# CLIFFORD AND THE 'SQUARE ROOT' IDEAS

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Abstract. This review article begins with a short history of the notions associated with spinors; it describes several distinct 'square root' ideas occurring in connection with Clifford algebras, spin groups and pure spinors. Applications of pure spinors to geometry and physics are briefly presented. An appendix contains a simple derivation of the Vahlen-Ahlfors, fractional-linear form of Möbius transformations.

## 1. Introduction

Spinors—and the structures underlying this notion, such as Clifford algebras, spin groups and their representations, and spin structures on manifolds—are a good example of the subtle relations and mutual influences between mathematics and physics. Their origins can be traced to work of mathematicians, but they owe their name and fame to physicists. One of the great achievements of 20th century physics is the elucidation of the role of fermions—particles with half integer spin, such as electrons and protons, requiring spinors for their quantum-mechanical description—in the observed stability of matter and the chemical properties of atoms. An essential physical law used to explain these properties is the *Pauli exclusion principle* ('no more than one fermion in any state') which, in turn, follows from the applicability of the Fermi-Dirac statistics to fermions: the quantum state of $k$ fermions of the same kind belongs to the $k$th exterior power of the vector space of one-particle states. A fundamental result of relativistic quantum theory is the theorem on the *connection between spin and statistics* 'explaining' why fermions obey that statistics and, therefore, satisfy the exclusion principle; see, e. g. [SW] and the references given there. One is tempted to say that the world around us, and life in particular, are so rich because Nature found it convenient to use, among its building blocks, entities requiring spinors in their description.

At first, spinors baffled physicists and mathematicians alike; in the words of Darwin (1928)[1]:*The relativity theory is based on nothing but the idea of invariance and develops from it the conception of tensors as a matter of necessity; and it is rather disconcerting to find that apparently something has slipped through the net, so that physical quantities exist, which it would be, to say the least, very artificial and inconvenient to express as tensors.*

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[1]In this style, I indicate references, mainly to the physics literature, that are listed in [BuT].

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Some people who had not accepted Darwin's wisdom expressed in the last sentence were criticized by É. Cartan; see the footnotes in the last section of [C4]. Since spinors are difficult to visualize, even though they cannot be expressed as tensors, it is natural to relate their properties to those of the more familiar vectors, tensors, quadratic forms and orthogonal transformations. Many of such heuristic considerations emphasize the appearance of the idea of a 'square' or of a 'square root' in the relations between vectors and spinors.

This article reviews some of the (distinct) square root ideas as well as a few applications of spinors. Proofs are replaced by references to the literature, whenever possible. Extensive information and bibliographies on the subject of Clifford algebras, spin structures, the Dirac operator, and on applications of spinors in geometry and physics can be found in recent books and reviews, such as [BGV, Bo, Ha, LM, PR]. The next section contains a rather cursory account of the history of notions related to spinors; this is intended to put in perspective the very significant contribution made by W. K. Clifford. Following Cartan [C4], and unlike in most contemporary expositions of the subject, spinor representations are treated here first for *odd*-dimensional spaces. This is justified by the fact that, even though the Clifford algebra $\mathrm{Cl}_{2m}$ is isomorphic to the even subalgebra $\mathrm{Cl}^{+}_{2m+1}$, the group $\mathrm{Spin}_{2m}(\mathbb{C})$ is properly contained in $\mathrm{Spin}_{2m+!}(\mathbb{C})$; any results established for the latter hold, by restriction, for the former, but not conversely. The Appendix outlines a simple, but apparently little known, derivation of the fractional-linear form of Möbius transformations of projective quadrics (in particular, of the $n$-sphere $\mathbb{S}_n$).

## 2. On the history of Clifford algebras and spinors

There is a prehistory of spinors: for example, in the papers by Euler [E] and Rodrigues [Ro] on the rational representation of rotations in $\mathbb{R}^3$ one can find—with a little effort and some good will—the map $\mathrm{SU}_2 \to \mathrm{SO}_3$. Hamilton represented rotations in terms of quaternions: every rotation in $\mathbb{R}^3$ is of the form $q \mapsto aqa^{-1}$, where $a \in \mathrm{Sp}_1$ and $q \in \mathbb{R}^3 \subset \mathbb{H}$ are a unit and a pure quaternion, respectively (this shows that the groups $\mathrm{Spin}_3$ and $\mathrm{Sp}_1$ are isomorphic). Cayley (1855) extended Hamilton's observation to $\mathbb{R}^4$; he proved, in essence, the isomorphism of the groups $\mathrm{Spin}_4$ and $\mathrm{Sp}_1 \times \mathrm{Sp}_1$. More information on that early period can be found in [C1].

Clifford [Cl1,2] introduced the algebras that now bear his name by considering the underlying vector space of a Grassmann algebra and endowing it with a new product, thus generalizing the algebras of complex numbers and of quaternions. In his language, the 'geometric algebras' over $\mathbb{R}$ are generated by $n$ 'units' $e_1, \dots, e_n$, such that

$$e_\mu e_\nu + e_\nu e_\mu = -2\delta_{\mu\nu}, \tag{1}$$

where $\mu, \nu = 1, \dots, n$. The minus sign in (1) was motivated by the algebras $\mathbb{C}$ and $\mathbb{H}$ over $\mathbb{R}$. Clifford knew most of the basic properties of his algebras, described now in modern texts: their $\mathbb{Z}_2$-grading and (semi-) simplicity, the structure of the center, their periodicity and relations between algebras associated with vector spaces of adjacent dimensions, etc. Using Clifford algebras, Lipschitz [Lp] introduced the groups that are now called Spin and Clifford; see, in this connection, the anonymous note [Corr]. In 1902 Vahlen found, in terms of 'Clifford numbers', the fractional-linear representation of the Möbius group in $n$ dimension; see the Appendix to this paper.

The modern period in the history of spinors begins probably with the papers by Cartan [C2, C3] containing a description of the spin representations of the Lie algebras of orthogonal groups. The discovery of the spin of the electron by Uhlenbeck and Goudsmit (1925) forced physicists to find mathematical tools to describe, within the framework of quantum mechanics, this new degree of freedom. When doing this, Pauli (1927) and Dirac (1928) rediscovered the spin representations and the Clifford algebras associated with three- and four-dimensional vector spaces, respectively. Dirac introduced, at that time, in the context of Minkowski space, the differential operator that now bears his name; the Dirac equation has been very successful in describing the quantum-relativistic behavior of electrons and other particles with spin $\frac{1}{2}$. Soon afterwards, Weyl [W] and Fock [F] developed a *local* theory of the Dirac operator in curved, (Lorentzian) spacetimes. Shortly after the appearance of the paper by Dirac, Ivanenko and Landau (1928) published another approach to the theory of the 'magnetic electron'. They proposed to use an equation based on the operator $d + \delta$, acting on (inhomogeneous) differential forms, where $\delta$ is the formal adjoint of the exterior derivative $d$. In view of the successes of the Dirac theory, their equation was ignored for a long time. The operator $d+\delta$, whose square is the Laplacian, was considered by Kähler [K] and, under the influence of his work, physicists used it to describe fermions on a lattice.

Brauer and Weyl (1935) gave a general construction, based on Clifford algebras, of the spin representations in any number of dimensions and showed how the tensor product of two such representations decomposes into irreducibles. The connections between spinors, totally isotropic spaces and projective geometry seem to have been clearly stated, for the first time, by Veblen [V1, V2] and developed in seminar lectures at Princeton, given jointly with Givens [VG]. These notes contain remarks that may have influenced Cartan in his work on pure spinors[2] [C4]. Chevalley based his *Algebraic theory of spinors* [Ch] on the notion of minimal, one-sided ideals of Clifford algebras, an idea considered earlier by Riesz [Ri] in the context of the Dirac equation in the theory of general relativity and, less explicitly, by several physicists; see, e. g., [Sa]. Chevalley developed the theory of spinors over an arbitrary field and provided complete proofs of many new and old results; in particular, on pure spinors and triality. Haefliger's note [H] seems to be the first publication containing the definition of a spin structure on a Riemannian space and the derivation of the obstruction to the existence of such a structure. On the basis of this definition, Atiyah and Singer [AS] developed a global theory of the Dirac operator $\mathcal{D}$ on a Riemannian space; they proved the index theorem for the operators $d + \delta$ and $\mathcal{D}$. This paper, together with [ABS] and the note by Lichnerowicz [Li] on harmonic spinors, begins the modern period of research on global aspects of spinor analysis and its applications to differential geometry and topology. This research is surveyed in [BGV, Bo, Gi, LM].

Penrose and, under his influence, many physicists (see [PR] for a review and references) have used spinors in the context of general relativity; the Newman-Penrose (1962) formalism, used to study solutions of Einstein's equations, is an extension of the method of moving frames to the principal spin bundle over a (Lorentzian) spacetime $\mathcal{M}$. Cartan (1922) noticed that, in such a spacetime, the Weyl tensor (of conformal curvature), at a point where it is $\neq 0$, defines 4 isotropic (optical, null) directions in the tangent space; sometimes these directions coincide;

[2] In fact, Cartan used the expression *spineur simple*; the name 'pure spinor' is due to Chevalley.

an enumeration of the possible coincidences leads to a an algebraic classification of the Weyl tensors of Lorentzian spacetimes; a spacetime is *algebraically degenerate* if at least two of those directions coincide; it is said to be of type $D$ if there are two distinct pairs of coinciding directions (example: the Schwarzschild metric). Since the manifold of isotropic directions at a point of $\mathcal{M}$ can be identified with the projective space of directions of Weyl (semi-) spinors, the structure group $\mathrm{SL}_2(\mathbb{C})$ of the spin bundle of $\mathcal{M}$ reduces to its Abelian subgroup $\mathbb{C}^\times = \mathbb{C} \setminus \{0\}$; the Newman-Penrose method has been very effective in finding solutions of Einstein's equations in this case.

