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Excerpts from my old lectures on spinors

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Words of the Masters

The relativity theory is based on nothing but the idea of invariance and develops from it the conception of tensors as a matter of necessity; and it is rather disconcerting to find that apparently something has slipped through the net, so that physical quantities exist, which it would be, to say the least, very artificial and inconvenient to express as tensors.

Charles Galton Darwin, *Proc. Roy. Soc. London* **A118** (1928) 654.
NB Charles Darwin was his grandfather

The orthogonal transformations are the automorphisms of Euclidean vector space. Only with the spinors do we strike that level in the theory of its representations on which Euclid himself, flourishing ruler and compass, so deftly moves in the realm of geometric figures.

H. Weyl, *The Classical Groups*, Princeton U.P., 1946

A little of history

The development of ideas connected with spinors shows, once more, how important and beneficial is the interaction between mathematics and physics: spinors were discovered by mathematicians, but their importance comes from the role they were shown to play in physics.

The initial misunderstandings and errors committed when trying to introduce spinors on manifolds, have shown how important it is to have clear notions and mathematical structures appropriate to the problems under consideration. In particular, the definition of spinors on manifolds – unlike that of tensors – requires the use of fiber bundles.

Euclid (around 300 BC) in Book X of *The Elements* gave the following solution

$$x = q^2 - p^2, \quad y = p^2 + q^2, \quad z = 2pq.$$

of the Pythagorean equation,

$$x^2 - y^2 + z^2 = 0 \qquad \Leftrightarrow \qquad \det \begin{pmatrix} y-x & z \\ z & y+x \end{pmatrix} = 0.$$

The solution can be written as

$$\begin{pmatrix} y-x & z \\ z & y+x \end{pmatrix} = 2 \begin{pmatrix} p \\ q \end{pmatrix} \begin{pmatrix} p & q \end{pmatrix}. \tag{1}$$

In $\mathbb{R}^3$, with a quadratic form of signature $(2,1)$, eq. (1) means:

null vector = $(\text{spinor})^2$.

Further lessons from Euclid

1. Generalize to higher dim: totally null multivector of max. dim. = $(\text{pure spinor})^2$. (Veblen, Givens [2, 3], Cartan [4], Chevalley [5]; see also [6] and [7]; pure essential in dim $\geqslant 7$).

2. Spin groups. Multiply (1) on the left by a real unimodular matrix, on the right by its transpose, take det to get

$$1 \to \mathbb{Z}_2 \to \mathsf{SL}(2,\mathbb{R}) = \mathsf{Spin}_0(2,1) \to \mathsf{SO}_0(2,1) \to 1$$

3. Euclid's solution contains the germ of the idea of Clifford algebras: multiplying the left side of (1) by $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ on the left and taking the square of the result, one obtains the representation of a quadratic form as the square of a linear form,

$$\begin{pmatrix} z & x+y \\ x-y & -z \end{pmatrix}^2 = (x^2 - y^2 + z^2) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

4. Replace z by $\sqrt{\text{-}1}\, z$ to get complex spinors, Pauli matrices, $\mathsf{Spin}(3) = \mathsf{SU}(2)$, etc.

5. Non-trivial topology involved. Since $x + \sqrt{\text{-}1}\, z = (q + \sqrt{\text{-}1}\, p)^2$, rotation by α in (p,q) plane

$$p' = p\cos\alpha + q\sin\alpha, \quad q' = -p\sin\alpha + q\cos\alpha$$

induces rotation by 2α in (x,z) plane.

$$x' = x\cos 2\alpha - z\sin 2\alpha, \quad z' = x\sin 2\alpha + z\cos 2\alpha$$

Take curve $0 \leqslant \alpha \leqslant \pi$ to conclude, with the physicists:
Spinors change sign when rotated by 2π.

XIX century

Hermann Grassmann and **William R. Hamilton** around 1840.

If $q \in \mathbb{H}$ is a pure quaternion, $\bar{q} = -q$, and $a \in \mathsf{Sp}(1) \subset \mathbb{H}$ is a unit quaternion, then $aq\bar{a}$ is also pure of the same norm as q so that $q \mapsto aq\bar{a}$ gives

$$1 \to \mathbb{Z}_2 \to \mathsf{Sp}(1) = \mathsf{SU}(2) \to \mathsf{SO}(3) \to 1.$$

Arthur Cayley (1855) noted also that $q \mapsto aq\bar{b}$, for $q \in \mathbb{H}$ and $a, b \in \mathsf{Sp}(1)$ gives rotations in 4 dimensions, thus

$$1 \to \mathbb{Z}_2 \to \mathsf{Sp}(1) \times \mathsf{Sp}(1) \to \mathsf{SO}(4) \to 1.$$

William Clifford was a most remarkable man. In his book *The common sense of the exact sciences,* published in New York in 1885, he gave an intuitive description of the future GRT and of relativistic cosmology:

(i) Our space is perhaps really possessed of a curvature varying from point to point, which we fail to appreciate because we are acquainted with only a small portion of space...

(ii) Our space may be really same (of equal curvature), but its degree of curvature may change as a whole with the time...

(iii) We may conceive our space to have everywhere a nearly uniform curvature, but that slight variations of the curvature may occur from point to point, and themselves vary with the time... We might even go so far as to assign to this variation of curvature of space 'what really happens in that phenomenon which we term the motion of matter'.

He defined 'geometric algebras' (1878), but did not consider groups constructed out of elements of these algebras; that was done, for the first time, by

Rudolf Lipschitz [8]. For this reason the name 'Clifford group', introduced by Chevalley, is a misnomer. Lipschitz's work, who died in 1903, was recalled by André Weil in a *Correspondence* published in 1959 [9].

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CORRESPONDENCE

We have received the following letter, purporting to come from an ultramundane correspondent:

SIR,

It is sometimes a matter of wonder, to us in Hades, that what we had believed to be our best work remains buried under thick layers of dust in your libraries, while the very talented young men in the mathematical world of the present day strive manfully against problems which are by no means as novel as they think.

For instance, it is not so long ago that the very remarkable algebraic systems discovered by my friend Professor Clifford shortly before leaving your world have again attracted the attention of your algebraists after many years of oblivion. When, during my lifetime, I first became interested in them, I, too, fancied that they were new; I soon found out my mistake, and hastened to acknowledge Professor Clifford's prior discovery. It is now a matter of great satisfaction to me to hear that his name has been given to them, as a fitting tribute to his memory among the living.

On the other hand, as Professor Clifford has told me himself, it had not occurred to him to apply these algebraic systems to the study of the substitutions which transform a sum of squares into a sum of squares (or, as my young friend and colleague Hermann Weyl would say, of the orthogonal group); he kindly insists that this idea was wholly mine. As you may well believe, we have often discussed this topic since I had the honour of joining the distinguished company of the mathematicians in the Elysian Fields; incidentally, without the many delightful conversations which I have had with him, I should hardly be able now to write to you in English (a feat which I could have accomplished only with great difficulty during my lifetime).

It is not, however, in order to assert my claims to fame in this matter that I am now asking for the hospitality of your journal. In what you are pleased to call the nether world, we are happily free from vainglorious feelings. But it may be useful to a few of your contemporaries to have their attention drawn upon some formulas contained in my memoir *Untersuchungen über die Summen von Quadraten* (a brief account of which may be found in the *Bulletin des Sciences Mathématiques* for 1886), since they would be sought in vain, unless I am much mistaken, in various learned volumes recently published on this very subject.

Unfortunately, it appears that there is now in your world a race of vampires, called referees, who clamp down mercilessly upon mathematicians unless they know the right passwords. I shall do my best to modernize my language and notations, but I am well aware of my shortcomings in that respect; I can assure you, at any rate, that my intentions are honourable and my results invariant, probably canonical, perhaps even functorial. But please allow me to assume that the characteristic is not 2.

