EINSTEIN–CARTAN THEORY

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Introduction

Notation

Standard notation and terminology of differential geometry and general relativity are used in this article. All considerations are local so that the four-dimensional *space-time M* is assumed to be a smooth manifold diffeomorphic to $\mathbb{R}^4$. It is endowed with a *metric tensor g* of signature $(1, 3)$ and a *linear connection* defining the *covariant differentiation* of tensor fields. Greek indices range from 0 to 3 and refer to space-time. Given a field of frames (e_μ) on M, and the dual field of coframes (θ^μ), one can write the metric tensor as $g = g_{\mu\nu}\theta^\mu\theta^\nu$, where $g_{\mu\nu} = g(e_\mu, e_\nu)$ and Einstein's summation convention is assumed to hold. Tensor indices are lowered with $g_{\mu\nu}$ and raised with its inverse $g^{\mu\nu}$. General-relativistic units are used so that both Newton's constant of gravitation and the speed of light are 1. This implies $\hbar = \ell^2$, where $\ell \approx 10^{-33}$ cm is the Planck length. Both mass and energy are measured in centimeters.

Historical remarks

The Einstein–Cartan Theory (ECT) of gravity is a modification of General Relativity Theory (GRT), allowing space-time to have torsion, in addition to curvature, and relating torsion to the density of intrinsic angular momentum. This modification was put forward in 1922 by Élie Cartan, before the discovery of spin. Cartan was influenced by the work of the Cosserat brothers (1909), who considered besides an (asymmetric) force stress tensor also a moments stress tensor in a suitably generalized continuous medium. Work done in the 1950s by physicists (Kondo, Bilby, Kröner and other authors) established the role played by torsion in the continuum theory of crystal dislocations. A recent review (Ruggiero and Tartaglia, 2003) describes the links between ECT and the classical theory of defects in an elastic medium.

Cartan assumed the linear connection to be metric and derived, from a variational principle, a set of gravitational field equations. He required, without justification, that the covariant divergence of the energy-momentum tensor be zero; this led to an algebraic constraint equation, bilinear in curvature and torsion, severely restricting the geometry. This misguided observation has probably discouraged Cartan from pursuing his theory. It is now known that conservation laws in relativistic theories of gravitation follow from the Bianchi identities and, in the presence of torsion, the divergence of the energy-momentum tensor need not vanish. Torsion is implicit in the 1928 Einstein theory of gravitation with teleparallelism. For a long time, Cartan's modified theory of gravity, presented in his rather abstruse notation, unfamiliar to physicists, did not attract any attention. In the late 1950s, the theory of gravitation with spin and torsion was independently rediscovered by Sciama and Kibble. The role of Cartan was recognized soon afterwards and ECT became the subject of much research; see Hehl et al. (1976) for a review and an extensive bibliography. In the 1970s it was recognized that ECT can be incorporated within supergravity. In fact, simple supergravity is equivalent to ECT with a massless, anticommuting Rarita–Schwinger field as the source. Choquet-Bruhat considered a generalization of ECT to higher dimensions and showed that the Cauchy problem for the coupled system of Einstein–Cartan and Dirac equations is well posed. Penrose (1982) has shown that torsion appears in a natural way when spinors are allowed to be rescaled by a *complex* conformal factor. ECT has been generalized by allowing nonmetric linear connections and additional currents, associated with dilation and shear, as sources of such a "metric-affine theory of gravity" (Hehl et al., 1995).

Physical motivation

Recall that, in Special Relativity Theory (SRT), the underlying Minkowski space-time admits, as its group of automorphisms, the full Poincaré group, consisting of translations *and* Lorentz transformations. It follows from the first Noether theorem that classical, special-relativistic field equations, derived from a variational principle, give rise to conservation laws of energy-momentum *and* angular momentum. Using

