Spin spaces, Lipschitz groups, and spinor bundles

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Abstract. It is shown that every bundle $\Sigma \to M$ of complex spinor modules over the Clifford bundle $\mathsf{Cl}(g)$ of a Riemannian space (M, g) with local model (V, h) is associated with an lpin ("Lipschitz") structure on M, this being a reduction of the $\mathsf{O}(h)$-bundle of all orthonormal frames on M to the Lipschitz group $\mathsf{Lpin}(h)$ of all automorphisms of a suitably defined spin space. An explicit construction is given of the total space of the $\mathsf{Lpin}(h)$-bundle defining such a structure. If the dimension m of M is even, then the Lipschitz group coincides with the complex Clifford group and the lpin structure can be reduced to a pin^c structure. If $m = 2n - 1$, then a spinor module Σ on M is of the Cartan type: its fibres are 2^n-dimensional and decomposable at every point of M, but the homomorphism of bundles of algebras $\mathsf{Cl}(g) \to \mathsf{End}\,\Sigma$ globally decomposes if, and only if, M is orientable. Examples of such bundles are given. The topological condition for the existence of an lpin structure on an odd-dimensional Riemannian manifold is derived and illustrated by the example of a manifold admitting such a structure, but no pin^c structure.

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Dedicated to the memory of Alfred Gray

1. Introduction

There are at least two approaches to objects appearing in classical differential geometry. One of them, most common now, can

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be traced back to the method of "moving frames" of Darboux and Cartan; it is based on the notion of a *principal* bundle with an infinitesimal connection in the sense of Ehresmann. Fields of geometric objects are defined as sections of associated bundles. Another approach assumes *vector* bundles with connections as the starting point; as emphasized by Lang [17], this approach is well adapted to the treatment of infinite-dimensional manifolds, modeled on Banach spaces. In the category of finite-dimensional smooth manifolds, the two approaches are essentially equivalent.

Spinors on Riemannian manifolds can also be introduced in two similar ways; however, the relation between the two approaches is subtler than in the case of tangent vectors. It is the purpose of this paper to describe this relation in the general setting of (not necessarily proper and orientable) Riemannian manifolds.

Recall first that if an orientable, proper Riemannian m-manifold (M, g) has a spin structure ("if M is spin"),

$$\begin{array}{ccccc} \mathsf{Spin}_m & \longrightarrow & Q_0 & & \\ {\scriptstyle \mathrm{Ad}}\big\downarrow & & \big\downarrow{\scriptstyle \chi_0} & & \\ \mathsf{SO}_m & \longrightarrow & P_0 & \xrightarrow{\pi} & M, \end{array} \tag{1}$$

then, given a spinor representation θ of Spin_m in a vector space S_0, one defines spinor fields as sections of the associated vector bundle $Q_0 \times_\theta S_0 \to M$. Another definition focuses on the vector bundle itself: it assumes the existence of a bundle $\Sigma \to M$ of modules over the bundle $\mathsf{Cl}(g)$ of Clifford algebras on M. This point of view can be traced back to early papers by physicists [7]; in particular, to the remarkable article by Schrödinger [20] which contains the first—to our knowledge—derivation of the formula for the square of the Dirac operator on Riemannian manifolds. There are two somewhat forgotten papers by Karrer [14, 15] on the description of spinors in terms of Clifford modules; they contain the relevant definitions in the language of contemporary mathematics; see also Chapter 3 in [3] and Chapter II in [18]. The second approach is more general in the sense that the fibres of Σ need not be isomorphic to a spinor space, carrying an irreducible representation of the Clifford algebra; for example, one can take for Σ the bundle $\wedge T^*M$ of exterior algebras. This possibility was considered by physicists, in an attempt to find the relativistic, quantum-mechanical equation of the electron, as early as 1928 [11]. Later, it has been developed by Kähler [12]. One easily sees that a bundle associated by a spinor representation with a spin structure is a bundle of modules over $\mathsf{Cl}(g)$ (see, e.g., Prop. 3.8 in [18]), but the converse is not true, even if $\mathsf{Cl}(g_x) \to \mathsf{End}\,\Sigma_x$ is the spinor representation for every $x \in M$ (see Examples 3, 6 and 10 in this paper).

To compare these two definitions, it is convenient to introduce a *category of spin spaces* (Section 3). We show that a spinor bundle Σ of spinor modules over $\mathsf{Cl}(g)$ is associated with an lpin structure on M, this being a reduction of the O_m-bundle of all orthonormal frames P to the *Lipschitz group* Lpin_m of automorphisms of a spin space. If m is *even*, then Lpin_m is essentially the *Clifford group*[1]. The case of m *odd* is somewhat more complicated because the complexification of the Clifford algebra of a real, odd-dimensional vector space is not simple and the adjoint representation of the corresponding Pin group does not cover the full orthogonal group. In this case, Lpin_m is the smallest group containing the Clifford group and such that the reflection $v \mapsto -v$ extends to an inner automorphism. In Sections 4 and 5 we recall the definition of a bundle of Clifford modules on a Riemannian manifold and of a suitably generalized spinor structure and spinor bundle. We show that every faithful spinor bundle is associated with an lpin structure and prove a theorem on the connection between the orientability of M and the decomposability of such a spinor bundle. An lpin structure on a manifold that is orientable or even-dimensional can be reduced to a pin^c structure. In Section 6 we establish the topological condition for the existence of an lpin structure on an odd-dimensional, non-orientable manifold, and give examples of such manifolds that admit an lpin structure, but no pin^c structure.

2. Notation and preliminaries

We use a notation and terminology which are largely standard in differential geometry [16] and spinor analysis [9, 18]. If S and S' are finite-dimensional complex vector spaces, then $\mathsf{Hom}(S, S')$ is the vector space of all complex-linear maps of S into S' and

[1] This name was introduced by Chevalley [6]. R. Lipschitz was the first to consider groups associated with Clifford algebras; see [24] and the references given there. For this reason, we find it appropriate to associate his name with one of the spinor groups.

$\mathsf{End}S = \mathsf{Hom}(S,S)$ is an algebra over $\mathbb{C}$. Every algebra under consideration here has a unit element; homomorphisms of algebras map units into units. We write $S^* = \mathsf{Hom}(S,\mathbb{C})$; if $f \in \mathsf{Hom}(S,S')$, then $f^* \in \mathsf{Hom}(S'^*,S^*)$ is defined by $\langle s, f^*(t')\rangle$ for every $s \in S$ and $t' \in S'^*$. A similar notation is used for real vector spaces.

A *quadratic space* is defined as a pair (V,h), where V is a real, finite-dimensional vector space and $h : V \to \mathbb{R}$ is a non-degenerate quadratic form. We denote by $\hat{h}$ the symmetric linear isomorphism of V onto V^*, associated with h. For our purposes, it is convenient to label with h the groups and algebras associated, in a natural manner, with the quadratic space (V,h); e.g. $\mathsf{O}(h)$ is the group of all its orthogonal automorphisms. The real *Clifford algebra* $\mathsf{Cl}(h)$ corresponding to (V,h) contains $\mathbb{R} \oplus V$ as a vector subspace and $v^2 = h(v)$ for every $v \in V$. If $(e_1,\dots,e_m)$ is an orthonormal frame in V, then $\eta = e_1 \dots e_m \in \mathsf{Cl}(h)$ and $-\eta$ are the *volume elements.* Their squares are either 1 or -1, depending on the signature of h; we define $\iota(h) \in \{1,\sqrt{-1}\}$ so that $\eta^2 = \iota(h)^2$. If $V = \mathbb{R}^m$ and h is the standard positive (resp., negative) definite quadratic form on that vector space, then we write Cl^+_m (resp., Cl^-_m) instead of $\mathsf{Cl}(h)$ and use similar conventions for the various groups associated with (V,h); see (1) for examples. The isometry $v \mapsto -v$ extends to an involutive automorphism α of the algebra defining its $\mathbb{Z}_2$-grading: $\mathsf{Cl}(h) = \mathsf{Cl}^0(h) \oplus \mathsf{Cl}^1(h)$. The even subalgebras of Cl^+_m and Cl^-_m, which are isomorphic, are denoted by Cl^0_m. The map $\mathbb{R}^m \to \mathsf{Cl}^0_{m+1}$, $v \mapsto ve_{m+1}$, extends to an isomorphism of algebras

$$i_m : \mathsf{Cl}^-_m \to \mathsf{Cl}^0_{m+1}. \tag{2}$$

Let $\mathbb{C}^\times = \mathbb{C} \setminus \{0\}$ be the multiplicative group of complex numbers. If G and H are groups and the sequences $1 \to \mathbb{Z}_2 \to G$ and $1 \to \mathbb{Z}_2 \to H$ are exact, then there is also the exact sequence $1 \to \mathbb{Z}_2 \to G \cdot H$, where

$$G \cdot H \quad \text{is the group} \quad (G \times H)/\mathbb{Z}_2.$$

One writes G^c instead of $\mathsf{U}_1 \cdot G$.

