DIFFERENTIAL FORMS
AND VECTOR ANALYSIS

We are used to work in the Cartesian coordinate system in which points of the space are identified by values of x, y and z. Associated with this system is the basis of three vectors

$$\mathbf{i}_x \equiv \mathbf{e}_x\,, \qquad \mathbf{i}_y \equiv \mathbf{e}_y\,, \qquad \mathbf{i}_z \equiv \mathbf{e}_z\,.$$

These three vectors have by definition unit lengths (we will use the symbol $\mathbf{e}_a$ for Cartesian and $\mathbf{e}_i$ for other unit length vectors) and are mutually orthogonal:

$$(\mathbf{e}_i|\mathbf{e}_j) \equiv \mathbf{e}_i\!\cdot\!\mathbf{e}_j = \delta_{ij}\,.$$

They also satisfy the rule

$$\mathbf{e}_i \times \mathbf{e}_j = \epsilon_{ijk}\,\mathbf{e}_k \equiv \mathbf{e}_k\,\epsilon_{kij}\,.$$

From these rules, the identity

$$\epsilon_{ijk}\epsilon_{klm} = \delta_{il}\,\delta_{jm} - \delta_{im}\delta_{jl}\,,$$

and the possibility of writing any vector $\mathbf{V}$ as a linear combination $\mathbf{V} = \mathbf{e}_i V^i_{(\mathbf{e}_i)} \equiv \mathbf{e}_i V^i$ all vector identities can easily be proven. For example

$$\begin{aligned}
\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) &= \mathbf{e}_i \times (\mathbf{e}_l \times \mathbf{e}_m) A^i B^l C^m \\
&= \mathbf{e}_i \times \mathbf{e}_k\,\epsilon_{klm}\,A^i B^l C^m \\
&= \mathbf{e}_j\,\epsilon_{jik}\,\epsilon_{klm}\,A^i B^l C^m \\
&= \mathbf{e}_j(\delta_{jl}\delta_{im} - \delta_{jm}\delta_{il})\,A^i B^l C^m \\
&= \mathbf{e}_j B^j (A^i C^i) - \mathbf{e}_j C^j (A^i B^i) \\
&\equiv \mathbf{B}\,(\mathbf{A}\!\cdot\!\mathbf{C}) - \mathbf{C}\,(\mathbf{A}\!\cdot\!\mathbf{B})\,.
\end{aligned}$$

Usually in this type of calculations one does not write explicitly the unit vectors $\mathbf{e}_i$. This makes the notation more economical but is possible only either if the vectors are decomposed into the Cartesian unit vectors $\mathbf{e}_x$, $\mathbf{e}_y$, $\mathbf{e}_z$, or (for vectors decomposed into unit vectors $\mathbf{e}_1(\xi)$, $\mathbf{e}_2(\xi)$, $\mathbf{e}_3(\xi)$ associated with some curvelinear coordinates ξ^1, ξ^2, ξ^3 - see below), or if no differentiations are involved: for instance, if in the example above $\mathbf{C}(\xi) = \mathbf{e}_i(\xi) C^i(\xi)$ and $\mathbf{B}(\xi) = \mathbf{e}_i(\xi)\partial/\partial\xi^i$, where the differential operator acts on everything standing to the right of it, then one cannot drop the vectors $\mathbf{e}_i(\xi)$ because they too should get differentiated.

In numerous special problems of classical electrodynamics it proves more convenient to use coordinate systems ξ^i other than the Cartesian ones. Curvelinear systems are introduced by giving three functions

$$\begin{aligned} x &= x(\xi^1, \xi^2, \xi^3)\,, \\ y &= y(\xi^1, \xi^2, \xi^3)\,, \\ z &= z(\xi^1, \xi^2, \xi^3)\,. \end{aligned}$$

Associated with each point of the space are then three vectors

$$\mathbf{i}_i(\xi) \equiv \frac{\partial}{\partial \xi^i} \equiv \mathbf{e}_x \frac{\partial x}{\partial \xi^i} + \mathbf{e}_y \frac{\partial y}{\partial \xi^i} + \mathbf{e}_z \frac{\partial z}{\partial \xi^i} \equiv \mathbf{e}_a \frac{\partial x^a}{\partial \xi^i}\,,$$

(the notation $\partial/\partial\xi^i$ used by differential geometers - różniczkowych omętrów zwanych gdzieniegdzie jeszcze różniczkowymi skoczybruzdami - should not terrify you as we will not use it). More precisely, with each point p (identified by the values of the coordinates ξ^i) of the space M (which should be thought of as a differential manifold M) there is associated a vector space T_pM (the tangent space) in which vectors attached to this point live. The vectors $\mathbf{i}_i(\xi)$ form a basis of the vector space T_pM attached to the point p labeled by ξ^1, ξ^2, ξ^3. The vectors $\mathbf{i}_1$, $\mathbf{i}_2$, $\mathbf{i}_3$ are not the same for different points p and for an arbitrary choice of the coordinates ξ^i are not of unit length and even not mutually orthogonal. Their scalar product defines the *metric tensor* $g_{ij}(\xi)$

$$g_{ij}(\xi) \equiv (\mathbf{i}_i|\mathbf{i}_j) = \frac{\partial x}{\partial \xi^i}\frac{\partial x}{\partial \xi^j} + \frac{\partial y}{\partial \xi^i}\frac{\partial y}{\partial \xi^j} + \frac{\partial z}{\partial \xi^i}\frac{\partial z}{\partial \xi^j} \equiv \frac{\partial x^a}{\partial \xi^i}\frac{\partial x^a}{\partial \xi^j}\,.$$

Here we work in the Euclidean three dimensional space and the metric tensor $g_{ij}(\xi)$ can be computed directly because we assume that the three functions $x(\xi)$, $y(\xi)$, $z(\xi)$ are given.[1] As in the usual algebra, any vector $\mathbf{V}$ attached to the point labeled by ξ^i or a vector field $\mathbf{V}(\xi)$ can be written in the form

$$\mathbf{V}(\xi) = \mathbf{i}_k(\xi) V^k_{(\mathbf{i})}(\xi) \equiv \mathbf{i}_k(\xi) V^k(\xi)\,,$$

where $V^k_{(\mathbf{i})}$ is the notation borrowed from my famous Algebra notes indicating explicitly that these are components of the vector $\mathbf{V}$ in the basis $\mathbf{i}_k$. The scalar product of two such vectors (vector fields) $\mathbf{V}$ and $\mathbf{W}$ is then given by

$$(\mathbf{V}|\mathbf{W}) = (\mathbf{i}_i|\mathbf{i}_k)\, V^i W^k = g_{ik} V^i W^k \equiv V_i W^i\,.$$

We have defined here *covariant* components $V_i \equiv g_{ij}V^j$ of the vector $\mathbf{V}$ (as opposed to its *contravariant* components V^i). Of course $V^i = g^{ij}V_j$ where g^{ij} is the matrix inverse with respect to the matrix g_{ij}. Mathematically V_k are components of a *linear*

[1]In General Relativity we do not assume this and try instead to reconstruct all features of the space-time from the metric tensor which in turn is determined by the differential Einstein's equations; the space-time is then in most cases non-Euclidean, that is it has a nonvanishing curvature - a characteristic which is independent of the choice of the coordinate system.

form or, if V_i depend on ξ^j, components of a field of forms called also a *differential one-form* $\hat{V} = V_i(\xi)d\xi^i$ (see below for the definition of the basis forms $d\xi^i$), associated with the vector $\mathbf{V}$ (with the vector field $\mathbf{V}(\xi)$) which on all vectors attached to the point ξ acts through the scalar product

$$\hat{V}(\cdot) \equiv (\mathbf{V}(\xi)|\cdot\,)\,.$$

All linear forms attached to the point ξ^i form a vector space (the adjoint vector space with respect to the vector space of vectors attached to this point) for which different bases can be chosen; the two natural bases will be defined below.

In the following we will be concerned with a special class of coordinate systems - the Lamé systems - singled out by the orthogonality (in each point of the space) of the three vectors $\mathbf{i}_i$. In such systems the metric tensor is diagonal:

$$g_{ij}(\xi) = h_i^2(\xi)\,\delta_{ij}\,, \qquad\qquad h_i = \sqrt{(\mathbf{i}_i|\mathbf{i}_i)} \equiv \|\mathbf{i}_i\|\,.$$

h_i are called the *Lamé coefficients*. Of course, $g^{ij}(\xi) = h_i^{-2}(\xi)\,\delta^{ij}$. In the Lamé systems, to make vector analysis easier, i.e. to make it similar to the vector analysis in the Cartesian coordinates, one introduces three normalized vectors

$$\mathbf{e}_i \equiv \frac{\mathbf{i}_i}{\|\mathbf{i}_i\|} = h_i^{-1}\,\mathbf{i}_i\,,$$

(no summation over i here) such that

$$(\mathbf{e}_i|\mathbf{e}_j) = h_i^{-1}h_j^{-1}(\mathbf{i}_i|\mathbf{i}_j) = h_i^{-1}h_j^{-1}g_{ij} = h_i^{-1}h_j^{-1}h_ih_j\delta_{ij} = \delta_{ij}\,.$$

(Of course, these vectors still depend on ξ, because their orientation in the space varies from point to point). Any vector $\mathbf{V}$ can be then decomposed in two ways (and, of course, in many other ways too)

$$\begin{aligned}\mathbf{V} &= \mathbf{i}_k V^k_{(\mathbf{i})} \equiv \mathbf{i}_k V^k \\ &= \mathbf{e}_k V^k_{(\mathbf{e})} \equiv \mathbf{e}_k \bar{V}^k\,.\end{aligned}$$

From the relation between the vectors $\mathbf{i}_k$ and $\mathbf{e}_k$ it follows that

$$V^k_{(\mathbf{e})} \equiv \bar{V}^k = h_k V^k \equiv h_k V^k_{(\mathbf{i})}\,,$$

(no summation over k here). The scalar product of two vectors can be then written as

$$(\mathbf{V}|\mathbf{W}) = \bar{V}^k\bar{W}^k = \bar{V}_k\bar{W}^k\,,$$

i.e. it looks as in the Cartesian system. The barred covariant components $\bar{V}_k$ of the vector $\mathbf{V}$ are identical to the contravariant ones

$$\bar{V}_k = \bar{V}^k\,,$$

and are related to the unbarred covariant components V_k of $\mathbf{V}$ by

$$\bar{V}_k = h_k^{-1} V_k \,,$$

(again no sum over k here). Thus, the whole point of introducing "physical" components $\bar{V}^k$ is to get rid of the metric tensor in the scalar product.

