian space M with a Lie group G acting on M transitively by isometries and let H be the isotropy group of a point of M.*

(i) If the linear isotropy representation $\tau : H \to \mathrm{O}(h)$ lifts to the homomorphism $\hat{\tau} : H \to \mathrm{Pin}(h)$, then there is a $\mathrm{Pin}(h)$-structure $Q \to P \to M$ such that $Q = G \times_{\hat{\tau}} \mathrm{Pin}(h)$.

(ii) If $\hat{\tau}$ and $\hat{\tau'}$ are two lifts of τ and the pin structures defined by these lifts are isomorphic, then $\hat{\tau} = \hat{\tau'}$.

(iii) If the group G is simply connected and M has a $\mathrm{Pin}(h)$-structure, then τ lifts to $\mathrm{Pin}(h)$ and the $\mathrm{Pin}(h)$-structure is as in (i).

PROOF. The proof is obtained as in the orientable case. For part (iii), one considers the lift of the action of G on M to an action of G on the the total space Q of the pin structure and observes that the lifted action commutes with that of the group $\mathrm{Pin}(h)$. $\square$

3.4 Products of pin manifolds

Theorem 2 *Let M' and M'' be two Riemannian spaces. Their product has a pin structure if, and only if, M' and M'' are pin manifolds and at least one of the factor spaces is orientable.*

PROOF. Assume first that the product $M = M' \times M''$ has a pin structure. Denote by w_i, w_i' and w_i'' the ith Stiefel–Whitney classes of the tangent bundles of the manifolds M, M' and M'', respectively; similarly, let w_1^-, w'^-_1 and w''^-_1 be the first classes of T^-M, T^-M' and T^-M''. We have (15) because M is pin and

$$w_2 = w_2' + w_2'' + w_1' w_1''$$

from the Whitney product property. Since $w_1{}^- = w_1'{}^- + w_1''{}^-$ and $2w_1'{}^- w_1''{}^- = 0$, condition (15) reduces to

$$\underline{w_2' + (w_1'{}^-)^2} + \underline{w_2'' + (w_1''{}^-)^2} + \underline{w_1' w_1''} = 0. \qquad (16)$$

Each of the three underlined terms in the last equation refers to a different space and thus it must vanish separately. The vanishing of the first two means that M' and M'' are both pin manifolds and $w_1'w_1'' = 0$ implies that at least one of them is orientable and, as such, has a spin structure. Conversely, eq. (16) implies (15): the product of a pin manifold and of a spin manifold is a pin manifold. $\square$

Consider two quadratic spaces (V_1, h_1), (V_2, h_2) and their orthogonal sum $(V_1 \oplus V_2, h_1 \oplus h_2)$. The injection $V_1 \to V_1 \oplus V_2$ extends to the monomorphism of groups, $\mathrm{Pin}(h_1) \to \mathrm{Pin}(h_1 \oplus h_2)$; similarly for $\mathrm{Pin}(h_2)$. Let $a_1, b_1 \in \mathrm{Pin}(h_1)$ and $a_2, b_2 \in \mathrm{Pin}(h_2)$; the 'twisted' multiplication law,

$$(a_1, a_2).(b_1, b_2) = (a_1 b_1, (-1)^{\deg a_2 \deg b_1} a_2 b_2), \tag{17}$$

makes $\mathrm{Pin}(h_1) \times \mathrm{Pin}(h_2)$ into a group such that the map

$$\mathrm{Pin}(h_1) \times \mathrm{Pin}(h_2) \to \mathrm{Pin}(h_1 \oplus h_2), \quad (a_1, a_2) \mapsto a_1 a_2,$$

is a homomorphism of groups with kernel $\mathbb{Z}_2$ generated by $(-1, -1)$ [6].