Cartesian coordinates (x^μ), abbreviating $\partial\varphi/\partial x^\rho$ to $\varphi_{,\rho}$ and denoting by $t^{\mu\nu}$ and $s^{\mu\nu\rho} = -s^{\nu\mu\rho}$ the tensors of energy-momentum and of intrinsic angular momentum (spin), respectively, one can write the conservation laws in the form

$$t^{\mu\nu}{}_{,\nu} = 0 \tag{1}$$

and

$$(x^\mu t^{\nu\rho} - x^\nu t^{\mu\rho} + s^{\mu\nu\rho})_{,\rho} = 0. \tag{2}$$

In the presence of spin, the tensor $t^{\mu\nu}$ need not be symmetric,

$$t^{\mu\nu} - t^{\nu\mu} = s^{\mu\nu\rho}{}_{,\rho}.$$

Belinfante and Rosenfeld have shown that the tensor

$$T^{\mu\nu} = t^{\mu\nu} + \tfrac{1}{2}(s^{\nu\mu\rho} + s^{\nu\rho\mu} + s^{\mu\rho\nu})_{,\rho}$$

is symmetric and its divergence vanishes.

In quantum theory, the irreducible, unitary representations of the Poincaré group correspond to elementary systems such as stable particles; these representations are labeled by the mass *and* spin.

In Einstein's GRT, the space-time M is curved; the Lorentz group — but not the Poincaré group — appears as the structure group acting on orthonormal frames in the tangent spaces of M. The energy-momentum tensor T appearing on the right side of the Einstein equation is necessarily symmetric. In GRT there is no room for translations and the tensors t and s.

By introducing torsion and relating it to s, Cartan restored the role of the Poincaré group in relativistic gravity: this group acts on the affine frames in the tangent spaces of M. Curvature and torsion are the surface densities of Lorentz transformations and translations, respectively. In a space with torsion, the Ricci tensor need not be symmetric so that an asymmetric energy-momentum tensor can appear on the right side of the Einstein equation.

Geometric preliminaries

Tensor-valued differential forms

It is convenient to follow Cartan in describing geometric objects as tensor-valued differential forms. To define them, consider a homomorphism $\sigma : \mathrm{GL}_4(\mathbb{R}) \to \mathrm{GL}_N(\mathbb{R})$ and an element $A = (A^\mu_\nu)$ of $\mathrm{End}\,\mathbb{R}^4$, the Lie algebra of $\mathrm{GL}_4(\mathbb{R})$. The derived representation of Lie algebras is given by

$\frac{\mathrm{d}}{\mathrm{d}t}\sigma(\exp At)|_{t=0} = \sigma^\nu_\mu A^\mu_\nu$. If (e_a) is a frame in $\mathbb{R}^N$, then $\sigma^\nu_\mu(e_a) = \sigma^{b\nu}_{a\mu}e_b$, where $a,b = 1,\ldots,N$.

A map $a = (a^\mu{}_\nu) : M \to \mathrm{GL}_4(\mathbb{R})$ transforms fields of frames so that

$$e'_\mu = e_\nu a^\nu_\mu \quad \text{and} \quad \theta^\nu = a^\nu_\mu \theta'^\mu. \tag{3}$$

A differential form φ on M, with values in $\mathbb{R}^N$, is said to be of type σ if, under changes of frames, it transforms so that $\varphi' = \sigma(a^{-1})\varphi$. For example, $\theta = (\theta^\mu)$ is a 1-form of type id. If now $A = (A^\mu_\nu) : M \to \mathrm{End}\,\mathbb{R}^4$, then one puts $a(t) = \exp tA : M \to \mathrm{GL}_4(\mathbb{R})$ and defines the variations induced by an infinitesimal change of frames,

$$\begin{aligned}\delta\theta &= \frac{\mathrm{d}}{\mathrm{d}t}(a(t)^{-1}\theta)|_{t=0} = -A\theta, \\ \delta\varphi &= \frac{\mathrm{d}}{\mathrm{d}t}(\sigma(a(t)^{-1})\varphi)|_{t=0} = -\sigma^\nu_\mu A^\mu_\nu \varphi.\end{aligned} \tag{4}$$