All manifolds and maps among them are assumed to be smooth. Manifolds are finite-dimensional, but not necessarily compact. If $f : M \to N$ is a map of manifolds, then $Tf : TM \to TN$ is the corresponding tangent (derived) map. A *Riemannian manifold* M is assumed to be connected; it has a metric tensor field g which is non-degenerate, but not necessarily definite; if it is, then M is said to be *proper* Riemannian. If $\pi : E \to M$ is a fibre bundle over a manifold M, then $E_x = \pi^{-1}(x) \subset E$ is the fibre over $x \in M$; in particular, $T_xM \subset TM$ is the tangent vector space to M at x. If the pair (f, h) is a a morphism of the principal bundles $G \to Q \to M$ and $G' \to Q' \to M$ so that $f : G \to G'$ is a morphism of Lie groups and $h : Q \to Q'$ is a morphism of fibre bundles such that $h(qa) = h(q)f(a)$ for every $q \in Q$ and $a \in G$, then one says that Q is an f*-reduction* of Q' to the Lie group G. Let $G \to Q \to M$ be a principal bundle and let θ be a representation of G in a complex vector space S; the total space of the vector bundle associated by θ with Q is the set $Q \times_\theta S$ of all classes $[(q, s)]$, where $(q, s) \in Q \times S$ and the equivalence relation is: $(q, s) \equiv (q', s')$ if, and only if, there is $a \in G$ such that $q' = qa$ and $s = \theta(a)s'$.

To reconstruct the principal bundle from a vector bundle, it is convenient to introduce a category $\mathcal{C}$ of isomorphisms of finite-dimensional vector spaces such that if $S \in \operatorname{Obj}\mathcal{C}$, then $G = \operatorname{Mor}(S, S)$ is a closed subgroup of $\mathsf{GL}(S)$. Given a vector bundle $\Sigma \to M$ with typical fibre $S \in \operatorname{Obj}\mathcal{C}$, such that $\Sigma_x \in \operatorname{Obj}\mathcal{C}$ for every $x \in M$, one defines the total space of the principal bundle to be

$$Q = \{q \in \operatorname{Mor}(S, \Sigma_x) | x \in M\}. \tag{3}$$

The group G acts in Q by composition and there is a natural projection $Q \to M$.

Throughout this paper, given a positive integer m, we define $\nu(m)$ to be the integer part of $\frac{1}{2}(m + 1)$; sometimes we write n instead of $\nu(m)$.

3. Spinor representations and spin spaces

For the purposes of this paper, it is convenient to have a precise definition of spinor groups and representations that, in the case of odd dimensions, slightly difers from the one prevailing in the literature.

(i) Recall that if the dimension m of a quadratic space (V, h) is *even*, $m = 2n$, then the algebra $\mathsf{Cl}(h)$ is central simple and has one, up to complex equivalence, faithful and irreducible *Dirac* representation in a complex, 2^n-dimensional vector space S. Restricted to

$\mathsf{Cl}^0(h)$, this representation decomposes into the direct sum of two complex-inequivalent, *half-spinor* (Weyl) representations σ_+ and σ_-.

(ii) If m is *odd*, $m = 2n-1$, then the algebra $\mathsf{Cl}^0(h)$ is central simple; it has a faithul and irreducible *Pauli* representation in a complex vector space S_0 of dimension 2^{n-1}; this representation extends to two complex-inequivalent representations σ and $\sigma \circ \alpha$ of the full algebra $\mathsf{Cl}(h)$ such that $\sigma(\eta) = \iota(h)\,\mathrm{id}_{S_0}$. These representations need not be faithful (example: Cl_1^+). The *Cartan* representation[2] $\sigma \oplus \sigma \circ \alpha$ of $\mathsf{Cl}(h)$ in the 2^n-dimensional vector space $S = S_0 \oplus S_0$ is faithful.

The following Proposition is an obvious consequence of (2) and our terminology:

Proposition 1. *Consider the sequence of homomorphisms of algebras*

$$\mathsf{Cl}_m^- \xrightarrow{\mathrm{inj}} \mathsf{Cl}_{m+1}^- \xrightarrow{i_{m+1}} \mathsf{Cl}_{m+2}^0 \xrightarrow{\theta} \mathsf{End}\, S.$$

If m is even (resp., odd) and θ is a Weyl (resp., Pauli) representation, then $\theta \circ i_{m+1}$ is a Pauli (resp., Dirac) representation and $\theta \circ i_{m+1} \circ \mathrm{inj}$ is a Dirac (resp., Cartan) representation.

Definition 1. Any representation of $\mathsf{Cl}(h)$ or $\mathsf{Cl}^0(h)$ equivalent to one of the representations described in (i) and (ii) is called a *spinor representation* of that algebra.

Definition 2. Let (k,l) be a pair of non-negative integers and let $m = k + l$. The *category $\mathcal{C}_{k,l}$ of spin spaces* is as follows. An *object* of $\mathcal{C}_{k,l}$ is a triple $\varsigma = (S, V, h)$ such that S is a complex, $2^{\nu(m)}$-dimensional vector space, $V \subset \mathsf{End}\, S$ is a real vector space of dimension m and $v^2 = h(v)\,\mathrm{id}_S$ for every $v \in V$, where h is a quadratic form on V of signature (k,l). *Morphisms* between two spin spaces $\varsigma = (S, V, h)$ and $\varsigma' = (S', V', h')$ of the same category are defined by

$$\mathrm{Mor}(\varsigma, \varsigma') = \{a \in \mathsf{Hom}(S, S') | a \text{ is invertible and } aVa^{-1} = V'\}.$$

[2] This representation rarely appears because it is decomposable; it is needed to define the Dirac operator on odd-dimensional, non-orientable pin manifolds [4, 23]. The names of Pauli and Dirac are associated by physicists with spinors in dimensions 3 and 4, respectively.

If $a \in \mathrm{Mor}(\varsigma, \varsigma')$, then the map $V \to V'$, given by $v \mapsto ava^{-1}$, is an isometry: there is a forgetful functor from the category of spin spaces to the corresponding category of quadratic spaces. If $(S, V, h) \in \mathrm{Obj}\, \mathcal{C}_{k,l}$, then $(S, \sqrt{-1}\, V, -h) \in \mathrm{Obj}\, \mathcal{C}_{l,k}$. Given a quadratic space (V, h) of even (resp., odd) dimension m, one constructs a spin space by considering a Dirac (resp., Cartan) representation of $\mathsf{Cl}(h)$ in S and *identifying* V with its image in $\mathsf{End}\, S$.

Proposition 2. *Let (S, V, h) be a spin space. The dimension of the complex vector space*

$$A(h) = \{w \in \mathsf{End}\, S | wv + vw = 0 \text{ for every } v \in V\}$$

is 1 *or* 2 *depending on whether m is* even *or* odd.

