Reflections and spinors on manifolds*

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Abstract

This paper reviews some recent work on (s)pin structures and the Dirac operator on hypersurfaces (in particular, on spheres), on real projective spaces and quadrics. Two approaches to spinor fields on manifolds are compared. The action of reflections on spinors is discussed, also for two-component (chiral) spinors.

Introduction

This paper contains a brief review of the work, done mostly in collaboration with Ludwik Dąbrowski [8], Michel Cahen, Simone Gutt [5] and Thomas Friedrich [11], on pin structures and the Dirac operator on higher-dimensional Riemannian manifolds; see also [18] and the references given there. In physics, there is now interest in higher dimensions motivated by research on unified theories, on supersymmetries, strings and their generalizations. There is also an intrinsic motivation: the Dirac operator is a fundamental object of an importance comparable to that of the Laplace operator and of the Maxwell, Yang-Mills and Einstein equations.

Reflections in a quadratic space generate the orthogonal group of automorphisms of that space; according to the Cartan-Dieudonné theorem, every orthogonal transformation in an m-dimensional quadratic space can be written as the product of a sequence of no more than m reflections. Reflections are of considerable interest in physics: invariance of electromagnetism under space reflections leads to selection rules; their violation is a striking feature of weak interactions. The PCT theorem describes a fundamental property of relativistic quantum field theories.

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There are several "spinorial" extensions of orthogonal groups; each of them can be used to define a "(s)pin structure" that is required to describe spinor fields on a curved manifold. For these structures to exist, the manifold should satisfy certain topological conditions; see [10] and the literature listed there. In this paper, I recall two definitions of spinor fields on manifolds and give several simple examples of spin and pin structures. The next section summarizes the definitions and terminology of Clifford algebras and their representations.

Clifford algebras and spinors

There are important differences—and similarities—between spinors associated with vector spaces of even and odd dimensions.

Let h be a quadratic form on a real vector space V of dimension m. The pair (V, h) is said to be a quadratic space. The Clifford algebra $\mathsf{Cliff}(h)$ associated with (V, h) is generated by elements of V subject to relations of the form $u^2 = h(u)$; see [1, 7, 10] and Ch. IX of [3]. Let α be the involutive automorphism of $\mathsf{Cliff}(h)$ such that $\alpha(1) = 1$ and $\alpha(v) = -v$ for every $v \in V$. This (main) automorphism defines a $\mathbb{Z}_2$-grading of the Clifford *algebra*, $\mathsf{Cliff}(h) = \mathsf{Cliff}^{\text{even}}(h) \oplus \mathsf{Cliff}^{\text{odd}}(h)$ and V is a vector subspace of the odd part. Let $(e_i), i = 1, \ldots, m$ be an orthonormal frame in V. As a *vector* space, the algebra $\mathsf{Cliff}(h)$ is $\mathbb{Z}$-graded, $\mathsf{Cliff}(h) = \oplus_{p=0}^{m} \mathsf{Cliff}^p(h)$, where $\mathsf{Cliff}^p(h)$ is the vector space spanned by all elements of the form $e_{i_1} \ldots e_{i_p}$ such that $1 \leqslant i_1 < \cdots < i_p \leqslant m$. In particular, $\mathsf{Cliff}^m(h)$ is spanned by the *volume element* $\eta = e_1 \ldots e_m$; its square is either 1 or -1, depending on the signature of h. If $u \in V$ is not null, $u^2 \neq 0$, then the linear map $V \to V$, $v \mapsto -uvu^{-1}$, is a reflection in the hyperplane orthogonal to u. Since $\eta v = (-1)^{m+1} v\eta$ for every $v \in V$, if m is *even*, then one can write $-uvu^{-1} = (u\eta)v(u\eta)^{-1}$. One says that $u \in V$ is a unit vector if $h(u) = 1$ or -1. The group $\mathsf{Pin}(h)$ is defined as the set of products $u_1 u_2 \ldots u_r$ of all sequences of unit vectors, with a composition induced by Clifford multiplication and $\mathsf{Spin}(h) = \mathsf{Pin}(h) \cap \mathsf{Cliff}^{\text{even}}(h)$. The *adjoint* representation Ad of $\mathsf{Pin}(h)$ in V is defined by $\mathrm{Ad}(a)v = ava^{-1}$, where $a \in \mathsf{Pin}(h)$ and $v \in V$. For m *even*, the map Ad is a homomorphism onto $\mathsf{O}(h)$ with kernel $\mathbb{Z}_2 = \{1, -1\}$; for m *odd*, Ad is a homomorphism onto $\mathsf{SO}(h)$. In both cases, to obtain a double cover of $\mathsf{O}(h)$ that coincides with Ad when restricted to $\mathsf{Spin}(h)$, one can use the *twisted adjoint* representation $\widetilde{\mathrm{Ad}}$ of $\mathsf{Pin}(h)$, defined by $\widetilde{\mathrm{Ad}}(a)v = \alpha(a)va^{-1}$. If h is of signature (k, l), $k + l = m$, then one writes $\mathsf{Cliff}_{k,l}$, $\mathsf{Pin}_{k,l}$, and $\mathsf{Spin}_{k,l}$ instead of $\mathsf{Cliff}(h)$, $\mathsf{Pin}(h)$ and $\mathsf{Spin}(h)$, respectively.