## 3. The universal linearization of quadratic forms

The idea of *linearization* pervades much of pure and applied mathematics: it suffices to mention the notions of a differential, of tensor and exterior products or the inverse scattering method. The construction of Clifford algebras solves the following universal problem: given a quadratic space $(V, h)$, where $V$ is a (finite-dimensional) vector space over a field $K$ and $h : V \to K$ is a quadratic form, find all Clifford maps, i. e. linear maps $f : V \to \mathcal{A}$, where $\mathcal{A}$ is an algebra over $K$, with a unit element $1_{\mathcal{A}}$, such that

$$f(v)^2 = h(v)1_{\mathcal{A}}$$

for every $v \in V$. The Clifford algebra of the quadratic space $(V, h)$ is defined as $\mathrm{Cl}(V, h) = \mathrm{T}(V)/\mathrm{J}(V, h)$, where $\mathrm{J}(V, h)$ is the ideal generated by all elements of the tensor algebra $\mathrm{T}(V) = \oplus_{p=0}^{\infty} \otimes^p V$ of the form $v \otimes v - h(v)$. The Clifford algebra of $(V, h)$ contains $K \oplus V$ as a vector subspace, the injection $V \to \mathrm{Cl}(V, h)$ is a Clifford map and every Clifford map $f : V \to \mathcal{A}$ extends to a homomorphism $\tilde{f} : \mathrm{Cl}(V, h) \to \mathcal{A}$ of algebras with units. In particular, if $f : V \to V \subset \mathrm{Cl}(V, h)$ is an isometry, $f \in \mathrm{O}(V, h)$, then $\mathrm{Cl}(f) = \tilde{f}$ is an automorphism of the algebra $\mathrm{Cl}(V, h)$; in other words, Cl is a functor from the category of quadratic spaces to that of algebras with unit elements. The isometry $v \mapsto -v$ extends to the *main automorphism* $\alpha$ of $\mathrm{Cl}(V, h)$. Since this automorphism is involutive, it defines a $\mathbb{Z}_2$-grading of the algebra, $\mathrm{Cl}(V, h) = \mathrm{Cl}^+(V, h) \oplus \mathrm{Cl}^-(V, h)$, where $\mathrm{Cl}^\pm(V, h) = \{a \in \mathrm{Cl}(V, h) | \alpha(a) = \pm a\}$. The *transposition* is an antiautomorphism $a \mapsto a^t$ of $\mathrm{Cl}(V, h)$ characterized by being a linear automorphism of the underlying vector space such that $1^t = 1$, $v^t = v$ for every $v \in V$ and $(ab)^t = b^t a^t$ for every $a, b \in \mathrm{Cl}(V, h)$.

Assuming, from now on, that the characteristic of $K$ is not 2, one can associate with the quadratic form $h$ the symmetric linear isomorphism $V \to V^*$ given by $v \mapsto v^h$, where $v^h \in V^*$ is the linear form on $V$ such that $2\langle v', v^h \rangle = h(v+v') - h(v) - h(v')$ for every $v, v' \in V$. The linear map $v^h$ extends to a derivation $i(v)$ of degree $-1$ of the $\mathbb{Z}$-graded (exterior) algebra $\wedge V$. Denoting by $e(v) \in \mathrm{End} \wedge V$ the exterior multiplication by $v$, one has $i(v)e(v) + e(v)i(v) = h(v)1_{\mathrm{End} \wedge V}$. Therefore, if $F(v) = e(v) + i(v)$, then $F : V \to \mathrm{End} \wedge V$ is a Clifford map and the linear map

$$c : \mathrm{Cl}(V, h) \to \wedge V, \quad \text{defined by} \quad c(a) = \tilde{F}(a)1_{\wedge V}, \tag{2}$$

is an isomorphism of vector spaces over $K$, natural with respect to isometries: if $f \in \mathrm{O}(V, h)$, then

$$c \circ \mathrm{Cl}(f) = \wedge f \circ c. \tag{3}$$

The map $c$ reduces to the identity on $K \oplus V$ and satisfies

$$c(va) = (e(v) + i(v))c(a) \tag{4}$$

for every $v \in V$ and $a \in \mathrm{Cl}(V,h)$ [Ch]. The algebra $\wedge V$ has a natural transposition antiautomorphism, $(v_1 \wedge \cdots \wedge v_p)^t = v_p \wedge \cdots \wedge v_1$, and $c(a^t) = c(a)^t$ for every $a \in \mathrm{Cl}(V,h)$. There is also the main automorphism $\alpha_V$ of $\wedge V$ such that $c \circ \alpha = \alpha_V \circ c$.

Let $V_0$ be a one-dimensional vector space with a generator $e_0$ and a quadratic form $h_0$ such that $h_0(e_0) = -1$. The Clifford map $V \to \mathrm{Cl}^+(V \oplus V_0, h \oplus h_0)$ given by $v \mapsto ve_0$ extends to an isomorphism of algebras $\mathrm{Cl}(V,h) \to \mathrm{Cl}^+(V \oplus V_0, h \oplus h_0)$.

In applications, one considers a *representation* $\gamma$ of $\mathrm{Cl}(V,h)$ in a finite-dimensional vector space $S$ of 'spinors'. Let $(e_\mu)$ be a linear basis in $V$ and let $h_{\mu\nu} = \langle e_\mu, e_\nu^h \rangle$, where $\mu, \nu = 1, \ldots, n$ and $n$ is the dimension of $V$. Using Einstein's summation convention over pairs of repeated indices, every $v \in V$ can be written, in terms of its components, as $v^\mu e_\mu$. The representation $\gamma$, restricted to $V$, is a Clifford map and

$$(\gamma_\mu v^\mu)^2 = h_{\mu\nu} v^\mu v^\nu \mathrm{id}_S.$$

If $(e_\mu)$ is an orthogonal basis, i. e. if $h_{\mu\nu} = 0$ for $\mu \neq \nu$, then $\mu_1 < \mu_2 < \cdots < \mu_p$ implies $c(e_{\mu_1} e_{\mu_2} \ldots e_{\mu_p}) = e_{\mu_1} \wedge e_{\mu_2} \wedge \cdots \wedge e_{\mu_p}$. Let $c_p(a)$ denote the component in $\wedge^p V$ of $c(a)$, $a \in \mathrm{Cl}(V,h)$. If $c(a) = c_p(a)$, then one says that $a \in \mathrm{Cl}(V,h)$ is of *degree* $p$. If $a$ is of degree $p$, then $a^t = (-1)^{\frac{1}{2}p(p-1)} a$.

## 4. Double-valuedness of spinor representations of orthogonal groups

In this section it is assumed that $K = \mathbb{R}$ and the quadratic form $h$ is non-degenerate. One says that $v$ is a *unit vector* if either $h(v) = 1$ or $h(v) = -1$. The *spin group* associated with $(V,h)$ is the set $\mathrm{Spin}(V,h)$ of products of the elements of all sequences consisting of an even number of unit vectors, with multiplication induced from $\mathrm{Cl}(V,h)$. The adjoint representation $\rho$ of $\mathrm{Spin}(V,h)$ in $V$, defined by $\rho(a)v = ava^{-1}$ for $a \in \mathrm{Spin}(V,h)$ and $v \in V$, gives the exact sequence of group homomorphisms,

$$1 \to \mathbb{Z}_2 \to \mathrm{Spin}(V,h) \xrightarrow{\rho} \mathrm{SO}(V,h) \to 1.$$

This shows that, with respect to a natural topology and differential structure, the group $\mathrm{Spin}(V,h)$ is a Lie group doubly covering the Lie group $\mathrm{SO}(V,h)$. A representation of the spin group lifts to a representation of the rotation group $\mathrm{SO}(V,h)$ if, and only if, it is trivial on the kernel of $\rho$; the *spin representations*, which are defined here as restrictions to $\mathrm{Spin}(V,h)$ of non-trivial representations of the algebra $\mathrm{Cl}(V,h)$, never lift; they are sometimes said to define double-valued representations of $\mathrm{SO}(V,h)$.

If $V = \mathbb{R}^{k+l}$ and the signature of $h$ is $(k,l)$, then one writes $\mathrm{Cl}_{k,l}$ instead of $\mathrm{Cl}(V,h)$; a similar notation is used for the orthogonal and spin groups. The number $k - l$ is the *index* of $h$. If $k$ or $l > 1$, then there are two unit vectors $u$ and $v$ orthogonal one to another and such that $h(u) = h(v)$. The function $t \mapsto \omega(t) = \cos t + uv \sin t$ maps the interval $[0, \pi]$ into a curve in $\mathrm{Spin}_{k,l}$ connecting 1 and $-1$; the image of this curve by $\rho$ is a non-contractible loop in $\mathrm{SO}_{k,l}$. Let $\pi_1(G^\circ)$ denote the fundamental group of the connected component $G^\circ$ of a Lie group $G$; one has

**Proposition 1.** *If $k$ and $l$ are non-negative integers and at least one of them is $> 1$, then the sequence*

$$1 \to \pi_1(\mathrm{Spin}^{\circ}_{k,l}) \to \pi_1(\mathrm{SO}^{\circ}_{k,l}) \to \mathbb{Z}_2 \to 1$$

*is exact: the group $\pi_1(\mathrm{SO}^{\circ}_{k,l})$ has a non-trivial $\mathbb{Z}_2$-grading.*

Note that $\mathrm{Spin}_{1,0} = \mathbb{Z}_2$ and $\mathrm{Spin}^{\circ}_{1,1}$ is isomorphic to $\mathbb{R}$. The groups $\mathrm{Spin}_{k,0}$ and $\mathrm{Spin}_{0,k}$, which are isomorphic, are denoted $\mathrm{Spin}_k$; they are connected for $k > 1$ and simply connected for $k > 2$. The groups $\mathrm{Spin}^{\circ}_{k,l}$, $k > 2$, are simply connected for $l = 1$, but not for $l > 1$. If the groups $\mathrm{Spin}_2$ and $\mathrm{SO}_2$ are both identified with $\mathrm{U}_1 \subset \mathbb{C}$, then $\rho$ becomes literally the map of taking the square.