Gradient

Consider a function S defined on the space (on the manifold). In coordinates ξ^i it is a function $S(\xi)$. At each point its total differential

$$dS = \frac{\partial S}{\partial \xi^i}\, d\xi^i \,,$$

is a linear form, or more precisely, a differential one-form. As every linear form, it is a device with a hole into which one inserts a vector and obtains in return a number; the action of such a form is linear. The differentials $d\xi^i$ form a basis in the space of one-forms; their action on any vector follows from the rule

$$d\xi^k(\mathbf{i}_j) = \delta^k{}_j \,,$$

and linearity. The factors $\partial S/\partial \xi^i$ are simply components of the one-form dS in the natural basis $d\xi^i$ of one-forms associated with the coordinates ξ^i. On a vector $\delta\boldsymbol{\xi} = \mathbf{i}_k \delta\xi^k$ of a small displacement by $\delta\xi^i$ the total differential dS gives

$$dS(\delta\boldsymbol{\xi}) = \frac{\partial S}{\partial \xi^i}\, d\xi^i(\mathbf{i}_k \delta\xi^k) = \frac{\partial S}{\partial \xi^i}\, d\xi^i(\mathbf{i}_k)\, \delta\xi^k = \frac{\partial S}{\partial \xi^i}\, \delta\xi^i \approx S(\xi + \delta\xi) - S(\xi) \,,$$

- the first approximation to the difference of S at ξ^i and the neighbouring point $\xi^i + \delta\xi^i$, that is $dS(\delta\boldsymbol{\xi})$ is what an average physicist, not mislead by mathematicians (and their complicated symbolics), would call dS.

In the Lamé systems one introduces also another basis $\hat{f}^i$ of one-forms singled out by their action on the $\mathbf{e}_i$ vectors:

$$\hat{f}^k(\mathbf{e}_j) = \delta^k{}_j \,.$$

From linearity it then follows that

$$\hat{f}^k = h_k \, d\xi^k \,,$$

because then

$$\hat{f}^k(\mathbf{e}_j) = \hat{f}^k(h_j^{-1}\mathbf{i}_j) = h_j^{-1} \hat{f}^k(\mathbf{i}_j) = h_j^{-1} h_k \, d\xi^k(\mathbf{i}_j) = h_j^{-1} h_k \, \delta^k{}_j = \delta^k{}_j \,.$$

The action of a linear form ${}^{(1)}\hat{\omega} = \omega_k d\xi^k$ attached to the point ξ (or a field of one-forms ${}^{(1)}\hat{\omega}(\xi)$ defined for each point of the manifold, if the components ω_k are functions of ξ^i) on a vector $\mathbf{V}$ attached to the same point ξ (or a vector field defined on the manifold) is given by

$$ {}^{(1)}\hat{\omega}(\mathbf{V}) = \omega_k d\xi^k(\mathbf{i}_j V^j) = \omega_k V^k \equiv h_k \bar{\omega}_k \, h_k^{-1} \bar{V}^k = \bar{\omega}_k \bar{V}^k \, . $$

This shows that components of a one-form can be treated as (covariant) components of a vector and the action of the one-form ${}^{(1)}\hat{\omega}$ on a vector $\mathbf{V}$ can be represented by the scalar product of $\mathbf{V}$ with the vector $\mathbf{i}_i \, \omega^i = \mathbf{e}_i \, \bar{\omega}^i$ associated with the form ${}^{(1)}\hat{\omega}$.

In Lamé systems the gradient (the "physical" gradient) of a function S is by definition the total differential dS referred to the basis $\hat{f}^k$:

$$ dS = \frac{\partial S}{\partial \xi^k} \, d\xi^k = \left(\frac{1}{h_k} \frac{\partial S}{\partial \xi^k} \right) \hat{f}^k \equiv \overline{(\nabla S)}_k \hat{f}^k \, . $$

The gradient of $S : M \longrightarrow \mathbb{R}$, or in other words, the total derivative of S, is a liner function mapping the vectors living in the tangent space $T_p M$ into $\mathbb{R}$:

$$ dS(\mathbf{V}) = V^l \, \frac{\partial S}{\partial \xi^k} \, d\xi^k(\mathbf{i}_l) = V^k \, \frac{\partial S}{\partial \xi^k} = \bar{V}^k \, \overline{(\nabla S)}_k \, . $$

Of course, in physical calculations the bars over "physical" components are omitted (as components of vectors and forms in the bases $\mathbf{i}_i$ and $d\xi^j$ never appear in such calculations).

Divergence

Divergence of a vector field $\mathbf{V}(\xi) = \mathbf{i}_k V^k(\xi)$ is in the most general case defined as follows: We associate with the vector field $\mathbf{V}$ the already introduced one-form $\hat{V}$:

$$ \hat{V} = V_i \, d\xi^i \equiv g_{ik} V^k \, d\xi^i \, , $$

and apply to it the Hodge star operation:

$$ *\hat{V} \equiv \frac{1}{2} \sqrt{g} \, \epsilon_{ijk} V^k \, d\xi^i \wedge d\xi^j \, , $$

where $g \equiv \det(g_{ij})$ and finally take the exterior derivative of the resulting two-form:

$$ \begin{aligned} d(*\hat{V}) &= \frac{1}{2} \, \epsilon_{ijk} \, \frac{\partial}{\partial \xi^l} \left(V^k \sqrt{g} \right) d\xi^l \wedge d\xi^i \wedge d\xi^j \\ &= \frac{\partial}{\partial \xi^k} \left(V^k \sqrt{g} \right) d\xi^1 \wedge d\xi^2 \wedge d\xi^3 \, . \end{aligned} $$

We have used here the relations

$$ d\xi^l \wedge d\xi^i \wedge d\xi^j = \epsilon_{lij} \, d\xi^1 \wedge d\xi^2 \wedge d\xi^3 \, , \qquad \text{and} \qquad \epsilon_{ijk} \epsilon_{lij} = 2 \, \delta_{kl} \, . $$

The "physical" divergence is the three-form $d(*\hat{V})$ but referred to the canonical volume form $\hat{f}^1 \wedge \hat{f}^2 \wedge \hat{f}^3$:

$$d(*\hat{V}) = \frac{1}{h_1 h_2 h_3} \frac{\partial}{\partial \xi^k} \left(h_1 h_2 h_3 \frac{\bar{V}^k}{h_k} \right) \hat{f}^1 \wedge \hat{f}^2 \wedge \hat{f}^3 .$$

i.e.

$$\mathrm{div}\mathbf{V} \equiv \boldsymbol{\nabla} \cdot \mathbf{V} = \frac{1}{h_1 h_2 h_3} \frac{\partial}{\partial \xi^k} \left(h_1 h_2 h_3 \, \frac{\bar{V}^k}{h_k} \right) .$$

Curl

Curl of a vector field $\mathbf{V}(\xi) = \mathbf{i}_k V^k(\xi)$ is defined as follows: first associate with $\mathbf{V}$ the form $\hat{V} = V_i d\xi^i$. Then take its exterior derivative

$$d\hat{V} = \frac{\partial}{\partial \xi^k} \left(g_{ij} V^j \right) d\xi^k \wedge d\xi^i ,$$

obtaining a two-form. Finally apply the Hodge star operation:

$$*(d\hat{V}) = \sqrt{g}\, g^{kl} g^{im} \frac{\partial}{\partial \xi^k} \left(g_{ij} V^j \right) \epsilon_{lmn} \, d\xi^n .$$

In a Lamé system, components of the resulting one-form in the basis $\hat{f}^i$ are just what is called the "physical" curl of $\mathbf{V}$:

$$*(d\hat{V}) = h_1 h_2 h_3 \, h_k^{-2} h_i^{-2} \, \epsilon_{kin} \frac{\partial}{\partial \xi^k} \left(h_i \bar{V}^i \right) h_n^{-1} \, \hat{f}^n \equiv \overline{(\boldsymbol{\nabla} \times \mathbf{V})}_n \, \hat{f}^n .$$

Simplifying a bit, the "physical" components of the curl of $\mathbf{V}$ are

$$\overline{(\boldsymbol{\nabla} \times \mathbf{V})}_n = \frac{h_n}{h_1 h_2 h_3} \, \epsilon_{kin} \frac{\partial}{\partial \xi^k} \, \left(h_k \bar{V}^k \right) .$$

Laplacian

The Laplacian acting on a function $S(\xi)$ is just the divergence of its gradient - it is the three-form:

$$d(*dS) = (\nabla^2 S) \, \hat{f}^1 \wedge \hat{f}^2 \wedge \hat{f}^3 .$$

Explicitly:

$$\begin{aligned} d\left(* \left(\frac{\partial S}{\partial \xi^i} \, d\xi^i \right) \right) &= \frac{1}{2} \, d \left(\frac{\partial S}{\partial \xi^i} \, \sqrt{g} \, g^{ik} \, \epsilon_{klm} \, d\xi^l \wedge d\xi^m \right) \\ &= \frac{1}{2} \frac{\partial}{\partial \xi^j} \left(\frac{\partial S}{\partial \xi^i} \, \sqrt{g} \, g^{ik} \right) \epsilon_{klm} \, d\xi^j \wedge d\xi^l \wedge d\xi^m \\ &= \frac{\partial}{\partial \xi^j} \left(\sqrt{g} \, g^{ij} \, \frac{\partial S}{\partial \xi^i} \right) d\xi^1 \wedge d\xi^2 \wedge d\xi^3 . \end{aligned}$$

We have used the same relations as in the derivation of the divergence. The "physical Laplacian" (in Lamé coordinate systems) is referred to the canonical volume three-form:

$$\nabla^2 S = \frac{1}{h_1 h_2 h_3} \frac{\partial}{\partial \xi^j} \left(\frac{h_1 h_2 h_3}{h_i^2} \frac{\partial S}{\partial \xi^i} \right) .$$

Examples

We illustrate all these considerations on two examples.

A. The spherical coordinates $(\xi^i, \xi^2, \xi^3) \equiv (r, \theta, \phi)$ are introduced through the well known relations

$$\begin{aligned} x &= r \sin\theta \cos\phi \,, \\ y &= r \sin\theta \sin\phi \,, \\ z &= r \cos\theta \,. \end{aligned}$$

One then has

$$\mathbf{i}_r = \begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{pmatrix}, \qquad \mathbf{i}_\theta = \begin{pmatrix} r\cos\theta\cos\phi \\ r\cos\theta\sin\phi \\ -r\sin\theta \end{pmatrix}, \qquad \mathbf{i}_\phi = \begin{pmatrix} -r\sin\theta\sin\phi \\ r\sin\theta\cos\phi \\ 0 \end{pmatrix}.$$

The Lamé coefficients read

$$h_r = \sqrt{(\mathbf{i}_r|\mathbf{i}_r)} = 1 \,, \qquad h_\theta = \sqrt{(\mathbf{i}_\theta|\mathbf{i}_\theta)} = r \,, \qquad h_\phi = \sqrt{(\mathbf{i}_\phi|\mathbf{i}_\phi)} = r\sin\theta \,,$$

and the vectors $\mathbf{e}_r$, $\mathbf{e}_\theta$, $\mathbf{e}_\phi$ have the form

$$\mathbf{e}_r = \begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{pmatrix}, \qquad \mathbf{e}_\theta = \begin{pmatrix} \cos\theta\cos\phi \\ \cos\theta\sin\phi \\ -\sin\theta \end{pmatrix}, \qquad \mathbf{e}_\phi = \begin{pmatrix} -\sin\phi \\ \cos\phi \\ 0 \end{pmatrix}.$$

The canonical volume three-form

$$\hat{f}^r \wedge \hat{f}^\theta \wedge \hat{f}^\phi = r^2 \sin\theta \, dr \wedge d\theta \wedge d\phi \,,$$

looks familiar for anybody who at least once has integrated something over a three dimensional domain using spherical coordinates, but what do these "$\wedge$'s" serve for?! Be patient and look below how the integration of differential forms is defined.