It is convenient to have an explicit construction of the pin structure on the product, in terms of the pin and spin structures on the factors. Let again M_1 and M_2 be pin manifolds, with local models (V_1, h_1) and (V_2, h_2), respectively, and assume that one of them, say M_2, is orientable. Let

$$\mathrm{Pin}(h_1) \to Q_1 \xrightarrow{\chi_1} P_1 \xrightarrow{\pi_1} M_1$$

and

$$\mathrm{Spin}(h_2) \to SQ_2 \xrightarrow{\chi_2} SP_2 \xrightarrow{\pi_2} M_2$$

be the pin and spin structures of the two spaces. The bundle $\mathrm{O}(h_1) \times \mathrm{SO}(h_2) \to P_1 \times SP_2 \xrightarrow{\pi_1 \times \pi_2} M_1 \times M_2$ is a restriction of the bundle of all orthonormal frames on the product space. Let Q be the quotient of the set $Q_1 \times SQ_2$ by the equivalence relation: $(q_1, q_2) \equiv (q_1', q_2')$ if, and only if, either $q_1 = q_1'$ and $q_2 = q_2'$ or $q_1 = q_1'(-1)$ and $q_2 = q_2'(-1)$. The group $(\mathrm{Pin}(h_1) \times \mathrm{Spin}(h_2))/\mathbb{Z}_2$ acts freely on Q,

$$[(q_1, q_2)].[(a_1, a_2)] = [(q_1 a_1, q_2 a_2)], \tag{18}$$

where $q_1 \in Q_1,\ q_2 \in SQ_2,\ a_1 \in \mathrm{Pin}(h_1)$ and $a_2 \in \mathrm{Spin}(h_2)$. The projection $\chi : Q \to P_1 \times SP_2,\ \chi([(q_1, q_2)]) = (\chi_1(q_1), \chi_2(q_2))$ has the equivariance property required of a 'restricted' pin structure. The total space of the $\mathrm{Pin}(h_1 \oplus h_2)$-structure on the product is $Q \times_f \mathrm{Pin}(h_1 \oplus h_2)$, where f is the homomorphism

$$f : (\mathrm{Pin}(h_1) \times \mathrm{Spin}(h_2))/\mathbb{Z}_2 \to \mathrm{Pin}(h_1 \oplus h_2), \quad f([(a_1, a_2)]) = a_1 a_2.$$

Note that if M_1 and M_2 are both non-orientable pin manifolds, then Q can be still defined as above, but (18) does not yield an action of the group $(\mathrm{Pin}(h_1) \times \mathrm{Pin}(h_2))/\mathbb{Z}_2$ on Q because of the twisting in (17).

Example. To illustrate Theorems 1 and 2 on a simple example, consider the product M of the real projective plane $\mathbb{P}_2 = \mathbb{S}_2/\mathbb{Z}_2$ by itself. The non-orientable space $\mathbb{P}_2$, given the proper Riemannian metric descending from $\mathbb{S}_2$, is symmetric and has a $\mathrm{Pin}(0,2)$-structure [4]. The group $G = \mathrm{Spin}(3) \times \mathrm{Spin}(3)$ acts transitively on M and the stabilizer H of an element of M has four connected components. Considering lifts of the linear isotropy representation $\tau : H \to \mathrm{O}(2)\times\mathrm{O}(2)$ to any one of the groups $\mathrm{Pin}(4,0)$, $\mathrm{Pin}(2,2)$ and $\mathrm{Pin}(0,4)$ one shows that none exist and, therefore, M has no pin structure.

3.5 The relation between pin structures on manifolds and on their universal covers

Let Π be the first homotopy group of a connected manifold M. The universal covering manifold $\tilde{M}$ of M is the total space of the principal Π-bundle $\xi : \tilde{M} \to M$; see, e.g., §14 in [8]. We write the left action of Π on $\tilde{M}$ as $(c,x) \mapsto cx$, so that $\xi(cx) = \xi(x)$ for every $c \in \Pi$ and $x \in \tilde{M}$. If M is a Riemannian space with a local model (V,h), then so is $\tilde{M}$. The principal $\mathrm{O}(h)$-bundle $\tilde{\pi} : \tilde{P} \to \tilde{M}$ of all orthonormal frames on $\tilde{M}$ can be identified with the bundle induced from $\pi : P \to M$ by ξ,