Call $e_1, \cdots, e_n$ the generators of Professor Clifford's algebraic system; this means that $e_i^2 = -1$ for all i, and $e_j e_i = -e_i e_j$ for $i < j$. For each set $I = \{i_1, \cdots, i_m\}$ of indices, written in their natural order

$$1 \leqq i_1 < i_2 < \cdots < i_m \leqq n ,$$

put

$$e(I) = e_{i_1} e_{i_2} \cdots e_{i_m} ,$$

with $e(I) = 1$ if $m = 0$; the set I and the unit $e(I)$ will be called *even* if m is even, *odd* if m is odd. Linear combinations of even (resp. odd) units will be called even (resp. odd) quantities.

Now take an alternating matrix $X = (x_{ij})$, and assume at first that the determinant of $E + X$ (where E is the unit matrix) is not 0. My learned and illustrious colleague Professor Cayley was, I believe, the first one to observe that, if X is such a matrix, the formula

$$U = (E - X)\cdot(E + X)^{-1} \tag{1}$$

defines an orthogonal matrix U, and that conversely X can be expressed in terms of U by the formula

$$X = (E - U)\cdot(E + U)^{-1} . \tag{2}$$

For each even set $J = \{j_1, \cdots, j_{2p}\}$ of indices (written, as always, in their natural order), put

$$x(J) = \frac{1}{2^p p!} \sum_H \varepsilon(J, H) x_{h_1 h_2} x_{h_3 h_4} \cdots x_{h_{2p-1} h_{2p}}$$

where the summation is extended to all permutations H of J, and $\varepsilon(J, H)$ is $+1$ or -1 according as the permutation is even or odd; put $x(J) = 1$ for $p = 0$. Consider the even quantity

$$\Omega = \sum_J x(J) e(J) \tag{3}$$

where the summation is extended to all even sets of indices J. On the other hand, take two vectors $\xi = (\xi_1, \cdots, \xi_n)$, $\eta = (\eta_1, \cdots, \eta_n)$ such that $\eta = U\xi$; by the definition of U, this can be written as quantity Ω, it remains valid if Ω is multiplied by a scalar factor. Therefore it holds whenever U is an orthogonal matrix of determinant $+1$, provided Ω is an even quantity, given by (3), such that (5) is equivalent to $\eta = U\xi$. I can still vividly recall my pleasure when I first came across this result in bygone days. But I fear that I am becoming garrulous, and that your patience with me may be exhausted by now.

I have the honour to be, etc.

R. LIPSCHITZ

XX century

Élie Cartan discovers [10] among fundamental representations of the Lie algebra $\mathfrak{so}(m,\mathbb{C})$, some that do not lift to representation(s) of $\mathsf{SO}(m,\mathbb{C})$; he probably does not consider them to be sufficiently important to deserve a special name.

1927: enter the physicists: **W. Pauli** [11] and **P. A. M. Dirac** [12] put forward differential equations for the electron; they rediscover and use Clifford algebras in 3 and 4 dimensions.

Only a few months after Dirac's paper of 1928 had appeared, **D. Ivanenko** and **L. Landau** proposed [13] to use the differential operator $\mathrm{d}+\delta$, $\delta = \star\,\mathrm{d}\star$ (not in this notation), to describe the *magnetischen Elektron*. The square of that operator is also the Laplacian (or $\Box$) , but instead of a 4-dim space of Dirac spinors, it requires, at a point, the 16-dim space of differential forms. In view of the successes of the Dirac equation, there was not much interest in that equation; the operator $\mathrm{d}+\delta$ was rediscovered around 1960 by Erich Kähler [14]. It is sometimes referred to as the Dirac–Kähler operator.

The word *spinor* appears in print, for the first time, in the title of the paper *Spinoranalyse* (1929) by **B. L. van der Waerden**; the author attributes it to Paul Ehrenfest; dotted indices introduced.

In 1929 **H. Weyl** [15] considers an equation for a particle of zero mass and spin 1/2, based on the use of two-component ('Weyl') spinors.

E. Wigner and **V. Fock** (1929) propose to introduce spinors in General Relativity under the assumption of 'teleparallelism'.

In 1933 **L. Infeld and B. L. van der Waerden** [16] show how to describe (two-component) spinor fields in GRT without teleparallelism; they write a formula for the covariant derivative.

In 1935 **R. Brauer and H. Weyl** [17] construct algebras of matrices that generalize to n dimensions those used by Pauli and Dirac; oddly enough, there is no reference to Clifford.

In 1936 **É. Cartan** gave a series of lectures, later published as *Leçons sur la théorie des spineurs* [4]; he introduced *simple* spinors, renamed by Chevalley as *pure*. At last, there is a reference to Clifford whose work Cartan knew from the time of writing his article *Nombres Complexes* for *Encycl. Sci. math.* (1908). Cartan criticized Infeld and van der Waerden for their attempt to describe spinors fields in a manner similar to the one used for tensor fields on Riemannian manifolds:

Certain physicists regard spinors as entities which are, in a sense, unaffected by the rotations which classical geometric quantities (vectors, etc.) can undergo, and of which the components in a given reference system are susceptible of undergoing linear transformations which are in a sense autonomous. See for example L. Infeld and B. L. van der Waerden...

COMPLEMENTS OF ALGEBRA

It is understood that much material on Clifford algebras was given at the preceding seminars of this series.

In algebra, mathematicians are interested mainly in the intrinsic properties of objects such as groups and algebras; physicists, in view of applications, need their representations. *Simple associative algebras, unlike groups, have only one, up to equivalence, faithful and irreducible representation.* The properties of these representations, that are the same for equivalent representations, are therefore intrinsic to simple algebras. Clifford algebras are either simple or semi-simple (sums of two simple); spinors, as carriers of these representations are unique.

Remarks on notation

I write $\bar{z}$ for complex (sometimes: quaternionic) conjugation (physicists often z^*). Transposition (duality) contravariant functor * from vector spaces to vector spaces: if $a \in \mathsf{Hom}(V, W)$, then $a^* \in \mathsf{Hom}(W^*, V^*)$ is such that if $v \in V$ and $w' \in W^*$, then

$$\langle a(v), w' \rangle = \langle v, a^*(w') \rangle.$$

Note the *evaluation map* (not a scalar product):

$$W \times W^* \to \mathbb{C}, \quad (w, w') \mapsto \langle w, w' \rangle = w'(w).$$

If h is a bilinear form on V and $v \in V$, then $h(v) \in V^*$ is such that $\langle v_1, h(v_2) \rangle = h(v_1, v_2)$ for all $v_1, v_2 \in V$. If $\Bbbk = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$, then $\Bbbk(n)$ is the real algebra of all $n \times n$ matrices with entries in $\Bbbk$. The algebra of octonions is denoted by $\mathbb{O}$.

Short review of representations of Clifford algebras

Recall that every Clifford algebra has a a canonical *automorphism* α such that $\alpha(v) = -v$ for every vector v and a canonical *antiautomorphism* β such that $\beta(v) = v$, $\beta(ab) = \beta(b)\beta(a)$. The grading is given by

$$\mathcal{C}\ell_{\pm}(V, h) = \{a \in \mathcal{C}\ell(V, h) \mid \alpha(a) = \pm a\},$$

and one often represents the grading of the Clifford algebra by writing

$$\mathcal{C}\ell_{+}(V, h) \to \mathcal{C}\ell(V, h).$$

Consider now *complex* representations of $\mathcal{C}\ell(V, h)$, where (V, h) is a *real* quadratic space of dimension $m = 2n$. There is the faithful and irreducible (Dirac) representation

$$\gamma : \mathcal{C}\ell(V,h) \to \mathsf{End}\, S, \quad \dim_{\mathbb{C}} S = 2^n.$$