For $m = 2n$, the algebra $\mathsf{Cliff}(h)$ is central simple and has one, up to equivalence, irreducible and faithful representation γ in a complex, 2^n-dimensional space S of *Dirac* spinors. Given such a representation, one identifies $\mathsf{Cliff}(h)$ with its image in $\mathsf{End}\, S$. Upon restriction to $\mathsf{Cliff}^{\text{even}}(h)$, this representation decomposes into the direct sum of two irreducible and complex-inequivalent representations in spaces of Weyl (chiral, reduced or half) spinors so that $S = S_+ \oplus S_-$. The Dirac operator changes the chirality of spinors. Introducing the $2^n \times 2^n$ Dirac matrices γ^i and writing $\gamma^i_\pm = \gamma^i \mid S_\pm$, one obtains the well-known decomposition of the Dirac operator, $\gamma^i \partial_i = \left(\begin{smallmatrix} 0 & \gamma^i_- \partial_i \\ \gamma^i_+ \partial_i & 0 \end{smallmatrix}\right)$.

For $m = 2n-1$, the algebra $\mathsf{Cliff}^{\text{even}}(h)$ is central simple and has one, up to equivalence, representation in a complex, 2^{n-1}-dimensional space of *Pauli* spinors. The full algebra has two complex-inequivalent, in general not faithful, representations in spaces of Pauli spinors. The direct sum of these representations is a decomposable, but faithful, representation of $\mathsf{Cliff}(h)$ in the 2^n-dimensional space of *Cartan* spinors. Similarly as in this case of an even-dimensional space, the Dirac operator interchanges here the spinors belonging to the two Pauli representations. Namely, let σ^i, where $i = 1, \ldots, 2n-1$, be the $2^{n-1} \times 2^{n-1}$ Pauli matrices. The (modified) Dirac operator, acting on Cartan spinor fields, can be written as $\left(\begin{smallmatrix} 0 & \sigma^i\partial_i \\ \sigma^i\partial_i & 0 \end{smallmatrix}\right)$. The Cartan representation is essential when one considers the Dirac operator on non-orientable, odd-dimensional manifolds [4, 17].

Spinors on manifolds

There are (at least) two approaches to spinors on manifolds; both of them can be traced to early work by mathematicians and physicists; see [14] and the references to the period 1928-1931 given there.

The classical approach

The first approach to be summarized here, initiated by Wigner, Weyl and Fock, consists in referring spinors to tetrads ("Vierbeine"); its modern formulation uses the notion of a (s)pin structure involving a "prolongation" of the bundle P of orthonormal frames to the principal (s)pin bundle Q. More precisely, given a Riemannian manifold M with a metric tensor of signature (k, l), a $\mathsf{Pin}_{k,l}$-structure on M is given by the maps

$$\begin{array}{ccccc} \mathsf{Pin}_{k,l} & \longrightarrow & Q & & \\ {\scriptstyle \widetilde{\mathrm{Ad}}}\downarrow & & \downarrow{\scriptstyle \chi} & & \\ \mathsf{O}_{k,l} & \longrightarrow & P & \xrightarrow{\pi} & M, \end{array} \tag{1}$$