$$\tilde{P} = \{(x,p) \in \tilde{M} \times P : \xi(x) = \pi(p)\}.$$

The projection $\tilde{\pi} : \tilde{P} \to \tilde{M}$ is given by $\tilde{\pi}(x,p) = x$ and there is the map $\eta : \tilde{P} \to P$ such that $\eta(x,p) = p$. The group $\mathrm{O}(h)$ acts on $\tilde{P}$ so that $((x,p),A) \mapsto (x,pA)$, where $A \in \mathrm{O}(h)$; the map η is equivariant: $\eta(x,pA) = \eta(x,p)A$. . There is a natural lift of the action of Π to $\tilde{P}$ given by $(c,(x,p)) \mapsto (cx,p)$. The lifted action commutes with that of $\mathrm{O}(h)$. We can now formulate

Theorem 3 *A Riemannian space M, with a local model (V,h), admits a pin structure (12) if, and only if, there exists a pin structure*

$$\mathrm{Pin}(h) \to \tilde{Q} \xrightarrow{\tilde{\chi}} \tilde{P} \xrightarrow{\tilde{\pi}} \tilde{M} \tag{19}$$

on its universal cover $\tilde{M}$ and an action of $\Pi = \pi_1(M)$ on $\tilde{Q}$, lifting the action of Π on $\tilde{P}$ and commuting with the action of $\mathrm{Pin}(h)$.

PROOF. Assume first that M has the pin structure (12). The $\mathbb{Z}_2$-bundle $\tilde{Q} \to \tilde{P}$ is induced from the bundle $Q \to P$ by the map $\eta : \tilde{P} \to P$,

$$\tilde{Q} = \{(x,q) \in \tilde{M} \times Q : \xi(x) = \varpi(q)\}, \quad \tilde{\chi}(x,q) = (x,\chi(q)).$$

The action of Π on $\tilde{Q}$ given by $c(x,q) = (cx,q)$ commutes with the action of $\mathrm{Pin}(h)$, given by $(x,q)a = (x,qa)$, where $(x,q) \in \tilde{Q}$ and $a \in \mathrm{Pin}(h)$. Conversely, assume that there is a pin structure (19) on the universal covering space of M and an action of Π on $\tilde{Q}$ such that, for every $c \in \Pi$, $a \in \mathrm{Pin}(h)$ and $\tilde{q} \in \tilde{Q}$ one has $(c\tilde{q})a = c(\tilde{q}a)$ and $\tilde{\chi}(c\tilde{q}) = c\tilde{\chi}(\tilde{q})$. We define $Q = \tilde{Q}/\Pi$; i.e. if $[\tilde{q}], [\tilde{q}'] \in Q$, then $[\tilde{q}] = [\tilde{q}']$ if, and only if, there is $c \in \Pi$ such that $\tilde{q}' = c\tilde{q}$. The projection $\chi : Q \to P$ is now given by $\chi([\tilde{q}]) = \eta(\tilde{\chi}(\tilde{q}))$. An action of $\mathrm{Pin}(h)$ on Q is defined by $[\tilde{q}]a = [\tilde{q}a]$ and seen to satisfy $\chi([\tilde{q}]a) = \chi([\tilde{q}])\rho(a)$ $\square$

4 Pin structures on non-orientable, real projective quadrics

Recall that the real projective quadric $Q_{k,l} = (\mathbb{S}_k \times \mathbb{S}_l)/\mathbb{Z}_2$ is acted upon transitively by the group $G = \mathrm{Spin}(k+1) \times \mathrm{Spin}(l+1)$. Proper quadrics, i.e. those for which $kl > 0$, are orientable if, and only if, $k + l$ is even; their spin structures have been determined in [3]. Some of those quadrics have no spin structure (example: $Q_{3,5}$). As an application of Theorem 1 we now show that all non-orientable quadrics have pin structures. They are described in:

Theorem 4 *Let k and l be positive integers, even and odd, respectively. Every quadric $Q_{k,l}$ has two pairs of inequivalent pin structures:*

$$\textit{for} \quad k + l \equiv 1 \bmod 4 \quad \textit{in signature} \quad (0, k+l) \;\; \textit{and} \;\; (l, k),$$

$$\textit{for} \quad k + l \equiv 3 \bmod 4 \quad \textit{in signature} \quad (k+l, 0) \;\; \textit{and} \;\; (k, l).$$