Using the Lamé coefficients given above it is straightforward to write down "physical" components of the gradient of a function S

$$\overline{(\boldsymbol{\nabla} S)}_r = \frac{\partial S}{\partial r} \,, \qquad \overline{(\boldsymbol{\nabla} S)}_\theta = \frac{1}{r}\frac{\partial S}{\partial \theta} \,, \qquad \overline{(\boldsymbol{\nabla} S)}_\phi = \frac{1}{r\sin\theta}\frac{\partial S}{\partial \phi} \,,$$

of the rotation of a vector field $\mathbf{V} = \mathbf{e}_r \bar{V}^r + \mathbf{e}_\theta \bar{V}^\theta + \mathbf{e}_\phi \bar{V}^\phi$

$$
\begin{aligned}
\overline{(\boldsymbol{\nabla}\times\mathbf{V})}_r &= \frac{1}{r\sin\theta}\left(\frac{\partial}{\partial\theta}\,(\bar{V}^\phi\sin\theta) - \frac{\partial\bar{V}^\theta}{\partial\phi}\right),\\
\overline{(\boldsymbol{\nabla}\times\mathbf{V})}_\theta &= \frac{1}{r\sin\theta}\,\frac{\partial\bar{V}^r}{\partial\phi} - \frac{1}{r}\,\frac{\partial}{\partial r}\,(r\bar{V}^\phi)\,,\\
\overline{(\boldsymbol{\nabla}\times\mathbf{V})}_\phi &= \frac{1}{r}\left(\frac{\partial}{\partial r}\,(r\bar{V}^\theta) - \frac{\partial\bar{V}^r}{\partial\theta}\right),
\end{aligned}
$$

as well as the "physical divergence" of $\mathbf{V}$:

$$
\boldsymbol{\nabla}\cdot\mathbf{V} = \frac{1}{r^2}\,\frac{\partial}{\partial r}\,(r^2\bar{V}^r) + \frac{1}{r\sin\theta}\,\frac{\partial}{\partial\theta}\,(\bar{V}^\theta\sin\theta) + \frac{1}{r\sin\theta}\,\frac{\partial\bar{V}^\phi}{\partial\phi}\,,
$$

and the "physical" Laplacian of a function $S(\xi)$:

$$
\nabla^2 S = \frac{1}{r^2}\,\frac{\partial}{\partial r}\left(r^2\frac{\partial S}{\partial r}\right) + \frac{1}{r^2\sin\theta}\,\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial S}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\,\frac{\partial^2 S}{\partial\phi^2}\,.
$$

B. Another, less familiar example are the *parabolic coordinates*

$$
\begin{aligned}
x &= \sqrt{\xi\eta}\,\cos\phi\,,\\
y &= \sqrt{\xi\eta}\,\sin\phi\,,\\
z &= \frac{1}{2}(\xi-\eta)\,,
\end{aligned}
$$

with $\xi,\eta \geq 0$ and $0 \leq \phi \leq 2\pi$. One then has

$$
\mathbf{i}_\xi = \begin{pmatrix} \frac{1}{2}\sqrt{\frac{\eta}{\xi}}\cos\phi \\ \frac{1}{2}\sqrt{\frac{\eta}{\xi}}\sin\phi \\ \frac{1}{2} \end{pmatrix}, \qquad \mathbf{i}_\theta = \begin{pmatrix} \frac{1}{2}\sqrt{\frac{\xi}{\eta}}\cos\phi \\ \frac{1}{2}\sqrt{\frac{\xi}{\eta}}\sin\phi \\ -\frac{1}{2} \end{pmatrix}, \qquad \mathbf{i}_\phi = \begin{pmatrix} -\sqrt{\xi\eta}\,\sin\phi \\ \sqrt{\xi\eta}\,\cos\phi \\ 0 \end{pmatrix}.
$$

It is easy to see that $\mathbf{i}_\xi\cdot\mathbf{i}_\eta = 0$, $\mathbf{i}_\xi\cdot\mathbf{i}_\phi = 0$, $\mathbf{i}_\eta\cdot\mathbf{i}_\phi = 0$ - the parabolic coordinates form an orthogonal system. The Lamé coefficients read

$$
h_\xi = \sqrt{(\mathbf{i}_\xi|\mathbf{i}_\xi)} = \sqrt{\frac{\xi+\eta}{4\xi}}, \qquad h_\eta = \sqrt{(\mathbf{i}_\eta|\mathbf{i}_\eta)} = \sqrt{\frac{\xi+\eta}{4\eta}}, \qquad h_\phi = \sqrt{(\mathbf{i}_\phi|\mathbf{i}_\phi)} = \sqrt{\xi\eta}\,,
$$

and the normalized vectors $\mathbf{e}_\xi$, $\mathbf{e}_\eta$, $\mathbf{e}_\phi$ have the form

$$
\mathbf{e}_\xi = \frac{1}{\sqrt{\xi+\eta}}\begin{pmatrix} \sqrt{\eta}\cos\phi \\ \sqrt{\eta}\sin\phi \\ \sqrt{\xi} \end{pmatrix}, \qquad \mathbf{e}_\theta = \frac{1}{\sqrt{\xi+\eta}}\begin{pmatrix} \sqrt{\xi}\cos\phi \\ \sqrt{\xi}\sin\phi \\ -\sqrt{\eta} \end{pmatrix}, \qquad \mathbf{e}_\phi = \begin{pmatrix} -\sin\phi \\ \cos\phi \\ 0 \end{pmatrix}.
$$

Having the above components of the vectors $\mathbf{e}_i$ (i.e. $\mathbf{e}_\xi$, $\mathbf{e}_\eta$ and $\mathbf{e}_\phi$) in the basis formed by the vectors $\mathbf{e}_a$, that is: $\mathbf{e}_x$, $\mathbf{e}_y$, $\mathbf{e}_z$, we have the transition matrix $R_{(\mathbf{e}_a\leftarrow\mathbf{e}_i)}$:

$$(\mathbf{e}_\xi,\, \mathbf{e}_\eta,\, \mathbf{e}_\phi) = (\mathbf{e}_x,\, \mathbf{e}_y,\, \mathbf{e}_z) \begin{pmatrix} \sqrt{\frac{\eta}{\xi+\eta}}\cos\phi & \sqrt{\frac{\xi}{\xi+\eta}}\cos\phi & -\sin\phi \\ \sqrt{\frac{\eta}{\xi+\eta}}\sin\phi & \sqrt{\frac{\xi}{\xi+\eta}}\sin\phi & \cos\phi \\ \sqrt{\frac{\xi}{\xi+\eta}} & -\sqrt{\frac{\eta}{\xi+\eta}} & 0 \end{pmatrix} .$$

It allows to find the $\mathbf{e}_a$ basis components of a vector field $\mathbf{V}$ from its $\mathbf{e}_i$ basis components:

$$\mathbf{V} = \mathbf{e}_i V^i_{(\mathbf{e}_i)} = \mathbf{e}_a [R_{(\mathbf{e}_a\leftarrow\mathbf{e}_i)}]^a{}_i V^i_{(\mathbf{e}_i)}\,,$$

that is, $V^a_{(\mathbf{e}_a)} = [R_{(\mathbf{e}_a\leftarrow\mathbf{e}_i)}]^a{}_i V^i_{(\mathbf{e}_i)}$. Since the vectors of the basis $\mathbf{e}_i$ differ from the vectors of the basis $\mathbf{i}_i$ only by normalization, it is easy to write down also the transition matrix $R_{(\mathbf{e}_a\leftarrow\mathbf{i}_i)}$. Indeed,

$$\mathbf{V} = \mathbf{e}_a [R_{(\mathbf{e}_a\leftarrow\mathbf{e}_i)}]^a{}_i V^i_{(\mathbf{e}_i)} = \mathbf{e}_a [R_{(\mathbf{e}_a\leftarrow\mathbf{e}_i)}]^a{}_i (h_i V^i_{(\mathbf{i}_i)}) \equiv \mathbf{e}_a [R_{(\mathbf{e}_a\leftarrow\mathbf{i}_i)}]^a{}_i V^i_{(\mathbf{i}_i)}\,,$$

so that the matrix $R_{(\mathbf{e}_a\leftarrow\mathbf{i}_i)}$ is obtained by multiplying the i-th column of the matrix $R_{(\mathbf{e}_a\leftarrow\mathbf{e}_i)}$ by h_i. Of course, the same matrix $R_{(\mathbf{e}_a\leftarrow\mathbf{i}_i)}$ can immediately be obtained from the components of the vectors $\mathbf{i}_i$ in the basis $\mathbf{e}_a$:

$$R_{(\mathbf{e}_a\leftarrow\mathbf{i}_i)} = \begin{pmatrix} \frac{1}{2}\sqrt{\frac{\eta}{\xi}}\cos\phi & \frac{1}{2}\sqrt{\frac{\xi}{\eta}}\cos\phi & -\sqrt{\xi\eta}\sin\phi \\ \frac{1}{2}\sqrt{\frac{\eta}{\xi}}\sin\phi & \frac{1}{2}\sqrt{\frac{\xi}{\eta}}\sin\phi & \sqrt{\xi\eta}\cos\phi \\ \frac{1}{2} & -\frac{1}{2} & 0 \end{pmatrix} .$$

Since the two bases $\mathbf{e}_i$ and $\mathbf{e}_a$ are orthonormal, the matrix $R_{(\mathbf{e}_i\leftarrow\mathbf{e}_a)}$, the inverse of $R_{(\mathbf{e}_a\leftarrow\mathbf{e}_i)}$ is just given by the transposition of $R_{(\mathbf{e}_a\leftarrow\mathbf{e}_i)}$:

$$R_{(\mathbf{e}_i\leftarrow\mathbf{e}_a)} = \begin{pmatrix} \sqrt{\frac{\eta}{\xi+\eta}}\cos\phi & \sqrt{\frac{\eta}{\xi+\eta}}\sin\phi & \sqrt{\frac{\xi}{\xi+\eta}} \\ \sqrt{\frac{\xi}{\xi+\eta}}\cos\phi & \sqrt{\frac{\xi}{\xi+\eta}}\sin\phi & -\sqrt{\frac{\eta}{\xi+\eta}} \\ -\sin\phi & \cos\phi & 0 \end{pmatrix} .$$