Indeed, put $\mathbb{C} \otimes V = K \oplus L$, where K and L are *maximal totally null* (mathematicians say: *isotropic*, a misnomer: there is nothing isotropic here) subspaces of dim n. If $k \in K$ and $l \in L$, then

$$h(k+l, k+l) = \mathbf{2}h(k,l).$$

Define the 2^n-dim space $S = \wedge K$ and $\gamma : K \oplus L \to \mathsf{End}\, S$, by

$$\gamma(k+l)s = \sqrt{\mathbf{2}}(k \wedge s + h(l) \lrcorner\, s), \quad s \in S,$$

so that $\gamma(k+l)^2 = h(k+l, k+l)\,\mathrm{id}_S$. The linear map γ has the Clifford property and extends to an isomorphism of algebras, $\gamma : \mathbb{C} \otimes \mathcal{C}\ell(V,h) \to \mathsf{End}\, S$. By restriction to $\mathcal{C}\ell(V,h)$, it gives a complex representation of the real algebra.
If $(e_1, \dots, e_{2n})$ is an orthonormal basis (frame), then $\gamma_\mu = \gamma(e_\mu)$ for $\mu = 1, \dots, 2n$ and $\eta = e_1 \dots e_{2n}$ is the volume element. Physicists define $\gamma_{2n+1} = \gamma(\eta)$ or $\sqrt{-1}\gamma(\eta)$ depending on whether $\eta^2 = 1$ or -1 so that $\gamma_{2n+1}^2 = \mathrm{id}_S$. The restriction of γ to the even subalgebra decomposes,

$$\gamma | \mathcal{C}\ell_+(V,h) = \gamma_+ \oplus \gamma_-, \quad \gamma_\pm : \mathcal{C}\ell_+(V,h) \to \mathsf{End}\, S_\pm,$$

and

$$S_\pm = \{ s \in S \mid \gamma_{2n+1} s = \pm s \} \qquad \text{chirality}$$

are two spaces of Weyl (chiral, reduced, or half-) spinors.
Relative to the split $S = S_+ \oplus S_-$ gamma matrices have the form

$$\gamma^\mu = \begin{pmatrix} 0 & \gamma_-^\mu \\ \gamma_+^\mu & 0 \end{pmatrix}, \quad \mu = 1, \dots, 2n, \qquad \gamma_{2n+1} = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}.$$

For a particle of spin 1/2 and mass 0 the Dirac equation $\gamma^\mu \partial/\partial x^\mu \psi = 0$ splits into Weyl equations $\gamma_\pm^\mu \partial/\partial x^\mu \psi_\pm = 0$. Criticized by Pauli: it violates invariance under reflections which change chirality. For some time, neutrinos were considered to be massless.
The *dual* (contragredient) repr. with respect to γ is

$$\check{\gamma} : \mathcal{C}\ell(V,h) \to \mathsf{End}\, S^*, \quad \check{\gamma}(a) = \gamma(\beta(a))^*.$$

For $\dim V$ even, the algebra is simple, therefore $\check{\gamma} \sim \gamma$ so that there is an *intertwiner* $B : S \to S^*$ such that

$$\check{\gamma}(a) = B\gamma(a)B^{-1} \quad \text{for every } a \in \mathcal{C}\ell(V,h).$$

Since β is involutive, $\beta \circ \beta = \mathrm{id}$, transposing the last equation, one obtains that $B^{-1}B^*$ is in the commutant of γ and from Schur's Lemma $B^{-1}B^* = \lambda\,\mathrm{id}$ and $B^{**} = B$ gives

$$\text{either } B^* = B \text{ or } B^* = -B.$$

The map B defines the **bi**linear form

$$(s_1, s_2) \mapsto \langle s_1, B(s_2) \rangle, s_1, s_2 \in S$$

which is invariant with respect to the action of the group

$$\mathsf{G}(V, h) = \{a \in \mathcal{C}\ell_+(V, h) \mid \beta(a)a = 1\}$$

and makes S into a quadratic or symplectic space.

Def of spin groups: algebra vs topology

algebra The group $\mathsf{Pin}(V, h) \subset \mathcal{C}\ell(V, h)$ is generated by all unit vectors; the map

$$\rho : \mathsf{Pin}(V, h) \to \mathsf{O}(V, h), \quad \rho(a)v = \alpha(a)va^{-1},$$

defines the exact sequence

$$1 \to \mathbb{Z}_2 \to \mathsf{Pin}(V, h) \xrightarrow{\rho} \mathsf{O}(V, h) \to 1.$$

(Explain: if u is an invertible vector, then $-uvu^{-1}$ is the reflection of v in the hyperplane orthogonal to u). If $V = \mathbb{R}^m$ the one writes $\mathsf{Pin}(m)$, etc.
Alternatively, the Pin group can be defined as

$$\{a \in \mathcal{C}\ell(V, h) \mid \beta(a)a \in \{1, -1\} \text{ and } aVa^{-1} \subset V\}$$

There is also the group (named *Clifford group* by Chevalley)

$$\{a \in \mathcal{C}\ell(V, h) \mid \beta(a)a \in \mathbb{C}^\times = \mathbb{C} \smallsetminus \{0\} \text{ and } aVa^{-1} \subset V\}$$

The spin group is defined as

$$\mathsf{Spin}(V, h) = \mathsf{Pin}(V, h) \cap \mathcal{C}\ell_+(v, h)$$

so that $\rho : \mathsf{Spin}(V, h) \to \mathsf{SO}(V, h)$, $\rho(a)v = ava^{-1}$, $v \in V$, defines the sequence

$$1 \to \mathbb{Z}_2 \to \mathsf{Spin}(V, h) \to \mathsf{SO}(V, h) \to 1.$$

Notation: if $V = \mathbb{R}^{p+q}$ and h has signature (p, q), then $\mathsf{Pin}(p, q)$ instead of $\mathsf{Pin}(V, h)$; $\mathsf{Spin}(p, 0) = \mathsf{Spin}(p)$, etc.

topology (For a topological group G, the connected subgroup containing 1 is denoted by G_0.) If $p \geqslant 3$, then the groups $\mathsf{Spin}(p)$ and $\mathsf{Spin}_0(1, p)$ are *simply-connected* so that they can be defined as the universal, connected and simply-connected covering groups of $\mathsf{SO}(p)$ and $\mathsf{SO}_0(1,p)$, but this does not extend to other dimensions and signatures: e.g. $\mathsf{Spin}(2) = \mathsf{U}(1)$. As a less trivial example, consider $\mathsf{Spin}_0(3,3)$. Let $V = \wedge^2\mathbb{R}^4 \subset \mathbb{R}(4)$. If $(\mathbf{E},\mathbf{B}) = F \in V$, then $\mathsf{Pf}\, F = \mathbf{E}\cdot\mathbf{B}$ defines a quadratic form of signature $(3,3)$ in V. If $a \in \mathsf{SL}(4,\mathbb{R})$, then $aFa^* \in V$ and $\mathsf{Pf}(aFa^*) = \mathsf{Pf}\, F$ so that $\mathsf{Spin}_0(3,3) = \mathsf{SL}(4,\mathbb{R})$ is not simply connected, $\mathsf{SO}(4)$ is its maximal compact subgroup. However, on p. 56 of *Spin Geometry* by Lawson and Michelsohn one finds

$$\mathsf{Spin}_0(3,3) = \widetilde{\mathsf{SL}(4,\mathbb{R})} = \text{simply-connected cover of } \mathsf{SL}(4,\mathbb{R})$$

But it is known that the latter cover is a group that has no finite dim faithful repr. (this can be shown as in §86 of [4]).

If $a \in \mathsf{Spin}(V,h)$, and (e_μ) is a basis in V, then $ae_\mu a^{-1}$ is a vector and there is a matrix $(\rho^\nu{}_\mu(a))$ such that $ae_\mu a^{-1} = e_\nu \rho^\nu{}_\mu(a)$, therefore

$$\gamma(a)\gamma_\mu\gamma(a^{-1}) = \gamma_\nu\rho^\nu{}_\mu(a).$$

This leads to if s_1, s_2 are spinors, then $\langle s_1, B\gamma_\mu s_2\rangle$ are components of a vector,

$$\langle\gamma(a^{-1})s_1, B\gamma_\mu\gamma(a^{-1})s_2\rangle = \langle s_1, B\gamma_\nu s_2\rangle\rho^\nu{}_\mu(a)$$

and one constructs similarly multivectors as bilinear forms of spinors.