Proof. If m is even, then $A(h)$ is spanned by $\Gamma = \sqrt{-1}\, \iota(h)\eta$. If m is odd, then it is convenient to represent the elements of $\mathsf{End}\, S$ in a block form, corresponding to the decomposition $S = S_0 \oplus S_0$, so that

$$v = \begin{pmatrix} \sigma(v) & 0 \\ 0 & -\sigma(v) \end{pmatrix} \in V, \quad \eta = \iota(h) \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \tag{4}$$

where σ is the representation defined in (ii) above and $I = \mathrm{id}_{S_0}$. The space $A(h)$ is then spanned by the pair $(\Gamma, \Gamma\eta)$, where

$$\Gamma = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}. \quad \square$$

Since both the Dirac and the Cartan representations are faithful, it follows from the universal property of Clifford algebras that, given a spin space (S, V, h), one can *identify* $\mathsf{Cl}(h)$ with the subalgebra of $\mathsf{End}\, S$ generated, over the reals, by $V \subset \mathsf{End}\, S$. After this identification, one has

$$\alpha(a) = \Gamma^{-1} a \Gamma \quad \text{for every} \quad a \in \mathsf{Cl}(h) \subset \mathsf{End}\, S, \tag{5}$$

where, and in the sequel, Γ is an element of $A(h)$ such that $\Gamma^2 = -\mathrm{id}_S$. If m is even, then, over the complex numbers, the real vector space V generates $\mathbb{C} \otimes \mathsf{Cl}(h) = \mathsf{End}\, S$; if m is odd, then, over $\mathbb{C}$, the vector space $V \oplus \mathbb{R}\Gamma \subset \mathsf{End}\, S$ generates the algebra $\mathsf{End}\, S$. The involutive automorphism $a \mapsto \eta a \eta^{-1}$ of $\mathsf{End}\, S$ defines a $\mathbb{Z}_2$-grading of the algebra, $\mathsf{End}\, S = \mathsf{End}^0 S \oplus \mathsf{End}^1 S$, so that

$$\mathsf{End}^{\varepsilon} S = \{a \in \mathsf{End}\, S | a\eta = (-1)^{\varepsilon} \eta a\}, \ \varepsilon = 0, 1. \tag{6}$$

Let (S, V, h) be a spin space. The group $\mathsf{Pin}(h)$ can be now defined as the subgroup of $\mathsf{GL}(S)$ generated by all unit vectors, i.e. by all elements $v \in V \subset \mathsf{End}\, S$ such that either $v^2 = \mathrm{id}_S$ or $v^2 = -\,\mathrm{id}_S$. The *twisted adjoint* representation $\widetilde{\mathrm{Ad}}$ of $\mathsf{Pin}(h)$ in V is defined by

$$\widetilde{\mathrm{Ad}}(a)v = \alpha(a)va^{-1}, \ a \in \mathsf{Pin}(h) \text{ and } v \in V.$$

The spin group is $\mathsf{Spin}(h) = \mathsf{Pin}(h) \cap \mathsf{Cl}^0(h)$ and $\mathsf{Spin}_0(h)$ is the connected component of $\mathsf{Pin}(h)$; if the form h is definite, then $\mathsf{Spin}(h) = \mathsf{Spin}_0(h)$.

The linear map $V \to \mathsf{End}\, S$, $v \mapsto \Gamma v$, has the Clifford property, $(\Gamma v)^2 = h(v)\,\mathrm{id}_S$. It extends to a representation of the Clifford algebra

$$\gamma : \mathsf{Cl}(h) \to \mathsf{End}\, S \quad \text{such that } \gamma \circ \alpha = \mathrm{Ad}(\Gamma^{-1}) \circ \gamma\ . \tag{7}$$

Since $\Gamma v = (\mathrm{id}_S + \Gamma)v(\mathrm{id}_S + \Gamma)^{-1}$, the representation (7) is equivalent to the inclusion representation $\mathsf{Cl}(h) \to \mathsf{End}\, S$. Note also that if the dimension of V is odd, then

$$\gamma(v) = \begin{pmatrix} 0 & \sigma(v) \\ \sigma(v) & 0 \end{pmatrix}$$

whereas, in the inclusion representation, a vector is represented by a "block-diagonal" matrix (4).

Let $\mathsf{Pin}_\gamma(h)$ be the image of $\mathsf{Pin}(h)$ by the monomorphism (7) of algebras; by restriction, it gives rise to the isomorphism of groups

$$\gamma : \mathsf{Pin}(h) \to \mathsf{Pin}_\gamma(h). \tag{8}$$

The isomorphic groups $\mathsf{Pin}(h)$ and $\mathsf{Pin}_\gamma(h)$ are differently situated in $\mathsf{End}\, S$ relative to V.

Proposition 3. *The groups* $\mathsf{Pin}(h)$ *and* $\mathsf{Pin}_\gamma(h)$ *provide equivalent extensions of* $\mathsf{O}(h)$ *by* $\mathbb{Z}_2$*: the diagram of group homomorphisms*

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbb{Z}_2 & \longrightarrow & \mathsf{Pin}(h) & \xrightarrow{\widetilde{\mathrm{Ad}}} & \mathsf{O}(h) & \longrightarrow & 1 \\ & & \Vert & & \downarrow{\scriptstyle \gamma} & & \Vert & & \\ 1 & \longrightarrow & \mathbb{Z}_2 & \longrightarrow & \mathsf{Pin}_\gamma(h) & \xrightarrow{\mathrm{Ad}} & \mathsf{O}(h) & \longrightarrow & 1 \end{array}$$

is commutative and its two horizontal sequences are exact.

Proof. The exactness of the upper sequence is classical [1]. A unit vector u is in $\mathsf{Pin}(h)$, Γu is in $\mathsf{Pin}_\gamma(h)$ and the equalities $\widetilde{\mathrm{Ad}}(u)v = -uvu^{-1} = \Gamma uv(\Gamma u)^{-1} = (\mathrm{Ad}\circ\gamma)(u)v$ complete the proof. $\square$

Recall that, for m odd, the adjoint representation Ad maps $\mathsf{Pin}(h)$ onto $\mathsf{SO}(h)$ with a four-element kernel. In every dimension,

$$\mathsf{Pin}_\gamma(h) \cap \mathsf{Cl}^0(h) = \mathsf{Spin}(h).$$

The *extensions* of $\mathsf{O}(h)$ by $\mathbb{Z}_2$ corresponding to $\mathsf{Pin}(h)$ and $\mathsf{Pin}(-h)$ are *inequivalent*, even in the case of h of neutral signature (k,k), when the *groups* $\mathsf{Pin}(h)$ and $\mathsf{Pin}(-h)$ are *isomorphic*. Restricted to $\mathsf{Spin}(h)$, the representations Ad and $\widetilde{\mathrm{Ad}}$ coincide; the groups $\mathsf{Spin}(h)$ and $\mathsf{Spin}(-h)$ are isomorphic and provide equivalent extensions of $\mathsf{SO}(h)$ by $\mathbb{Z}_2$.

Definition 3. Let (S,V,h) be a spin space. A closed subgroup G of $\mathsf{GL}(S)$ is called a *spinor group* if it contains the group $\mathsf{Spin}_0(h)$ and is such that $aVa^{-1} = V$ for every $a \in G$. A representation of G equivalent to the evaluation representation of G in S—or to its subrepresentation— is called a *spinor representation* of this group.

If the group G is contained in $\mathsf{Cl}(h) \subset \mathsf{End}S$, then its spinor representation is equivalent to the restriction to G of the representation described in either (i) or (ii), depending on the parity of m. The groups $\mathsf{Spin}(h)$, $\mathsf{Pin}(h)$, $\mathsf{Pin}(-h)$, $\mathsf{Pin}_\gamma(h)$, $\mathsf{Pin}^c(h) = \mathsf{U}_1 \cdot \mathsf{Pin}(h)$ and $\mathbb{C}^\times \cdot \mathsf{Pin}(h)$ are spinor groups.

Definition 4. The *Lipschitz group* is the "largest" spinor group,

$$\mathsf{Lpin}(h) = \{a \in \mathsf{GL}(S) | aVa^{-1} = V\}.$$

Example 1. Consider $(S,V,h) \in \mathcal{C}_{1,0}$ so that $S = \mathbb{C}^2$ and $V = \mathbb{R}$. The injection $V \to \mathsf{End}S$ is given by $t \mapsto \begin{pmatrix} t & 0 \\ 0 & -t \end{pmatrix}$ and

$$\mathsf{Lpin}_{1,0} = \{a \in \mathsf{GL}_2(\mathbb{C}) \,|\, \text{either } a = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} \text{ or } a = \begin{pmatrix} 0 & \lambda \\ \mu & 0 \end{pmatrix}\}.$$

Let $\psi : \mathbb{Z}_2 \to \mathrm{Aut}(\mathbb{C}^\times \times \mathbb{C}^\times)$ be the homomorphism given by

$$\psi(-1)(\lambda,\mu) = (\mu,\lambda).$$

The group $\mathsf{Lpin}_{1,0}$ is isomorphic to the semi-direct product $\mathbb{Z}_2 \times_\psi (\mathbb{C}^\times \times \mathbb{C}^\times)$.