such that $\chi(qa) = \chi(q)\widetilde{\mathrm{Ad}}(a)$, $(q, a) \mapsto qa$ denotes the action map of $\mathsf{Pin}_{k,l}$ in the principal spin bundle Q, etc. If $\mathsf{Pin}_{k,l}$ in (1) is replaced by $\mathsf{Pin}_{l.k}$, then one obtains the definition of a $\mathsf{Pin}_{l,k}$-structure. They are both referred to as pin structures; if the manifold is orientable and has a pin structure, then it has a spin structure. The diagram describing a spin structure is shortened to

$$\mathsf{Spin}_{k,l} \to Q \to P \to M. \tag{2}$$

The spinor connection is obtained as the lift of the Levi-Civita connection from P to Q. This approach, standard in mathematics [10], is sometimes criticized by physicists who say that they have no use for *principal bundles* and are willing to consider only spinor fields.

One can present this approach in a language familiar to physicists, by referring everything to local sections of the bundle $Q \to M$ and using the terminology of *gauge fields*. For simplicity, consider an even-dimensional manifold, $k + l = 2n$, put $G = \mathsf{Pin}_{k,l}$ and let a representation of $\mathsf{Cliff}_{k,l}$ in S be given by the Dirac matrices $\gamma_i \in \mathsf{End}\,S$. A spinor field is now a map $\psi : M \to S$; given a function $U : M \to G$, one defines the gauge-transformed spinor field as $\psi' = U^{-1}\psi$, $\psi'(x) = U(x)^{-1}\psi(x)$ for $x \in M$. A spinor connection ("gauge potential") is a 1-form ω on M with values in the Lie algebra of G, i.e. in $\mathsf{Cliff}^2_{k,l} \subset \mathsf{End}\ S$;

therefore, it can be written as $\omega = \frac{1}{4}\gamma^i\gamma^j\omega_{ij}$, where $\omega_{ij} = -\omega_{ji}$ are 1-forms. The covariant ("gauge") derivative of ψ is

$$D\psi = \mathrm{d}\psi + \omega\psi. \tag{3}$$

The gauge transformation U induces a change of the connection, $\omega \mapsto \omega' = U^{-1}\omega U + U^{-1}\mathrm{d}U$ so that $(D\psi)' = U^{-1}D\psi$. Since the dimension of M is even, the adjoint representation is onto $\mathsf{O}_{k,l}$ and one can define, for every $a \in \mathsf{Pin}_{k,l} \subset \mathsf{GL}(S)$, the (orthogonal) matrix $(\rho^i{}_j(a))$ by $a^{-1}\gamma^i a = \rho^i{}_j(a)\gamma^j$, so that $a^{-1}\gamma_i a = \gamma_j\rho^j{}_i(a^{-1})$. From the Lemma

$$\text{if} \quad a \in \mathsf{Cliff}^p_{k,l}, \quad \text{then} \quad g^{ij}\gamma_i a\gamma_j = (-1)^p(n-2p)a,$$

taking into account that $U^{-1}\mathrm{d}U$ is in the Lie algebra of G—therefore of degree $p = 2$—one obtains

$$g^{ij}U^{-1}\gamma_i U\mathrm{d}(U^{-1}\gamma_j U) = 4U^{-1}\mathrm{d}U$$

so that

$$\omega'^i{}_j = \rho^i{}_k(U^{-1})\omega^k{}_l\rho^l{}_j(U) + \rho^i{}_k(U^{-1})\mathrm{d}\rho^k{}_j(U).$$

Let (e_i) be a field of orthonormal frames on M and let (e^i) denote the dual field of coframes. Since, by definition, $\omega_{ij} + \omega_{ji} = 0$, the 1-forms (ω_{ij}) define a *metric* linear connection. Its *torsion* $\mathrm{d}e^i + \omega^i{}_j \wedge e^j$ need not be zero.

The action of the Dirac operator $\mathcal{D}$ on a spinor field is $\mathcal{D}\psi = \gamma^i e_i \lrcorner D\psi$, where $\lrcorner$ denotes contraction.