Periodicity in $\dim V$ from symmetry of B

If $\beta(a) = a$, then from the def of B one obtains

$$(B\gamma(a))^* = (B^*B^{-1})B\gamma(a)$$

so that the symmetry of B can be obtained by a 'dimension count', by taking into account $B\gamma(a) \in \mathsf{End}(S,S^*) = S\otimes S$,

$$S\otimes S = \wedge^2 S \oplus \vee^2 S,\ \dim\vee^2 S > \dim\wedge^2 S$$

Defining

$$\mathcal{C}\ell^\pm(V,h) = \{a \in \mathcal{C}\ell(V,h) \mid \beta(a) = \pm a\}$$

and

$$d(m) = \dim\mathcal{C}\ell^+(V,h) - \dim\mathcal{C}\ell^-(V,h),$$

one has

$$B^* = B \quad\Leftrightarrow\quad d(m) > 0$$

and from the exercise below one obtains

$$B^* = (-1)^{\frac{1}{2}n(n-1)} B. \tag{2}$$

Exercise: prove

$$d(m) = 2^{m/2}(\cos \tfrac{1}{4}m\pi + \sin \tfrac{1}{4}m\pi),$$

so that $\operatorname{sgn} d(m+8) = \operatorname{sgn} d(m)$.
This is an early appearance, in algebra, of periodicity of period 8 in *dimension* of V. It holds for quadratic spaces over **any field of numbers** of characteristic $\neq 2$.

B restricted to Weyl spinors

From $\beta(\eta) = (-1)^n \eta$ one obtains

$$\gamma^*_{2n+1} = (-1)^n B \gamma_{2n+1} B^{-1} \tag{3}$$

Putting

$$S^*_\pm = \{s' \in S^* \mid \gamma^*_{2n+1} s' = \pm s'\} \quad \text{and} \quad B_\pm = B|S_\pm$$

and using (2) and (3) one obtains (note periodicity m mod 8)

n mod 4	B	symmetry	examples
1	$\begin{pmatrix} 0 & B_- \\ B_+ & 0 \end{pmatrix}$	$B^*_\pm = B_\mp$	
2	$\begin{pmatrix} B_+ & 0 \\ 0 & B_- \end{pmatrix}$	$B^*_\pm = -B_\pm$	relativity in dim 4
3	$\begin{pmatrix} 0 & B_- \\ B_+ & 0 \end{pmatrix}$	$B^*_\pm = -B_\mp$	twistors in dim 6
4	$\begin{pmatrix} B_+ & 0 \\ 0 & B_- \end{pmatrix}$	$B^*_\pm = B_\pm$	triality in dim 8

Inductive construction of representations

For every $n = 0, 1, 2, \dots$ one constructs a set of matrices

$$(n) \qquad \gamma_1^{(n)}, \dots, \gamma_{2n+1}^{(n)} \in \mathbb{R}(2^n) \qquad \text{so that}$$

(i) $\gamma_1^{(0)} = 1$.
(ii) given the set (n), one defines

$$\gamma_\mu^{(n+1)} = \begin{pmatrix} 0 & \gamma_\mu^{(n)} \\ \gamma_\mu^{(n)} & 0 \end{pmatrix}, \quad 1 \leqslant \mu \leqslant 2n+1,$$

$$\gamma_{2n+2}^{(n+1)} = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}, \quad \gamma_{2n+3}^{(n+1)} = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$$

where $I \in \mathbb{R}(2^n)$ is the unit matrix.
For every n, one has

$$\gamma_\mu^{(n)}\gamma_\nu^{(n)} + \gamma_\nu^{(n)}\gamma_\mu^{(n)} = 2g_{\mu\nu}^{(n)} I$$

where $\mu \neq \nu \Rightarrow g_{\mu\nu}^{(n)} = 0$, $g_{\mu\mu}^{(n)} = (-1)^{\mu+1}$. Notation: if (V, h) is a real quadratic space and h is of signature (p, q) then the algebra $\mathcal{C}\ell_+(V, h) \to \mathcal{C}\ell(V, h)$ is written as $\mathcal{C}\ell_+(p, q) \to \mathcal{C}\ell(p, q)$.
Proposition: Let (e_μ) be an orthonormal basis in $\mathbb{R}^{2n}$ such that $e_\mu e_\nu + e_\nu e_\mu = 2g_{\mu\nu}^{(n)}$. The linear map

$$\mathbb{R}^{2n} \to \mathbb{R}(2^n) \text{ such that } e_\mu \mapsto \gamma_\mu^{(n)},\ \mu = 1, \dots, 2n$$

has the Clifford property and extends to an isomorphism of algebras, $\mathcal{C}\ell(n, n) \to \mathbb{R}(2^n)$.
The linear map $\mathbb{R}^{2n-1} \to \mathbb{R}(2^n)$ such that $e_\mu \mapsto \gamma_\mu^{(n)}\gamma^{(n)}{}_{2n}$, $\mu = 1, \dots, 2n-1$, has the Clifford property and extends to a representation of the algebra $\mathcal{C}\ell(n, n-1)$ in $\mathbb{R}(2^n)$. This representation is faithful, but decomposable: the matrices $\frac{1}{2}(I + \gamma_{2n+1}^{(n)})$ and $\frac{1}{2}(I - \gamma_{2n+1}^{(n)})$ are projectors on two invariant subspaces of $\mathbb{R}(2^n)$.
Multiplying some of the matrices by $\sqrt{-1}$ one obtains complex representations of all the algebras $\mathcal{C}\ell(p, q)$.

Complex (charge) conjugation

Consider a **real** $2n$-dim (V, h). The repr γ can be now complex-conjugated,

$$\bar{\gamma} : \mathcal{C}\ell(V, h) \to \mathsf{End}\, \bar{S}, \quad \bar{\gamma}(a) = \overline{\gamma(a)}.$$

From simplicity $\bar{\gamma} \sim \gamma$ so that there is an intertwiner $C : S \to \bar{S}$ so that

$$\bar{\gamma}(a)C = C\gamma(a), \quad \forall a \in \mathcal{C}\ell(V, h). \tag{4}$$

The composition $A = \bar{B}C : S \to \bar{S}^*$ is used to form a sesquilinear map

$$(s_1, s_2) \mapsto \langle \bar{s}_1, As_2 \rangle$$

Invoking irreducibility of γ and Schur's lemma, one shows that by rescaling one can achieve $A^\dagger = A$. Physicists use A to construct real multivectors out of spinors; e.g. $\langle \bar{s}, As \rangle$ is real (but physicists write this as $\bar{s}s$, the map A being absorbed in the definition of $\bar{s}$.)

Dirac predicts existence of anti-particles

if $\psi : V \to S$ is a solution of

$$(\gamma^\mu(\frac{\partial}{\partial x^\mu} - \mathrm{i}\, eA_\mu) - m)\psi = 0$$

then the charge conjugate function $\psi_c = C^{-1}\bar{\psi}$ is a solution of

$$(\gamma^\mu(\frac{\partial}{\partial x^\mu} + \mathrm{i}\, eA_\mu) - m)\psi = 0$$

Real and quaternionic structures in S

Iterating (4) and using Schur's Lemma one obtains that, after rescaling,
(i) either $\bar{C}C = I \quad$ (i.e. id_S) and then the space

$$\mathrm{Re}\, S = \{s \in S \mid \bar{s} = Cs\}$$

is real and the repr γ restricts to $\mathrm{Re}\, S$ (Majorana spinors),

$$\gamma(a)\,\mathrm{Re}\, S \subset \mathrm{Re}\, S$$

and there is an isomorphism of algebras

$$\mathcal{C}\ell(V,h) \to \mathbb{R}(2^n).$$

(ii) or $\bar{C}C = -I$, and then there is a right quaternionic structure in S obtained by putting

$$s\,\mathrm{i} = \sqrt{-1}s, \quad s\,\mathrm{j} = C^{-1}\bar{s}, \quad s\,\mathrm{k} = (s\,\mathrm{i})\,\mathrm{j}\,.$$

Eq. (4) then implies that the representation is quaternionic,

$$\gamma(a)(sq) = (\gamma(a)s)q, \quad q \in \mathbb{H}$$

so that there is an isomorphism of algebras over $\mathbb{R}$,

$$\mathcal{C}\ell(V,h) \to \mathbb{H}(2^{n-1}).$$

One shows

$$\bar{C}C = \begin{cases} I & \text{if } q-p \equiv 0 \text{ or } 6 \bmod 8 \\ -I & \text{if } q-p \equiv 2 \text{ or } 4 \bmod 8 \end{cases}$$

The graded structure of $\mathcal{C}\ell(V,h)$ depends on the volume element: if $\eta^2 = 1$, then $\mathcal{C}\ell_+(V,h)$ is the sum of two simple algebras. If $\eta^2 = -1$, then this algebra is $\mathbb{C}(2^{n-1})$.