PROOF. Following §6 of [3], we introduce two orthonormal frames $(e_1, \ldots, e_{k+1})$ and $(f_1, \ldots, f_{l+1})$ in $\mathbb{R}^{k+1}$ and $\mathbb{R}^{l+1}$, respectively. Considered as elements of Clifford algebras, the vectors satisfy

$$e_\alpha e_\beta + e_\beta e_\alpha = \pm 2\delta_{\alpha\beta} \quad \text{and} \quad f_\mu f_\nu + f_\nu f_\mu = \pm 2\delta_{\mu\nu},$$

where the choice of signs depends on the signature under consideration and $\alpha, \beta = 1, \ldots, k+1$; $\mu, \nu = 1, \ldots, l+1$.

The stabilizer of $[(e_{k+1}, f_{l+1})]$ is the group $H = H_0 \cup H_1$, where $H_0 = \mathrm{Spin}(k) \times \mathrm{Spin}(l)$ and H_1 is generated in G by H_0 and the element $(e_1 e_{k+1}, f_1 f_{l+1})$. The linear isotropy representation $\tau : H \to \mathrm{O}(k) \times \mathrm{SO}(l)$ is given by $\tau(a, b) = (\rho(a), \rho(b))$ for $(a, b) \in H_0$ and $\tau(e_1 e_{k+1}, f_1 f_{l+1}) = (-\rho(e_1), -\rho(f_1))$, where ρ denotes the twisted adjoint representation. It is now appropriate to consider lifts to the groups $\mathrm{Pin}(k, l)$, $\mathrm{Pin}(l, k)$, $\mathrm{Pin}(k + l, 0)$, and $\mathrm{Pin}(0, k + l)$. For $(a, b) \in H_0$ one has $\hat{\tau}(a, b) = ab$. The element $(-\rho(e_1), -\rho(f_1))$ is covered by two elements of the Pin group, namely by $\pm e_1 f_1 \,\mathrm{vol}$, where $\mathrm{vol} = e_1 \ldots e_k f_1 \ldots f_l$. Since $(e_1 e_{k+1}, f_1 f_{l+1})^2 = (-1, -1) \overset{\hat{\tau}}{\mapsto} 1$, one has to have $(e_1 f_1 \,\mathrm{vol})^2 = 1$. Since vol is now in the center of the Pin group, the last condition reduces to $e_1^2 f_1^2 \mathrm{vol}^2 = -1$. The squares occurring above depend on the signature; their evaluation leads to the conclusion of the theorem. An independent check of this result is provided by the computation of the Stiefel–Whitney classes. According to §3 of [3], the tangent bundle of $Q_{k,l}$ decomposes into the direct sum of two vector bundles T' and T'' of fiber dimension k and l, respectively. The odd-dimensional subbundle is orientable, $w_1'' = 0$, whereas the even-dimensional one is not, $w_1 = w_1' \neq 0$. The second class of $TQ_{k,l}$ is

$$w_2 = w_2' + w_2'' = \tfrac{1}{2}(k(k+1) + l(l+1))w_1^2 = \begin{cases} w_1^2 \text{ for } k+l \equiv 1 \bmod 4, \\ 0 \ \text{ for } k+l \equiv 3 \bmod 4. \end{cases}$$

Since now $w_1'w_1'' = 0$, according to the Karoubi theorem, the quadric has a pin structure if $w_2 + w_1^- w_1 = 0$. For $k + l \equiv 1 \bmod 4$, this gives $w_1^- = w_1$, i.e. the metric restricted to T' should be negative-definite. For $k + l \equiv 3 \bmod 4$, condition (14) implies $w_1^- = 0$ and the metric restricted to T' should be positive-definite. □