Hodge duals

Since M is diffeomorphic to $\mathbb{R}^4$, one can choose an orientation on M and restrict the frames to agree with that orientation so that only transformations with values in $\mathrm{GL}_4^+(\mathbb{R})$ are allowed. The metric then defines the Hodge dual of differential forms. Put $\theta_\mu = g_{\mu\nu}\theta^\nu$. The forms η, η_μ, $\eta_{\mu\nu}$, $\eta_{\mu\nu\rho}$ and $\eta_{\mu\nu\rho\sigma}$ are defined to be the duals of 1, θ_μ, $\theta_\mu \wedge \theta_\nu$, $\theta_\mu \wedge \theta_\nu \wedge \theta_\rho$ and $\theta_\mu \wedge \theta_\nu \wedge \theta_\rho \wedge \theta_\sigma$, respectively. The 4-form η is the volume element; for a holonomic coframe $\theta^\mu = \mathrm{d}x^\mu$ it is given by $\sqrt{-\det(g_{\mu\nu})}\,\mathrm{d}x^0 \wedge \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3$. In SRT, in Cartesian coordinates, one can define the tensor-valued 3-forms

$$t^\mu = t^{\mu\nu}\eta_\nu \quad \text{and} \quad s^{\mu\nu} = s^{\mu\nu\rho}\eta_\rho \tag{5}$$

so that equations (1) and (2) become

$$\mathrm{d}t^\mu = 0 \quad \text{and} \quad \mathrm{d}j^{\mu\nu} = 0,$$

where

$$j^{\mu\nu} = x^\mu t^\nu - x^\nu t^\mu + s^{\mu\nu}. \tag{6}$$

For an isolated system, the 3-forms t^μ and $j^{\mu\nu}$, integrated over the 3-space $x^0 =$const., give the system's total energy-momentum vector and angular momentum bivector, respectively.

Linear connection, its curvature and torsion

A linear connection on M is represented, with respect to the field of frames, by the field of 1-forms

$$\omega^\mu_\nu = \Gamma^\mu_{\rho\nu}\theta^\rho$$

so that the covariant derivative of e_ν in the direction of e_μ is $\nabla_\mu e_\nu = \Gamma^\rho_{\mu\nu} e_\rho$. Under a change of frames (3), the connection forms transform as follows:

$$a^\mu_\rho \omega'^\rho_\nu = \omega^\mu_\rho a^\rho_\nu + \mathrm{d}a^\mu_\nu.$$

If $\varphi = \varphi^a e_a$ is a k-form of type σ, then its *covariant exterior derivative*

$$D\varphi^a = \mathrm{d}\varphi^a + \sigma^{a\mu}_{b\nu} \omega^\nu_\mu \wedge \varphi^b$$

is a $(k+1)$-form of the same type. For a 0-form one has $D\varphi^a = \theta^\mu \nabla_\mu \varphi^a$. The infinitesimal change of ω, defined similarly as in (4), is $\delta\omega^\mu_\nu = DA^\mu_\nu$. The 2-form of curvature $\Omega = (\Omega^\mu{}_\nu)$, where

$$\Omega^\mu_\nu = \mathrm{d}\omega^\mu_\nu + \omega^\mu_\rho \wedge \omega^\rho_\nu,$$

is of type ad: it transforms with the adjoint representation of $\mathrm{GL}_4(\mathbb{R})$ in $\mathrm{End}\,\mathbb{R}^4$. The 2-form of torsion $\Theta = (\Theta^\mu)$, where

$$\Theta^\mu = \mathrm{d}\theta^\mu + \omega^\mu_\nu \wedge \theta^\nu,$$

is of type id. These forms satisfy the *Bianchi identities*

$$D\Omega^\mu_\nu = 0 \quad \text{and} \quad D\Theta^\mu = \Omega^\mu_\nu \wedge \theta^\nu.$$