Chevalley theorem

Graded ('super')tensor product of $\mathbb{Z}_2$-graded algebras $\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1$ and
$\mathcal{B} = \mathcal{B}_0 \oplus \mathcal{B}_1$: $\mathcal{A}\hat{\otimes}\mathcal{B}$ as a vector space is $\mathcal{A} \otimes \mathcal{B}$,
$\deg(a \otimes b) = \deg a + \deg b \mod 2$
If $b \in \mathcal{B}_\epsilon$ and $a' \in \mathcal{A}_{\epsilon'}$, then

$(a \otimes b)(a' \otimes b') = (-1)^{\epsilon\epsilon'} aa' \otimes bb'$
$\mathcal{C}\ell(V,h)\hat{\otimes}\,\mathcal{C}\ell(W,g)$ is isomorphic to $\mathcal{C}\ell(V \oplus W, h \oplus g)$
Proof: the map
$V \oplus W \to \mathcal{C}\ell(V,h)\hat{\otimes}\,\mathcal{C}\ell(W,g), \quad (v,w) \mapsto v \otimes 1 + 1 \otimes w$ has the Clifford property,
$(v \otimes 1 + 1 \otimes w)^2 = v^2 + w^2$, etc.

Brauer–Wall groups

The dependence of C on $q - p$ mod 8 provides a better known periodicity of properties of algebras $\mathcal{C}\ell(\underset{+}{p}, \underset{-}{q})$ with respect to the **index** $q - p$. Together with periodicity with respect to the **dimension**, this gives rise to the *spinorial* (Clifford) *chessboard* [18]: periodicity of the structure of $\mathcal{C}\ell(p,q)$ with respect to both p and q. Define the algebras $\mathcal{C}\ell(p_1,q_1)$ and $\mathcal{C}\ell(p_2,q_2)$ to be of the same **type** if their indices are equal mod 8, $q_1 - p_1 \equiv q_2 - p_2$ mod 8. This defines an equivalence relation in the set of all Clifford algebras over real quadratic spaces. The **class** of $\mathcal{C}\ell(p,q)$ is the set $[\mathcal{C}\ell(p,q)]$ of all Clifford algebras of the same type as $\mathcal{C}\ell(p,q)$. The graded tensor product induces in the set of classes a multiplication

$$[\mathcal{C}\ell(p_1,q_1)].[\mathcal{C}\ell(p_2,q_2)] = [\mathcal{C}\ell(p_1,q_1)\hat{\otimes}\,\mathcal{C}\ell(p_2,q_2)]$$
$$=[\mathcal{C}\ell(p_1+p_2, q_1+q_2)] \text{ by Chevalley thm}$$

This multiplication is associative and $[\mathcal{C}\ell(0,8)] = [\mathcal{C}\ell(1,1)]$ is its neutral element. Every element is invertible:

$$[\mathcal{C}\ell(p,q)]^{-1} = [\mathcal{C}\ell(p',q')]$$

where (p',q') is such that $p + p' \equiv 0$ mod 8 and $q + q' \equiv 0$ mod 8.Therefore, the set of all such classes is a group, the *Brauer–Wall group* of the field $\mathbb{R}$ underlying the vector spaces V.
NB the original def of this group uses (all) $\mathbb{Z}_2$-graded associative algebras over a field. It turns out that every class defined by a similar equivalence relation among $\mathbb{Z}_2$-graded associative algebras contains Clifford algebras so that these algebras suffice to determine the Brauer–Wall group of a field.
There is a general definition of the Brauer–Wall groups for associative, graded central simple algebras over any field of characteristic $\neq 2$ [19, 20].

The real spinorial clock $\mathbb{Z}_8$

provides an explicit description of the Brauer–Wall group of $\mathbb{R}$.

$$\begin{tikzcd}
\mathbb{R} \arrow[r, "7"] & \mathbb{R}\oplus\mathbb{R} \arrow[r, "0"] & \mathbb{R} \arrow[d, "1"] \\
\mathbb{C} \arrow[u, "6"] & & \mathbb{C} \arrow[d, "2"] \\
\mathbb{H} \arrow[u, "5"] & \mathbb{H}\oplus\mathbb{H} \arrow[l, "4"'] & \mathbb{H} \arrow[l, "3"']
\end{tikzcd}$$

Classes of algebras are represented on the clock as follows:

$$\mathbb{R}\oplus\mathbb{R} \to \mathbb{R} \quad \text{means} \quad [\mathbb{R}\oplus\mathbb{R} \to \mathbb{R}(2)],$$

$$\mathbb{H} \to \mathbb{C} \quad \text{means} \quad [\mathbb{H} \underset{\text{Pauli}}{\to} \mathbb{C}(2)], \quad \text{etc.}$$

Recipe for determining $\mathcal{C}\ell_+(p,q) \to \mathcal{C}\ell(p,q)$: write $q-p = 8\mu+\nu$, $0 \leqslant \nu \leqslant 7$, from the clock read off $\mathcal{A}_0 \xrightarrow{\nu} \mathcal{A}$, multiply $\mathcal{A}_0$ and $\mathcal{A}$ by $\mathbb{R}(N_0)$ and $\mathbb{R}(N)$ with N_0 and N chosen as to get the dimensions right.

Exercise: Show that $\mathbb{R} \to \mathbb{C}$ generates the Brauer–Wall group $\mathbb{Z}_8$. Hint:

$(\mathbb{R} \to \mathbb{C})\hat{\otimes}(\mathbb{R} \to \mathbb{C}) = \mathbb{C} \to \mathbb{H}, \quad \text{etc.}$

Note that the complex clock is simpler: it has a two-hour dial.

Radon-Hurwitz numbers

From the clock, one has: the algebra $\mathcal{A}(m) = \mathcal{C}\ell_+(0,m)$ for $m \equiv 3 \bmod 4$, and $\mathcal{C}\ell(0,m)$ otherwise, is simple and has an irreducible representation in a real vector space of dimension $2^{\chi(m)}$, where $\chi(m)$ is the mth Radon–Hurwitz number, given by (recall representations $\mathbb{C} \to \mathbb{R}(2)$ and $\mathbb{H} \to \mathbb{R}(4)$):

$$\begin{array}{rcccccccccc}
m & = & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
\chi(m) & = & 1 & 2 & 2 & 3 & 3 & 3 & 3 & 4 \\
\mathcal{A}(m) & = & \mathbb{C} & \mathbb{H} & \mathbb{H} & \mathbb{H}(2) & \mathbb{C}(4) & \mathbb{R}(8) & \mathbb{R}(8) & \mathbb{R}(16)
\end{array}$$

Since $\mathcal{A}(m+8) = \mathcal{A}(m) \otimes \mathbb{R}(4)$, one has $\chi(m+8) = \chi(m)+4$. The map $\chi : \mathbb{N} \to \mathbb{N}$ is surjective.