Proposition 4. *The group*

$$K(h) = \{a \in \mathsf{GL}(S) | av = va \ \textit{ for every } \ v \in V\}$$

is isomorphic to $\mathbb{C}^\times$ *or* $\mathbb{C}^\times \times \mathbb{C}^\times$ *depending on whether* m *is even or odd.*

Proof. If m is even, then $K(h) = \mathbb{C}^\times$. If m is odd, then $K(h) = \mathbb{C}^\times(\mathrm{id}_S + \iota(h)\eta) \times \mathbb{C}^\times(\mathrm{id}_S - \iota(h)\eta)$. □

Proposition 5. *Let* G *be a spinor group. The homomorphism* $\mathrm{Ad} : G \to \mathsf{O}(h)$, $\mathrm{Ad}(a)v = ava^{-1}$, *where* $a \in G$ *and* $v \in V$, *is surjective if, and only if,*

$$\mathsf{Pin}_\gamma(h) \subset K(h) \cdot G \tag{9}$$

Proof. If (9) holds, then, for every unit vector $v \in V$, there is $k \in K(h)$ and $a \in G$ such that $\Gamma v = ka$; therefore $\mathrm{Ad}(a) = \mathrm{Ad}(\Gamma v)$ and $\mathrm{Ad}(G)$ contains the reflection in every hyperplane. Conversely, if $a \in G$ is such that $\mathrm{Ad}(a)$ is the reflection in the hyperplane orthogonal to a unit vector v, then $\mathrm{Ad}(a^{-1}) \circ \mathrm{Ad}(\Gamma v) = \mathrm{id}_V$ so that $a^{-1}\Gamma v \in K(h)$. □

In particular, the Lipschitz group satisfies (9); since this group contains also all invertible vectors, the kernel of $\mathrm{Ad} : \mathsf{Lpin}(h) \to \mathsf{O}(h)$ is $K(h)$.

If a is in a spinor group G and η is a volume element, then $a\eta a^{-1}$ is also a volume element. Therefore, $a\eta a^{-1}$ equals either η or $-\eta$ and the grading (6) induces a $\mathbb{Z}_2$-grading of the spinor group,

$$G = G^0 \cup G^1, \text{ where } G^\varepsilon = G \cap \mathsf{End}^\varepsilon S, \ \varepsilon = 0, 1. \tag{10}$$

The real orthogonal group $\mathsf{O}(h)$ is $\mathbb{Z}_2$-graded by the determinant and the map $\mathrm{Ad} : G \to \mathsf{O}(h)$ preserves the grading.

If m is even and $G \subset \mathsf{Cl}(h)$, then $G^\varepsilon = G \cap \mathsf{Cl}^\varepsilon(h)$. For m odd, the grading (10) is *not* induced from that of the Clifford algebra: an element of $\mathsf{Lpin}(h)$ is odd if, and only if, it is of the form

$$\begin{pmatrix} 0 & \lambda\sigma(a) \\ \mu\sigma(a) & 0 \end{pmatrix}, \text{ for some } \lambda, \mu \in \mathbb{C}^\times \text{ and } a \in \mathsf{Spin}(h). \tag{11}$$

In this grading, the elements (4) of $\mathsf{Lpin}(h)$ are *even.*

There is a complex quadratic form h_0 on $A(h)$ such that, for every, $w \in A(h)$, one has $w^2 = h_0(w)\,\mathrm{id}_S$. If $a \in \mathsf{Lpin}(h)$ and $w \in A(h)$, then $\kappa(a)w = awa^{-1}$ is also in $A(h)$; this defines a homomorphism κ of the Lipschitz group to the complex orthogonal group $\mathsf{O}(h_0, \mathbb{C})$.

Theorem 1. *Let (V, h) be a quadratic space of dimension m.*
(i) *If m is* even, *then there is a split exact sequence*

$$1 \to \mathbb{C}^\times \cdot \mathsf{Spin}(h) \to \mathsf{Lpin}(h) \xrightarrow{\kappa} \mathbb{Z}_2 \to 1$$

and the group $\mathsf{Lpin}(h)$ is isomorphic to $\mathbb{C}^\times \cdot \mathsf{Pin}(h)$.
(ii) *If m is* odd, *then there is the exact sequence*

$$1 \to \mathbb{C}^\times \cdot \mathsf{Spin}(h) \to \mathsf{Lpin}(h) \xrightarrow{\kappa} \mathbb{Z}_2 \times_\varphi \mathbb{C}^\times \to 1, \tag{12}$$

where the homomorphism $\varphi : Z_2 \to \mathrm{Aut}\,\mathbb{C}^\times$, defining the semi-direct product group structure, is given by $\varphi(-1)z = z^{-1}$, $z \in \mathbb{C}^\times$. There is an isomorphism of groups

$$(\mathsf{Pin}(h) \times_\psi (\mathbb{C}^\times \times \mathbb{C}^\times))/\mathbb{Z}_2 \to \mathsf{Lpin}(h) \tag{13}$$

such that

$$\text{if } a \in \mathsf{Pin}(h) \smallsetminus \mathsf{Spin}(h), \text{ then } \psi(a)(\lambda, \mu) = (\mu, \lambda).$$

Proof. If $a \in \ker\kappa \subset \mathsf{Lpin}(h)$, then a is even with respect to the grading (10) of $G = \mathsf{Lpin}(h)$ and commutes with Γ.
(i) For m even, the group $\mathsf{O}(h_0, \mathbb{C})$ is $\mathsf{O}_1(\mathbb{C}) = \mathbb{Z}_2$, the map κ is the grading homomorphism and $\mathsf{Lpin}(h) = \mathbb{C}^\times \cdot \mathsf{Pin}(h)$. Let $u \in V$ be a unit vector, put $v = u$ if $h(u) = 1$ and $v = \sqrt{-1}u$ if $h(u) = -1$. The map $f : \mathbb{Z}_2 \to \mathsf{Lpin}(h)$ such that $f(1) = 1$ and $f(-1) = v$ is a splitting homomorphism.
(ii) Let now m be odd. The set of all odd elements of $\mathsf{Lpin}(h)$ generates the group; every odd element of $\mathsf{Lpin}(h)$ is of the form (11). The group $\mathsf{O}(h_0, \mathbb{C})$ is now isomorphic to $\mathsf{O}_2(\mathbb{C})$ and can be identified with the semi-direct product $\mathbb{Z}_2 \times_\varphi \mathbb{C}^\times$. Under this identification, the homomorphism κ maps (11) to $(-1, \lambda\mu^{-1})$. The injection of $\mathbb{C}^\times \cdot \mathsf{Spin}(h)$ into $\mathsf{Lpin}(h)$ is given by

$$[(\lambda, a)] \mapsto \begin{pmatrix} \lambda\sigma(a) & 0 \\ 0 & \lambda\sigma(a) \end{pmatrix} \text{ for } \lambda \in \mathbb{C}^\times \text{ and } a \in \mathsf{Spin}(h).$$

For $a \in \mathsf{Pin}(h)$ odd, the isomorphism (13) sends

$$[(a, \lambda, \mu)] = [(-a, -\lambda, -\mu)]$$

to (11). $\square$

4. Bundles of Clifford modules and spinor structures

4.1. Clifford bundles and modules

Let (M, g) be a Riemannian manifold and let g_x be the restriction of g to the vector space T_xM. The Clifford algebra associated with the quadratic space (T_xM, g_x) is $\mathsf{Cl}(g_x)$ and $\mathsf{Cl}(g) = \bigcup_x \mathsf{Cl}(g_x)$ is the total space of the *Clifford bundle* of (M, g) [1].

Definition 5. A bundle of *Clifford modules* on the Riemannian space (M, g) is a complex vector bundle $\Sigma \to M$ with a homomorphism of bundles of algebras

$$\tau : \mathsf{Cl}(g) \to \mathsf{End}\,\Sigma. \tag{14}$$

In other words, for every $x \in M$, the vector space Σ_x is a left module over the algebra $\mathsf{Cl}(g_x)$. Restricted to $TM \subset \mathsf{Cl}(g)$, the map τ is a *Clifford morphism*, i.e. a homomorphism of vector bundles such that

$$\tau(v)^2 = g_x(v)\,\mathrm{id}_{\Sigma_x} \quad \text{for every } x \in M \text{ and } v \in T_xM.$$

It follows from the universal property of Clifford algebras that, conversely, given a vector bundle Σ over M and a Clifford morphism $TM \to \mathsf{End}\,\Sigma$, one can extend it to a homomorphism (14) of bundles of algebras.