5 Spinor fields and Dirac operators on pin manifolds

5.1 Bundles of spinors and their sections

Let M be a Riemannian space with a $\mathrm{Pin}(h)$-structure (12) and let γ be a spinor representation of the group $\mathrm{Pin}(h)$ in S. The complex vector bundle $\pi_E : E \to M$, with typical fiber S, associated with Q by γ, is the *bundle of spinors of type* γ. If the dimension m of M is even (resp., odd), then E is called a bundle of Dirac (resp., Cartan) spinors. For m odd, one can also define the bundle of Pauli spinors over M. Similarly, if m is even and M has a spin structure, then there are two bundles of Weyl spinors over M. A *spinor field* of type γ on M is a section of π_E. The (vector) space of such sections is known to be in a natural and bijective correspondence with the set of all maps $\psi : Q \to S$ equivariant with respect to the action of $\mathrm{Pin}(h)$. Denoting by $\delta(a) : Q \to Q$ the map $q \mapsto qa$, we can write the 'transformation law' of ψ as

$$\psi \circ \delta(a) = \gamma(a^{-1})\psi, \tag{20}$$

for every $a \in \mathrm{Pin}(h)$. It is convenient to refer to ψ itself as a spinor field of type γ on M. Depending on whether E is a bundle of Dirac, Weyl, Cartan or Pauli spinors, one refers to its sections as Dirac, Weyl, Cartan or Pauli spinor fields, respectively.

5.2 *Covariant differentiation of spinor fields*

Let (12) be a pin structure on a Riemannian space M with a local model (V, h). The Levi-Civita connection form on P lifts to the spin connection 1-form ω on Q with values in the Lie algebra of the group $\mathrm{Spin}(h)$. For every $q \in Q$, there is the orthonormal frame $\chi(q) : V \to T_{\varpi(q)}M$. The *basic* horizontal V^*-valued vector field ∇ is defined on Q by the spin connection as follows. For every $q \in Q$ the linear map $\nabla(q) : V \to T_q Q$ is such that

$$T_q\varpi(\nabla(q)) = \chi(q) \quad \text{and} \quad \langle v, \nabla(q)\rangle \lrcorner\, \omega = 0,$$

where $\lrcorner$ denotes the inner product (contraction). The field ∇ transforms according to

$$\nabla(qa) = T_q\delta(a) \circ \nabla(q) \circ \rho(a).$$

Let $\psi : Q \to S$ be a spinor field of type γ. Its *covariant derivative* is a map $\nabla\psi : Q \to \mathrm{Hom}(V, S)$ such that, for every $v \in V$ and $q \in Q$, one has

$$\langle v, (\nabla\psi)(q)\rangle = \langle v, \nabla(q)\rangle \lrcorner\, \mathrm{d}\psi.$$

The covariant derivative transforms according to

$$(\nabla\psi)(qa) = \sigma(a^{-1})(\nabla\psi)(q),$$

where σ is the representation of $\mathrm{Pin}(h)$ in $\mathrm{Hom}(V, S)$ given by (9).

5.3 *The classical and the modified Dirac operators*

Using the notation of the preceding paragraph and (10), the *classical Dirac operator* can be written as

$$D^{\mathrm{cl}}\psi = \tilde{\gamma} \circ \nabla\psi.$$

According to (11), the classical Dirac operator maps a spinor field of type γ into a spinor field of type $\gamma \circ \alpha$.

Let Γ be a Dirac intertwiner; the *modified Dirac operator* is defined by

$$D = \mathrm{i}\,\Gamma D^{\mathrm{cl}}. \tag{21}$$

It preserves the type of the spinor field and the corresponding eigenvalue equation, $D\psi = \lambda\psi$, is meaningful on non-orientable pin manifolds for Cartan or Dirac spinor fields.

Remark 1. If the dimension of M is *even*, then one can use the adjoint vector representation of $\mathrm{Pin}(h)$ in the definition of the pin structure on M. The classical Dirac operator preserves then the type of spinor fields and there is no need for its modification.

Remark 2. If the dimension of M is *odd* and one considers Cartan spinors, then one has a freedom in choosing Γ, as described in Part (ii) of Lemma 3. If $D' = \mathrm{i}\,\Gamma' D^{\mathrm{cl}}$ and Ω is as in (5), then $D'\Omega = \Omega D$. Therefore, the spectra of the operators D and D' coincide.