$\diamondsuit$The representations $\gamma : \mathcal{A}(m) \to \mathbb{R}(\chi(m))$ are generated by matrices $\gamma_\mu \in \mathbb{R}(2^{\chi(m)})$ which are antisymmetric, $\gamma_\mu^* = -\gamma_\mu$, $\mu = 1,\ldots,m$.

They can be constructed inductively from the unit matrix and the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

by putting
for $m = 1$: $\gamma_1 = \sigma_2$,
for $m = 2$: $\gamma_1 = I \otimes \sigma_2$, $\gamma_2 = \sigma_2 \otimes \sigma_1$,
for $m = 3$: γ_1 and γ_2 as above, and $\gamma_3 = \sigma_2 \otimes \sigma_3$
etc. See the review [21].

Vector fields on spheres

There is are no non-vanishing tangent vector fields on even-dim spheres, but every odd-dim sphere has at least one such field. Clifford algebras provide a construction of the maximal number of such fields, linearly independent at all points of the sphere.
For even N, let m be the largest number such that $N = 2^{\chi(m)}p$ where p is odd. Let $I \in \mathbb{R}(p)$, $v = (v^1, \dots, v^m) \in \mathbb{R}^m$ and

$$\gamma(v) = \gamma_\mu v^\mu \otimes I \in \mathbb{R}(N)$$

where γ_μ are as in $\diamondsuit$ so that $\gamma(v)$ is antisymmetric. For every $v \neq 0$ and $x \in \mathbb{S}_{N-1}$, the vector $\gamma(v)x \neq 0$ is orthogonal to x, therefore tangent to the sphere: there are m vector fields tangent to the sphere and linearly independent at all points. Using topological methods, J. F. Adams [22] has shown that this is the maximum number of such vector fields.
From the table of Radon-Hurwitz numbers one obtains that the spheres $\mathbb{S}_1 \subset \mathbb{C}$, $\mathbb{S}_3 \subset \mathbb{H}$ and $\mathbb{S}_7 \subset \mathbb{O}$ are the only ones that are parallelizable and that spheres of dimension 1 mod 4 have only one nonvanishing tangent vector field.

Examples of Clifford algebras

There are two 'natural' quadratic forms: det on $\mathbb{C}(2)$ and the pfaffian on $\wedge^2\mathbb{C}^4 =$; they give rise to interesting Clifford algebras in dimensions 4 (physicists' spinors) and 6 (twistors).
Twistor theory is based on the following construction:
Let (T, ε), $\varepsilon \in \wedge^4 T$, be a 4-dim unimodular (twistor) space. The 6-dim space $W = \wedge^2 T$ has a scalar product h (the Pfaffian) such that for all $w_1, w_2 \in W$,

$$\tfrac{1}{2} w_1 \wedge w_2 = h(w_1, w_2)\varepsilon.$$

The form ε defines the Hodge dual map $\star : \wedge^2 T \to \wedge^2 T^*$. If $w \in W$, then there are maps

$$w : T^* \to T, \quad \star w : T \to T^*, \quad \text{and} \quad w \circ \star w = -h(w, w)\,\mathrm{id}_T$$

so that the map

$$W \to \mathsf{End}(T \oplus T^*), \quad w \mapsto \begin{pmatrix} 0 & w \\ -\star w & 0 \end{pmatrix}$$

has the Clifford property for the quadratic space (W, h) and extends to an isomorphism of algebras

$$\gamma : \mathcal{C}\ell(\wedge^2 T, h) \to \mathsf{End}(T \oplus T^*).$$

4-dim space(-time) is identified with the quadric

$$\{\operatorname{dir} w \in \mathbb{C}P_5 \mid h(w, w) = 0\}.$$

SPINORS ON MANIFOLDS

The early papers on spinors on manifolds, by both mathematicians and physicists, lacked rigour. Fibre bundles in topology appear in 1935-40 (Whitney, Hopf, Stiefel, Steenrod) and in differential geometry somewhat later (Ehresmann 1943). Cartan criticised "certain physicists", but the chapter on spinors in Riemannian geometry of his book of 1938 lacks precision and clarity.
Physicists encountered – and solved – some topological problems connected with spinors on manifolds already in the 1930s (Schrödinger [23], Pauli): spinor fields on spheres appeared to be double-valued. They used coordinates (ϕ, θ) on $\mathbb{S}_2$ with meridian $\phi = 0$ removed. If an orthonormal frame $\overset{\uparrow}{\rightarrow}$ is moved along the equator from $\phi = 0+$ to $\phi = 2\pi-$, then the frame is rotated by 2π and, as a result, a spinor changes sign...
Much of differential geometry can be presented with rigour without the use of fibre bundles (vector fields as derivations of the algebra of smooth fns, etc).
But spinors on manifolds do require fibre bundles.

Two approaches to spinor fields on manifolds

Physicists tried to do "spinor analysis on manifolds" in two ways:

1p. either referring everything to orthonormal frames and using constant gamma matrices (Wigner, Fock 1929), or
2p. introducing point-dependent gamma matrices,

$$\gamma_\mu(x)\gamma_\nu(x) + \gamma_\nu(x)\gamma_\mu(x) = 2g_{\mu\nu}(x)$$

(Tetrode 1928 [24], Schrödinger 1932 [25], Bergmann 1957)
These approaches have led to precise, bundle-theoretic formulations:

1m. (S)pin structure on (M, g) is 'reduction' (poor name, here group is 'enlarged') of the $\mathsf{O}(m)$ bundle P of orthonormal frames to a $\mathsf{Pin}(m)$-bundle $\pi : Q \to M$ (Borel and Hirzebruch 1959)

$$\begin{array}{ccccc} \mathsf{Pin}(m) & \longrightarrow & Q & \xrightarrow{\Psi} & S \\ \downarrow & & \downarrow \chi & & \\ \mathsf{O}(m) & \longrightarrow & P & & \\ & & \downarrow & & \\ & & M & & \end{array}$$

Spinor field is given as a map Ψ such that $\Psi(qa) = \gamma(a^{-1})\Psi(q)$.
When M is orientable, one defines a Spin structure and extends it, if necessary, to a Pin str. For the purpose of describing charged fermions, on uses $\mathsf{Spin}^c = (\mathsf{Spin} \times \mathsf{U}(1))/\mathbb{Z}_2$)-structures. (Weaker obstructions: $\mathbb{C}\mathsf{P}_{2n}$ have no Spin str., but have a Spin^c-str.)
The spinor bundle is the associated bundle
$\Sigma = (Q \times S)/\,\mathsf{Spin}(m)$.
2m. Guido Karrer (1973) gave another definition of Σ based on the use of the Clifford functor. Given (M, g), for every $x \in M$ there is the quadratic space (T_xM, g_x) and one constructs the **Clifford bundle** over M

$$\mathcal{C}\ell(M, g) = \cup_{x\in M}\, \mathcal{C}\ell(T_xM, g_x)$$

This is a natural (functorial with respect to isometries) construction, no obstacles.
A spinor bundle is now a 'representation bundle' $\Sigma \to M$ with a bundle map

$$\gamma : \mathcal{C}\ell(M, g) \to \mathsf{End}\,\Sigma$$

such that the restriction of γ to the fibre over x is a representation of the Clifford algebra $\mathcal{C}\ell(T_xM, g_x)$ in Σ_x. There are obstructions; such a Σ may not exist and, if it does, may be not unique. **Spinor bundles are not natural.**
A vector field defines a section X of the Clifford bundle and the composition $\gamma \circ X$ provides the physicists' point-dependent gammas.
It is easy to go from 1m to 2m by constructing Σ as an associated bundle. Going from 2m to 1m is more subtle, especially when M is odd-dim and non-orientable [26]. Improve.