Finally, the matrix $R_{(\mathbf{i}_i\leftarrow\mathbf{e}_a)}$ (the inverse of $R_{(\mathbf{e}_a\leftarrow\mathbf{i}_i)}$) can be quickly obtained from $R_{(\mathbf{e}_i\leftarrow\mathbf{e}_a)}$:

$$\mathbf{e}_a V^a_{(\mathbf{e}_a)} = \mathbf{e}_i [R_{(\mathbf{e}_i\leftarrow\mathbf{e}_a)}]^i{}_a V^a_{(\mathbf{e}_a)} = (\mathbf{i}_i h_i^{-1}) [R_{(\mathbf{e}_i\leftarrow\mathbf{e}_a)}]^i{}_a V^a_{(\mathbf{e}_a)}\,,$$

so $R_{(\mathbf{i}_i\leftarrow\mathbf{e}_a)}$ is obtained by dividing the i-th row of $R_{(\mathbf{e}_i\leftarrow\mathbf{e}_a)}$ by h_i:

$$R_{(\mathbf{i}_i\leftarrow\mathbf{e}_a)} = \begin{pmatrix} \frac{2\sqrt{\xi\eta}}{\xi+\eta}\cos\phi & \frac{2\sqrt{\xi\eta}}{\xi+\eta}\sin\phi & \frac{2\xi}{\xi+\eta} \\ \frac{2\sqrt{\xi\eta}}{\xi+\eta}\cos\phi & \frac{2\sqrt{\xi\eta}}{\xi+\eta}\sin\phi & -\frac{2\eta}{\xi+\eta} \\ -\frac{1}{\sqrt{\xi\eta}}\sin\phi & \frac{1}{\sqrt{\xi\eta}}\cos\phi & 0 \end{pmatrix} .$$

In the parabolic coordinates we have two sets of basis one-forms: $d\xi$, $d\eta$, $d\phi$ dual to the basis vectors $\mathbf{i}_\xi$, $\mathbf{i}_\eta$, $\mathbf{i}_\phi$, and one-forms $\hat{f}^\xi$, $\hat{f}^\eta$, $\hat{f}^\phi$ dual to the basis vectors $\mathbf{e}_\xi$, $\mathbf{e}_\eta$, $\mathbf{e}_\phi$. The canonical volume form is

$$\hat{f}^\xi \wedge \hat{f}^\eta \wedge \hat{f}^\phi = \frac{1}{4}(\xi+\eta)\, d\xi \wedge d\eta \wedge d\phi\,.$$

Of course, the factor $\frac{1}{4}(\xi+\eta)$ is the traditional Jacobian of the change of variables from the Cartesian ones to the parabolic ones.

The "physical" (barred) gradient components in this coordinate system are

$$\left(\frac{1}{h_\xi}\frac{\partial}{\partial\xi}, \frac{1}{h_\eta}\frac{\partial}{\partial\eta}, \frac{1}{h_\phi}\frac{\partial}{\partial\phi}\right) = \left(\sqrt{\frac{4\xi}{\xi+\eta}}\frac{\partial}{\partial\xi}, \sqrt{\frac{4\eta}{\xi+\eta}}\frac{\partial}{\partial\eta}, \frac{1}{\sqrt{\xi\eta}}\frac{\partial}{\partial\phi}\right).$$

In the traditional approach to get this result one would have to write

$$\mathrm{grad} = \mathbf{e}_x \frac{\partial}{\partial x} + \mathbf{e}_y \frac{\partial}{\partial y} + \mathbf{e}_z \frac{\partial}{\partial z} = \mathbf{e}_x \left(\frac{\partial\xi}{\partial x}\frac{\partial}{\partial\xi} + \frac{\partial\eta}{\partial x}\frac{\partial}{\partial\eta} + \frac{\partial\phi}{\partial x}\frac{\partial}{\partial\phi}\right) + \dots$$

expressing next the derivatives $\partial\xi(x,y,z)/\partial x$ computed using the relations (inverse to those defining the parabolic coordinates)

$$\begin{aligned}
\xi &= \ \ z + \sqrt{x^2+y^2+z^2}\,,\\
\eta &= -z + \sqrt{x^2+y^2+z^2}\,,\\
\phi &= \ \ \mathrm{arctg}(y/x)\,,
\end{aligned}$$

back through the variables[2] ξ, η and ϕ and finally writing the basis vectors $\mathbf{e}_x$, $\mathbf{e}_y$, $\mathbf{e}_z$ as linear combinations of $\mathbf{e}_\xi$, $\mathbf{e}_\eta$, $\mathbf{e}_\phi$ using the matrix $R_{\mathbf{e}_i\leftarrow\mathbf{e}_a}$ and finally grouping together terms multiplying each of these vectors. All this requires a lot of work!

It is instructive to express the one-forms dx^a (i.e. dx, dy and dz) through the canonical forms $\hat{f}^i$ (that is $\hat{f}^\xi$, $\hat{f}^\eta$, $\hat{f}^\phi$). To this end we write

$$\begin{aligned}
{}^{(1)}\omega(\mathbf{V}) &= \omega^{(\hat{f})}_i \hat{f}^i(\mathbf{e}_j) V^j_{\mathbf{e}_i} = \omega^{(\hat{f})}_i \hat{f}^i(\mathbf{e}_a)[R_{\mathbf{e}_a\leftarrow\mathbf{e}_i}]^a{}_j V^j_{\mathbf{e}_i}\\
&= \omega^{(\hat{f})}_i [P^{\hat{f}\to dx^a}]^i{}_b\, dx^b(\mathbf{e}_a)[R_{\mathbf{e}_a\leftarrow\mathbf{e}_i}]^a{}_j V^j_{\mathbf{e}_i}\,.
\end{aligned}$$

---

[2]A trick allowing to simplify this work is to realize that the required derivatives $\partial\xi/\partial x$ etc. form the Jacobian matrix of the transformation of variables $(\xi,\eta,\phi)\to(x,y,z)$ which is inverse to the Jacobian matrix of the transformation $(x,y,z)\to(\xi,\eta,\phi)$:

$$\begin{pmatrix} \partial\xi/\partial x & \partial\xi/\partial y & \partial\xi/\partial z \\ \partial\eta/\partial x & \partial\eta/\partial y & \partial\eta/\partial z \\ \partial\phi/\partial x & \partial\phi/\partial y & \partial\phi/\partial z \end{pmatrix} = \begin{pmatrix} \partial x/\partial\xi & \partial x/\partial\eta & \partial x/\partial\phi \\ \partial y/\partial\xi & \partial y/\partial\eta & \partial y/\partial\phi \\ \partial z/\partial\xi & \partial z/\partial\eta & \partial z/\partial\phi \end{pmatrix}^{-1}.$$

In this way one gets the derivatives $\partial\xi/\partial x$ expressed directly in terms of ξ, η, ϕ. But this still requires inverting a complicated 3×3 matrix...

Since ${}^{(1)}\omega(\mathbf{V})$ can also be written as $\omega_i^{(\hat f)}\,\hat f^i$, it follows that

$$P^{\hat f \to dx^a} = [R_{\mathbf{e}_a \leftarrow \mathbf{e}_i}]^{-1} = R_{\mathbf{e}_i \leftarrow \mathbf{e}_a}\,.$$

Therefore

$$dx^a = [P^{dx^a \to \hat f}]^a{}_i \hat f^i = [R_{\mathbf{e}_a \leftarrow \mathbf{e}_i}]^a{}_i \hat f^i\,.$$

For the parabolic coordinated one gets in this way

$$dx = \hat f^\xi \sqrt{\frac{\eta}{\xi+\eta}}\,\cos\phi + \hat f^\eta \sqrt{\frac{\xi}{\xi+\eta}}\,\cos\phi - \hat f^\phi \sin\phi\,,$$

etc. This is of course a complicated way of obtaining the result which can be found by treating $x = x(\xi, \eta, \phi)$ as a function of the parabolic coordinates and taking its exterior derivative:

$$\begin{aligned} dx = dx(\xi,\eta,\phi) &\equiv d\Big(\sqrt{\xi\eta}\,\cos\phi\Big) \\ &= \frac{1}{2}\sqrt{\frac{\eta}{\xi}}\,\cos\phi\, d\xi + \frac{1}{2}\sqrt{\frac{\xi}{\eta}}\,\sin\phi\, d\eta - \sqrt{\xi\eta}\,\sin\phi\, d\phi\,, \end{aligned}$$

and then inserting here $d\xi = h_\xi^{-1}\hat f^\xi$, etc.

We now find the divergence in the parabolic coordinates. We assume a vector field $\mathbf{V}$ is given by its components $V^i_{(\mathbf{e}_i)}$, that is $\bar V^\xi$, $\bar V^\eta$, $\bar V^\pi$ are known. We have found that the divergence of $\mathbf{V}$ is the three-form

$$\begin{aligned} d(*\hat V) = \frac{\partial}{\partial \xi^k}\left(\sqrt{g}\,V^k\right) d\xi^1\wedge d\xi^2 \wedge d\xi^3 &= \frac{1}{h_\xi h_\eta h_\phi}\frac{\partial}{\partial \xi^k}\left(h_\xi h_\eta h_\phi\,\frac{\bar V^k}{h_k}\right)\hat f^\xi \wedge \hat f^\eta \wedge \hat f^\phi \\ &\equiv (\mathrm{div}\mathbf{V})\,\hat f^\xi \wedge \hat f^\eta \wedge \hat f^\phi\,. \end{aligned}$$

The "physical" divergence (the coefficient of the canonical volume three-form) therefore reads

$$\begin{aligned} \mathrm{div}\mathbf{V} &= \frac{4}{\xi+\eta}\left[\frac{\partial}{\partial\xi}\left(h_\eta h_\phi\,\bar V^\xi\right) + \frac{\partial}{\partial\eta}\left(h_\xi h_\phi\,\bar V^\eta\right) + \frac{\partial}{\partial\phi}\left(h_\xi h_\eta\,\bar V^\phi\right)\right] \\ &= \frac{2}{\xi+\eta}\left[\frac{\partial}{\partial\xi}\left(\sqrt{\xi(\xi+\eta)}\,\bar V^\xi\right) + \frac{\partial}{\partial\eta}\left(\sqrt{\eta(\xi+\eta)}\,\bar V^\eta\right) + \frac{\partial}{\partial\phi}\left(\frac{\xi+\eta}{2\sqrt{\xi\eta}}\,\bar V^\phi\right)\right]. \end{aligned}$$

Let us check this on a simple example. Let $\mathbf{V} = x\,\mathbf{e}_x$, so that every physicist knows that $\mathrm{div}\mathbf{V} = 1$. Using the matrix $R_{\mathbf{e}_i \leftarrow \mathbf{e}_a}$ we get

$$\mathbf{V} = x\,\mathbf{e}_x = \sqrt{\xi\eta}\,\cos\phi\left(\mathbf{e}_\xi\sqrt{\frac{\eta}{\xi+\eta}}\,\cos\phi + \mathbf{e}_\eta\sqrt{\frac{\xi}{\xi+\eta}}\,\cos\phi - \mathbf{e}_\phi\,\sin\phi\right).$$