The following examples are well known:

Example 2. The bundle of *exterior algebras* on M. Put $\Sigma = \wedge T^*M$ and define τ by $\tau(v)\omega = v \lrcorner \omega + \hat{g}_x(v) \wedge \omega$ for $v \in T_xM$ and $\omega \in \Sigma_x$.

Example 3. Let (M, g) be an *almost Hermitean space* and let J be the associated orthogonal almost complex structure. Define $N = \{n \in \mathbb{C} \otimes TM \,|\, J(n) = \sqrt{-1}\,n\}$ and put $\Sigma = \wedge N$. The map τ given, for every $n \in N$ and $\omega \in \Sigma$, by $\tau(n + \bar{n})\omega = \sqrt{2}\,(\hat{g}(\bar{n}) \lrcorner \omega + n \wedge \omega)$ is a Clifford morphism.

Example 4. Let (M, g) be a Riemannian manifold of dimension m with (V, h) as its local model. Consider a pin structure on M defined, as in [2], to be an $\widetilde{\mathrm{Ad}}$-reduction of the bundle P of all orthonormal frames on M to the group $\mathsf{Pin}(h)$,

$$\begin{array}{ccccc} \mathsf{Pin}(h) & \longrightarrow & \widetilde{Q} & & \\ {\scriptstyle \widetilde{\mathrm{Ad}}}\downarrow & & \downarrow{\scriptstyle \widetilde{\chi}} & & \\ \mathsf{O}(h) & \longrightarrow & P & \xrightarrow{\pi} & M, \end{array} \tag{15}$$

so that $\pi \circ \widetilde{\chi} : \widetilde{Q} \to M$ is a principal $\mathsf{Pin}(h)$-bundle and $\widetilde{\chi}(qa) = \widetilde{\chi}(q) \circ \widetilde{\mathrm{Ad}}(a)$ for every $q \in \widetilde{Q}$ and $a \in \mathsf{Pin}(h)$. Let $\theta : \mathsf{Cl}(h) \to \mathsf{End}S$ be either the Dirac (m even) or the Cartan (m odd) evaluation representation. The associated bundle $\Sigma = \widetilde{Q} \times_\theta S$ is a bundle of Clifford modules. The Clifford morphism is defined as follows: if $v \in T_x M, q \in \widetilde{Q}_x$ and $s \in S$, then

$$\tau(v)[(q, s)] = [(q, \Gamma\widetilde{\chi}(q)^{-1}(v)s)]. \tag{16}$$

To check that τ is correctly defined by (16), take $a \in \mathsf{Pin}(h) \subset \mathsf{GL}(S)$ and use (5) to compute

$$\begin{aligned} \tau(v)[(qa, a^{-1}s)] &= [(qa, \Gamma\widetilde{\chi}(qa)^{-1}(v)a^{-1}s)] \\ &= [(qa, \Gamma\widetilde{\mathrm{Ad}}(a^{-1})(\widetilde{\chi}(q)^{-1}(v))a^{-1}s)] \\ &= \tau(v)[(q, s)]. \end{aligned}$$

Note that Γ in (16) is essential to undo the twisting implied by the use of $\widetilde{\mathrm{Ad}}$.

4.2. Spinor structures

Consider a Riemannian, not necessarily orientable, manifold (M, g) and a spin space (S, V, h) such that (V, h) is a local model of the Riemannian manifold. Let G be a spinor group in the sense of Definition 3 such that $\mathsf{Pin}_\gamma(h) \subset K(h) \cdot G$ and let P be the bundle of all orthonormal frames on M.

Definition 6. A *spinor G-structure* on (M, g) is an Ad-reduction Q of the $\mathsf{O}(h)$-bundle P to the group G,

$$\begin{array}{ccccc} G & \longrightarrow & Q & & \\ {\scriptstyle \mathrm{Ad}}\big\downarrow & & \big\downarrow{\scriptstyle \chi} & & \\ \mathsf{O}(h) & \longrightarrow & P & \xrightarrow{\pi} & M, \end{array} \tag{17}$$

so that $\pi \circ \chi : Q \to M$ is a principal G-bundle and $\chi(qa) = \chi(q) \circ \mathrm{Ad}(a)$ for every $q \in Q$ and $a \in G$.

For $G = \mathsf{Pin}_\gamma(h)$, $\mathsf{Pin}^c_\gamma(h)$ and $\mathsf{Lpin}(h)$ one shortens the expression "spinor G-structure" to pin, pin^c and lpin structure, respectively. A pin structure on an non-orientable Riemannian manifold is often defined as in Example 4, as an $\widetilde{\mathrm{Ad}}$-reduction $\widetilde{Q}$ of P to the group $\mathsf{Pin}(h)$, where $\widetilde{\mathrm{Ad}}$ is the *twisted* adjoint representation. That definition is equivalent to ours, given by an Ad-reduction of P to $\mathsf{Pin}_\gamma(h)$. To show this explicitly, let us consider the pin structure (15). We construct a pin structure in the sense of Definition 6 for $G = \mathsf{Pin}_\gamma(h)$ by defining Q to be the bundle associated with the bundle $\widetilde{Q}$ by the isomorphism (8) so that

$$Q = \widetilde{Q} \times_\gamma \mathsf{Pin}_\gamma(h) \quad \text{and} \quad \chi([(q, \gamma(a))]) = \widetilde{\chi}(qa)$$

for $q \in Q$ and $a \in \mathsf{Pin}(h)$. A proof of the equivalence of these two definitions is based on Proposition 3. In view of these observations, we restrict ourselves to spinor structures defined in terms of the (untwisted) adjoint representation, even in the case of non-orientable, odd-dimensional Riemannian manifolds.

On an orientable Riemannian manifold, after reducing P to the group $\mathsf{SO}(h)$, one can consider spinor G-structures such that $\mathrm{Ad}(G) = \mathsf{SO}(h)$. For $G = \mathsf{Spin}(h)$ and $\mathsf{Spin}^c(h)$, one obtains the usual notion of spin and spin^c structure, respectively.

5. Spinor bundles

Definition 7. A bundle of Clifford modules (14) on the Riemannian manifold (M, g) is said to be a *spinor bundle* over M if, for every $x \in M$, the restriction of τ to $\mathsf{Cl}(g_x)$ is equivalent to a spinor representation of the algebra in the sense of Definition 1.

If, moreover, τ is injective, then the spinor bundle is said to be *faithful.*

Example 5. Let $\mathbb{R}^{m+1}$ be embedded in Cl^+_{m+1} so that $\mathbb{S}_m = \{x \in \mathbb{R}^{m+1} | x^2 = 1\}$ and $T\mathbb{S}_m = \{(x, y) \in \mathbb{S}_m \times \mathbb{R}^{m+1} | xy + yx = 0\}$. The Riemannian metric g_m on the m-sphere is defined by $g_m(x, y) = y^2$. Depending on whether m is even or odd, take $\theta : \mathsf{Cl}^0_{m+1} \to \mathsf{End}\, S_m$ to be either the Pauli or the Dirac representation so that $\dim_{\mathbb{C}} S_m = 2^{\nu(m)}$. Therefore, for m even (resp., odd) and every $x \in \mathbb{S}_m$, the Clifford map $\mathbb{R}^m \to \mathsf{End}\, S_m$ given by $y \mapsto \sqrt{-1}\theta(xy)$, where $xy + yx = 0$, extends to a Dirac (resp., Cartan) representation of Cl^+_m in S_m. The trivial vector bundle $\Sigma_m = \mathbb{S}_m \times S_m$ is made into a faithful spinor bundle by defining $\tau_m : \mathsf{Cl}(g_m) \to \mathsf{End}\, \Sigma_m$ so that $\tau_m(x, y)(x, s) = (x, \sqrt{-1}\theta(xy)s)$.

The bundles of Clifford modules described in Examples 3 and 4 are also spinor bundles. Since there are Hermitean manifolds that are not spin (e.g. the even-dimensional complex projective spaces), Example 3 shows that there are spinor bundles on orientable manifolds that are not associated with a spin structure.