Remark 3. If M is a *spin* manifold and ψ is a spinor field, then $\Gamma\psi$ is a spinor field of the same type. Since $(1 + \mathrm{i}\,\Gamma)^{-1} = \frac{1}{2}(1 - \mathrm{i}\,\Gamma)$ and

$$D = (1 + \mathrm{i}\,\Gamma) D^{\mathrm{cl}} (1 + \mathrm{i}\,\Gamma)^{-1},$$

if $D^{\mathrm{cl}}\psi = \lambda\psi$, then $D\psi' = \lambda\psi'$, where $\psi' = (1 + \mathrm{i}\,\Gamma)\psi$.

Remark 4. Since Γ anticommutes with D^{cl}, on a spin manifold the spectra of D^{cl} and D are both *symmetric*: if λ is an eigenvalue, then so is $-\lambda$.

Remark 5. If M is an *odd*-dimensional *spin* manifold, then the interesting object is the *Pauli operator*, the restriction of D^{cl} to Pauli spinor fields. If φ is an eigenfunction of the Pauli operator, then the Cartan spinor fields $(\varphi, 0)$ and $(0, \varphi)$ are eigenfunctions of D^{cl} with opposite eigenvalues.

5.4 The spectrum of the Dirac operator on a product of pin manifolds

Consider, as in Section 3.4, two pin manifolds M_1 and M_2 and assume that the second is orientable. Their product M has a pin structure determined by the double cover $Q = (Q_1 \times SQ_2)/\mathbb{Z}_2$ of the restriction $P_1 \times SP_2$ of the bundle of orthonormal frames of M to the group $\mathrm{O}(h_1) \times \mathrm{SO}(h_2)$. Consider the canonical projections $Q_1 \times SQ_2 \xrightarrow{\mathrm{pr}} Q$, $Q_1 \times SQ_2 \xrightarrow{\mathrm{pr}_1} Q_1$ and $Q_1 \times SQ_2 \xrightarrow{\mathrm{pr}_2} SQ_2$. The spin connection forms ω_1 and ω_2 on Q_1 and SQ_2, respectively, define a form ω on Q such that $\mathrm{pr}^*\omega = \mathrm{pr}_1^*\omega_1 + \mathrm{pr}_2^*\omega_2$. The 1-form ω has values in the Lie algebra of $\mathrm{Spin}(h_1 \oplus h_2)$ and defines the spin connection form on Q. If $\psi_1 : Q_1 \to S_1$ and $\psi_2 : SQ_2 \to S_2$ are spinor fields of type γ_1 and γ_2 on M_1 and M_2, respectively, then their tensor product, $\psi_1 \otimes \psi_2 : Q \to S_1 \otimes S_2$, is well-defined by $(\psi_1 \otimes \psi_2)([(q_1, q_2)]) = \psi_1(q_1) \otimes \psi_2(q_2)$ and is a spinor field on M of type γ given by Prop. 1. Denoting by ∇, ∇_1 and ∇_2 the basic horizontal vector fields on Q, Q_1 and SQ_2, respectively, we can write the Leibniz rule for the covariant derivative as $\nabla(\psi_1 \otimes \psi_2) = (\nabla_1\psi_1) \otimes \psi_2 + \psi_1 \otimes \nabla_2\psi_2$. Using an analogous notation for the (modified) Dirac operators, and assuming that the representations γ_1 and γ_2 are either Dirac or Cartan, we obtain, by virtue of (6), (7) and (21),

$$D(\psi_1 \otimes \psi_2) = D_1\psi_1 \otimes \Gamma_2\psi_2 + \psi_1 \otimes D_2\psi_2. \tag{22}$$

The Pythagoras Theorem for the Dirac operator *Let M be the product of a pin manifold M_1 and of a spin manifold M_2. If λ_1 and λ_2 are eigenvalues of the Dirac operators on M_1 and M_2, respectively, then $\sqrt{\lambda_1^2+\lambda_2^2}$ and $-\sqrt{\lambda_1^2+\lambda_2^2}$ are eigenvalues of the Dirac operator on M.*