We read off that

$$\bar{V}^\xi = \eta\sqrt{\frac{\xi}{\xi+\eta}}\,\cos^2\phi \qquad \bar{V}^\eta = \xi\sqrt{\frac{\eta}{\xi+\eta}}\,\cos^2\phi \qquad \bar{V}^\phi = -\sqrt{\xi\eta}\,\sin\phi\,\cos\phi\,.$$

Inserting these components in the formula written above we find

$$\begin{aligned}
\mathrm{div}\mathbf{V} &= \frac{2}{\xi+\eta}\left[\frac{\partial}{\partial\xi}\left(\xi\eta\cos^2\phi\right) + \frac{\partial}{\partial\eta}\left(\xi\eta\cos^2\phi\right) + \frac{\partial}{\partial\phi}\left(-\frac{1}{2}(\xi+\eta)\sin\phi\cos\phi\right)\right] \\
&= \frac{2}{\xi+\eta}\left[(\xi+\eta)\cos^2\phi - \frac{1}{2}(\xi+\eta)(\cos^2\phi - \sin^2\phi)\right] = 1\,.
\end{aligned}$$

The rotation of a vector field $\mathbf{V} = \mathbf{e}_i\bar{V}^i$ is the one-form

$$*(d\hat{V}) = \frac{h_k}{h_1h_2h_3}\,\epsilon_{ijk}\,\frac{\partial}{\partial\xi^i}\left(h_j\bar{V}^j\right)\hat{f}^k\,.$$

In the parabolic coordinates we find:

$$\begin{aligned}
\overline{(\boldsymbol{\nabla}\times\mathbf{V})}^\xi &= \frac{2}{\sqrt{\xi(\xi+\eta)}}\left[\frac{\partial}{\partial\eta}\left(\sqrt{\xi\eta}\,\bar{V}^\phi\right) - \frac{\partial}{\partial\phi}\left(\sqrt{\frac{\xi+\eta}{4\eta}}\,\bar{V}^\eta\right)\right] \\
\overline{(\boldsymbol{\nabla}\times\mathbf{V})}^\eta &= \frac{2}{\sqrt{\eta(\xi+\eta)}}\left[\frac{\partial}{\partial\phi}\left(\sqrt{\frac{\xi+\eta}{4\xi}}\bar{V}^\xi\right) - \frac{\partial}{\partial\xi}\left(\sqrt{\xi\eta}\,\bar{V}^\phi\right)\right] \\
\overline{(\boldsymbol{\nabla}\times\mathbf{V})}^\phi &= \frac{4\sqrt{\xi\eta}}{\xi+\eta}\left[\frac{\partial}{\partial\xi}\left(\sqrt{\frac{\xi+\eta}{4\eta}}\,\bar{V}^\eta\right) - \frac{\partial}{\partial\eta}\left(\sqrt{\frac{\xi+\eta}{4\xi}}\,\bar{V}^\xi\right)\right].
\end{aligned}$$

Let us check this formula computing curl of the vector field

$$\mathbf{V} = -y\,\mathbf{e}_x = -\sqrt{\xi\eta}\,\sin\phi\left(\mathbf{e}_\xi\,\sqrt{\frac{\eta}{\xi+\eta}}\,\cos\phi + \mathbf{e}_\eta\,\sqrt{\frac{\xi}{\xi+\eta}}\,\cos\phi - \mathbf{e}_\phi\,\sin\phi\right).$$

From the formulae given above one finds

$$\begin{aligned}
\overline{(\boldsymbol{\nabla}\times\mathbf{V})}^\phi &\propto \frac{\partial}{\partial\xi}\left(-\sqrt{\frac{\xi+\eta}{4\eta}}\,\xi\sqrt{\frac{\eta}{\xi+\eta}}\,\sin\phi\cos\phi\right) \\
&\quad -\frac{\partial}{\partial\eta}\left(-\sqrt{\frac{\xi+\eta}{4\xi}}\,\eta\sqrt{\frac{\xi}{\xi+\eta}}\,\sin\phi\cos\phi\right) \\
&= \frac{\partial}{\partial\xi}\left(-\frac{1}{2}\,\xi\sin\phi\cos\phi\right) - \frac{\partial}{\partial\eta}\left(-\frac{1}{2}\,\eta\sin\phi\cos\phi\right) = 0\,,
\end{aligned}$$

$$
\begin{aligned}
\overline{(\boldsymbol{\nabla}\times\mathbf{V})}^{\xi} &= \frac{2}{\sqrt{\xi(\xi+\eta)}}\left[\frac{\partial}{\partial\eta}\left(\sqrt{\xi\eta}\,\sqrt{\xi\eta}\sin^2\phi\right)\right. \\
&\qquad\left. -\frac{\partial}{\partial\phi}\left(-\sqrt{\frac{\xi+\eta}{4\eta}}\,\xi\sqrt{\frac{\eta}{\xi+\eta}}\,\sin\phi\cos\phi\right)\right] \\
&= \frac{2}{\sqrt{\xi(\xi+\eta)}}\left[\xi\,\sin^2\phi+\frac{1}{2}\,\xi\,(\cos^2\phi-\sin^2\phi)\right] = \sqrt{\frac{\xi}{\xi+\eta}}\,,
\end{aligned}
$$

and finally

$$
\begin{aligned}
\overline{(\boldsymbol{\nabla}\times\mathbf{V})}^{\eta} &= \frac{2}{\sqrt{\xi(\xi+\eta)}}\left[\frac{\partial}{\partial\phi}\left(-\sqrt{\frac{\xi+\eta}{4\xi}}\,\eta\sqrt{\frac{\xi}{\xi+\eta}}\,\sin\phi\cos\phi\right)\right. \\
&\qquad\left. -\frac{\partial}{\partial\xi}\left(\sqrt{\xi\eta}\,\sqrt{\xi\eta}\,\sin^2\phi\right)\right] \\
&= \frac{2}{\sqrt{\xi(\xi+\eta)}}\left[-\frac{1}{2}\,\eta\,(\cos^2\phi-\sin^2\phi)-\eta\sin^2\phi\right] = -\sqrt{\frac{\eta}{\xi+\eta}}\,.
\end{aligned}
$$

This is of course what one should get, because in the Cartesian coordinates $\boldsymbol{\nabla}\times\mathbf{V} = \mathbf{e}_z$, that is

$$
\boldsymbol{\nabla}\times\mathbf{V} = \mathbf{e}_z = \mathbf{e}_\xi\sqrt{\frac{\xi}{\xi+\eta}} - \mathbf{e}_\eta\sqrt{\frac{\eta}{\xi+\eta}}\,.
$$

Of course, strictly speaking curl is a one-form, but the barred components of the vector are simply equal to the barred components (that is components in the basis $\hat{f}^i$ of one-forms) of the one-form $\hat{W}$ associated with the given vector $\mathbf{W}$ (and for this reason an average physicist perceive curl as a vector).

Finally the Laplacian. It is a three-form. For a physicst it is the coefficient

$$
\boldsymbol{\nabla}^2 S = \frac{1}{h_1h_2h_3}\,\frac{\partial}{\partial\xi^j}\left(\frac{h_1h_2h_3}{h_j^2}\,\frac{\partial S}{\partial\xi^j}\right)
$$

of the canonical volume three-form. In the parabolic coordinates this is

$$
\begin{aligned}
\boldsymbol{\nabla}^2 S &= \frac{4}{\xi+\eta}\left[\frac{\partial}{\partial\xi}\left(\frac{\xi+\eta}{4}\,\frac{4\xi}{\xi+\eta}\,\frac{\partial S}{\partial\xi}\right)+\frac{\partial}{\partial\eta}\left(\frac{\xi+\eta}{4}\,\frac{4\eta}{\xi+\eta}\,\frac{\partial S}{\partial\eta}\right)\right. \\
&\qquad\left. +\frac{\partial}{\partial\phi}\left(\frac{\xi+\eta}{4}\,\frac{1}{\xi\eta}\,\frac{\partial S}{\partial\phi}\right)\right] \\
&= \frac{4}{\xi+\eta}\left[\frac{\partial}{\partial\xi}\left(\xi\,\frac{\partial S}{\partial\xi}\right)+\frac{\partial}{\partial\eta}\left(\eta\,\frac{\partial S}{\partial\eta}\right)+\frac{\xi+\eta}{4\xi\eta}\,\frac{\partial^2 S}{\partial\phi^2}\right].
\end{aligned}
$$

This can be easily tested on the function $S = x^2+y^2+z^2 = \frac{1}{4}(\xi+\eta)^2$. Of course $\boldsymbol{\nabla}^2 S = 6$ both in the Cartesian and in the parabolic coordinates, as it should be.

Integration of p-forms over p-dimensional domains

A p-form ${}^{(p)}\hat{\omega} = \omega_{i_1 \dots i_p}(\xi)\, d\xi^{i_1} \wedge \dots \wedge d\xi^{i_p}$ can be integrated over a p-dimensional domain (a p-dimensional submanifold) Ω_p of the d-dimensional space (d-dimensional manifold). The integral

$$\int_{\Omega_p} {}^{(p)}\hat{\omega} = \int_{\Omega_p} \omega_{i_1 \dots i_p}(\xi)\, d\xi^{i_1} \wedge \dots \wedge d\xi^{i_p} ,$$

is defined as follows. We have first to parametrize the domain Ω_p with p parameters τ^1, ..., τ^p:

$$\begin{aligned} \xi^1 &= \xi^1(\tau^1, \dots, \tau^p) , \\ \xi^2 &= \xi^2(\tau^1, \dots, \tau^p) , \\ &\dots\dots\dots\dots\dots\dots \\ \xi^d &= \xi^d(\tau^1, \dots, \tau^p) . \end{aligned}$$

One then has p vector fields $\mathbf{t}_{(1)}$, ..., $\mathbf{t}_{(p)}$:

$$\mathbf{t}_{(i)} = \mathbf{i}_1 \frac{\partial \xi^1}{\partial \tau^i} + \dots + \mathbf{i}_d \frac{\partial \xi^d}{\partial \tau^i} , \qquad i = 1, \dots, p.$$

all of which are tangent to the submanifold Ω_p. It is easy to see that $\mathbf{t}_{(i)}$ is tangent to the curve traced in Ω_p by varying the parameter τ^i keeping all other τ's fixed. The ordering of the tangent vectors $\mathbf{t}_{(1)}, \dots, \mathbf{t}_{(p)}$ fixes the relative (with respect to the orientation of the "big" space M defined in turn by the ordering of the coordinates $\xi^1, \dots, \xi^d$) orientation of the submanifold Ω_p. By definition

$$\int_{\Omega_p} {}^{(p)}\hat{\omega} = \int d\tau^1 \dots \int d\tau^p\, \omega_{i_1 \dots i_p}(\xi(\tau))\, d\xi^{i_1} \wedge \dots \wedge d\xi^{i_p}(\mathbf{t}_{(1)}, \dots, \mathbf{t}_{(p)}) .$$