Definition 8. Two spinor bundles

$$\tau_1 : \mathsf{Cl}(g) \to \mathsf{End}\, \Sigma_1 \quad \text{and} \quad \tau_2 : \mathsf{Cl}(g) \to \mathsf{End}\, \Sigma_2$$

on M are said to be *equivalent* if there is an isomorphism of vector bundles $i : \Sigma_1 \to \Sigma_2$ intertwining τ_1 and τ_2 so that, for every $x \in M$, $s \in \Sigma_x$ and $u \in T_x M$ one has $i(\tau_1(u)s) = \tau_2(u)i(s)$.

Example 6. Let $\mathbb{P}_m = \mathbb{S}_m/\mathbb{Z}_2$ be the real projective m-space. The action of $\mathbb{Z}_2$ on $\mathbb{S}_m$ lifts to its tangent bundle and $T\mathbb{P}_m$ can be identified with $T\mathbb{S}_m/\mathbb{Z}_2$. The metric on the sphere descends to the corresponding projective space. If θ is one of the representations referred to in Example 5 and now $\Sigma_m = \mathbb{P}_m \times S_m$, then the formula $\tau^{\pm}_m([(x, y)])([x], s) = ([x], \pm\sqrt{-1}\theta(xy)s)$ defines on Σ_m two structures of faithful spinor bundles over $\mathbb{P}_m$. The spinor bundles τ^+_m and τ^-_m are inequivalent.

Since for $m \equiv 1 \bmod 4$, $m > 1$, the space $\mathbb{P}_m$ has no spin structure, the above construction provides another example of a spinor bundle that is not associated with a spin structure. This example can be generalized to covering spaces with a finite cyclic group of deck transformations: let M be an odd-dimensional Riemannian manifold with the fundamental group isomorphic to $\mathbb{Z}_p$. If the universal covering space of M admits a spin structure, then there exists a spinor bundle over M.

Theorem 2. *Let (M, g) be a Riemannian manifold with an lpin structure* (17)*, $G = \mathsf{Lpin}(h) \subset \mathsf{GL}(S)$, and let $\theta : \mathsf{Lpin}(h) \to \mathsf{GL}(S)$ be the evaluation representation. There is a natural structure of a spinor bundle on the associated vector bundle $\Sigma = Q \times_\theta S \to M$.*

Proof. The Clifford morphism $\tau : TM \to \mathsf{End}\,\Sigma$ is defined by

$$\tau(v)[(q, s)] = [(q, \chi(q)^{-1}(v)s)],$$

where $v \in T_xM$, $q \in Q_x$ and $s \in S$. □

The following theorem shows that, conversely, every spinor bundle can be so obtained.

Theorem 3. *Every faithful spinor bundle $\Sigma \to M$ on a Riemannian manifold (M, g) with local model (V, h) is isomorphic to the bundle associated, by the spinor representation, with an lpin structure on that manifold.*

Proof. Since the bundle is assumed to be faithful, τ is injective and one can identify TM with its image by τ in $\mathsf{End}\,\Sigma$. One then constructs the total space Q of an lpin structure by taking for the fibre Q_x the set of all isomorphisms of the spin space (S, V, h) onto the spin space (Σ_x, T_xM, g_x). The map $\chi : Q_x \to P_x$ is given by $\chi(q) : V \to T_xM$, $\chi(q)v = qvq^{-1}$. If q and $q' \in Q_x$, then $q^{-1}q' \in \mathsf{Lpin}(h)$; the group $\mathsf{Lpin}(h)$ acts freely and transitively on Q_x and $\chi(qa) = \chi(q) \circ \mathrm{Ad}(a)$ for every $a \in \mathsf{Lpin}(h)$. It remains to check that the associated bundle of spinors $Q \times_\theta S \to M$ is isomorphic to $\Sigma \to M$: such an isomorphism is given by $[(q, s)] \mapsto q(s)$, where $q \in Q$ and $s \in S$. □

For every spinor bundle (14) and $x \in M$, the restriction $\tau_x = \tau|\mathsf{Cl}(g_x)$ is a spinor representation of the Clifford algebra $\mathsf{Cl}(g_x)$ in the vector space Σ_x. Therefore, if the dimension m of M is *odd*, then this representation decomposes into two irreducibles. Similarly, if m is *even*, then the restriction τ_x^0 of τ_x to the even subalgebra $\mathsf{Cl}^0(g_x)$ decomposes into the sum of two half-spinor representations. Put $\mathsf{Cl}^0(g) = \bigcup_x \mathsf{Cl}^0(g_x)$ and define

$$\mathsf{Cl}^{\mathrm{d}}(g) = \begin{cases} \mathsf{Cl}(g) & \text{if } m \text{ is odd,} \\ \mathsf{Cl}^0(g) & \text{if } m \text{ is even.} \end{cases} \tag{18}$$

Definition 9. Let Σ and $\mathcal{A}$ be a vector bundle and a bundle of algebras over M, respectively. Let $\tau : \mathcal{A} \to \mathsf{End}\,\Sigma$ be a morphism of bundles of algebras. We say that τ is *decomposable* if there are two vector bundles Σ_+ and Σ_- of positive fibre dimensions such that $\Sigma = \Sigma_+ \oplus \Sigma_-$ and, for every $x \in M$, one has $\tau(\mathcal{A}_x)\Sigma_{\pm x} \subset \Sigma_{\pm x}$.

Theorem 4. *Let* (14) *be a spinor bundle on an m-dimensional Riemannian manifold M and let τ^{d} be the restriction of τ to the bundle of algebras* $\mathsf{Cl}^{\mathrm{d}}(g)$ *as in* (18). *A necessary and sufficient condition for τ^{d} to be* decomposable *is that M be* orientable.

Proof. Let the quadratic space (V, h) be a local model of (M, g). If M is orientable, then there is a volume map $\mathrm{vol} : M \to \mathsf{Cl}^{\mathrm{d}}(g)$ such that $\mathrm{vol}(x)^2 = \iota(h)^2$ for every $x \in M$. Defining

$$\Sigma_\pm = (\mathrm{id} \pm \iota(h)\,\mathrm{vol})\Sigma \quad \text{so that} \quad \Sigma = \Sigma_+ \oplus \Sigma_- \tag{19}$$

and noting that $\mathrm{vol}(x)$ is in the center of $\mathsf{Cl}^{\mathrm{d}}(g_x)$, one obtains the required decomposition. Conversely, assume that M is not orientable. Let $\mathrm{Vol} \subset \mathsf{Cl}^{\mathrm{d}}(g)$ be the set of normalized volume elements at all points of M; the map $\mathrm{Vol} \to M$ is a double cover. Non-orientability of M is equivalent to the statement that the set Vol is connected. If τ^{d} decomposes, $\tau^{\mathrm{d}} = \tau^{\mathrm{d}}_+ \oplus \tau^{\mathrm{d}}_-$ and $\Sigma_+ = \tau^{\mathrm{d}}_+(\mathsf{Cl}^{\mathrm{d}}(g))\Sigma$, then $\tau^{\mathrm{d}}_+(\mathrm{Vol}_x) = \{\iota(h)\,\mathrm{id}_{\Sigma_{+x}}, -\iota(h)\,\mathrm{id}_{\Sigma_{+x}}\}$ so that τ^{d}_+ maps the connected set Vol onto a set with two components. The contradiction shows that τ^{d} does not decompose. $\square$

Remark. A Cartan spinor bundle $\Sigma \to M$ associated with a pin structure on an odd-dimensional manifold M always admits a decomposition into two Pauli bundles, $\Sigma = \Sigma'_+ \oplus \Sigma'_-$, corresponding to the decomposition of the Cartan representation, $\gamma = \sigma \oplus \sigma \circ \alpha$. This decomposition of Σ is given by $[(q, s)] = [(q, s'_+)] + [(q, s'_-)]$, where $s'_\pm = \frac{1}{2}(\mathrm{id}_S \pm \iota(h)\gamma(\eta))s$. If $v \in T_xM$, then $\tau(v)$ maps Σ'_{+x} into Σ'_{-x}. Therefore, on an orientable odd-dimensional M, this decomposition is different from ("transversal to") the decomposition (19). The following example illustrates this remark.