Let $\mathcal{M}$ be an oriented manifold with a metric tensor of signature $(k, l)$. Let $\pi : \mathcal{P} \to \mathcal{M}$ be its $\mathrm{SO}_{k,l}$-bundle of orthonormal frames of coherent orientation. A *spin structure* on $\mathcal{M}$ is a principal $\mathrm{Spin}_{k,l}$-bundle $\varpi : \mathcal{Q} \to \mathcal{M}$, together with a morphism $\chi : \mathcal{Q} \to \mathcal{P}$ of principal bundles over $\mathcal{M}$ associated with $\rho$: for every $a \in \mathrm{Spin}_{k,l}$ and $q \in \mathcal{Q}$ one has $\chi(qa) = \chi(q)\rho(a)$ and $\pi \circ \chi = \varpi$. The topological obstruction to the existence of a spin structure on a pseudo-Riemannian space has been found by Karoubi [Ka]. Given a spin structure $\mathcal{Q} \to \mathcal{P} \to \mathcal{M}$ and a spin representation $\gamma : \mathrm{Spin}_{k,l} \to \mathrm{End}\, S$, one defines a *spinor field* of type $\gamma$ on $\mathcal{M}$ to be a (smooth) map $\psi : \mathcal{Q} \to S$ such that, for every $a \in \mathrm{Spin}_{k,l}$ and $q \in \mathcal{Q}$, one has $\psi(qa) = \gamma(a^{-1})\psi(q)$. Let $\omega$ be the map defined in the preceding paragraph; since $\gamma(-1) = -\mathrm{id}_S$, one has, for a spinor field, $\psi(q\omega(\pi)) = -\psi(q\omega(0))$; the frame $\chi(q)\rho(\omega(t))$ results from the frame $\chi(q)$ by a rotation by the angle $2t$; for this reason one sometimes says that 'a rotation by $2\pi$ induces a change of the sign of a spinor'.

## 5. Tensor squares of spinors are multivectors

The tensor product of two spin representations of $\mathrm{Spin}(V, h)$ is a representation of $\mathrm{SO}(V, h)$; this simple fact underlies the physicists' construction of (real) multivectors from spinors.

**5. 1. Complex vector spaces.** Consider the complex vector space $W = \mathbb{C}^n$ with a quadratic form $h$ defined by $h(z) = (z^1)^2 - (z^2)^2 + \cdots + (-1)^{n+1}(z^n)^2$, where $z = (z^\mu e_\mu) \in \mathbb{C}^n$. Denote by $\mathrm{Cl}_n$ the Clifford algebra $\mathrm{Cl}(W, h)$. Let $\eta = e_1 \cdots e_n \in \mathrm{Cl}_n$ be an (oriented) volume element in $W$. Clearly, $\eta v = (-1)^{n+1} v\eta$ for every $v \in W$. The Kähler definition of the Hodge map $\star : \wedge W \to \wedge W$ reads

$$\star c(a) = c(\eta a), \quad \text{where} \quad a \in \mathrm{Cl}_m,$$

and leads to

$$\star\star = \mathrm{id}_{\wedge W} \quad \text{and} \quad i(v) \circ \star = (-1)^{n+1} \star \circ\, e(v)$$

for every $v \in W$.

The complex spin group $\mathrm{Spin}_n(\mathbb{C})$ is defined as the subset of $\mathrm{Cl}^+_n$ consisting of products of all sequences of an even number of vectors with squares $= 1$; if $a \in \mathrm{Spin}_n(\mathbb{C})$, then $a^t a = 1$. A representation of the algebra $\mathrm{Cl}_n$ in a vector space defines, by restriction, a representation of the group $\mathrm{Spin}_n(\mathbb{C})$; one uses the same name and letter for a representation of the algebra and of its restriction to the group. If $a \in \mathrm{Spin}_n(\mathbb{C})$ and $b \in \mathrm{Cl}_n$, then $\mathrm{Cl}(\rho(a))b = aba^{-1}$ so that (3) gives

$$c(aba^{-1}) = \wedge\rho(a) \circ c(b). \tag{5}$$

*5. 1. 1.* Consider first an *odd*-dimensional complex vector space $W = \mathbb{C}^{2m+1}$ ($m = 1, 2, \ldots$). The even subalgebra $\mathrm{Cl}^+_{2m+1}$ is simple and, since it is complex $2^{2m}$-dimensional, it has one, up to equivalence, irreducible and faithful 'Pauli' representation $\sigma$ in a $2^m$-dimensional complex space $S$ of spinors. By putting $\sigma_\pm(\eta) = \pm\mathrm{id}_S$ and $\sigma_\pm|\mathrm{Cl}^+_{2m+1} = \sigma$, one extends $\sigma$ to the representations $\sigma_+$ and $\sigma_- = \sigma_+ \circ \alpha$ of $\mathrm{Cl}_{2m+1}$ in $S$; these representations are irreducible and inequivalent, but not faithful: the kernel of $\sigma_-$ (resp., $\sigma_+$) is the vector space of self-dual (resp., antiself-dual) elements of $\mathrm{Cl}_{2m+1}$.

The representations $\sigma_\pm$ can be described explicitly as follows. Consider a Witt decomposition $W = N \oplus P \oplus \mathbb{C}e_{2m+1}$, where $N$ and $P$ are $m$-dimensional—therefore maximal—totally isotropic subspaces of $W$ and $e_{2m+1}$ is a unit vector orthogonal to $V = N \oplus P$. One takes $S = \wedge N$ and, writing an element of $W$ as $n + p + ze_{2m+1}$, where $n \in N$, $p \in P$ and $z \in \mathbb{C}$, one puts $\sigma_\pm(n + p + ze_{2m+1}) = \pm(\sqrt{2}(e(n) + i(p)) + z\alpha_N)$.

If $f \in \mathrm{End}\, S$, then $f^* \in \mathrm{End}\, S^*$ is defined by $\langle \varphi, f^*(\varphi')\rangle = \langle f(\varphi), \varphi'\rangle$ for every $\varphi \in S$ and $\varphi' \in S^*$. Let $\beta$ be the antiautomorphism of $\mathrm{Cl}_{2m+1}$ defined by $\beta(a) = a^t$ for $m$ even and $\beta(a) = \alpha(a)^t$ for $m$ odd so that $\beta(\eta) = \eta$ for every $m$. The two representations $a \mapsto \sigma_\pm(\beta(a))^*$ of $\mathrm{Cl}_{2m+1}$ in $S^*$ are equivalent to the corresponding representations $\sigma_\pm$: there exists an isomorphism $B : S \to S^*$ such that $\sigma_\pm(\beta(a))^* = B\sigma_\pm(a)B^{-1}$ for every $a \in \mathrm{Cl}_{2m+1}$. Iterating and using Schur's lemma one obtains $B^* = \varepsilon B$, where either $\varepsilon = 1$ or $\varepsilon = -1$. To determine $\varepsilon$, note that $(B\sigma(a))^* = \varepsilon B\sigma(a^t)$ for every $a \in \mathrm{Cl}^+_{2m+1}$. Since $\dim\{f : S \to S^* | f^* = f\} > \dim\{f : S \to S^* | f^* = -f\}$, one has $\varepsilon = \mathrm{sgn}\,(\dim A^+_m - \dim A^-_m)$, where $A^\pm_m = \{a \in \mathrm{Cl}^+_{2m+1} | a^t = \pm a\}$. Moreover, $\dim A^+_m - \dim A^-_m = \sum_{p=0}^n (-1)^p \binom{2m+1}{2p} = 2^m\sqrt{2}\cos(2m+1)\frac{\pi}{4}$ so that [C4]

$$B^* = (-1)^{\frac{1}{2}m(m+1)} B. \tag{6}$$

For every $a \in \mathrm{Cl}^+_{2m+1}$ one has

$$\sigma(a)^* = B\sigma(a^t)B^{-1}. \tag{7}$$

The isomorphism $B$ defines also non-degenerate quadratic forms $B \otimes B^{-1}$ and $B \otimes B$ on $\mathrm{End}\, S$ and $S \otimes S$, respectively. Namely, if $f \in \mathrm{End}\, S$, then $(B \otimes B^{-1})(f) = \mathrm{tr}\,(B^{-1} \circ f^* \circ B \circ f)$; if $\varphi, \psi \in S$, then $(B \otimes B)(\varphi \otimes \psi) = \langle \varphi, B\varphi\rangle\langle \psi, B\psi\rangle$; the linear map $S \otimes S \to \mathrm{End}\, S$ defined by $\varphi \otimes \psi \mapsto \varphi \otimes B\psi$ is an isometry for these quadratic forms. Similarly, the algebra $\mathrm{Cl}^+_{2m+1}$ has a quadratic form $H$, $H(a) = 2^{-m}\mathrm{tr}\,\sigma(a^t a)$ and $\sigma$ is an isometry of $(\mathrm{Cl}_{2m+1}, H)$ onto $(\mathrm{End}\, S, 2^{-m} B \otimes B^{-1})$. The even exterior algebra $\wedge^+ W$ has a quadratic form $\wedge^+ h$ obtained by extension of $h$; the isomorphism (2) restricted to $\mathrm{Cl}^+_{2m+1}$ is an isometry equivariant with respect to the action of the spin group. This leads to

**Proposition 2.** *Let $\sigma$ be a faithful and irreducible representation of the even Clifford algebra $\mathrm{Cl}^+_{2m+1}$, associated with $W = \mathbb{C}^{2m+1}$, in a $2^m$-dimensional complex vector space $S$. There then exists an isomorphism $B : S \to S^*$ such that* (6) *and* (7) *hold. The bilinear map*

$$E : S \times S \to \wedge^+ W, \quad E(\varphi, \psi) = c \circ \sigma^{-1}(\varphi \otimes B\psi),$$

*(i) satisfies*

$$E(\psi, \varphi) = (-1)^{\frac{1}{2}m(m+1)} E(\varphi, \psi)^t;$$

*(ii) if* $v \in W$*, then*

$$E(\sigma_+(v)\varphi, \psi) = (i(v) + e(v)) \star E(\varphi, \psi);$$

*(iii) if* $a \in \mathrm{Spin}_{2m+1}(\mathbb{C})$*, then*

$$E(\sigma(a)\varphi, \sigma(a)\psi) = \wedge\rho(a) \circ E(\varphi, \psi);$$

*(iv) the linear map* $S \otimes S \to \wedge^+ W$*, associated with* $E$*, is an isometry of the quadratic space* $(S \otimes S,\ 2^{-m} B \otimes B)$ *onto* $(\wedge^+ W,\ \wedge^+ h)$ *which is equivariant with respect to the action of the group* $\mathrm{Spin}_{2m+1}(\mathbb{C})$*.*

To prove (i), put $b = \sigma^{-1}(\varphi \otimes B\psi)$; then $\psi \otimes B\varphi = B^{-1} \circ (\varphi \otimes B\psi)^* \circ B^* = (-1)^{\frac{1}{2}m(m+1)}\sigma(b^t)$. The proof of (ii) is based on $\sigma_+(v) = \sigma(\eta v)$, on the definition of the Hodge dual and on (4). Part (iii) follows from $\sigma(a)\varphi \otimes B\sigma(a)\psi = \sigma(a) \circ (\varphi \otimes B\psi) \circ \sigma(a^t)$, $a^t a = 1$ and and (5). Part (iv) follows from the preceding remarks; see also the proof of Prop. 4. 2. in [BuT].