The domain of integration over the parameters τ follows of course from the parametrization of Ω_p (this is an ordinary iterated integral). Since (see the definition of the action of a general p-form on p vectors)

$$\begin{aligned} d\xi^{i_1} \wedge \dots \wedge d\xi^{i_p}(\mathbf{t}_{(1)}, \dots, \mathbf{t}_{(p)}) &= \sum_\pi \mathrm{sgn}(\pi)\, d\xi^{i_1}(\mathbf{t}_{\pi(1)}) \dots d\xi^{i_p}(\mathbf{t}_{\pi(p)}) \\ &= \sum_\pi \mathrm{sgn}(\pi) \frac{\partial \xi^{k_1}}{\partial \tau^{\pi(1)}} \dots \frac{\partial \xi^{k_p}}{\partial \tau^{\pi(p)}}\, d\xi^{i_1}(\mathbf{i}_{k_1}) \dots d\xi^{i_p}(\mathbf{i}_{k_p}) \\ &= \sum_\pi \mathrm{sgn}(\pi) \frac{\partial \xi^{i_1}}{\partial \tau^{\pi(1)}} \dots \frac{\partial \xi^{i_p}}{\partial \tau^{\pi(p)}} \equiv \frac{\partial(\xi^{i_1}, \dots, \xi^{i_p})}{\partial(\tau^1, \dots, \tau^p)} . \end{aligned}$$

$$d\xi^{i_1} \wedge \dots \wedge d\xi^{i_p}(\mathbf{t}_{(1)}, \dots, \mathbf{t}_{(p)}) = \sum_\pi \mathrm{sgn}(\pi)\, d\xi^{i_1}(\mathbf{t}_{\pi(1)}) \dots d\xi^{i_p}(\mathbf{t}_{\pi(p)})$$

$$= \sum_{\pi} \operatorname{sgn}(\pi) \frac{\partial \xi^{k_1}}{\partial \tau^{\pi(1)}} \dots \frac{\partial \xi^{k_p}}{\partial \tau^{\pi(p)}} d\xi^{i_1}(\mathbf{i}_{k_1}) \dots d\xi^{i_p}(\mathbf{i}_{k_p})$$
$$= \sum_{\pi} \operatorname{sgn}(\pi) \frac{\partial \xi^{i_1}}{\partial \tau^{\pi(1)}} \dots \frac{\partial \xi^{i_p}}{\partial \tau^{\pi(p)}} \equiv \frac{\partial(\xi^{i_1}, \dots, \xi^{i_p})}{\partial(\tau^1, \dots, \tau^p)} .$$

The final, practical formula for the integral reads

$$\int_{\Omega_p} {}^{(p)}\hat{\omega} = \int d\tau^1 \dots \int d\tau^p \, \omega_{i_1 \dots i_p}(\xi(\tau)) \frac{\partial(\xi^{i_1}, \dots, \xi^{i_p})}{\partial(\tau^1, \dots, \tau^p)} .$$

Stokes theorem

The fundamental Stokes theorem states that

$$\int_{\Omega_p} d({}^{(p-1)}\hat{\omega}) = \int_{\partial \Omega_p} {}^{(p-1)}\hat{\omega} ,$$

where $\partial \Omega_p$ is the $p-1$-dimensional boundary of the domain Ω_p.

Exterior derivative of a zero-form, i.e. of a function $S(\xi)$ is a one-form dS which can be integrated over a curve Γ_{AB} going from a point A to a point B. The Stokes theorem reduces then to the trivial statement that

$$\int_{\Gamma_{AB}} dS = \int_{\partial \Gamma_{AB}} S \equiv S(B) - S(A) ,$$

because the boundary of the curve Γ_{AB} consists of the points A and B.

What is the physical interpretation of an integral of a one form ${}^{(1)}\hat{\omega} = \omega_i d\xi^i$ over a curve Γ_{AB}? Let's see. To evaluate the integral we parametrize the curve with some parameter $\tau \in [\tau_A, \tau_B]$: $\xi^i = \xi^i(\tau)$, where $\xi^i(\tau_A) = \xi^i_A$ and $\xi^i(\tau_B) = \xi^i_B$. Since the curve is a one dimansional manifold, there is a single tangent vector $\mathbf{t}$ and

$$\int_{\Gamma_{AB}} {}^{(1)}\hat{\omega} = \int_{\tau_A}^{\tau_B} d\tau \, \omega_i(\xi(\tau)) \, d\xi^i(\mathbf{t})$$
$$= \int_{\tau_A}^{\tau_B} d\tau \, \omega_i(\xi(\tau)) \, d\xi^i \left(\mathbf{i}_k \frac{d\xi^k}{d\tau} \right) = \int_{\tau_A}^{\tau_B} d\tau \, \omega_k(\xi(\tau)) \frac{d\xi^k}{d\tau} .$$

To get the physical interpretation let's assume ξ^i are coordinates of a Lamé system and rewrite the integrand differently:

$$\int_{\Gamma_{AB}} {}^{(1)}\hat{\omega} = \int_{\tau_A}^{\tau_B} d\tau \, \omega_i(\xi(\tau)) \frac{1}{h_i} \hat{f}^i(\mathbf{t})$$
$$= \int_{\tau_A}^{\tau_B} d\tau \, \omega_i(\xi(\tau)) \frac{1}{h_i} \bar{t}^i = \int_{\tau_A}^{\tau_B} d\tau \, \bar{\omega}_i(\xi(\tau)) \, \bar{t}^i ,$$

where $\bar{t}^i$ are the components of the tangent vector in the basis of ortonormal vectors $\mathbf{e}_i$, which is dual to the basis $\hat{f}^i$ of one-forms. Since the contraction $\bar{\omega}_i\bar{t}^i$ can be treated as a scalar product of two vectors, it is clear that

$$\int_{\Gamma_{AB}} {}^{(1)}\hat{\omega} = \int_{\Gamma_{AB}} d\mathbf{l}\cdot\mathbf{V}\,,$$

where $\mathbf{V}$ is a vector field associated with the one-form ${}^{(1)}\hat{\omega}$ and $d\mathbf{l} = d\tau\mathbf{t}$. Thus, it is just the ordinary integral of the vector field associated $\mathbf{V}$ along the curve Γ_{AB}.

The alternative way to see this is as follows: we rewrite the integrand in the form

$$\begin{aligned}\omega_i d\xi^i\left(\mathbf{i}_k\,\frac{d\xi^k}{d\tau}\right) &= \bar{\omega}_i\,\hat{f}^i\left(\mathbf{e}_x\,\frac{\partial x}{\partial\xi^k}\frac{d\xi^k}{d\tau} + \mathbf{e}_y\,\frac{\partial y}{\partial\xi^k}\frac{d\xi^k}{d\tau} + \mathbf{e}_z\,\frac{\partial z}{\partial\xi^k}\frac{d\xi^k}{d\tau}\right)\\ &= \bar{\omega}_i\,\hat{f}^i\left(\mathbf{e}_x\,\frac{dx}{d\tau} + \mathbf{e}_y\,\frac{dy}{d\tau} + \mathbf{e}_z\,\frac{dz}{d\tau}\right).\end{aligned}$$

We have used here the definition of the vectors $\mathbf{i}_k$ and the ordinary chain differentiation rule. On the other hand, in the Lamé systems one can also write

$$\bar{\omega}_i\,\hat{f}^i = \bar{\omega}_x\hat{f}^x + \bar{\omega}_y\hat{f}^y + \bar{\omega}_z\hat{f}^z\,,$$

because both $(\bar{\omega}_x,\,\bar{\omega}_y,\,\bar{\omega}_z)$ and $(\hat{f}^x,\,\hat{f}^y,\,\hat{f}^z) \equiv (dx, dy, dz)$ are related to $(\bar{\omega}_1,\,\bar{\omega}_2,\,\bar{\omega}_3)$ and $(\hat{f}^1,\,\hat{f}^2,\,\hat{f}^3)$ (associated with the coordinates ξ^i) by the same orthogonal transformation. Introducing a vector field $\mathbf{V}$ with the Cartesian components $V^x = \bar{\omega}_x$, $V^y = \bar{\omega}_y$, $V^z = \bar{\omega}_z$ we get

$$\int_{\Gamma_{AB}} {}^{(1)}\hat{\omega} = \int_{\tau_A}^{\tau_B} d\tau\,\frac{d\mathbf{r}(\tau)}{d\tau}\cdot\mathbf{V}(\mathbf{r}(\tau)) = \int_{\Gamma_{AB}} d\mathbf{l}\cdot\mathbf{V}\,,$$

where $\mathbf{V}$ is a vector field associated with the one-form ${}^{(1)}\hat{\omega}$. The last equality is obvious from ordinary mechanics: $d\mathbf{l} \equiv d\tau\ (d\mathbf{r}(\tau)/d\tau)$ is just the vector of the displacement along the curve Γ_{AB} corresponding to the change of the parameter from τ to $\tau + d\tau$; the integral of the scalar product of $d\mathbf{l}$ and $\mathbf{V}(\tau)$ is just what one calls the integral of $\mathbf{V}$ along the curve Γ_{AB}.

Thus, to compute the integral of a vector field $\mathbf{V}$ along a curve Γ one takes this field decomposed into vectors $\mathbf{i}_k$ associated with some curvelinear coordinates ξ^i and integrates the form $\hat{V} = V_i\,d\xi^i \equiv g_{ij}V^j\,d\xi^i$.