For a differential form φ of type σ there holds the identity

$$D^2\varphi^a = \sigma^{a\nu}_{b\mu} \Omega^\mu_\nu \wedge \varphi^b. \tag{7}$$

The tensors of curvature and torsion are given by

$$\Omega^\mu_\nu = \tfrac{1}{2} R^\mu{}_{\nu\rho\sigma} \theta^\rho \wedge \theta^\sigma$$

and

$$\Theta^\mu = \tfrac{1}{2} Q^\mu{}_{\rho\sigma} \theta^\rho \wedge \theta^\sigma,$$

respectively. With respect to a holonomic frame, $\mathrm{d}\theta^\mu = 0$, one has

$$Q^\mu{}_{\rho\sigma} = \Gamma^\mu_{\rho\sigma} - \Gamma^\mu_{\sigma\rho}.$$

In SRT, the Cartesian coordinates define a radius-vector field $X^\mu = -x^\mu$, pointing towards the origin of the coordinate system. The differential equation it satisfies generalizes to a manifold with a linear connection:

$$DX^\mu + \theta^\mu = 0. \tag{8}$$

By virtue of (7), the integrability condition of (8) is

$$\Omega^\mu_\nu X^\nu + \Theta^\mu = 0.$$

Integration of (8) along a curve defines the Cartan displacement of X; if this is done along a small closed circuit spanned by the bivector Δf, then the radius vector changes by about

$$\Delta X^\mu = \tfrac{1}{2}(R^\mu{}_{\nu\rho\sigma} X^\nu + Q^\mu{}_{\rho\sigma})\Delta f^{\rho\sigma}.$$

This holonomy theorem — rather imprecisely formulated here — shows that torsion bears to translations a relation similar to that of curvature to linear homogeneous transformations.

In a space with torsion it matters whether one considers the potential of the electromagnetic field to be a scalar-valued 1-form φ or a covector-valued 0-form (φ_μ). The first choice leads to a field $\mathrm{d}\varphi$ that is invariant with respect to the gauge transformation $\varphi \mapsto \varphi + \mathrm{d}\chi$. The second gives $\tfrac{1}{2}(\nabla_\mu \varphi_\nu - \nabla_\nu \varphi_\mu)\theta^\mu \wedge \theta^\nu = (D\varphi_\mu) \wedge \theta^\mu = \mathrm{d}\varphi - \varphi_\mu \Theta^\mu$, a gauge-dependent field.

Metric-affine geometry

A *metric-affine* space (M, g, ω) is defined to have a metric and a linear connection that need not dependent on each other. The metric alone determines the torsion-free Levi-Civita connection $\mathring\omega$ characterized by

$$\mathrm{d}\theta^\mu + \mathring\omega^\mu_\nu \wedge \theta^\nu = 0 \quad \text{and} \quad \mathring{D} g_{\mu\nu} = 0.$$

Its curvature is

$$\mathring\Omega^\mu_\nu = \mathrm{d}\mathring\omega^\mu_\nu + \mathring\omega^\mu_\rho \wedge \mathring\omega^\rho_\nu.$$

The 1-form of type ad,

$$\kappa^\mu_\nu = \omega^\mu_\nu - \mathring\omega^\mu_\nu, \tag{9}$$

determines the torsion of ω and the covariant derivative of g,

$$\Theta^\mu = \kappa^\mu_\nu \wedge \theta^\nu, \quad Dg_{\mu\nu} = -\kappa_{\mu\nu} - \kappa_{\nu\mu}.$$

The curvature of ω can be written as

$$\Omega^\mu_\nu = \mathring\Omega^\mu_\nu + \mathring{D}\kappa^\mu_\nu + \kappa^\mu_\rho \wedge \kappa^\rho_\nu. \tag{10}$$

The *transposed* connection $\tilde\omega$ is defined by

$$\tilde\omega^\mu_\nu = \omega^\mu_\nu + Q^\mu{}_{\nu\rho}\theta^\rho$$

so that, with respect to a holonomic frame, one has $\tilde\Gamma^\mu_{\nu\rho} = \Gamma^\mu_{\rho\nu}$. The torsion of $\tilde\omega$ is opposed to that of ω.