Spinor fields according to Schrödinger and Karrer

The second approach can be traced back to work by Tetrode; it has been clearly formulated by Schrödinger [14][1] and Karrer [12]; it sometimes appears in texts written by physicists; see, e.g., [2]. Consider a Riemannian manifold (M, g) and let g_x denotes the quadratic form induced by g in the vector space T_xM tangent to M at x. One assumes now the existence of a representation γ of the Clifford bundle $\mathsf{Cliff}(g) = \bigcup_{x\in M}\mathsf{Cliff}(g_x)$ in a *vector bundle* $\Sigma \to M$ of spinors so that $\gamma(u)^2 = g(u,u)\mathrm{id}_{T_xM}$ for every $u \in T_xM$. One then introduces a spinor covariant derivative on Σ, compatible with a metric covariant derivative on TM. Such a structure is weaker (more general) than a classical spin structure. For example, it exists on every almost Hermitean manifold even though some of these manifolds—such as the even-dimensional complex projective spaces—do not admit a spin structure. The precise relation between those two approaches, in the case of even-dimensional orientable manifolds is described in [11]: the second method is equivalent to the introduction of a spin^c-structure on M.

In the physicist's local approach one introduces—following Schrödinger—a spinor field as a smooth map $\psi : M \to S$. The set of all such fields is a module $\mathcal{S}$ over the ring $\mathcal{C}$ of smooth functions on the Riemannian manifold M assumed here to be of dimension $m = 2n$. Let ∇ be a covariant derivative in the module $\mathcal{V}$ of vector fields: if $u, v \in \mathcal{V}$, then $\nabla_u v \in \mathcal{V}$ is the covariant derivative of v in the direction of u. The representation γ mentioned above associates with a vector field u and a spinor field ψ another spinor field

[1] I thank Engelbert Schücking for having drawn my attention to this remarkable paper. It contains a derivation of the formula for the square of the Dirac operator on Riemannian manifolds.

$\gamma(u)\psi$; the map $\mathcal{V} \times \mathcal{S} \to \mathcal{S}$, $(u, \psi) \mapsto \gamma(u)\psi$, is bilinear and $\gamma(u) f\psi = f\gamma(u)\psi$ for every $f \in \mathcal{C}$; moreover, it has the Clifford property:

$$\gamma(u)^2\psi = g(u, u)\psi. \tag{4}$$

One postulates now the existence of a *spinor covariant derivative* $\nabla^{\mathrm{s}}_u : \mathcal{S} \to \mathcal{S}$ compatible with ∇ in the sense that

$$\nabla^{\mathrm{s}}_u(\gamma(v)\psi) = \gamma(\nabla_u v)\psi + \gamma(u)\nabla^{\mathrm{s}}_u\psi \quad \text{for } u, v \in \mathcal{V} \text{ and } \psi \in \mathcal{S}. \tag{5}$$

Let (e_μ), where $\mu = 1, \ldots, m$, be a field of frames and let (e^μ) be the dual field of coframes; they need not be orthonormal; e.g., given local coordinates (x^μ), one can take $e^\mu = \mathrm{d}x^\mu$. The field $\gamma_\mu = \gamma(e_\mu) : M \to \mathsf{End}\, S$ satisfies

$$\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2g_{\mu\nu}, \quad \text{where} \quad g_{\mu\nu} = g(e_\mu, e_\nu).$$

The coefficients of the linear connection defined by ∇ can be read off from

$$\nabla_{e_\nu} e_\mu = e_\rho \Gamma^\rho_{\mu\nu}.$$

The spinor covariant derivative of ψ in the direction of u can be written as the contraction

$$\nabla^{\mathrm{s}}_u\psi = u \lrcorner D\psi,$$

where $D\psi$ has the form (3). The compatibility condition (5) is equivalent to

$$\mathrm{d}\gamma_\mu + [\omega, \gamma_\mu] - \Gamma^\rho_{\mu\sigma}\gamma_\rho e^\sigma = 0. \tag{6}$$

The metricity of ∇ can be justified by covariant-differentiating both sides of (4) and using (5). If ω is a solution of (6) and A is a complex-valued 1-form on M, then $\omega + \mathrm{i}A\,\mathrm{id}_S$ is another solution. Therefore, the connection ω can be interpreted as including an interaction with the electromagnetic field (of potential equal to the real part of A). Such an interpretation has been clearly formulated by Fock [9] and Schrödinger [14]: they may be considered as precursors of the idea of spin^c-structures.