And how to get a flux of a vector field $\mathbf{V}$ through a surface Σ? To get a hint let's look at the Stokes theorem and compare it with the ordinary Gauss theorem for a closed surface $\Sigma = \partial\Omega$ (Ω being a three-dimensional domain):

$$\int_\Omega \mathrm{div}\mathbf{V}\,d(\mathrm{Volume}) \equiv \int_\Omega d(*\hat{V}) = \text{ Stokes Th. } = \int_{\partial\Omega} *\hat{V}\,.$$

This shows that $*\hat{V}$ must be the right object to integrate over Σ, i.e.

$$\int_\Sigma *\hat{V}$$

should give the flux of the vector field $\mathbf{V}$ through the surface Σ. Indeed,

$$\begin{aligned}\int_\Sigma *\hat{V} &= \int_\Sigma \frac{1}{2}\sqrt{g}\,\epsilon_{ijk}\,V^k\,d\xi^i \wedge d\xi^j \\ &= \iint d\tau^1 d\tau^2\,\frac{1}{2}\,h_1h_2h_3\,\epsilon_{ijk}\,h_k^{-1}\bar{V}^k\,\frac{1}{h_ih_j}\,\hat{f}^i \wedge \hat{f}^j\left(\mathbf{t}_{(1)},\mathbf{t}_{(2)}\right).\end{aligned}$$

Due to the presence of the totally antisymmetric symbol ϵ_{ijk}, the three Lamé coefficients $h_k^{-1}h_i^{-1}h_j^{-1}$ must be simply $h_1^{-1}h_2^{-1}h_3^{-1}$ and they cancel out the factors $h_1h_2h_3$, so that

$$\int_\Sigma *\hat{V} = \iint d\tau^1 d\tau^2\,\bar{V}^k\,\frac{1}{2}\,\epsilon_{ijk}\left(\bar{t}^i_{(1)}\bar{t}^j_{(2)} - \bar{t}^i_{(2)}\bar{t}^j_{(1)}\right) = \iint d\tau^1 d\tau^2\,\bar{V}^k\,\epsilon_{ijk}\,\bar{t}^i_{(1)}\bar{t}^j_{(2)}.$$

The factor $\epsilon_{ijk}\,(d\tau^1\bar{t}^i_{(1)})(d\tau^2\bar{t}^j_{(2)})$ is nothing else than the vector perpendicular to the parallelogram spanned by the vectors $d\tau^1\mathbf{t}_{(1)}$ and $d\tau^2\mathbf{t}_{(2)}$ of the infinitesimal displacements corresponding to varying the two parameters from τ^1 and τ^2 to $\tau^1+d\tau^1$ and $\tau^2+d\tau^2$ respectively, and has the length equal to the area of this parallelogram. It follows that the expression under the integral is just what one physically interprets as the flux of $\mathbf{V}$ through the small element of area of the surface Σ. This completes the demonstration.

Finally let us clarify how the usual Stokes theorem, stating that

$$\int_\Sigma d\mathbf{s}\cdot \mathrm{rot}\mathbf{A} = \int_{\Gamma=\partial\Sigma} d\mathbf{l}\cdot\mathbf{A}$$

arises in this picture. Rotation of a vector field $\mathbf{A}$, as defined above, is the one-form $*d\hat{A}$, and is, hence, not suitable for integrationg over two-dimentional surfaces like Σ. Instead, this one-form (in Cartesian, or Lamé systems) should be treated as a one-form $\hat{V}$ associated with a vector fiels $\mathbf{V} \equiv \mathrm{rot}\mathbf{A}$; to compute its flux through a surface Σ it has to be coverted to a two-form by applying to the the Hodge star operation. Since $**=$id, one interates over Σ the two-form $d\hat{A}$ and the usual Stokes theorem follows then readily.

Example

As an example let us compute the flux of the electric field $\mathbf{E}$ produced by a uniformly charged ball (of radius R and total charge Q) through a flat disc also of radius R, tangent to the ball.

Outside the ball the electric field has the form as if it was produced by the point charge Q located in the center of the ball. We will work with the spherical coordinates $\xi^1 = r$, $\xi^2 = \theta$, $\xi^3 = \phi$ with the origin ($r = 0$) in the center of the ball. In these coordinates only the radial component of the electric field is nonzero: $E^r = \bar{E}^r = k_1 Q/r^2$ (because $h_r = 1$). According to the general considerations the flux is given by

$$\begin{aligned}\text{Flux} = \int_{\text{disc}} *\hat{E} &= \int_{\text{disc}} \frac{1}{2} \sqrt{g}\, \epsilon_{ijk}\, E^k \, d\xi^i \wedge d\xi^j \\ &= \int_{\text{disc}} r^2 \sin\theta \left(\frac{k_1 Q}{r^2} \right) d\theta \wedge d\phi\,.\end{aligned}$$

We have used $\sqrt{g} = h_r h_\theta h_\phi = r^2 \sin\theta$ and the specific form of the components of the electric field $\mathbf{E}$.

We parametrize the disc by the parameters $\alpha \in [0, \frac{\pi}{4}]$ and $\beta \in [0, 2\pi]$:

$$\begin{aligned} r &= R/\cos\alpha\,, \\ \theta &= \alpha\,, \\ \phi &= \beta\,, \end{aligned}$$

so that the tangent vectors read

$$\begin{aligned} \mathbf{t}_{(\alpha)} &= \mathbf{i}_r \frac{\partial r}{\partial \alpha} + \mathbf{i}_\theta \frac{\partial \theta}{\partial \alpha} + \mathbf{i}_\phi \frac{\partial \phi}{\partial \alpha} = \mathbf{i}_r \frac{R \sin\alpha}{\cos^2\alpha} + \mathbf{i}_\theta\,, \\ \mathbf{t}_{(\beta)} &= \mathbf{i}_r \frac{\partial r}{\partial \beta} + \mathbf{i}_\theta \frac{\partial \theta}{\partial \beta} + \mathbf{i}_\phi \frac{\partial \phi}{\partial \beta} = \mathbf{i}_\phi\,. \end{aligned}$$

Hence,

$$d\theta \wedge d\phi \left(\mathbf{t}_{(\alpha)}, \mathbf{t}_{(\beta)} \right) = d\theta \left(\mathbf{t}_{(\alpha)} \right) \, d\phi \left(\mathbf{t}_{(\beta)} \right) - d\theta \left(\mathbf{t}_{(\beta)} \right) \, d\phi \left(\mathbf{t}_{(\alpha)} \right) = 1 - 0 = 1\,,$$

and

$$\text{Flux} = \int_0^{2\pi} d\beta \int_0^{\pi/4} d\alpha\, k_1 Q\, \sin\alpha = k_1 Q\, \pi (2 - \sqrt{2})\,.$$

Lengths, areas, volumes, etc.

All these quantities are encoded in the metric tensor g_{ij}. Let us consider first a hypersurface (a curve, a surface, etc.) embedded in the ordinary $\mathbb{R}^n$ space with canonical Cartesian coordinates x^a and a system of orthogonal vectors $\mathbf{e}_a$ (forming in fact a basis of a tangent space at each point of $\mathbb{R}^n$: since the space is "flat" - whatever this imprecise term may mean here - the corresponding basis vectors attached at different points of $\mathbb{R}^n$ can be simply identified). In this case we (arbitrarily) ascribe the unit length to these vectors and this "physical" unit is used to measure everything. The measure of the intrinsic "volume" $A_{(k)}$ (length, area, volume etc.) of a k-dimensional hypersurface $\Sigma_{(k)} \subset \mathbb{R}^n$ is then universally given by the formula

$$A_{(k)} = \int_{\Delta_{(k)}} d\tau^1 \dots d\tau^k \sqrt{g^{(k)}}\,,$$

in which $g^{(k)}$ is the determinant of the metric tensor (the subscript k indicates that it is a $k \times k$ matrix) induced (for a more precise formula, see below) on $\Sigma_{(k)}$ by the canonical $n \times n$ metric tensor $g^{(n)}_{ab} = \delta_{ab} = \mathbf{e}_a \cdot \mathbf{e}_b$ of $\mathbb{R}^n$. $\Delta_{(k)}$ is the domain of the parameters τ^1, ...,τ^n covering $\Sigma_{(k)}$.

Length. A curve is specified by giving the functions $x^a = x^a(\tau)$, where τ is a real parameter. The vector $\mathbf{t}$ tangent to the curve at each its point is then given by

$$\mathbf{t} = \mathbf{e}_a \frac{dx^a}{d\tau}\,,$$

and the metric tensor $g^{(1)}$ on the curve induced from $\mathbb{R}^n$ is given by

$$g^{(1)} = \mathbf{t}\cdot\mathbf{t} = \frac{dx^a}{d\tau}\frac{dx^a}{d\tau}\,.$$

Hence, the length L of a segment of the curve delimited by some values τ_1 and τ_2 of the parameter τ is given by

$$L = \int_{\tau_1}^{\tau_2} d\tau \sqrt{g^{(1)}} = \int_{\tau_1}^{\tau_2} d\tau \sqrt{\dot{x}^a\dot{x}^a}\,.$$

where $\dot{x}^a \equiv dx^a/d\tau$. This formula is physically obvious: $|\mathbf{t}\, d\tau| \equiv d\tau\sqrt{g^{(1)}}$ is the length of the infinitesimal vector of a displacement from a given point on a curve in the direction tangent to it, when τ changes from τ to $\tau + d\tau$.

Area. A two-dimensional surface $\Sigma_{(2)}$ is specified by giving the functions $x^a = x^a(\tau^1, \tau^2)$ of two real parameters τ^1 and τ^2. At each point of $\Sigma_{(2)}$ there are then two tangent vectors

$$\mathbf{t}_{(1)} = \mathbf{e}_a \frac{dx^a}{d\tau^1}\,, \qquad \mathbf{t}_{(2)} = \mathbf{e}_a \frac{dx^a}{d\tau^2}\,,$$

in terms of which the metric tensor $g^{(2)}_{ij}$ is given by

$$g^{(2)}_{ij} = \mathbf{t}_{(i)} \cdot \mathbf{t}_{(j)} = \frac{\partial x^a}{\partial \tau^i} \frac{\partial x^b}{\partial \tau^j} \mathbf{e}_a \cdot \mathbf{e}_b = \frac{\partial x^a}{\partial \tau^i} \frac{\partial x^a}{\partial \tau^j} \,.$$

To justify the formula

$$A_{(2)} = \int_{\Delta_{(2)}} d\tau^1 d\tau^2 \sqrt{g^{(2)}} \,,$$

in which $\Delta_{(2)}$ is the appropriate domain of τ^1 and τ^2, we notice that

$$g^{(2)} \equiv \det(g^{(2)}_{ij}) = \mathbf{t}^2_{(1)} \, \mathbf{t}^2_{(2)} - (\mathbf{t}_{(1)} \cdot \mathbf{t}_{(2)})^2 = (\mathbf{t}_{(1)} \times \mathbf{t}_{(2)})^2 \,.$$

Since $|\mathbf{t}_{(1)} d\tau^1 \times \mathbf{t}_{(2)} d\tau^2|$, as has been already discussed, is the area of an infinitesimal parallelogram spanned by two infinitesimal displacements $\mathbf{t}_{(1)} d\tau^1$ and $\mathbf{t}_{(2)} d\tau^2$ from a point on $\Sigma_{(2)}$ in the directions tangent to $\Sigma_{(2)}$, the formula obviously gives what one usually takes for area.