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Riemann–Cartan geometry

A *Riemann–Cartan* space is a metric-affine space with a connection that is metric,

$$Dg_{\mu\nu} = 0. \tag{11}$$

The metricity condition implies $\kappa_{\mu\nu} + \kappa_{\nu\mu} = 0$ and $\Omega_{\mu\nu} + \Omega_{\nu\mu} = 0$. In a Riemann–Cartan space the connection is determined by its torsion Q and the metric tensor. Let $Q_{\rho\mu\nu} = g_{\rho\sigma}Q^{\sigma}{}_{\mu\nu}$, then

$$\kappa_{\mu\nu} = \tfrac{1}{2}(Q_{\mu\sigma\nu} + Q_{\nu\mu\sigma} + Q_{\sigma\mu\nu})\theta^{\sigma}. \tag{12}$$

The transposed connection of a Riemann–Cartan space is metric if, and only if, the tensor $Q_{\rho\mu\nu}$ is completely antisymmetric. Let $\tilde{\nabla}$ denote the covariant derivative with respect to $\tilde{\omega}$. By definition, a *symmetry* of a Riemann–Cartan space is a diffeomorphism of M preserving both g and ω. The one-parameter group of local transformations of M, generated by the vector field v, consists of symmetries of (M, g, ω) if, and only if,

$$\tilde{\nabla}^{\mu}v^{\nu} + \tilde{\nabla}^{\nu}v^{\mu} = 0 \tag{13}$$

and

$$D\tilde{\nabla}_{\nu}v^{\mu} + R^{\mu}{}_{\nu\rho\sigma}v^{\rho}\theta^{\sigma} = 0. \tag{14}$$

In a Riemannian space, the connections ω and $\tilde{\omega}$ coincide and (14) is a consequence of the Killing equation (13). The metricity condition implies

$$D\eta_{\mu\nu\rho} = \eta_{\mu\nu\rho\sigma}\Theta^{\sigma}. \tag{15}$$

The Einstein–Cartan theory of gravitation

An identity resulting from local invariance

Let (M, g, ω) be a metric-affine space-time. Consider a Lagrangian L which is an invariant 4-form on M, depends on g, θ, ω, φ, and the first derivatives of $\varphi = \varphi^{a}e_{a}$. The general variation of the Lagrangian is

$$\delta L = L_{a} \wedge \delta\varphi^{a} + \tfrac{1}{2}\tau^{\mu\nu}\delta g_{\mu\nu} + \delta\theta^{\mu} \wedge t_{\mu} \\ - \tfrac{1}{2}\delta\omega_{\nu}^{\mu} \wedge s^{\nu}{}_{\mu} + \text{an exact form} \tag{16}$$

so that $L_{a} = 0$ is the Euler–Lagrange equation for φ. If the changes of the functions θ, ω, g and φ are induced by an infinitesimal change of the frames (4) then $\delta L = 0$ and (16) gives the identity

$$g_{\mu\rho}\tau^{\rho\nu} - \theta^{\nu} \wedge t_{\mu} + \tfrac{1}{2}Ds^{\nu}{}_{\mu} - \sigma_{a\mu}^{b\nu}L_{a} \wedge \varphi^{b} = 0.$$

It follows from the identity that the two sets of Euler–Lagrange equations obtained by varying L with respect to the triples $(\varphi, \theta, \omega)$ and (φ, g, ω) are equivalent. In the sequel, the first triple is chosen to derive the field equations.

Projective transformations and the metricity condition

Still under the assumption that (M, g, ω) is a metric-affine space-time, consider the 4-form

$$8\pi K = \tfrac{1}{2}g^{\nu\rho}\eta_{\mu\rho} \wedge \Omega^{\mu}{}_{\nu} \tag{17}$$

which is equal to ηR, where $R = g^{\mu\nu}R_{\mu\nu}$ is the Ricci scalar; the Ricci tensor $R_{\mu\nu} = R^{\rho}{}_{\mu\rho\nu}$ is, in general, asymmetric. The form (17) is invariant with respect to *projective transformations* of the connection,

$$\omega_{\nu}^{\mu} \mapsto \omega_{\nu}^{\mu} + \delta_{\nu}^{\mu}\lambda, \tag{18}$$