Examples

Hypersurfaces in $\mathbb{R}^{m+1}$

Every hypersurface M in the Euclidean space $\mathbb{R}^{m+1}$, defined by an isometric immersion $f : M \to \mathbb{R}^{m+1}$, has a $\mathsf{Pin}_{0,m}$-structure, canonically defined by f. Moreover, the *associated bundle* Σ of spinors on M is *trivial*: it is isomorphic to the Cartesian product $M \times S$ and a spinor field can be (globally!) described by a funtion $\psi : M \to S$. A Dirac (resp., Cartan) spinor field on an even (resp., odd) dimensional hypersurface is the restriction of a Pauli (resp., Dirac) field on the surrounding space. In terms of the trivialization of Σ, the Dirac operator assumes a rather simple form [17]. Let ν_i, where $i = 1, \ldots, m+1$, be the Cartesian components of a unit, normal vector field on M. Let $m = 2n$ or $2n - 1$; introduce the Dirac $2^n \times 2^n$ matrices γ_i satisfying

$$\gamma_i\gamma_j + \gamma_j\gamma_i = -2\delta_{ij}, \quad i, j = 1, \ldots, m+1.$$

Then

$$\mathcal{D} = \tfrac{1}{2}(\gamma^k \nu_k)(\gamma^i \gamma^j (\nu_j \partial_i - \nu_i \partial_j) - \operatorname{div} \nu), \tag{7}$$

where $\operatorname{div}\nu$ is the intrinsic divergence of ν, $\operatorname{div}\nu = \sum_{i,j}(\delta_{ij} - \nu_i\nu_j)\partial_i\nu_j$. If M is the hyperplane of equation $x^{m+1} = 0$, then $\nu_i = \delta_i^{m+1}$ and $\mathcal{D} = \sum_{\mu=1}^{m} \gamma^\mu \partial_\mu$. Formula (7) has been used to find, in a simple manner, the spectrum and the eigenfunctions of the Dirac operator on spheres [18].

Spheres, projective spaces, and quadrics

1. The spin structures on *spheres* are well-known: for every $m > 1$, the m-dimensional sphere $\mathbb{S}_m$ has a unique spin structure; in the style of (2) it is given by the sequence of maps

$$\mathsf{Spin}_m \to \mathsf{Spin}_{m+1} \to \mathsf{SO}_{m+1} \to \mathbb{S}_m.$$

The spectrum of $\mathcal{D}$ on $\mathbb{S}_m$ is of the form:

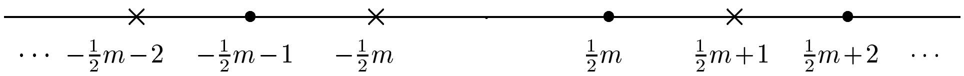

Fig. 1. The spectrum of the Dirac operator on the m-sphere.

The eigenfunctions $\psi : \mathbb{S}_m \to S$ are either symmetric or antisymmetric; these two types of symmetries are indicated here by bullets and crosses; which of the eigenfunctions (bullets or crosses) are even depends on the trivialization of the bundle of spinors; only the relative parity matters.

2. The real projective spaces $\mathbb{R}\mathrm{P}_m = \mathbb{S}_m/\mathbb{Z}_2$ are orientable iff m is odd; for $m > 1$ there are either two inequivalent (s)pin structures or none [8]:

$m \equiv$	0	1	2	3	mod 4
structure:	$\mathsf{Pin}_{m,0}$	no	$\mathsf{Pin}_{0,m}$	Spin_m	

Since $\mathbb{R}\mathrm{P}_m$ is locally isometric to $\mathbb{S}_m$, the spectrum of $\mathcal{D}$ on such a space can be obtained from that of the sphere: the symmetric eigenfunctions descend to one (s)pin structure on $\mathbb{R}\mathrm{P}_m$ and the antisymmetric functions—to the other; these spectra are thus asymmetric.