Example: **Spin structures and spinor fields on spheres.**

$$\begin{tikzcd} \mathsf{Spin}(m) \arrow[r] \arrow[d] & \mathsf{Spin}(m+1) = Q \arrow[d] \\ \mathsf{SO}(m) \arrow[r] & \mathsf{SO}(m+1) = P \arrow[d] \\ & \mathbb{S}_m \end{tikzcd}$$

Here $\mathsf{SO}(m+1) \to P$ is given by
$a \mapsto$ frame $(ae_2, ae_3, \ldots, ae_{m+1})$ at $ae_1 \in \mathbb{S}_m$.
$\mathsf{SO}(m) \to \mathsf{SO}(m+1)$ is $b \mapsto \begin{pmatrix} 1 & 0 \\ 0 & b \end{pmatrix}$, etc.
On the circle, there are two spin structures
(recall $\mathsf{Spin}(1) = \mathbb{Z}_2$, $\mathsf{Spin}(2) = \mathsf{U}(1) \xrightarrow{\text{square}} \mathsf{U}(1) = \mathsf{SO}(2)$, $P = \mathsf{U}(1) \xrightarrow{\text{id}} \mathsf{U}(1) = \mathbb{S}_1$)

$$\begin{tikzcd} \mathbb{Z}_2 \arrow[r] \arrow[d] & \mathsf{U}(1) \arrow[d, "\text{square}"'] \\ 1 \arrow[r] & \mathsf{U}(1) \arrow[d] \\ & \mathbb{S}_1 \end{tikzcd} \qquad \begin{tikzcd} \mathbb{Z}_2 \arrow[r] \arrow[d] & \mathbb{Z}_2 \times \mathsf{U}(1) \arrow[d] \\ 1 \arrow[r] & \mathsf{U}(1) \arrow[d] \\ & \mathbb{S}_1 \end{tikzcd}$$

The triviality of spinor bundles on spheres

Proposition: The bundle of complex spinors Σ, associated with the principal bundle

$$\mathsf{Spin}(2n) \to Q = \mathsf{Spin}(2n+1) \xrightarrow{\pi} \mathbb{S}_{2n}$$

is a trivial vector bundle.
To show this, recall isomorphism of algebras

$$\mathcal{C}\ell(2n, \mathbb{C}) \to \mathcal{C}\ell_+(2n+1, \mathbb{C})$$

obtained by extending the Clifford map

$$\mathbb{C}^{2n} \to \mathcal{C}\ell_+(2n+1, \mathbb{C}), \quad v \mapsto ve_1 \ldots e_{2n+1}$$

The simple algebras $\mathcal{C}\ell(2n, \mathbb{C}) = \mathcal{C}\ell_+(2n+1, \mathbb{C})$ have a repr. in S, $\dim S = 2^n$; by restriction, one obtains representations of groups,

$$\gamma' : \mathsf{Spin}(2n+1) \to \mathsf{GL}(S), \quad \gamma = \gamma' | \mathsf{Spin}(2n), \quad \dim_{\mathbb{C}} S = 2^n.$$

γ is the complex ‘Dirac representation’.
The definition of Σ is (recall $Q = \mathsf{Spin}(2n+1)$)

$$\Sigma = (Q \times S)/\,\mathsf{Spin}(2n),$$
$$(q, s) \equiv (q', s') \Leftrightarrow \exists_{a \in \mathsf{Spin}(2n)}\ q' = qa \ \ \& \ \ s' = \gamma(a^{-1})s.$$

An isomorphism of bundles over M is

$$\Sigma \to M \times S, \quad [(q, s)] \mapsto (\pi(q), \gamma'(q)s).$$

This is well defined because, by virtue of $\gamma'(qa) = \gamma'(q)\gamma(a)$ for $q \in \mathsf{Spin}(2n+1)$ and $a \in \mathsf{Spin}(2n)$, one has

$$(\pi(qa), \gamma'(qa)\gamma(a^{-1})s) = (\pi(q), \gamma'(q)s)$$

The idea of this construction can be traced back to Schrödinger (1937). The result is slightly surprising: the tangent bundle $T\,\mathbb{S}_{2n}$ is not trivial and there is not even a single non-vanishing tangent vector field on $\mathbb{S}_{2n}$.
There is a more general result: if M is a hypersurface (not necessarily orientable) in a spin manifold, then M (with induced metric) has a pin structure such that the associated bundle of spinors is trivial.

Unlike tensor bundles, spinor bundles are not natural, even in the category of Riemannian manifolds and isometries. Failure to recognize this was the basis of controversies concerning the definition and properties of spinors on manifolds.
In particular, the Lie derivative is a notion intimately connected with natural bundles. Recall that a **natural bundle** is a functor F from the category of manifolds and diffeomorphisms to that of bundles such that $\pi_M : F(M) \to M$ is a bundle and if $\varphi : M \to N$ is a diffeomorphism, then $F(\varphi) : F(M) \to F(N)$ is an isomorphism of bundles covering φ. If A is a section of $\pi_N : F(N) \to N$, i.e. a field on N of geometric objects of type F, then $\varphi^* A = F(\varphi^{-1}) \circ A \circ \varphi$ is its pull-back by φ to M. The vertical bundle $VF(M)$ is the subbundle of the tangent bundle $TF(M)$ consisting of all vertical vectors, i.e. vectors that are annihilated by $T\pi_M$.
Let $(\varphi_t, t \in \mathbb{R})$ be the flow generated by a vector field X on M and let A be a section of π_M. The curve $t \mapsto (\varphi_t^* A)(x)$ is vertical for every $x \in M$ and the Lie derivative $\mathcal{L}(X)A$ is now defined as the section of the vector bundle $VF(M) \to M$ such that $(\mathcal{L}(X)A)(x)$ is the vector tangent to $t \mapsto (\varphi_t^* A)(x)$ at $t = 0$. If F is a natural vector bundle, then

$$\frac{\mathrm{d}}{\mathrm{d}t}(\varphi_t^* A) = \varphi_t^* \mathcal{L}(X) A$$

so that $\mathcal{L}(X)A = 0$ is the n. a. s. cond. for invariance of A with respect to the flow Some authors proposed an expression $\mathcal{L}(X)\varphi$ for the "Lie derivative" of a spinor field constructed with the help of covariant derivatives of φ, but it satisfied the condition $\mathcal{L}([X,Y]) = \mathcal{L}(X)\mathcal{L}(Y) - \mathcal{L}(Y)\mathcal{L}(X)$ only for vector fields X and Y that generate isometries and $\varphi_t^* A$ failed to be defined for a general X.

Covariant differentiation of spinor fields

1m Spinor structure $\mathsf{Spin}(m) \to Q \xrightarrow{\chi} P \xrightarrow{\pi} M$. Since χ is a local diffeomorphism, the distribution of horizontal subspaces of TP lifts to a similar distribution on Q. In other words, a metric linear connection form $\omega : TP \to \mathfrak{so}(m) \subset \mathsf{End}\,\mathbb{R}^m$ on P lifts to a connection form $\sigma : TQ \to \mathfrak{spin}(m) \subset \mathsf{End}\,S$ on Q and defines the covariant exterior derivative
$D\Psi = \mathrm{d}\Psi + \sigma\Psi$ of a spinor field $\Psi : Q \to S$.
The soldering form on P lifts to θ^μ on Q: if $X \in T_q Q$, $\chi(q) = (e_1, \ldots, e_m)$, and

$$T(\pi \circ \chi)(X) = X^\mu e_\mu$$

then θ^μ is such that$X \lrcorner\, \theta^\mu = X^\mu$ and

$$D\Psi = \theta^\mu \nabla_\mu \Psi$$

The Dirac operator is

$$\nabla\!\!\!/ = \gamma^\mu \nabla_\mu.$$

A pragmatic derivation of σ is as follows. For any spinor fields φ and ψ, the functions $\langle \varphi, B\gamma^\mu \psi \rangle$ are components of a vector field. Its covariant derivative can be evaluated either as

$$D\langle \varphi, B\gamma^\mu \psi \rangle = \mathrm{d}\langle \varphi, B\gamma^\mu \psi \rangle + \omega^\mu{}_\nu \langle \varphi, B\gamma^\nu \psi \rangle$$

or as

$$D\langle \varphi, B\gamma^\mu \psi \rangle = \langle D\varphi, B\gamma^\mu \psi \rangle + \langle \varphi, B\gamma^\mu D\psi \rangle$$

This gives

$$\begin{aligned} \omega^\mu{}_\nu \langle \varphi, B\gamma^\nu \psi \rangle =& \langle \sigma\varphi, B\gamma^\mu \psi \rangle + \langle \varphi, B\gamma^\mu \sigma\psi \rangle \\ =& \langle \varphi, (\sigma^* B\gamma^\mu + B\gamma^\mu \sigma)\psi \rangle \end{aligned}$$

where σ^* is the transpose of σ in $\mathsf{End}\,S$. From the arbitrariness of φ and ψ one obtains

$$\omega^\mu{}_\nu \gamma^\nu = \sigma^* B\gamma^\mu + B\gamma^\mu \sigma.$$

A solution of the last equation is $\sigma = \frac{1}{2}\omega_{\mu\nu}\gamma^\mu\gamma^\nu$.