where λ is an arbitrary 1-form. Projectively related connections have the same (unparametrized) geodesics. If the total Lagrangian for gravitation interacting with the matter field φ is $K + L$, then the field equations, obtained by varying it with respect to φ, θ and ω are: $L_{a} = 0$,

$$\tfrac{1}{2}g^{\rho\sigma}\eta_{\mu\nu\rho} \wedge \Omega^{\nu}{}_{\sigma} = -8\pi t_{\mu}, \tag{19}$$

and

$$D(g^{\mu\rho}\eta_{\rho\nu}) = 8\pi s^{\mu}{}_{\nu}, \tag{20}$$

respectively. Put $s_{\mu\nu} = g_{\mu\rho}s^{\rho}{}_{\nu}$. If

$$s_{\mu\nu} + s_{\nu\mu} = 0, \tag{21}$$

then $s^{\nu}{}_{\nu} = 0$ and L is also invariant with respect to (18). One shows that, if (21) holds, then, among the projectively related connections satisfying (20), there is precisely one that is metric. To implement properly the metricity condition in the variational principle one can use the Palatini approach with constraints (Kopczyński, 1975). Alternatively, following Hehl, one can use (9) and (12) to eliminate ω and obtain a lagrangian depending on φ, θ and the tensor of torsion.

The Sciama–Kibble field equations

From now on the metricity condition (11) is assumed so that (21) holds and the Cartan field equation (20) is

$$\eta_{\mu\nu\rho} \wedge \Theta^{\rho} = 8\pi s_{\mu\nu}. \tag{22}$$

Introducing the asymmetric energy-momentum tensor $t_{\mu\nu}$ and the spin density tensor $s_{\mu\nu\rho} = g_{\rho\sigma}s^\sigma{}_{\mu\nu}$ similarly as in (5), one can write the Einstein–Cartan equations (19) and (22) in the form given by Sciama and Kibble,

$$R_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}R = 8\pi t_{\mu\nu} \qquad (23)$$

$$Q^\rho{}_{\mu\nu} + \delta^\rho_\mu Q^\sigma{}_{\nu\sigma} - \delta^\rho_\nu Q^\sigma{}_{\mu\sigma} = 8\pi s^\rho{}_{\mu\nu}. \qquad (24)$$

Equation (24) can be solved to give

$$Q^\rho{}_{\mu\nu} = 8\pi(s^\rho{}_{\mu\nu} + \tfrac{1}{2}\delta^\rho_\mu s^\sigma{}_{\nu\sigma} + \tfrac{1}{2}\delta^\rho_\nu s^\sigma{}_{\sigma\mu}) \qquad (25)$$

Therefore, torsion vanishes in the absence of spin and then (23) is the classical Einstein field equation. In particular, there is no difference between the Einstein and Einstein–Cartan theories in empty space. Since practically all tests of relativistic gravity are based on consideration of Einstein's equations in empty space, there is no difference, in this respect, between the Einstein and the Einstein–Cartan theories: the latter is as viable as the former.

In any case, the consideration of torsion amounts to a slight change of the energy-momentum tensor that can be also obtained by the introduction of a new term in the Lagrangian. This observation was made in 1950 by Weyl in the context of the Dirac equation.

In Einstein's theory one can also satisfactorily describe spinning matter without introducing torsion (Bailey and Israel, 1975).

Consequences of the Bianchi identities: conservation laws

Computing the covariant exterior derivatives of both sides of the Einstein–Cartan equations, using (15) and the Bianchi identities one obtains

$$8\pi Dt_\mu = \tfrac{1}{2}\,\eta_{\mu\nu\rho\sigma}\Theta^\nu \wedge \Omega^{\rho\sigma} \qquad (26)$$

and

$$8\pi Ds_{\mu\nu} = \eta_{\nu\sigma} \wedge \Omega^\sigma{}_\mu - \eta_{\mu\sigma} \wedge \Omega^\sigma{}_\nu. \qquad (27)$$