3. The real projective quadrics are defined as conformal compactifications of pseudo-Euclidean spaces; they generalize the Penrose construction of compactified Minkowski space-time. The quadric $\mathbb{S}_{k,l} = (\mathbb{S}_k \times \mathbb{S}_l)/\mathbb{Z}_2$ admits two natural metrics, descending from the spheres: a proper Riemannian metric and a pseudo-Riemannian one, of signature (k,l). The quadrics $\mathbb{S}_{k,0}$ and $\mathbb{S}_{0,k}$ can be identified with $\mathbb{S}_k$; a quadric is said to be *proper* if $kl \neq 0$. A proper quadric is orientable iff its dimension is even. If $kl > 1$, then $\mathrm{H}^1(\mathbb{S}_{k,l}, \mathbb{Z}_2) = \mathbb{Z}_2$; therefore, for $kl > 1$, the quadric has either 2 inequivalent (s)pin structures or none. The following table, based on [4], summarizes the results on the existence of (s)pin structures on $\mathbb{S}_{k,l}$ for $kl > 1$:

$k+l=2n$	n	proper Riem.	pseudo-Riem.
either k or $l=1$	any	yes	yes
k and l even	even	no	yes
k and l even	odd	yes	no
k and l odd	even	no	no
k and l odd	odd	yes	yes
k even and l odd			
$k+l \equiv 1 \bmod 4$		$\mathsf{Pin}_{0,k+l}$	$\mathsf{Pin}_{l,k}$
$k+l \equiv 3 \bmod 4$		$\mathsf{Pin}_{k+l,0}$	$\mathsf{Pin}_{k,l}$

For example, the quadric $\mathbb{S}_{3,5}$ has no spin structure for either of the metric tensors.

The spectrum of the Dirac operator on the projective quadrics can be obtained from that of the spheres; this is facilitated by the following Lemma: if λ_i is an eigenvalue of the Dirac operator on a Riemannian spin manifold M_i, $i=1,2$, then the numbers $\sqrt{\lambda_1^2+\lambda_2^2}$ and $-\sqrt{\lambda_1^2+\lambda_2^2}$ are eigenvalues of the Dirac operator on $M_1 \times M_2$ [5].

Action of space and time reflections on spinors

Charge conjugation

Space and time reflections seem to be closely related to charge conjugation. Consider the Dirac equation

$$(\gamma^\mu(\partial_\mu - \mathrm{i}qA_\mu) - m)\psi = 0$$

for the Dirac wave function $\psi : \mathbb{R}^{2n} \to S$ of a particle of mass m and charge q moving in a $2n$-dimensional flat space-time with a metric tensor of signature $(2n-1,1)$ and an electromagnetic field with potential A_μ. Let $C : S \to \bar{S}$ be the isomorphism such that $\overline{\gamma_\mu} = C\gamma_\mu C^{-1}$, where bar denotes complex conjugation; the *charge conjugate* spinor field $\mathsf{C}\psi = \overline{C\psi}$ satisfies the equation

$$(\gamma^\mu(\partial_\mu + \mathrm{i}qA_\mu) - m)\mathsf{C}\psi = 0.$$

If $\psi \sim \exp(-\mathrm{i}Et)$, then $\mathsf{C}\psi \sim \exp(+\mathrm{i}Et)$: charge conjugation is said to transform particles into antiparticles.

Wigner's time inversion

Let $\psi : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{C}$ be a wave function in non-relativistic quantum theory. The time-reversed wave function $\mathsf{T}_W\psi$ is defined by [20]

$$(\mathsf{T}_W\psi)(\mathbf{r},t) = \overline{\psi(\mathbf{r},-t)}.$$

If ψ is a solution of the Schrödinger equation with a time-independent, real potential, then so is $\mathsf{T}_W\psi$. If $\psi \sim \exp(-\mathrm{i}Et)$, then also $\mathsf{T}_W\psi \sim \exp(-\mathrm{i}Et)$.

Time inversion of spinor fields: Feynman *versus* Wigner

In the relativistic theory, there are two ways of defining time inversion [13]. Recall first the general statement about the relativistic invariance of the free Dirac equation in Minkowski space. Let S denote, as before, the complex vector space of Dirac spinors. A representation of the Clifford algebra $\mathsf{Cliff}_{3,1}$ in S being given in terms of the Dirac matrices γ_μ, one can identifiy the group $\mathsf{Pin}_{3,1}$ with a subgroup of $\mathsf{GL}(S)$ and embed $\mathbb{R}^4$ in $\mathsf{End}\, S$ by $x \mapsto x^\mu \gamma_\mu$, as usual. There is the exact sequence

$$1 \to \mathbb{Z}_2 \to \mathsf{Pin}_{3,1} \xrightarrow{\mathrm{Ad}} \mathsf{O}_{3,1} \to 1,$$