The component of $E$ in $\wedge^{2p} W$ is denoted by $E_{2p}$. In particular, $E_0(\varphi, \psi) = 2^{-m}\langle \varphi, B\psi \rangle$. The bilinear form $E_0$ is invariant with respect to the action of $\mathrm{Spin}_{2m+1}(\mathbb{C})$. More generally, if $a \in \mathrm{Cl}^+_{2m+1}$ and $a^t a = 1$, then $\langle \sigma(a)\varphi, B\sigma(a)\psi \rangle = \langle \varphi, B\psi \rangle$. According to part (i) of Prop. 2, one has

$$E_{2p}(\psi, \varphi) = (-1)^{\frac{1}{2}m(m+1)+p} E_{2p}(\varphi, \psi).$$

Putting $\nu$ equal to the integer part of $\frac{1}{2}(m+1)$, one obtains that

$$\text{if } \nu - p \text{ is odd, then } E_{2p}(\varphi, \varphi) = 0. \tag{8}$$

*5. 1. 2.* Consider now the *even*-dimensional subspace $V = \mathbb{C}^{2m}$ of $W$ orthogonal to the unit vector $e_{2m+1}$. The algebra $\mathrm{Cl}_{2m}$ can be identified with a subalgebra of $\mathrm{Cl}_{2m+1}$; the Clifford map $V \to \mathrm{Cl}^+_{2m+1}$, $v \mapsto \eta v$, extends to an isomorphism of algebras $j : \mathrm{Cl}_{2m} \to \mathrm{Cl}^+_{2m+1}$. Since $j(v)^t = (-1)^m j(v)$, one has $j(a)^t = j(\beta(a))$ for every $a \in \mathrm{Cl}_{2m}$. The element $\eta e_{2m+1} = e_1 \cdots e_{2m}$ is a volume in $V$. The composition $\gamma = \sigma \circ j$ is the 'Dirac' representation of the algebra $\mathrm{Cl}_{2m}$ in $S$. The automorphisms $\gamma_\mu = \gamma(e_\mu)$ ($\mu = 1, \ldots, 2m$) of $S$ generalize the classical Dirac matrices; one has $\sigma_\pm(e_\mu) = \pm\gamma_\mu$ for $\mu = 1, \ldots, 2m$ and $\sigma_\pm(e_{2m+1}) = \pm\mathit{\Gamma}$, where $\mathit{\Gamma} = \gamma_1 \cdots \gamma_{2m}$ is the 'chirality' automorphism of $S$. It follows from these definitions that

$$\gamma^*_\mu = (-1)^m B\gamma_\mu B^{-1} \quad \text{and} \quad \mathit{\Gamma}^* = (-1)^m B\mathit{\Gamma} B^{-1}. \tag{9}$$

Let $k : \wedge V \to \wedge^+ W$ be the isomorphism of vector spaces such that $k \circ c = c \circ j$; explicitly, it is given by $k(w) = w$ for $w \in \wedge^+ V$ and $k(w) = \star w$ for $w$ odd, $w \in \wedge^- V$. Let the Hodge dual in $V$ be denoted by $*$, i. e. $c(\eta e_{2m+1} a) = *c(a)$ for $a \in \mathrm{Cl}_{2m}$. Consider the bilinear map

$$F = k^{-1} \circ E : S \times S \to \wedge V.$$

Denoting by $F_p(\varphi,\psi)$ the component of $F(\varphi,\psi)$ in $\wedge^p V$, one obtains, as a corollary of part (i) of Prop. 2,

$$F_p(\psi,\varphi) = (-1)^{\frac{1}{2}(m-p)(m-p+1)} F_p(\varphi,\psi). \tag{10}$$

Putting $v = e_{2m+1}$ in part (ii) and using $\sigma_+(e_{2m+1}) = \sigma(\eta e_{2m+1})$ leads to

$$F(\varGamma\varphi,\psi) = *F(\varphi,\psi) \quad \text{and} \quad F(\varphi,\varGamma\psi) = (-1)^m \alpha \circ *F(\varphi,\psi). \tag{11}$$

If $\wedge V$ is given the quadratic form $\wedge h$ such that

$$\wedge h(v_1 \wedge \cdots \wedge v_p) = h(v_1)\cdots h(v_p),$$

where $v_1,\ldots,v_p \in V$, then $k$ becomes an isometry of $(\wedge V, \wedge h)$ onto $(\wedge^+ W, \wedge^+ h)$. As a corollary from Prop. 2 one obtains that the map

$$S \otimes S \to \wedge V, \ \text{ given by } \ \varphi \otimes \psi \mapsto c \circ \gamma^{-1}(\varphi \otimes B\psi), \tag{12}$$

is an isometry of the corresponding quadratic spaces, equivariant with respect to the action of $\mathrm{Spin}_{2m}(\mathbb{C})$.

Restricted to the even subalgebra $\mathrm{Cl}^+_{2m}$, the representation $\gamma$ decomposes into the direct sum of two irreducible and inequivalent, but not faithful, representations $\gamma_+$ and $\gamma_-$ in $2^{m-1}$-dimensional spaces $S_+$ and $S_-$ of 'Weyl' or 'chiral' spinors. One has $S_\pm = \{\varphi \in S : \varGamma\varphi = \pm\varphi\}$ and $S = S_+ \oplus S_-$. If $\varphi$ and $\psi$ are both Weyl spinors with respect to $\gamma$, then (9) and (10) give

$$\text{if} \quad \chi(\varphi)+\chi(\psi)+m+p \equiv 1 \mod 2, \quad \text{then} \quad F_p(\varphi,\psi)=0. \tag{13}$$

In particular, if $\varphi$ is a Weyl spinor, then $F_p(\varphi,\varphi)=0$ unless $p \equiv m \mod 4$.

If the representation $\gamma$ comes from a representation $\sigma$ constructed in terms of a Witt decomposition, so that $S = \wedge N$, then $\varGamma = \alpha_N$ and $S_\pm = \wedge^\pm N$.

**5. 2. Real vector spaces.** Consider now the real vector space $\mathbb{R}^n$ with a quadratic form $h$ of signature $(k,l)$, $k+l=n$. If $(e_1,\ldots,e_n)$ is a frame orthonormal with respect to $h$, then the volume element $\eta = e_1\cdots e_n$ satisfies $\eta^2 = (-1)^{\frac{1}{2}(l-k)(l-k+1)}$. The complexification of the Clifford algebra $\mathrm{Cl}_{k,l}$ is isomorphic with $\mathrm{Cl}_n$.

*5. 2. 1.* Consider first the case of $n$ odd, $n = 2m+1$. The Pauli representation of $\mathrm{Cl}^+_{2m+1}$, restricted to $\mathrm{Cl}^+_{k,l}$, yields a representation $\sigma$ of this real algebra in a $2^m$-dimensional, complex vector space $S$. The representation $\sigma$ can be extended to the representations $\sigma_+$ and $\sigma_-$ of $\mathrm{Cl}_{k,l}$ in $S$ by putting $\sigma_\pm(\eta) = \pm\mathrm{id}_S$ when $\eta^2=1$ and $\sigma_\pm(\eta) = \pm\sqrt{-1}\,\mathrm{id}_S$ when $\eta^2=-1$. Recall also that with every complex vector space $S$ one can associate its 'complex conjugate' space $\bar{S}$: the space $\bar{S}$ has the same set of elements as $S$, but the product of $z \in \mathbb{C}$ by a vector in $\bar{S}$ is equal to the product of $\bar{z}$ by the same vector in $S$. Denoting by $\bar{\varphi}$ the vector $\varphi$, considered as an element of $\bar{S}$, one has $\overline{z\varphi} = \bar{z}\bar{\varphi}$ for every $\varphi \in S$ and $z \in \mathbb{C}$. If $f : S_1 \to S_2$ is a linear map of complex vector spaces, then the linear map $\bar{f} : \bar{S_1} \to \bar{S_2}$ is defined by $\overline{f}(\overline{\varphi}) = \overline{f(\varphi)}$; the maps $\varphi \mapsto \bar{\varphi}$ and $f \mapsto \bar{f}$ are semi-linear, etc. With every representation $\tau$ of a *real* algebra $\mathcal{A}$ in a *complex* vector space $S$ one can associate the complex conjugate representation $\bar{\tau}$ in $\bar{S}$, given by $\overline{\tau}(a) = \overline{\tau(a)}$ for every $a \in \mathcal{A}$. Since $\mathrm{Cl}^+_{k,l}$ is central simple for $k+l$ odd, its representations $\sigma$ and $\bar{\sigma}$ are complex-equivalent; if $\eta^2 = 1$, then the representations $\overline{\sigma_\pm}$ are equivalent to the corresponding representations $\sigma_\pm$; if $\eta^2 = -1$, then $\overline{\sigma_+}$ is equivalent to $\sigma_-$. In every case there is a linear isomorphism

$$C : S \to \overline{S} \quad \text{such that} \quad \overline{\sigma_\mu} = (-1)^{\frac{1}{2}(l-k)(l-k+1)} C \sigma_\mu C^{-1}, \quad \text{where} \quad \sigma_\mu = \sigma_+(e_\mu)$$

for $\mu = 1, \dots, k+l = 2m+1$. An argument similar to the one used with respect to $B$ in Sec. 5. 1 shows that $C$ can be rescaled so that either $\bar{C}C = \mathrm{id}_S$ or $\bar{C}C = -\mathrm{id}_S$. Moreover, one obtains from (7) and

$$\overline{\sigma(a)} = C\sigma(a)C^{-1}, \quad a \in \mathrm{Cl}^+_{k,l}, \tag{14}$$

that $C^{-1}\bar{B}^{-1}\bar{C}^* B^*$ is in the commutant of $\sigma$; therefore, one can rescale $B$ so that

$$B = C^* \bar{B} C \tag{15}$$

and then the sesquilinear form

$$A : S \times S \to \mathbb{C}, \quad \text{given by} \quad A(\varphi, \psi) = \langle \bar{\varphi}, \bar{B} C \psi \rangle$$

is either Hermitean or anti-Hermitean.