Example 7. Let M be the real projective quadric $(\mathbb{S}_1 \times \mathbb{S}_2)/\mathbb{Z}_2$ with a proper Riemannian metric descending from $\mathbb{S}_1 \times \mathbb{S}_2$. This is a non-orientable 3-manifold with a pin structure [4]. A typical element of TM can be written as $[(x, y, \xi, \eta)]$, where

$$x, \xi \in \mathbb{R}^2 \subset \mathsf{Cl}_2^+,\ y, \eta \in \mathbb{R}^3 \subset \mathsf{Cl}_3^+,\ x\xi + \xi x = 0,\ y\eta + \eta y = 0,$$

and $[(x, y, \xi, \eta)] = [(-x, -y, -\xi, -\eta)]$. Let $\theta : \mathsf{Cl}_2^+ \to \mathsf{End} S_1$ and $\sigma : \mathsf{Cl}_3^0 \to \mathsf{End} S_2$ be, respectively, the Dirac and the Pauli representations in the complex, 2-dimensional spaces of spinors S_1 and S_2. Let (e_1, e_2, e_3) and (f_1, f_2) be orthonormal bases in $\mathbb{R}^3$ and $\mathbb{R}^2$, respectively. There is a trivial spinor bundle $\Sigma = M \times (S_1 \otimes S_2)$ on M such that

$$\begin{aligned}&\tau([(x, y, \xi, \eta)])([(x, y)], s_1 \otimes s_2)\\ &\quad = ([(x, y)], \theta(\xi)s_1 \otimes \sigma(ye_1e_2e_3)s_2 + s_1 \otimes \sigma(y\eta)s_2).\end{aligned}$$

The map $\varpi : M \to \mathsf{End}\Sigma$ defined by

$$\varpi([(x, y)])([(x, y)], s_1 \otimes s_2) = ([(x, y)], \theta(xf_1f_2)s_1 \otimes \sigma(ye_1e_2e_3)s_2)$$

is involutive and $\varpi([(x, y)])$ anticommutes with $\tau([(x, y, \xi, \eta)])$. Putting $\Sigma_\pm = (\mathrm{id}_\Sigma \pm \varpi)\Sigma$, one obtains the decomposition referred to in the Remark. On the other hand, the spinor bundle Σ is not decomposable in the sense of Definition 9.

6. Topological conditions

In this section we restrict our considerations to *proper* Riemannian spaces. Following the notation of Section 2, we write $\mathsf{Pin}_m^+ = \mathsf{Pin}_{m,0}$ and $\mathsf{Pin}_m^- = \mathsf{Pin}_{0,m}$. These two groups provide inequivalent extensions of O_m by $\mathbb{Z}_2$. It is known that the groups $\mathsf{U}_1 \cdot \mathsf{Pin}_m^+$ and $\mathsf{U}_1 \cdot \mathsf{Pin}_m^-$ are isomorphic and give equivalent extensions of O_m by U_1; therefore, it is legitimate to denote both of these groups by Pin_m^c. Similarly, the isomorphic groups $\mathsf{Lpin}_{m,0}$ and $\mathsf{Lpin}_{0,m}$ are denoted by Lpin_m. It follows from the isomorphism (13) that Lpin_m contains $(\mathsf{Pin}_m \times_\psi (\mathsf{U}_1 \times U_1))/\mathbb{Z}_2$ as its maximal compact subgroup.

If $E \to M$ is a real vector bundle, then $w_i(E) \in \mathrm{H}^i(M, \mathbb{Z}_2)$ denotes its ith Stiefel-Whitney class. The manifold M is orientable if, and only if, $w_1(TM) = 0$.

Lemma. *For every real vector bundle E over a manifold M the class $w_1(E)^2 \in \mathrm{H}^2(M, \mathbb{Z}_2)$ is the* mod 2 *reduction of an element of $\mathrm{H}^2(M, \mathbb{Z})$.*

Proof. Assume the bundle E to have m-dimensional fibres and consider the line bundle $F = \wedge^m E$. The direct sum $F \oplus F$ is an orientable bundle of fibre dimension 2. Its Euler class $e(F \oplus F)$ is an integral cohomology class and $w_2(F \oplus F)$ is the mod 2 reduction of $e(F \oplus F)$. The Whitney product theorem gives $w_2(F \oplus F) = w_1(F)^2$. The equality $w_1(E) = w_1(F)$ is established by the "splitting method" used in [10] in the proof of Theorem 4.4.3 for the Chern classes. □

There are well-known topological obstructions to the existence of the various pin and spin structures on a a Riemannian space. Recall that the necessary and sufficient conditions for the existence of these structures are as follows [13]:

(i) spin structure: $w_1(TM) = 0$ and $w_2(TM) = 0$;

(ii) pin$^+$ structure: $w_2(TM) = 0$;

(iii) pin$^-$ structure: $w_1(TM)^2 + w_2(TM) = 0$;

(iv) spin^c structure: $w_1(TM) = 0$ and there exists a cohomology class $c \in \mathrm{H}^2(M, \mathbb{Z})$ such that $w_2(TM) \equiv c \bmod 2$;

(v) pin^c structure: there exists a cohomology class $c \in \mathrm{H}^2(M, \mathbb{Z})$ such that $w_2(TM) \equiv c \bmod 2$.

We shall now determine the topological conditions for the existence of an lpin structure. According to Theorem 1, one has to distinguish two cases depending on the parity of the dimension of the manifold. If $\dim M = m$ is even, then the group Lpin_m contains Pin^c_m as a maximal compact subgroup and, therefore, the existence of an lpin structure on an even-dimensional manifold is equivalent to the existence of a pin^c structure. The subtler case of m odd is described in the following

Theorem 5. *A manifold M of odd dimension admits an lpin structure if, and only if, there exists an element $c \in \mathrm{H}^2(M, \mathbb{Z})$ and a real vector bundle E over M, of fibre dimension 2, such that*

$$w_2(TM) + w_2(E) \equiv c \bmod 2. \tag{20}$$

Proof. The set of all elements given in (11) generates the group Lpin_m. Consider the homomorphism

$$\pi : \mathsf{Lpin}_m \to \mathsf{O}_m \times (\mathbb{Z}_2 \times_\varphi \mathbb{C}^\times) \times \mathbb{C}^\times \tag{21}$$

mapping an odd element (11) of Lpin_m onto the triple

$$(\widetilde{\mathrm{Ad}}(a), (-1, \lambda\mu^{-1}), \lambda\mu).$$

One easily checks that π provides a two-fold covering. Consider the subgroups H_0, H_1, and H_2 of Lpin_m defined by

$$\begin{aligned} H_0 &= \{\begin{pmatrix} \sigma(a) & 0 \\ 0 & \sigma(a) \end{pmatrix} \mid a \in \mathsf{Pin}_m\}, \\ H_1 &= \{\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \begin{pmatrix} 0 & -\lambda^{-1} \\ \lambda & 0 \end{pmatrix} \mid \lambda \in \mathbb{C}^\times\}, \\ H_2 &= \{\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \mid \lambda \in \mathbb{C}^\times\}. \end{aligned}$$

If $i \neq j$, $i, j = 0, 1, 2$, then $H_i \cap H_j = \mathbb{Z}_2$ and every two elements of two different subgroups commute. Moreover, the restrictions of the covering homomorphism (21) to H_0, H_1 and H_2 are two-fold coverings of the groups O_m, $\mathbb{Z}_2 \times_\varphi \mathbb{C}^\times$ and $\mathbb{C}^\times$, respectively.
An lpin structure on the manifold M defines via the covering π:

(i) a two-dimensional real vector bundle E associated with the representation of the group
$\mathbb{Z}_2 \times_\varphi \mathbb{C}^\times = \mathsf{O}_2 \times \mathbb{R}_+ \subset \mathsf{GL}_2(\mathbb{R})$;

(ii) an oriented two-dimensional real vector bundle F associated with the representation of the group
$\mathbb{C}^\times = \mathsf{SO}_2 \times \mathbb{R}_+ \subset \mathsf{GL}_2^+(\mathbb{R})$.

Suppose that the bundles E and F are given. Fix a covering $\{U_\alpha\}$ of the manifold M and denote by

$g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathsf{O}_m$	the transition functions of the tangent bundle TM,
$h_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathbb{Z}_2 \times_\varphi \mathbb{C}^\times$	the transition functions of the bundle E,
$k_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathbb{C}^\times$	the transition functions of the bundle F.