where $\mathrm{Ad}(U)x = UxU^{-1}$. There is a similar, but inequivalent, extension of $\mathsf{O}_{3,1}$ by $\mathbb{Z}_2$ corresponding to $\mathsf{Pin}_{1,3}$, as well as several other extensions described in [7]; see also [6] and the references given there. Every $U \in \mathsf{Pin}_{3,1}$ acts on spinor fields by sending a solution $\psi : \mathbb{R}^4 \to S$ of the free Dirac equation to another solution $\mathsf{U}\psi$,

$$(\mathsf{U}\psi)(x) = U(\psi(\mathrm{Ad}(U^{-1})x)).$$

In particular, with γ_4 and $\gamma_1\gamma_2\gamma_3 \in \mathsf{Pin}_{3,1}$, there are associated the space and time inversion operators P and T, respectively. The operator T is the *geometrical* time inversion; if $\psi \sim \exp(-\mathrm{i}Et)$, then $\mathsf{T}\psi \sim \exp(+\mathrm{i}Et)$. One can justify the interpretation of T as the time inversion operator by the Feynman idea of viewing antiparticles as particles travelling backwards in time. Physicists favour nowadays the *Wigner time inversion*

$$\mathsf{T}_W = \mathsf{T} \circ \mathsf{C}.$$

Since charge conjugation is involutory, $\mathsf{C}^2 = \mathrm{id}$, the product of operators considered in the PCT theorem is equivalent to $\mathsf{P} \circ \mathsf{T}$. This is the space-time reflection R corresponding to γ_5. The space-time reflection is in the connected component of the identity of the “complex” Lorentz group ($=\mathsf{SO}_4(\mathbb{C})$). The idea underlying the PCT theorem is that, in a quantum field theory invariant only with respect to the connected component of the Poincaré group, holomorphic functions such as the vacuum expectation values of field operators, are also invariant with respect to the “complex rotation” R [16].

In non-relativistic quantum mechanics, there is no place for T because that theory is obtained as a limit of the relativistic theory implying, for a free particle, the removal of all negative energy states.

Space and time inversion of Weyl spinors

Complex conjugation appears also in the realization of space and time reflections in the space of Weyl (chiral) spinors proposed by Staruszkiewicz [15]; see also [6].

Recall that the connected component of the group $\mathsf{Spin}_{3,1}$ is isomorphic to $\mathsf{SL}_2(\mathbb{C})$. It has two inequivalent representations in the spaces of 2-component (Weyl) dotted and undotted spinors.[2] These representations are complex conjugate one to another; their

[2] This terminology is due to van der Waerden; many people follow now Penrose and use the names: primed and unprimed “reduced” spinors; see [19] for references and further comments.

direct sum defines the representation in the space of Dirac spinors,

$$a \mapsto \begin{pmatrix} a & 0 \\ 0 & \bar{a} \end{pmatrix}, \quad a \in \mathsf{SL}_2(\mathbb{C}).$$

This decomposable representation is a restriction to $\mathsf{SL}_2(\mathbb{C})$ of the representation of $\mathsf{Pin}_{3,1}$ determined by the Dirac matrices

$$\gamma_1 = \begin{pmatrix} 0 & \mathrm{i}\sigma_1 \\ -\mathrm{i}\sigma_1 & 0 \end{pmatrix}, \quad \gamma_2 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \quad \gamma_3 = \begin{pmatrix} 0 & \mathrm{i}\sigma_3 \\ -\mathrm{i}\sigma_3 & 0 \end{pmatrix}, \quad \gamma_4 = \begin{pmatrix} 0 & \sigma_2 \\ -\sigma_2 & 0 \end{pmatrix}.$$

In this representation one has $C = \gamma_2$ and a Dirac spinor of the form

$$\psi = \begin{pmatrix} u \\ \bar{u} \end{pmatrix}, \quad \text{where} \quad u \in \mathbb{C}^2, \tag{8}$$

is real in the sense that it satisfies the Majorana condition, $\mathsf{C}\psi = \psi$. Space and time inversions, as defined in the previous section, induce the following transformations of the Weyl part u of the Majorana spinor (8),

$$\mathsf{P} : u \mapsto \sigma_2 \bar{u} \quad \text{and} \quad \mathsf{T} : u \mapsto \mathrm{i}\sigma_2 \bar{u},$$

respectively.

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