According to a terminology used in physics, one says that $\varphi_{\mathrm{c}} = C^{-1}\bar{\varphi}$ is the *charge conjugate* of the spinor $\varphi \in S$. The charge conjugate $\varphi'_{\mathrm{c}}$ of $\varphi' \in S^*$ is defined so that $\langle \varphi_{\mathrm{c}}, \varphi'_{\mathrm{c}} \rangle = \overline{\langle \varphi, \varphi' \rangle}$; by virtue of (14), if $\varphi' = B\psi$, then $\varphi'_{\mathrm{c}} = B\psi_{\mathrm{c}}$. There are two cases to consider [C1, BuT]:
(i) The *real* case: if $l - k \equiv 1$ or $7 \mod 8$, then $\bar{C}C = \mathrm{id}_S$; there is then the $2^m$-dimensional real vector space

$$S_{\mathbb{R}} = \{\varphi \in S \varphi_{\mathrm{c}} = \varphi\}.$$

and a decomposition of $S$ into complementary subspaces of 'Majorana' spinors, $S = S_{\mathbb{R}} \oplus \sqrt{-1} S_{\mathbb{R}}$. The representation $\sigma$ is real: $\sigma(a)S_{\mathbb{R}} \subset S_{\mathbb{R}}$ for every $a \in \mathrm{Cl}^+_{k,l}$. The automorphisms $\sigma_\mu = \sigma_+(e_\mu)$ are real (resp., pure imaginary) for $l - k \equiv 7 \mod 8$ (resp., $l - k \equiv 1 \mod 8$). The algebra $\mathrm{Cl}^+_{k,l}$ is isomorphic to the matrix algebra $\mathbb{R}(2^m)$. The algebra $\mathrm{Cl}_{k,l}$ is isomorphic to $\mathbb{R}(2^m) \oplus \mathbb{R}(2^m)$ (resp., $\mathbb{C}(2^m)$) for $l - k \equiv 7 \mod 8$ (resp., $l - k \equiv 1 \mod 8$). The form $A$ restricted $S_{\mathbb{R}}$ is real and has the same symmetry as $B$.
(ii) The *quaternionic* case: if $l - k \equiv 3$ or $5 \mod 8$, then $\bar{C}C = -\mathrm{id}_S$ and $S$ can be given the structure of a right module over $\mathbb{H}$. Explicitly, denoting by $\mathrm{i}, \mathrm{j}$ and $\mathrm{k} = \mathrm{ij}$ the quaternionic units, one puts $\varphi\mathrm{i} = \sqrt{-1}\varphi$ and $\varphi\mathrm{j} = \varphi_{\mathrm{c}}$. The algebra $\mathrm{Cl}^+_{k,l}$ is isomorphic to the matrix algebra $\mathbb{H}(2^{m-1})$. The algebra $\mathrm{Cl}_{k,l}$ is isomorphic to $\mathbb{H}(2^{m-1}) \oplus \mathbb{H}(2^{m-1})$ (resp., $\mathbb{C}(2^m)$) for $l - k \equiv 3 \mod 8$ (resp., $l - k \equiv 5 \mod 8$).

The vector space $\mathcal{A} = \{f \in \mathrm{End}\, S \bar{f} C = Cf\}$ is a real algebra spanned by all elements of the form $\varphi \otimes \varphi' + \varphi_{\mathrm{c}} \otimes \varphi'_{\mathrm{c}}$, where $\varphi \in S$ and $\varphi' \in S^*$. The representation $\sigma$ factors through the injection $\mathcal{A} \to \mathrm{End}\, S$. Moreover, in the real case, the algebra $\mathcal{A}$ is isomorphic to $S_{\mathbb{R}} \otimes_{\mathbb{R}} S_{\mathbb{R}}$. In the quaternionic case, it is isomorphic to the tensor product over $\mathbb{H}$ of the right $\mathbb{H}$-module $S$ by the left $\mathbb{H}$-module $S^*$. In each case one has

$$\overline{E(\varphi,\psi)} = E(\varphi_{\rm c},\psi_{\rm c}).$$

The homogeneous components of the multivector $E(\varphi_{\rm c},\varphi)$ are either real or imaginary, as can be seen from part (i) of Prop. 2 and $\varphi_{\rm cc} = C\bar{C}\varphi$.

The quadratic form $H$ on $\mathrm{Cl}^+_{k,l}$, $k+l = 2m+1$, $H(a) = 2^{-m}\mathrm{tr}\,\sigma(a^t a)$, is real; its signature can be evaluated as follows. Consider the polynomial $\varsigma(\xi,\eta) = \frac{1}{2}(1+\xi)^k(1+\eta)^l + \frac{1}{2}(1-\xi)^k(1-\eta)^l = \sum_{p,q;\,p+q \text{ even}} \binom{k}{p}\binom{l}{q}\xi^p\eta^q$. The index of $H$ equals $\varsigma(1,-1)$. Therefore, $H$ is positive-definite if, and only if, either $k=0$ or $l=0$; if both $k$ and $l$ are positive, then $H$ is neutral.[3]

*5. 2. 2.* Consider now the *even*-dimensional subspace $V$ of $\mathbb{R}^{k+l}$, orthogonal to a unit vector $u$. Depending on whether $u^2 = 1$ or $-1$, the signature of the restriction $h_V$ of $h$ to $V$ is $(k-1,l)$ or $(k,l-1)$. The map $V \to \mathrm{Cl}^+_{k,l}$, $v \mapsto v\eta$, extends to an isomorphism of algebras $j : \mathrm{Cl}(V,\eta^2 h_V) \to \mathrm{Cl}^+_{k,l}$: if $\eta^2 = 1$ (resp., $\eta^2 = -1$), then $\mathrm{Cl}^+_{k,l}$ is isomorphic to both $\mathrm{Cl}_{k-1,l}$ and $\mathrm{Cl}_{k,l-1}$ (resp., $\mathrm{Cl}_{l-1,k}$ and $\mathrm{Cl}_{l,k-1}$).

If $k+l = 2m$ and $\gamma : \mathrm{Cl}_{k,l} \to \mathrm{End}\, S$ is a Dirac representation in a complex space $S$ of dimension $2^m$, then there is $C : S \to \bar{S}$ such that $\overline{\gamma(a)} = C\gamma(a)C^{-1}$ for every $a \in \mathrm{Cl}_{k,l}$. According to previous remarks, one can rescale $C$ so that either $\bar{C}C = \mathrm{id}_S$ (for $l-k \equiv 0$ or $6 \mod 8$) or $\bar{C}C = -\mathrm{id}_S$ (for $l-k \equiv 2$ or $4 \mod 8$). It is convenient to define the chirality automorphism as $\Gamma = (-1)^{\frac{1}{4}(l-k)(l-k+1)}\gamma_1\cdots\gamma_{2m}$ so that

$$\Gamma^2 = \mathrm{id}_S \quad \text{and} \quad \bar{\Gamma} = (-1)^{\frac{1}{2}(l-k)(l-k+1)} C\Gamma C^{-1}. \tag{16}$$

The representation $\gamma$, restricted to $\mathrm{Cl}^+_{k,l}$ decomposes as in the complex case, $\gamma = \gamma_+ \oplus \gamma_-$. The representation $\check{\gamma}$ of $\mathrm{Cl}^+_{k,l}$ in $S^*$, contragredient to $\gamma$, is defined by $\check{\gamma}(a) = \gamma(a^t)^*$, $a \in \mathrm{Cl}^+_{k,l}$. It also decomposes, $\check{\gamma} = \check{\gamma}_+ \oplus \check{\gamma}_-$, where $\check{\gamma}_\pm : \mathrm{Cl}^+_{k,l} \to \mathrm{End}\, S^*_\pm$, $S^*_\pm = \{\varphi \in S\Gamma^*\varphi = \pm\varphi\}$. There is also a similar decomposition $\bar{\gamma} = \bar{\gamma}_+ \oplus \bar{\gamma}_-$ of $\bar{\gamma}$ restricted to $\mathrm{Cl}^+_{k,l}$. The representations $\gamma_+$ and $\gamma_-$ are not complex-equivalent; each of the representations $\check{\gamma}_+$, $\check{\gamma}_-$, $\bar{\gamma}_+$ and $\bar{\gamma}_-$ is equivalent to either $\gamma_+$ or $\gamma_-$. In particular, denoting by $\sim$ the equivalence relation of representations, one obtains from (9)

$$\check{\gamma}_\pm \sim \gamma_\pm \quad \text{for } m \text{ even and} \quad \check{\gamma}_\pm \sim \gamma_\mp \quad \text{for } m \text{ odd}$$

and from (16)

$$\bar{\gamma}_\pm \sim \gamma_\pm \quad \text{for } l-k \equiv 0 \text{ or } 4 \mod 8 \text{ and} \quad \bar{\gamma}_\pm \sim \gamma_\mp \quad \text{for } l-k \equiv 2 \text{ or } 6 \mod 8.$$

*5. 2. 3.* Many authors of papers on spinors use a notation with indices similar to that codified for tensors by Schouten. This van der Waerden-Penrose notation for Weyl spinors associated with a $2m$-dimensional real vector space can be briefly described as follows. Consider linear bases: $(e_A)$ in $S_+$, $(e_{A'})$ in $S_-$, $(e_{\dot{A}})$ in $\bar{S}_+$,

[3] I am indebted to P. Lounesto for having pointed out to me a mistake in the formulation of this result given in [BuT].