Cartan required the right side of (26) to vanish. If, instead, one uses the field equations (19) and (22) to evaluate the right sides of (26) and (27), one obtains,

$$Dt_\mu = Q^\rho{}_{\mu\nu}\theta^\nu \wedge t_\rho - \tfrac{1}{2}R^\rho{}_{\sigma\mu\nu}\theta^\nu \wedge s^\sigma{}_\rho \qquad (28)$$

and

$$Ds_{\mu\nu} = \theta_\nu \wedge t_\mu - \theta_\mu \wedge t_\nu. \qquad (29)$$

Let v be a vector field generating a group of symmetries of the Riemann–Cartan space (M, g, ω) so that equations (13) and (14) hold. Equations (28) and (29) then imply that the 3-form

$$j = v^\mu t_\mu + \tfrac{1}{2}\,\tilde{\nabla}^\nu v^\mu s_{\mu\nu}$$

is closed, $\mathrm{d}j = 0$. In particular, in the limit of SRT, in Cartesian coordinates x^μ, to a constant vector field v there corresponds the projection onto v of the energy-momentum density. If $A^{\mu\nu}$ is a constant bivector, then $v^\mu = A^\mu{}_\nu x^\nu$ gives $j = j^{\mu\nu}A_{\mu\nu}$, where $j^{\mu\nu}$ is as in (6).

Spinning fluid and the generalized Mathisson–Papapetrou equation of motion

As in classical general relativity, the right sides of the Einstein–Cartan equations need not necessarily be derived from a variational principle; they may be determined by phenomenological considerations. For example, following Weyssenhoff, consider a spinning fluid characterized by

$$t^{\mu\nu} = P^\mu u^\nu \quad \text{and} \quad s^{\mu\nu\rho} = S^{\mu\nu}u^\rho,$$

where $S^{\mu\nu} + S^{\nu\mu} = 0$ and u is the unit, timelike velocity field. Let $U = u^\mu\eta_\mu$ so that

$$t_\mu = P_\mu U \quad \text{and} \quad s_{\mu\nu} = S_{\mu\nu}U.$$

Define the *particle derivative* of a tensor field φ^a in the direction of u by

$$\dot{\varphi}^a\eta = D(\varphi^a U).$$

For a scalar field φ, the equation $\dot{\varphi} = 0$ is equivalent to the conservation law $\mathrm{d}(\varphi U) = 0$. Define $\rho = g_{\mu\nu}P^\mu u^\nu$, then (29) gives an equation of motion of spin

$$\dot{S}_{\mu\nu} = u_\nu P_\mu - u_\mu P_\nu$$

so that

$$P_\mu = \rho u_\mu + \dot{S}_{\mu\nu}u^\nu.$$

From (28) one obtains the equation of translatory motion,

$$\dot{P}_\mu = (Q^\rho{}_{\mu\nu}P_\rho - \tfrac{1}{2}R^{\rho\sigma}{}_{\mu\nu}S_{\rho\sigma})u^\nu,$$

which is a generalization to the Einstein–Cartan theory of the Mathisson–Papapetrou equation for

point particles with an intrinsic angular momentum.

From ECT to GRT: the effective energy-momentum tensor

Inside spinning matter, one can use (12) and (25) to eliminate torsion and replace the Sciama–Kibble system by a single Einstein equation with an effective energy-momentum tensor on the right side. Using the split (10) one can write (23) as

$$\mathring{R}_{\mu\nu} - \tfrac{1}{2}\, g_{\mu\nu}\mathring{R} = 8\pi T_{\mu\nu}^{\text{eff}}. \tag{30}$$

Here $\mathring{R}_{\mu\nu}$ and $\mathring{R}$ are, respectively, the Ricci tensor and scalar formed from g. The term in (10) that is quadratic in κ contributes to T^{eff} an expression quadratic in the components of the tensor $s_{\mu\nu\rho}$ so that, neglecting indices, one can write symbolically

$$T^{\text{eff}} = T + s^2. \tag{31}$$

The symmetric tensor T is the sum of t and a term coming from $\mathring{D}\kappa^\mu{}_\nu$ in (10),

$$T^{\mu\nu} = t^{\mu\nu} + \tfrac{1}{2}\,\mathring{\nabla}_\rho(s^{\nu\mu\rho} + s^{\nu\rho\mu} + s^{\mu\rho\nu}). \tag{32}$$

It is remarkable that the Belinfante–Rosenfeld symmetrization of the canonical energy-momentum tensor appears as a natural consequence of ECT. From the physical point of view, the second term on the right side of (31), can be thought of as providing a spin-spin contact interaction, reminiscent of the one appearing in the Fermi theory of weak interactions.