Let $g^*_{\alpha\beta}, h^*_{\alpha\beta}, k^*_{\alpha\beta}$ be their lifts into the covering groups:

$$g^*_{\alpha\beta},\ h^*_{\alpha\beta},\ k^*_{\alpha\beta} : U_\alpha \cap U_\beta \to H_0,\ H_1,\ H_2.$$

We consider the map

$$\Phi^*_{\alpha\beta} = g^*_{\alpha\beta} h^*_{\alpha\beta} k^*_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathsf{Lpin}_m.$$

The cocycle $\Phi^*_{\alpha\beta}\Phi^*_{\beta\gamma}\Phi^{*-1}_{\alpha\gamma}$ represents in the Čech cohomology group $\mathrm{H}^2(M,\mathbb{Z}_2)$ the obstruction to the reduction of the structure group $\mathsf{O}_m \times (\mathbb{Z}_2 \times_\varphi \mathbb{C}^\times) \times \mathbb{C}^\times$ to the structure group Lpin_m (see, e.g., [21]). Since pairs of elements belonging to different subgroups H_0, H_1, H_2 commute, we obtain

$$\Phi^*_{\alpha\beta}\Phi^*_{\beta\gamma}\Phi^{*-1}_{\alpha\gamma} = \left(g^*_{\alpha\beta} g^*_{\beta\gamma} g^{*-1}_{\alpha\gamma}\right) \left(h^*_{\alpha\beta} h^*_{\beta\gamma} h^{*-1}_{\alpha\gamma}\right) \left(k^*_{\alpha\beta} k^*_{\beta\gamma} k^{*-1}_{\alpha\gamma}\right).$$

The cocycle $g^*_{\alpha\beta} g^*_{\beta\gamma} g^{*-1}_{\alpha\gamma}$ represents the obstruction for the existence of a pin structure on the manifold M, i.e., the characteristic class $w_1(TM)^2 + w_2(TM)$. Similarly, the two other cocycles define $w_1(E)^2 + w_2(E)$ and $w_2(F)$, respectively. Therefore, the vector bundle $TM \oplus E \oplus F$ admits a reduction of the structure group to the group Lpin_m if and only if the condition

$$(22) \quad w_1(TM)^2 + w_2(TM) + w_1(E)^2 + w_2(E) + w_2(F) = 0$$

in holds. Since F is an oriented vector bundle its Stiefel-Whitney class $w_2(F)$ is the mod 2 reduction of its Euler class $e(F)$. Moreover, according to the Lemma, $w_1(TM)^2$ is also the mod 2 reduction of some integral cohomology class. Therefore, the existence of an lpin structure implies condition (20). Conversely, suppose that $w_2(TM) + w_2(E)$ is the mod 2 reduction of some integral cohomology class. Using the Lemma we conclude that $w_1(TM)^2 + w_2(TM) + w_1(E)^2 + w_2(E)$ is also the mod 2 reduction of some integral cohomology class $c \in H^2(M,\mathbb{Z})$. There exists an oriented vector bundle F with fibres of real dimension two such that its Euler class $e(F)$ coincides with the cohomology class $c \in H^2(M,\mathbb{Z})$ (see, e.g., [19]). For this vector bundle F equation (22) holds; therefore, M admits an lpin structure. □

Example 8. Let M be a manifold of dimension $m = 2n-1$ isometrically immersed in the Euclidean space $\mathbb{R}^{2n+1}$. Since the codimension is two, there is a natural choice of a two-dimensional bundle E, namely the normal bundle of M. The Whitney theorem gives

$$w_1(E) \;=\; w_1(TM), \quad w_1(TM)^2 + w_2(TM) + w_2(E) = 0.$$

Therefore, every submanifold of codimension two of the Euclidean space admits an lpin structure. Moreover, in this case, there is an explicit construction of the corresponding spinor bundle, similar to the one known for hypersurfaces [23]. Refer to Proposition 1 and consider a Pauli representation $\theta : \mathsf{Cl}^0_{2n+1} \to \mathsf{End}S$ so that S is complex 2^n-dimensional. For every $x \in M$, the space T_xM can be identified with a $(2n-1)$-dimensional vector subspace of $\mathbb{R}^{2n+1}$. Let $\eta = e_1 \dots e_{2n+1}$ be a volume element for $\mathbb{R}^{2n+1}$. Choose $\iota \in \{1, \sqrt{-1}\}$ so that $\eta^2 = \iota^2$. One makes $\Sigma = M \times S \to M$ into a spinor bundle by putting $\tau(v)(x,s) = (x, \iota\theta(v\eta)s)$ for $v \in T_xM \subset \mathbb{R}^{2n+1}$ and $s \in S$.

Example 9. The latter example is a special case of a more general situation where a codimension two immersion induces an lpin structure on the submanifold. Let N be a manifold of dimension $2n+1$ and assume that its second Stiefel-Whitney class $w_2(TN)$ is the mod 2 reduction of some integral cohomology class $c \in \mathrm{H}^2(N, \mathbb{Z})$. For every codimension two submanifold M of N the formula

$$w_2(TM) + w_2(E) \equiv c \bmod 2$$

holds, where E denotes again the two-dimensional normal bundle of the submanifold M. The condition on the manifold N is satisfied, for example, in the following two cases: $N = X \times Y$, where X is a complex manifold and Y is parallelizable; N is a Sasakian manifold. Therefore, every submanifold of codimension two in these spaces admits an lpin structure.

Example 10. Throughout this example M denotes the *Grassmann manifold* $G_{5,2}$ of all (non-oriented) 2-dimensional linear subspaces of the 5-dimensional real vector space $\mathbb{R}^5$.

The six-dimensional manifold M is non-orientable, connected and compact [19]. Its homology groups are known [22]:

$$\mathrm{H}_1(M,\mathbb{Z}) = \mathbb{Z}_2, \qquad \mathrm{H}_2(M,\mathbb{Z}) = \mathbb{Z}_2.$$

Using these results we find the $\mathbb{Z}$- and the $\mathbb{Z}_2$-cohomology groups:

$$\mathrm{H}^1(M,\mathbb{Z}) = 0, \qquad \mathrm{H}^2(M,\mathbb{Z}) = \mathbb{Z}_2,$$

$$\mathrm{H}^1(M, \mathbb{Z}_2) = \mathbb{Z}_2, \qquad \mathrm{H}^2(M, \mathbb{Z}_2) = \mathbb{Z}_2 \oplus \mathbb{Z}_2.$$

Denote by γ the canonical 2-dimensional vector bundle over M. Its first Stiefel-Whitney class $w_1(\gamma)$ is the unique non-trivial element in $\mathrm{H}^1(M, \mathbb{Z}_2)$:

$$\mathrm{H}^1(M, \mathbb{Z}_2) = \{0, w_1(\gamma)\}.$$

Explicitly,

$$\mathrm{H}^2(M, \mathbb{Z}_2) = \{0, w_1(\gamma)^2, w_2(\gamma), w_1(\gamma)^2 + w_2(\gamma)\}.$$

Consider the restriction map $r : \mathbb{Z}_2 = \mathrm{H}^2(M, \mathbb{Z}) \to \mathrm{H}^2(M, \mathbb{Z}_2) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$. Since $w_1(\gamma)^2$ is the restriction of an integral cohomology class, the image of $\mathrm{H}^2(M, \mathbb{Z})$ by r is the set $\{0, w_1(\gamma)^2\}$. The tangent bundle TM is isomorphic to the tensor product $\gamma \otimes \gamma^\perp$, where $\gamma^\perp$ denotes the 3-dimensional vector bundle over M whose fibre at the point (= plane in $\mathbb{R}^5$) $\Pi \in M$ is the 3-space $\Pi^\perp$. A computation of the Stiefel-Whitney classes yields

$$w_1(TM) = w_1(\gamma), \quad w_2(TM) = w_1(\gamma)^2 + w_2(\gamma).$$

Consequently, the non-orientable manifold M does not admit a pin^c-structure. However, for the bundle $E = \gamma$ over M we have

$$w_2(TM) + w_2(E) = w_1(\gamma)^2$$

and this class is the mod 2 reduction of an integral cohomology class $c \in \mathrm{H}^2(M, \mathbb{Z})$. Consider now the 7-dimensional manifold $M' = M \times \mathbb{R}$ or $M' = M \times \mathbb{S}_1$. Again, M' is a non-orientable manifold without a pin^c-structure. However, it satisfies the condition of Theorem 5; therefore, it admits an lpin structure.

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