$(e^A)$ in $S_+^*$, etc., where $e_{\dot{A}} = \overline{e_A}$, $(e^A)$ is the basis dual to $(e_A)$ and the indices range from 1 to $2^{m-1}$. Instead of saying 'let $\varphi \in S_+^* \otimes S_+ \otimes S_-$' one says 'consider a spinor $\varphi_A{}^{BC'}$'. This notation is convenient when one works with several spinor fields and considers their tensor products and contractions: such is the case of applications of spinor analysis in the theory of relativity [PR]. The Dirac 'matrices' $\gamma^\mu = h^{\mu\nu}\gamma_\nu$ change chirality of Weyl spinors; Penrose writes $\gamma^\mu(e_A) = e_{B'}\sigma^{\mu B'}{}_A$. If $\mathcal{M}$ is a Riemannian manifold with a spin structure $\mathcal{Q} \to \mathcal{P} \to \mathcal{M}$ and $\nabla_\mu$ are the (horizontal) vector fields on $\mathcal{Q}$ defining covariant differentiation of spinor fields, then the Dirac operator $\gamma^\mu\nabla_\mu$, acting on a Weyl spinor field $\psi = e_A\psi^A : \mathcal{Q} \to S_+$, yields the spinor field $e_{B'}\nabla^{B'}{}_A\psi^A : \mathcal{Q} \to S_-$, where $\nabla^{B'}{}_A = \sigma^{\mu B'}{}_A\nabla_\mu$. Relative to the basis $(e_A, e_{B'})$ of $S$, the matrices of the linear maps $B$ and $C$ are either block-diagonal or block-antidiagonal, depending on whether they preserve or change the chirality. In particular, if $m$ is even, then $B$ yields an isomorphism of $S_\pm$ onto $S_\pm^*$, $B(e_A) = B_{AB}e^B$ and $B(e_{A'}) = B_{A'B'}e^{B'}$. If $m$ is odd, then $B : S_\pm \to S_\mp^*$, $B(e_A) = B_{AB'}e^{B'}$, etc. Depending on the parity of $m$, the matrices $(B_{AB})$ and $(B_{A'B'})$ or $(B_{AB'})$ and $(B_{A'B})$, and their inverses, are used to lower and raise spinorial indices. Similarly, for $l - k \equiv 0$ or $4 \mod 8$ one has $C(e_A) = C_A{}^{\dot{B}}e_{\dot{B}}$, $C(e_{A'}) = C_{A'}{}^{\dot{B}'}e_{\dot{B}'}$ and the invertible matrices $(C_A{}^{\dot{B}})$ and $(C_{A'}{}^{\dot{B}'})$ are used to convert undotted into dotted indices, etc. Originally, van der Waerden (1929) emphasized the representations $\gamma_+$ and $\bar{\gamma}_+$ and introduced dotted indices for the latter. Penrose (1960) pointed out that it suffices to consider the representations $\gamma_+$ and $\gamma_-$ and used primed indices for spinors in $S_-$. In Minkowski space one has $l - k = 2$ so that $\bar{\gamma}_+ \sim \gamma_-$: primed and dotted indices are equivalent.

## 6. Pure spinors

Pythagoras and Euclid could have solved the equation $x^2 - y^2 + z^2 = 0$ by noting that it is equivalent to the statement that the symmetric matrix

$$\begin{pmatrix} y+z & x \\ x & y-z \end{pmatrix}$$

is of rank $< 2$ and, therefore, there is a 'spinor' $(p, q)$ such that $x = 2pq$, $y = p^2 + q^2$ and $z = p^2 - q^2$. Interpreting now the triple $(x, y, z)$ as an element of a real vector space, one can rephrase the solution of the Pythagoras equation to read 'the (tensor) square of a real, two-component spinor equals an isotropic vector in $\mathbb{R}^3$ with a quadratic form of signature $(2,1)$'. This observation has interesting generalizations to higher dimensions, giving rise to the notion of pure spinors.

**6. 1. The complex case.** Continuing to use the notation of Sec. 5. 1, consider a non-zero spinor $\varphi \in S$ associated with $W = \mathbb{C}^{2m+1}$. The vector space

$$N(\varphi) = \{v \in W | \sigma(\eta v)\varphi = 0\}$$

is totally isotropic; its dimension is called the *nullity* of $\varphi$. A spinor $\varphi \neq 0$ is said to be *pure* if its nullity is maximal, i. e. equal to $m$. Let $W = N \oplus P \oplus \mathbb{C}u$ be a Witt decomposition and let $(n_1, \dots, n_m)$ be a basis of $N$. Put $a_N = n_1 \cdots n_m$ for $m$ even and $a_N = \eta n_1 \cdots n_m$ for $m$ odd, so that $a_N \in \mathrm{Cl}_{2m+1}^+$. Since $\sigma$ is faithful, there is a spinor $\varphi_0$ such that $\varphi = \sigma(a_N)\varphi_0 \neq 0$ and then $N(\varphi) = N$ so that $\varphi$ is pure. If $\psi$ is another spinor such that $N(\psi) = N$, then there is $z \in \mathbb{C}^\times$ such that $\psi = z\varphi$:

there is a bijective correspondence between the set (in fact, a compact, connected, complex manifold $\Sigma_{2m+1}$ of complex dimension $\frac{1}{2}m(m+1)$) of directions of pure spinors and the set of maximal isotropic subspaces of $W$. If $a \in \mathrm{Spin}_{2m+1}(\mathbb{C})$ and $\varphi$ is pure, then $\sigma(a)\varphi$ is also pure; the induced action of $\mathrm{Spin}_{2m+1}(\mathbb{C})$ on $\Sigma_{2m+1}$ is transitive.

Let $\varphi \neq 0$ be a spinor of nullity $q$. Let $(n_1, \dots, n_q)$ be a basis of $N(\varphi)$. It follows from part (ii) of Prop. 2 and (8) that $n \in N(\varphi)$ implies $e(n)E_{2p}(\varphi, \varphi) = 0$ and $i(n)E_{2p}(\varphi, \varphi) = 0$ for every $p$ such that $p \equiv \nu \mod 2$. Therefore, there is a $(2p-q)$-vector $E'_{2p-q}$ such that $E_{2p}(\varphi, \varphi) = n_1 \wedge \cdots \wedge n_q \wedge E'_{2p-q}$ and $i(n)E'_{2p-q} = 0$ for every $n \in N(\varphi)$. This implies that if $2p < q$ or $2p > 2m+1-q$, then $E_{2p}(\varphi, \varphi) = 0$. In particular, if $\varphi$ is pure, then $E_{2p}(\varphi, \varphi) \neq 0$ if, and only if, $p = \nu$. A more precise result is contained in

**Proposition 3.** *A spinor $\varphi \neq 0$ is pure if, and only if, $p \neq \nu$ implies $E_{2p}(\varphi, \varphi) = 0$. If $\varphi$ is pure, then there is a basis $(n_1, \dots, n_m)$ of $N(\varphi)$ such that*

$$\text{if } m \text{ is even, then} \quad E_m(\varphi, \varphi) = n_1 \wedge \cdots \wedge n_m;$$

$$\text{if } m \text{ is odd, then} \quad \star E_{m+1}(\varphi, \varphi) = n_1 \wedge \cdots \wedge n_m.$$

The somewhat difficult proof of the 'if' part of Prop. 3 appears in [C4, Ch]; recently, a new proof has been given by Urbantke [U] with the help of 'quartic identities' used by physicists in connection with the classification of interaction terms occurring in the Fermi theory of weak nuclear forces. By considering the multivectors $E_{2p}(\varphi, \varphi)$ for low values of $m$, one obtains, as a corollary of (8) and Prop. 3, that all spinors associated with $W$ of dimension 3 and 5 are pure; in dimensions 7 and 9 pure spinors belong to the cone of equation $E_0(\varphi, \varphi) = 0$.

If $\varphi$ is pure and $v \in W$ is non-isotropic, $v^2 \neq 0$, then $\sigma(\eta v)\varphi$ is also pure: $N(\sigma(\eta v)\varphi) = vN(\varphi)v^{-1}$. In particular, if $N(\varphi)$ is orthogonal to the unit vector $e_{2m+1}$, then $N(\sigma(\eta e_{2m+1})\varphi) = N(\varphi)$. Therefore, $\sigma(\eta e_{2m+1})\varphi = \pm\varphi$. Introducing. as in Sec. 5. 1. 2, the $2m$-dimensional space $V$ orthogonal to $e_{2m+1}$, one obtains, as a corollary of Prop. 3 and (11):

**Proposition 4.** *Let $W = V \oplus \mathbb{C}e_{2m+1}$ and $\gamma$ be the Dirac representation of $\mathrm{Cl}_{2m}$ in $S$, $\gamma(v) = \sigma(\eta v)$ for $v \in V$. If $\varphi$ is pure, then $N(\varphi) \subset V$ if, and only if, $\varphi$ is a Weyl spinor with respect to $\gamma$. Assuming that $\varphi$ is such a spinor, one has $F_p(\varphi, \varphi) = 0$ for every $p \neq m$ and there is a basis $(n_1, \dots, n_m)$ in $N(\varphi)$ so that*

$$F_m(\varphi, \varphi) = n_1 \wedge \cdots \wedge n_m.$$

*The $m$-vector $F_m(\varphi, \varphi)$ is either self-dual ($\Gamma\varphi = \varphi$) or antiself-dual ($\Gamma\varphi = -\varphi$).*