It is clear from (30), (31) and (32) that whenever terms quadratic in spin can be neglected — in particular in the linear approximation — ECT is equivalent to GRT. To obtain essentially new effects, the density of spin squared should be comparable to the density of mass. For example, to achieve this, a nucleon of mass m should be squeezed so that its radius r_{Cart} be such that

$$\left(\frac{\ell^2}{r_{\text{Cart}}^3}\right)^2 \approx \frac{m}{r_{\text{Cart}}^3}.$$

Introducing the Compton wavelength $r_{\text{Compt}} = \ell^2/m \approx 10^{-13}$ cm, one can write

$$r_{\text{Cart}} \approx (\ell^2 r_{\text{Compt}})^{1/3}.$$

The "Cartan radius" of the nucleon, $r_{\text{Cart}} \approx 10^{-26}$ cm, so small when compared to its physical radius under normal conditions, is much larger than the Planck length. Curiously enough, the energy ℓ^2/r_{Cart} is of the order of the energy at which, according to some estimates, the grand unification of interactions is presumed to occur.

Cosmology with spin and torsion

In the presence of spinning matter, T^{eff} need not satisfy the positive energy conditions, even if T does. Therefore, the classical singularity theorems of Penrose and Hawking can here be overcome. In ECT, there are simple cosmological solutions without singularities. The simplest such solution, found in 1973 by Kopczyński, is as follows. Consider a Universe filled with a spinning dust such that $P^\mu = \rho u^\mu$, $u^\mu = \delta_0^\mu$, $S_{23} = \sigma$, $S_{\mu\nu} = 0$ for $\mu + \nu \neq 5$ and both ρ and σ are functions of $t = x^0$ alone. These assumptions are compatible with the Robertson–Walker line element $\mathrm{d}t^2 - \mathcal{R}(t)^2(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2)$, where $(x, y, z) = (x^1, x^2, x^3)$ and torsion is determined from (25). The Einstein equation (23) reduces to the modified Friedmann equation,

$$\tfrac{1}{2}\dot{\mathcal{R}}^2 - M\mathcal{R}^{-1} + \tfrac{3}{2}S^2\mathcal{R}^{-4} = 0, \tag{33}$$

supplemented by the conservation laws of mass and spin,

$$M = \tfrac{4}{3}\pi\rho\mathcal{R}^3 = \text{const.}, \quad S = \tfrac{4}{3}\pi\sigma\mathcal{R}^3 = \text{const.}$$

The last term on the left side of (33) plays the role of a repulsive potential, effective at small values of $\mathcal{R}$; it prevents the solution from vanishing. It should be noted, however, that even a very small amount of shear in u results in a term counteracting the repulsive potential due to spin. Neglecting shear and making the (unrealistic) assumption that matter in the Universe at $t = 0$ consists of about 10^{80} nucleons of mass m with aligned spins, one obtains the estimate $\mathcal{R}(0) \approx 1$ cm and a density of the order of m^2/ℓ^4, very large, but much smaller than the Planck density $1/\ell^2$.

Tafel (1975) found large classes of cosmological solutions with a spinning fluid, admitting a group of symmetries transitive on the hypersurfaces of constant time. The models corresponding to symmetries of Bianchi types I, VII_0 and V are non-singular, provided that the influence of spin exceeds that of shear.

Summary

The Einstein-Cartan theory is a viable theory of gravitation that differs very slightly from the Einstein theory; the effects of spin and torsion can be significant only at densities of matter that are very high, but nevertheless much smaller than the Planck density at which quantum gravitational effects are believed to dominate. It is possible that the Einstein–Cartan theory will prove to be a better classical limit of a future quantum theory of gravitation than the theory without torsion.

Further Reading

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