PROOF. Assume first that the representations γ_1 and γ_2 defining the type of spinor fields on M_1 and M_2, respectively, are either Dirac or Cartan. If D_1 and D_2 are the (modified) Dirac operators on M_1 and M_2, respectively, then formula (22) applies and its consequence, $D^2 = D_1^2 \otimes \mathrm{id}_{S_2} + \mathrm{id}_{S_1} \otimes D_2^2$, suffices to prove the Theorem. To see in detail how the eigenfunctions of D are constructed from those D_1 and D_2, consider spinor fields ψ_i on M_i satisfying $D_i\psi_i = \lambda_i\psi_i,\ i = 1,2$. The spinor fields on M,

$$\psi_\pm = \psi_1 \otimes (\lambda_2 \pm \sqrt{\lambda_1^2+\lambda_2^2} + \lambda_1 \mathit{\Gamma}_2)\psi_2,$$

are then easily seen to satisfy $D\psi_\pm = \pm\sqrt{\lambda_1^2+\lambda_2^2}\,\psi_\pm$.

If both M_1 and M_2 are odd-dimensional spin manifolds, then one can assume γ_1 and γ_2 to be Pauli representations and take D_1 and D_2 to be the corresponding Pauli operators. Let $\varphi \in \mathbb{C}^2$ and let ψ_i be Pauli spinor fields on $M_i,\ i = 1,2$. According to Prop. 2, the Dirac operator on M acts on the spinor field $\varphi \otimes \psi_1 \otimes \psi_2$ as follows:

$$D^{\mathrm{cl}}(\varphi \otimes \psi_1 \otimes \psi_2) = \sigma_1(\varphi) \otimes D_1\psi_1 \otimes \psi_2 + \sigma_2(\varphi) \otimes \psi_1 \otimes D_2\psi_2.$$

Let (e_1, e_2) be the canonical frame in $\mathbb{C}^2$ so that $\sigma_1(e_1) = e_2,\ \sigma_2(e_1) = \mathrm{i}\,e_2$, etc. If $D_i\psi_i = \lambda_i\psi_i$ and

$$\psi_\pm = (\sqrt{\lambda_1 - \mathrm{i}\,\lambda_2}\; e_1 \pm \sqrt{\lambda_1 + \mathrm{i}\,\lambda_2}\; e_2) \otimes \psi_1 \otimes \psi_2,$$

then $D^{\mathrm{cl}}\psi_\pm = \pm\sqrt{\lambda_1^2+\lambda_2^2}\,\psi_\pm$. $\square$

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References

[1] P. Budinich and A. Trautman, *The spinorial chessboard*, Trieste Notes in Physics (Springer–Verlag, Berlin, 1988).

[2] M. Cahen and S. Gutt, Spin structures on compact simply connected Riemannian symmetric spaces, *Simon Stevin Quart. J. Pure and Appl. Math.* **62** (1988) 209–242.

[3] M. Cahen, S. Gutt and A. Trautman, Spin structures on real projective quadrics, *J. Geom. Phys.* **10** (1993) 127–154.

[4] L. Dąbrowski and A. Trautman, Spinor structures on spheres and projective spaces, *J. Math. Phys.* **27** (1986) 2022–28.

[5] S. Gutt, *On Dirac operators and pin manifolds*, Lecture given at the Workshop on *Spinors, Twistors and Conformal Invariants* at the Erwin Schrödinger Institute in Vienna, 19–25 September 1994.

[6] M. Karoubi, Algèbres de Clifford et K-théorie, *Ann. Sci. Éc. Norm. Sup.* 4ème sér. 1 (1968) 161–270.

[7] H. B. Lawson, Jr. and M. L. Michelsohn 1989, *Spin Geometry* (Princeton U. P., Princeton, 1989).

[8] N. Steenrod, *The Topology of Fibre Bundles* (Princeton U. P., Princeton, 1951).

[9] A. Trautman, Spinors and the Dirac operator on hypersurfaces. I. General theory, *J. Math. Phys.* **33** (1992) 4011–4019.

[10] A. Trautman, *The Dirac operator on pin manifolds*, Lecture given at the Workshop on *Spinors, Twistors and Conformal Invariants* at the Erwin Schrödinger Institute in Vienna, 19–25 September 1994.