In other words, in even-dimensional complex vector spaces, tensor squares of pure spinors define self- or antiself-dual decomposable multivectors of the middle degree. A pure spinor $\varphi$ associated with the representation $\gamma : \mathrm{Cl}_{2m} \to \mathrm{End}\, S$ can be characterized, without reference to the odd-dimensional space $W$, by $\dim\{v \in \mathbb{C}^{2m} \,|\, \gamma(v)\varphi = 0\} = m$; it follows that it is a Weyl spinor. The set of directions of pure spinors associated with $\gamma$ is a $\frac{1}{2}m(m-1)$-dimensional complex compact manifold $\Sigma_{2m} = \mathrm{O}_{2m}/\mathrm{U}_m$; it has two connected components, $\Sigma^+_{2m}$ and $\Sigma^-_{2m}$, corresponding to pure spinors of opposite chiralities. An argument similar to the one used in odd dimensions shows that, in dimensions 2, 4 and 6 all Weyl spinors are pure. In dimension 8 pure spinors lie on the cones of equation $F_0(\varphi,\varphi)=0$ in $S_+$ and $S_-$; in dimension 10 pure spinors are characterized by the equation $F_1(\varphi,\varphi)=0$ in $S_\pm$ and generic Weyl spinors have nullity 1; for $m=4$ and $m>5$ generic Weyl spinors have zero nullity. Spinors belonging to one orbit of the group $\mathrm{Spin}_{2m}(\mathbb{C})$ have the same nullity, but the converse is not true: in dimensions $\geqslant 12$ nullity of Weyl spinors provides a rather coarse classification of the orbits. There are no Weyl spinors of nullity $q$ such that $m-4<q<m$. All homogeneous polynomial invariants of the spin group vanish on spinors of positive nullity [TT].

**6. 2. The real case.** Changing somewhat the notation, consider now a $2m$-dimensional *real* vector space $V$, with a quadratic form $h$ of signature $(2\kappa+\varepsilon, 2\lambda+\varepsilon)$, where $\kappa$ and $\lambda$ are non-negative integers and $\varepsilon=0$ or 1. The cases $\varepsilon=0$ and $\varepsilon=1$ are referred to as *pseudo-euclidean* and *pseudo-lorentzian*, respectively; the prefix 'pseudo' is dropped when either $\kappa$ or $\lambda=0$. The *complexification* of $V$, $W=\mathbb{C}\otimes_{\mathbb{R}} V = V\oplus\sqrt{-1}V$, is given a quadratic form $\mathbb{C}\otimes h$ which is a natural extension of $h$ so that $\mathrm{Cl}(\mathbb{C}\otimes V,\mathbb{C}\otimes h)=\mathbb{C}\otimes\mathrm{Cl}(V,h)$. If $N\subset W$, then $\bar N\subset W$ is defined as the set $\{u+\sqrt{-1}v|u-\sqrt{-1}v\in N\}$. If $N$ is a (complex) vector subspace of $W$, then there are (real) vector subspaces $K$ and $L$ of $V$ such that

$$N\cap\bar N=\mathbb{C}\otimes K \quad\text{and}\quad N+\bar N=\mathbb{C}\otimes L. \tag{17}$$

.

If $\varphi$ is a pure spinor, then so is $\varphi_{\mathrm{c}}$ and (14) implies $N(\varphi_{\mathrm{c}})=\overline{N(\varphi)}$. Put $N=N(\varphi)$; the dimension $r$ of $K$ is called the *real index* of the pure spinor $\varphi$. One shows that the real index can assume every of the following values: $\varepsilon,\varepsilon+2,\ldots,\varepsilon+2\min\{\kappa,\lambda\}$, and only these values. Pure spinors with the least value of the real index are generic in the sense that the manifolds of directions of such spinors, of positive and negative chiralities, are open and dense in $\Sigma^+_{2m}$ and $\Sigma^-_{2m}$, respectively. Let $\varphi$ be a pure spinor of real index $r$; the vector space $K$ is totally isotropic and $K^\perp=L\supset K$. The quadratic form $h$ descends to a quadratic form of signature $(2\kappa+\varepsilon-r, 2\lambda+\varepsilon-r)$ on the quotient $L/K$. Moreover, there is an orthogonal complex structure $J$ on $L/K$ defined as follows: let $w \mod K$ denote the class in $L/K$ containing $w\in L$; every element of $L/K$ is of the form $v+\bar v \mod K$, where $v\in N$; put $J(v+\bar v \mod K)=\sqrt{-1}(v-\bar v) \mod K$.

The most interesting case, from the point of view of applications in geometry and physics, is that of pure spinors in a general position, i. e. of pure spinors of real index $\varepsilon$. If $\varphi$ is a generic pure spinor associated with a pseudo-euclidean quadratic space $(V,h)$, then the sum $N(\varphi)+\overline{N(\varphi)}$ is direct, $K=\{0\}$, and $J$ is an orthogonal complex structure in $V=L$. The pair $(h,J)$ defines a Hermitean form on $V$, considered as a complex vector space of dimension $\kappa+\lambda$. The imaginary part of that Hermitean form can be identified with $F_2(\varphi_{\mathrm{c}},\varphi)/F_0(\varphi_{\mathrm{c}},\varphi)$. If $\varphi$ is a generic pure spinor associated with a pseudo-lorentzian quadratic space and $N=N(\varphi)$, then $K$ is a real isotropic line, a 'light ray' for physicists; the quotient $L/K$ is a 'screen space', if one insists on the optical analogy. The line $\mathbb{C}\otimes K$ is generated by $F_1(\varphi_{\mathrm{c}},\varphi)$ [KoT, T].

**6. 3. Applications.** The preceding algebraic constructions can be applied, 'pointwise', to fibres of appropriate bundles. From now on, all manifolds, bundles and maps are assumed to be smooth. Consider a $2m$-dimensional, paracompact, connected manifold $\mathcal{M}$ with a metric tensor of signature $(2\kappa+\varepsilon, 2\lambda+\varepsilon)$ and a spin structure $\mathcal{Q} \overset{\chi}{\to} \mathcal{P} \to \mathcal{M}$. The last assumption is, in fact, not essential because all that is needed here are bundles of 'projectivized' spinors (i. e. bundles of directions of spinors) which are associated with the $\mathrm{SO}_{2\kappa+\varepsilon,2\lambda+\varepsilon}$-bundle $\mathcal{P} \to \mathcal{M}$. It is often, however, convenient to work with spinor fields and to represent them by the corresponding maps from $\mathcal{Q}$ to the typical fibre $S$ equivariant with respect to the action of the group $G = \mathrm{Spin}_{2\kappa+\varepsilon,2\lambda+\varepsilon}$, as described in Sec. 4. If $\mathcal{E} \to \mathcal{M}$ is a fiber bundle, then $\mathcal{E}_x$ denotes its fibre over $x \in \mathcal{M}$. Recall that vector fields, differential forms, etc. on $\mathcal{M}$ can be equivalently described as sections of appropriate bundles or as equivariant maps from $\mathcal{Q}$ to a suitable vector space. In paricular, a complex vector tangent to $\mathcal{M}$ at $x$ can be identified with a map $v : \mathcal{Q}_x \to W = \mathbb{C} \otimes V$ such that $v(qa) = a^{-1}v(q)a$ for every $q \in \mathcal{Q}_x$ and $a \in G$. With the pair $(\varphi, \psi)$ of spinor fields one associates the multivector-valued field $\mathcal{F}(\psi, \varphi) : \mathcal{P} \to \wedge W$, given by $\mathcal{F}(\psi, \varphi)(p) = F(\psi(q), \varphi(q))$, where $q \in \mathcal{Q}$, $p = \chi(q)$ and $F$ is as in Sec. 5. 1. 2. The manifold $\mathcal{M}$ being orientable, it is meaningful to consider spinors of the same chirality over $\mathcal{M}$. Generalizing the constructions due to Atiyah, Hitchin and Singer [AHS], Penrose [P], and O'Brian and Rawnsley [OBR], one defines the total space of the *twistor bundle* $\mathcal{T}_r$ of *real index* $r$ as consisting of directions of all pure spinors on $\mathcal{M}$ of one, say positive, chirality and of real index $r$. If a section of the twistor bundle exists, then it can be represented (in 'homogeneous coordinates') by a nowhere vanishing field $\varphi : \mathcal{Q} \to S$ of pure spinors; such a section defines a complex vector bundle $\mathcal{N} \to \mathcal{M}$: this is a subbundle of the complexified tangent bundle $\mathcal{W} = \mathbb{C} \otimes T\mathcal{M} \to \mathcal{M}$ such that the fibre of $\mathcal{N} \to \mathcal{M}$ at $x \in \mathcal{M}$ is $\mathcal{N}_x = \{v \in \mathcal{W}_x | \gamma(v(q))\varphi(q) = 0 \text{ for } q \in \mathcal{Q}_x\}$, where $\gamma$ is the Dirac representation of $\mathrm{Cl}_{2\kappa+\varepsilon,2\lambda+\varepsilon}$ in $S$.

The considerations of Sec. 6. 3 can be applied, pointwise, to the fibers of the bundle $\mathcal{N} \to \mathcal{M}$, whose fibers are maximal, totally isotropic subspaces of the complexified tangent spaces of $\mathcal{M}$, even if this bundle does not come from a section of the twistor bundle. In any case, given $\mathcal{N} \to \mathcal{M}$, one constructs the real vector bundles $\mathcal{K} \subset \mathcal{L} \subset T\mathcal{M} \to \mathcal{M}$ and a field $\mathcal{J}$ of complex structures in the fibers of the quotient bundle $\mathcal{L}/\mathcal{K} \to \mathcal{M}$; for example, $\mathbb{C} \otimes \mathcal{K}_x = \mathcal{N}_x \cap \overline{\mathcal{N}_x}$, etc. It is convenient to say that the triple $(\mathcal{K}, \mathcal{L}, \mathcal{J})$ is the *flag geometry* associated with $\mathcal{N} \to \mathcal{M}$.