Spectrum of the Laplace and Dirac operators on spheres

Consider unit sphere $\mathbb{S}_m \subset \mathbb{R}^{m+1}$, $m \geqslant 2$,

$$\mathrm{d}l^2{}_{\mathbb{R}^{m+1}} = g_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu = \mathrm{d}r^2 + r^2\,\mathrm{d}l^2_{\mathbb{S}_m}$$

leads to ($g = \det(g_{\mu\nu})$)

$$\Delta_{\mathbb{R}^{m+1}}\varphi = \frac{1}{\sqrt{g}}(\sqrt{g}g^{\mu\nu}\varphi_{,\mu})_{,\nu} = \frac{1}{r^m}\frac{1}{\partial r}(r^m\frac{\partial\varphi}{\partial r}) + \frac{1}{r^2}\Delta_{\mathbb{S}_m}\varphi.$$

Let φ be a harmonic polynomial homogeneous of degree l in $(x^1, \ldots, x^{m+1})$. From the last eq., φ restricted to $\mathbb{S}_m$ is an eigenfunction of $\Delta_{\mathbb{S}_m}$ with eigenvalue $-l(l-1+m)$ (the case $m = 2$ is familiar).

Consider now the eigenvalue problem for the Dirac operator $\nabla\!\!\!/$ on the sphere. The Dirac operator in $\mathbb{R}^{m+1}$ is

$$\gamma^\mu\partial_\mu = \gamma_\mu\frac{x^\mu}{r}(\frac{1}{r^{m/2}}\frac{1}{\partial r}r^{m/2} + \frac{1}{r}\nabla\!\!\!/)$$

Assume now φ to be harmonic polynomial homogeneous of degree $l+1$, $l = 1, 2, \ldots$. Then $\psi = \gamma^\mu\partial_\mu\varphi$ is of degree l and $\gamma^\mu\partial_\mu\psi = 0$ gives

$$\nabla\!\!\!/\psi = -(l + \frac{m}{2})\psi.$$

But the spectrum is symmetric (e.g. for $m = 2n$ one has $\gamma_{2n+1}\nabla\!\!\!/ + \nabla\!\!\!/\gamma_{2n+1} = 0$); therefore, the spectrum of the Dirac operator on $\mathbb{S}_m$ contains (in fact, it is) $\pm(l + m/2)$, $l = 0, 1, \ldots$. Zero is not an eigenvalue.

References

[1] P. Deligne, P. Etingof, D. S. Freed, L. C. Jeffrey, D. Kazhdan, J. W. Morgan, D. R. Morrison, and E. Witten, editors. *Quantum fields and strings: a course for mathematicians. Vol. 1, 2.* American Mathematical Society, Providence, RI; Institute for Advanced Study (IAS), Princeton, NJ, 1999. Material from the Special Year on Quantum Field Theory held at the Institute for Advanced Study, Princeton, NJ, 1996–1997.

[2] O. Veblen and J. W. Givens. Geometry of complex domains. Notes from a seminar conducted in 1935-36 at the Institute for Advanced Study in Princeton by O. Veblen and J. von Neumann (mimeographed and reissued, Princeton, 1955).

[3] J. W. Givens. Tensor coördinates of linear spaces. *Ann. of Math.*, 38:355–385, 1937.

[4] É. Cartan. *Leçons sur la théorie des spineurs.* Actualités Scientifiques et Industrielles, No. 643 et 701. Hermann, Paris, 1938. English transl.: *The Theory of Spinors*, Hermann, Paris 1966.

[5] C. Chevalley. *The Algebraic Theory of Spinors.* Columbia Univ. Press, New York, 1954.

[6] P. Budinich and A. Trautman. Fock space description of simple spinors. *J. Math. Phys.*, 30:2125–2131, 1988.

[7] A. Trautman and K. Trautman. Generalized pure spinors. *J. Geom. Phys.*, 15:1–22, 1994.

[8] R. O. Lipschitz. *Untersuchungen über die Summen von Quadraten.* Max Cohen und Sohn, Bonn, 1886.

[9] A. Weil. Correspondence, from an ultramundane correspondent. In *Œuvres Scientifiques (Collected Papers), Vol. II (1951-64)*. Springer, 1979. reprinted from *Ann. of Math.* **69** (1959) 247–251.

[10] É. Cartan. Les groupes projectifs qui ne laissent invariante aucune multiplicité plane. *Bull. Soc. Math.*, 41:53–96, 1913.

[11] W. Pauli. Zur Quantenmechanik des magnetischen Elektrons. *Z. Physik*, 43:601–623, 1927.

[12] P. A. M. Dirac. The quantum theory of the electron. *Proc. Roy. Soc. (London) A*, 117:610–624, 1928.

[13] D. Ivanenko and L. D. Landau. Zur Theorie des magnetischen Elektrons. *Z. Physik*, 48:340–348, 1928.

[14] E. Kähler. Innerer und äusserer Differentialkalkül. *Abh. Akad. Wiss. Berlin (Math.-Phys.)*, 4:1–32, 1960.

[15] H. Weyl. Elektron und Gravitation I. *Z. Physik*, 56:330–352, 1929. Submitted on 8 May 1929.

[16] L. Infeld and B. L. van der Waerden. Die Wellengleichung des Elektrons in der Allgemeinen Relativitätstheorie. *Sitzungsber. Preuss. Akad. Wiss., Phys.-Math. Kl.*, 9:380–402, 1933.

[17] R. Brauer and H. Weyl. Spinors in n dimensions. *Amer. J. Math.*, 57(2):425–449, 1935.

[18] P. Budinich and A. Trautman. *The Spinorial Chessboard*. Trieste Notes in Physics. Springer-Verlag, Berlin, 1988.

[19] C. T. C. Wall. Graded Brauer groups. *Journal für die reine und angewandte Mathematik*, 213:187–199, 1964.

[20] C.T.C. Wall. Graded algebras, antiinvolutions, simple groups and symmetric spaces. *Bull. Amer. Math. Soc.*, 74:198–202, 1968.

[21] A. Trautman. Clifford algebras and their representations. In J.-P. Françoise, G. L. Naber and S.T. Tsou, editor, *Encyclopedia of Mathematical Physics Vol. 1*, page 518. Elsevier, Oxford, 2006.

[22] J. F. Adams. Vector fields on spheres. *Ann. of Math. (2)*, 75:603–632, 1962.

[23] E. Schrödinger. Die Mehrdeutigkeit der Wellenfunktion. *Helv. Phys. Acta*, 32:49–55, 1938.

[24] H. Tetrode. Allgemein-relativistische Quantentheorie des Elektrons. *Z. Physik*, 50:336–346, 1928.

[25] E. Schrödinger. Diracsches Elektron im Schwerefeld I. *Sitzungsber. Preuss. Akad. Wiss., Phys.-Math. Kl.*, XI:105–28, 1932.

[26] Th. Friedrich and A. Trautman. Spin spaces, Lipschitz groups, and spinor bundles. *Ann. Glob. Anal. Geom.*, 18:221–240, 2000.