Let $\mathcal{S}(\mathcal{N})$ be the module of sections of the vector bundle $\mathcal{N} \to \mathcal{M}$; a similar notation is used for other vector bundles over $\mathcal{M}$. One says that the complex bundle $\mathcal{N}$ is *integrable* if

$$[\mathcal{S}(\mathcal{N}), \mathcal{S}(\mathcal{N})] \subset \mathcal{S}(\mathcal{N}). \tag{18}$$

In the general case, when $r > 0$, (18) implies

$$[\mathcal{S}(\mathcal{K}), \mathcal{S}(\mathcal{K})] \subset \mathcal{S}(\mathcal{K}), \tag{19}$$

and

$$[\mathcal{S}(\mathcal{K}), \mathcal{S}(\mathcal{L})] \subset \mathcal{S}(\mathcal{L}). \tag{20}$$

Therefore, the bundle $\mathcal{K}$ defines a *foliation* on $\mathcal{M}$. From now on, this foliation is assumed to be *regular* in the sense that the set $\mathcal{M}' = \mathcal{M}/\mathcal{K}$ of all its leaves has the structure of a smooth, $(2m - r)$-dimensional manifold such that the canonical map $\mathcal{M} \to \mathcal{M}'$ is a submersion. The following proposition rephrases well-known facts concerning complex and Cauchy–Riemann (CR) geometries [J].

**Proposition 5.** *Let $\mathcal{N} \to \mathcal{M}$ be a bundle on a $2m$-dimensional (pseudo-) Riemannian manifold $\mathcal{M}$. Assume that the fibers of $\mathcal{N}$ are maximal, totally isotropic subspaces of the fibers of the complexified tangent bundle of $\mathcal{M}$. Let $(\mathcal{K}, \mathcal{L}, \mathcal{J})$ be the flag geometry associated with the bundle $\mathcal{N} \to \mathcal{M}$ and let $r$ be the dimension of the fibers of the bundle $\mathcal{K} \to \mathcal{M}$. Then*
*(i) if $r = 0$, then $\mathcal{J}$ defines an almost complex structure on $\mathcal{M}$ and* (18) *is its classical integrability condition, equivalent to the vanishing of the Nijenhuis tensor of $\mathcal{J}$;*
*(ii) if $r > 0$, condition* (18) *implies the Frobenius integrability condition* (19) *of $\mathcal{K}$; the leaves of the foliation defined by $\mathcal{K}$ are totally isotropic; they are geodetic by virtue of* (20)*; the quotient manifold $\mathcal{M}' = \mathcal{M}/\mathcal{K}$ inherits, by projection, a subbundle $\mathcal{L}'$ of its tangent bundle and $\mathcal{J}$ descends to $\mathcal{M}'$, defining there a CR geometry.*

Assume now that the bundle $\mathcal{N} \to \mathcal{M}$ is defined by a section of the twistor bundle $\mathcal{T}_\varepsilon$, i. e. by a field of generic pure spinors, $r = \varepsilon$. In the pseudo-Euclidean case, a section of the twistor bundle $\mathcal{T}_0$ defines an orthogonal almost complex structure $\mathcal{J}$ on $\mathcal{M}$; therefore, together with $g$, it defines a pseudo-Hermitean form of signature $(\kappa, \lambda)$ at each point of $\mathcal{M}$. The resulting geometry can be called *almost pseudo-Hermitean*; it becomes pseudo-Hermitean if $\mathcal{J}$ is integrable.

In the pseudo-Lorentzian case, a section of the twistor bundle $\mathcal{T}_1$ defines an *almost optical geometry*, i. e. a flag geometry $(\mathcal{K}, \mathcal{L}, \mathcal{J})$ on $\mathcal{M}$ with a metric tensor $g$ of signature $(2\kappa + 1, 2\lambda + 1)$ such that $\mathcal{K} = \mathcal{L}^\perp$ is a real line bundle and $\mathcal{J}$ is orthogonal with respect to the metric induced by $g$ in the fibres of $\mathcal{L}/\mathcal{K}$. An almost optical geometry satisfying the integrability condition (18) is called *optical*: the trajectories of $\mathcal{K}$ are then isotropic geodesics and the quotient $\mathcal{M}' = \mathcal{M}/\mathcal{K}$ is a $(2m-1)$-dimensional CR manifold. In the proper Lorentzian case ($p = 1, q = 0$) the integrability condition is equivalent to the statement that the trajectories of $\mathcal{K}$ form a congruence of *isotropic geodesics without shear*. Such congruences play a fundamental role in the study of *algebraically special* gravitational fields; see, e. g., [PR, RT2] and the numerous references given there. Cauchy-Riemann structures appeared, in the context of relativity theory, for the first time in relation with Penrose's twistor spaces [P]; their connection with curved Lorentzian spaces, admitting non-shearing congruences of isotropic geodesics, has been made explicit in [RT1]. The analogies between Hermitean and optical geometries, apparent from the above considerations, and Cartan's results on 3-dimensional CR spaces, have been used in the study of solutions of Einstein's equations; see [N] and the references listed there.

## Appendix: A derivation of the Vahlen-Ahlfors formula

Let $K = \mathbb{R}$ or $\mathbb{C}$ and let $V$ be an $n$-dimensional vector space over $K$ with a non-degenerate quadratic form $h : V \to K$. Consider the vector space $V' = V \oplus K^2$ with the quadratic form $h'$ such that, for every $v' = (v, \lambda, \mu) \in V'$, where $v \in V$ and $\lambda, \mu \in K$, one has $h'(v') = h(v) + \lambda\mu$. Let $\operatorname{dir} v'$ be the direction (line) in $V'$ containing $v' \neq 0$ and let

$$Q = \{ \operatorname{dir} v' \in \mathbb{P}(V') | h'(v') = 0 \}$$

be the *projective quadric* associated with $V$ [CGuT]. The quadric is a compact submanifold, of dimension $n$, of the projective space $\mathbb{P}(V')$. In particular, if $K = \mathbb{R}$

and $h$ is of signature $(k,l)$, where $k+l=n$, then $Q$ is diffeomorphic to $(\mathbb{S}_k \times \mathbb{S}_l)/\mathbb{Z}_2$. If either $k$ or $l$ is 0, then $Q = \mathbb{S}_n$. The geometry of $V'$ induces a *conformal structure* in $Q$. For every $v \in V$, the vector $i(v) = (v, -h(v), 1) \in V'$ is isotropic. The map

$$j : V \to Q, \quad \text{given by} \quad j(v) = \operatorname{dir} i(v),$$

is a conformal diffeomorphism of $V$ on its image in $Q$; the image is open and dense in $Q$. Let $G$ be the connected component of the group $\mathrm{Spin}(V', h')$. The action of $G$ on $V'$ induces a transitive action of $G$ on $Q$ given by

$$\operatorname{dir} v' \mapsto f(A) \operatorname{dir} v' = \operatorname{dir} A v' A^{-1}. \tag{21}$$

Denoting by $\mathrm{Aut}\, Q$ the *Möbius group*, i. e. the connected component of the group of conformal automorphisms of $Q$, one sees that $f : G \to \mathrm{Aut}\, Q$ is an epimorphism of groups. The action of $G$ on $Q$ induces *local* conformal transformations of $V$. Namely, the set

$$V_A = \{\, v \in V \,|\, f(A) j(v) \in j(V) \,\}.$$

is open and dense in $V$ and the map

$$F(A) : V_A \to V_{A^{-1}}, \quad \text{defined by} \quad j(F(A)v) = f(A) j(v), \tag{22}$$

is a conformal diffeomorphism. The Clifford map $V' \to K(2) \otimes_K \mathrm{Cl}(V, h)$,

$$(v, \lambda, \mu) \mapsto \begin{pmatrix} v & \lambda \\ \mu & -v \end{pmatrix}, \tag{23}$$

extends to the isomorphism of algebras with units, $\mathrm{Cl}(V', h') \to K(2) \otimes_K \mathrm{Cl}(V, h)$. From now on these algebras are identified. Therefore, a typical element of $G$ is of the form

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tag{24}$$

where $a$, $d$ and $b$, $c$ are suitable, even and odd elements of $\mathrm{Cl}(V, h)$, respectively. The spinor norm of $A \in G$ is 1; therefore,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \begin{pmatrix} d^t & -b^t \\ -c^t & a^t \end{pmatrix},$$

where $a^t$ is the transpose of $a \in \mathrm{Cl}(V, h)$, etc.

**Proposition 6.** *Let $A \in G$ be as in* (24). *Then*

$$V_A = \{\, v \in V \,|\, cv + d \text{ is invertible in } \mathrm{Cl}\,(V, h) \,\}$$

*and the local conformal transformation* (22) *is given by the Vahlen-Ahlfors formula,*

$$F(A)v = (av + b)(cv + d)^{-1}. \tag{25}$$

To prove the proposition, let $v \in V_A$ and put $u = F(A)v$. According to (23), the isotropic vector $i(v)$ is represented in $K(2) \otimes_K \mathrm{Cl}(V, h)$ by the matrix

$$\begin{pmatrix} v & -h(v) \\ 1 & -v \end{pmatrix} = \begin{pmatrix} v \\ 1 \end{pmatrix} \begin{pmatrix} 1 & -v \end{pmatrix};$$

there is a similar representation of $i(u)$. The definitions (21) and (22) imply that the vectors $i(u)$ and $Ai(v)A^{-1}$ are parallel: there is $\kappa \in K$, $\kappa \neq 0$, such that

$$\kappa \begin{pmatrix} u & -h(u) \\ 1 & -u \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} v & -h(v) \\ 1 & -v \end{pmatrix} \begin{pmatrix} d^t & -b^t \\ -c^t & a^t \end{pmatrix}.$$

Therefore, $\kappa = (cv + d)(cv + d)^t$, so that $cv + d$ is invertible and (25) holds.

Moreover, if $h(dv) = h_{\mu\nu} dv^\mu dv^\nu$ denotes the quadratic differential form giving $V$ the structure of a (flat) Riemannian space, then $h(du) = \kappa^{-2} h(dv)$.

The Appendix is based on [RT3], where references to the relevant papers by Vahlen and Ahlfors are given.

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