For example if a surface $\Sigma_{(2)}$ in $\mathbb{R}^3$ is specified by the function $z = f(x, y)$, then x and y themselves furnish a natural parametrization of $\Sigma_{(2)}$. The components of the two tangent vectors are

$$\mathbf{t}_{(1)} = \begin{pmatrix} 1 \\ 0 \\ f'_x \end{pmatrix}, \qquad \mathbf{t}_{(2)} = \begin{pmatrix} 0 \\ 1 \\ f'_y \end{pmatrix},$$

and

$$g^{(2)} = (1 + f'^2_x)(1 + f'^2_y) - (f'_x f'_y)^2 \,.$$

One recovers in this way the well-known formula

$$A = \int dx \, dy \sqrt{1 + f'^2_x + f'^2_y} \,.$$

Volume. In this case there are three vectors, $\mathbf{t}_{(1)}$, $\mathbf{t}_{(2)}$ and $\mathbf{t}_{(3)}$ and

$$\begin{aligned} g^{(3)} \equiv \det(g^{(3)}_{ij}) &= \begin{vmatrix} \mathbf{t}_{(1)} \cdot \mathbf{t}_{(1)} & \mathbf{t}_{(1)} \cdot \mathbf{t}_{(2)} & \mathbf{t}_{(1)} \cdot \mathbf{t}_{(3)} \\ \mathbf{t}_{(2)} \cdot \mathbf{t}_{(1)} & \mathbf{t}_{(2)} \cdot \mathbf{t}_{(2)} & \mathbf{t}_{(2)} \cdot \mathbf{t}_{(3)} \\ \mathbf{t}_{(3)} \cdot \mathbf{t}_{(1)} & \mathbf{t}_{(3)} \cdot \mathbf{t}_{(2)} & \mathbf{t}_{(3)} \cdot \mathbf{t}_{(3)} \end{vmatrix} \\ &= \mathbf{t}^2_{(1)} \, \mathbf{t}^2_{(2)} \, \mathbf{t}^2_{(3)} \; + 2(\mathbf{t}_{(1)} \cdot \mathbf{t}_{(2)})(\mathbf{t}_{(2)} \cdot \mathbf{t}_{(3)})(\mathbf{t}_{(3)} \cdot \mathbf{t}_{(1)}) \\ &\qquad - \mathbf{t}^2_{(1)}(\mathbf{t}_{(2)} \cdot \mathbf{t}_{(3)})^2 - \mathbf{t}^2_{(2)}(\mathbf{t}_{(1)} \cdot \mathbf{t}_{(3)})^2 - \mathbf{t}^2_{(3)}(\mathbf{t}_{(3)} \cdot \mathbf{t}_{(3)})^2 \,. \end{aligned}$$

By a direct calculation it is easy[3] to show that

$$g^{(3)} = |\mathbf{t}_{(1)} \cdot (\mathbf{t}_{(2)} \times \mathbf{t}_{(3)})|^2.$$

[3]To simplify the task, one can assume, that the Cartesian vectors $\mathbf{e}_a$ are rotated in such a way, that $\mathbf{t}_{(1)}$ has nonzero only the first component, $\mathbf{t}_{(2)}$ only the first and the second one and $\mathbf{t}_{(3)}$ only the first three. Then $|\mathbf{t}_{(1)} \cdot (\mathbf{t}_{(2)} \times \mathbf{t}_{(3)})|^2 = (t^x_{(1)} t^y_{(2)} t^z_{(3)})^2$.

Since $|\mathbf{t}_{(1)}d\tau^1 \cdot (\mathbf{t}_{(2)}\tau^2 \times \mathbf{t}_{(3)}d\tau^3)|$ is evidently the volume of a parallelogram spanned by three vectors of infinitesimal displacements, that formula is again justified.

In a more general setting, the metric tensor $g^{(n)}_{ij}(\xi)$ of an n-dimensional manifold M_n covered by a system of coordinates $\xi^1, \dots, \xi^n$ (not necessarily an orthogonal ones) always *defines* all metric relations on the manifold. $g^{(n)}_{ij}(\xi)$ fixes the lengths squared and scalar products of the system of vectors $\partial/\partial\xi^i \equiv \mathbf{i}_i$ tangent to the manifold $M_{(n)}$ without any reference to an underlying $\mathbb{R}^n$ space (and its canonical vectors $\mathbf{e}_a$). Any k-dimensional submanifold $\Sigma_{(k)} \subset M_{(n)}$ is then specified by giving the functions $\xi^i = \xi^i(\tau^1, \dots, \tau^k)$ and the metric tensor $g^{(k)}_{ij}(\tau)$ is then given by

$$g^{(k)}_{rs}(\tau) = g^{(n)}_{ij}(\xi(\tau))\, \frac{\partial\xi_i}{\partial\tau^r}\, \frac{\partial\xi_j}{\partial\tau^s}\,.$$

It is clear that $g^{(k)}_{rs}(\tau)$ is the matrix of scalar products of vectors

$$\mathbf{t}_{(r)} \equiv \mathbf{i}_i\, \frac{\partial\xi^i}{\partial\tau^r}\,,$$

tangent to the submanifold $\Sigma_{(k)}$. The scalar products of $\mathbf{t}_{(r)}$ are determined by the scalar products of the vectors $\mathbf{i}_i$ associated with the coordinates ξ^i of the manifold $M_{(n)}$.

Useful formulae

1. A p-form ${}^{(p)}\hat{\omega}$ is a p-linear *totally antisymmetric* mapping of p vectors into $\mathbb{R}$. p-forms form a vector space; for their basis one can take $\begin{pmatrix} d \\ p \end{pmatrix}$ (d is the space dimension) antisymmetrized tensor products of p basic one-forms $d\xi^i$:

$$d\xi^{i_1} \wedge d\xi^{i_2} \wedge \ldots \wedge d\xi^{i_p} \equiv \sum_{\pi} \mathrm{sgn}(\pi)\, d\xi^{i_{\pi(1)}} \otimes d\xi^{i_{\pi(2)}} \otimes \ldots \otimes d\xi^{i_{\pi(p)}} .$$

π is a permutation and $\mathrm{sgn}(\pi)$ its sign. Action of a general p-form

$${}^{(p)}\hat{\omega} \equiv \omega_{i_1 i_2 \ldots i_p}\, d\xi^{i_1} \wedge d\xi^{i_2} \wedge \ldots \wedge d\xi^{i_p} ,$$

on p vectors $\mathbf{V}_{(1)} = \mathbf{i}_{k_1} V^{k_1}_{(1)}$, …, $\mathbf{V}_{(p)} = \mathbf{i}_{k_p} V^{k_p}_{(p)}$ is given by:

$$\begin{aligned} {}^{(p)}\hat{\omega}(\mathbf{V}_{(1)}, \ldots \mathbf{V}_{(p)}) &= V^{k_1}_{(1)} \ldots V^{k_p}_{(p)}\, \omega_{i_1 \ldots i_p}\, d\xi^{i_1} \wedge \ldots \wedge d\xi^{i_p}(\mathbf{i}_{k_1}, \ldots \mathbf{i}_{k_p}) \\ &= \sum_{\pi} \mathrm{sgn}(\pi)\, V^{k_1}_{(1)} \ldots V^{k_p}_{(p)}\, \omega_{i_1 i_2 \ldots i_p}\, d\xi^{i_1}(\mathbf{i}_{k_{\pi(1)}}) d\xi^{i_2}(\mathbf{i}_{k_{\pi(2)}}) \ldots d\xi^{i_p}(\mathbf{i}_{k_{\pi(p)}}) \\ &= \sum_{\pi} \mathrm{sgn}(\pi)\, V^{k_1}_{(1)} \ldots V^{k_p}_{(p)}\, \omega_{k_{\pi(1)} \ldots k_{\pi(p)}} . \end{aligned}$$

2. Exterior derivative of a p-form ${}^{(p)}\hat{\omega} = \omega_{i_1 \ldots i_p}(\xi)\, d\xi^{i_1} \wedge d\xi^{i_2} \wedge \ldots \wedge d\xi^{i_p}$ is a $p+1$-form:

$$d({}^{(p)}\hat{\omega}) = \frac{\partial \omega_{i_1 \ldots i_p}(\xi)}{\partial \xi^k}\, d\xi^k \wedge d\xi^{i_i} \wedge d\xi^{i_2} \wedge \ldots \wedge d\xi^{i_p} .$$

3. Action of the Hodge star operation which maps p-forms into $(d-p)$-forms on the basic p-forms $d\xi^{i_1} \wedge \ldots \wedge d\xi^{i_p}$ is defined by

$$*(d\xi^{i_1} \wedge \ldots \wedge d\xi^{i_p}) = \frac{1}{(d-p)!} \sqrt{g}\, g^{i_1 j_1} \ldots g^{i_p j_p} \epsilon_{j_1 \ldots j_p l_1 \ldots l_{d-p}} d\xi^{l_1} \wedge \ldots \wedge d\xi^{l_{d-p}} .$$

In $d = 3$ dimensions one then has on zero-forms:

$$*(1) = \frac{1}{3!} \sqrt{g}\, \epsilon_{ijk}\, d\xi^i \wedge d\xi^j \wedge d\xi^k \equiv \sqrt{g}\, d\xi^1 \wedge d\xi^2 \wedge d\xi^3 ,$$

(we have used $\epsilon_{ijk}\epsilon_{ijk} = 3!$); on basic one-forms

$$*(d\xi^i) = \frac{1}{2} \sqrt{g}\, g^{ik}\, \epsilon_{klm}\, d\xi^l \wedge d\xi^m ;$$

on basic two-forms

$$*(d\xi^i \wedge d\xi^j) = \sqrt{g}\, g^{ik}\, g^{jl}\, \epsilon_{klm}\, d\xi^m ;$$

and on the three-form

$$*(d\xi^1 \wedge d\xi^2 \wedge d\xi^3) = \frac{1}{0!}\,\sqrt{g}\,g^{1i}g^{1j}g^{1k}\epsilon_{ijk} = \frac{1}{\sqrt{g}}\,.$$

Action on general one- and two-forms follows from linearity of the $*$ operation.

We can also check that $**=$Id:

$$\begin{aligned}
((d\xi^i)) &= *\left(\sqrt{g}\,g^{ik}\,\epsilon_{klm}\,d\xi^l \wedge d\xi^m\right)\\
&= \frac{1}{2}\,\sqrt{g}\,g^{ik}\,\epsilon_{klm}\;*\left(d\xi^l \wedge d\xi^m\right)\\
&= \frac{1}{2}\,\sqrt{g}\,g^{ik}\,\epsilon_{klm}\,\sqrt{g}\,g^{lj}\,g^{ms}\epsilon_{jsp}\,d\xi^p\\
&= \frac{1}{2}\,g\left(g^{ik}\,g^{lj}\,g^{ms}\,\epsilon_{klm}\right)\epsilon_{jsp}\,d\xi^p = d\xi^i
\end{aligned}$$

because the expression in the last bracket is just $\det(g^{kl})\,\epsilon^{ijs}$ (and $\det(g^{kl})$ is the inverse of $g=\det(g_{ij})$) and $\epsilon^{ijs}\epsilon_{jsp} = 2\delta^i{}_p$.

Similarly,

$$\begin{aligned}
((d\xi^i \wedge d\xi^j)) &= *\left(\sqrt{g}\,g^{ik}\,g^{jl}\,\epsilon_{klm}\,d\xi^m\right)\\
&= \sqrt{g}\,g^{ik}\,g^{jl}\,\epsilon_{klm}\,\frac{1}{2}\,\sqrt{g}\,g^{mp}\,\epsilon_{prs}\,d\xi^r \wedge d\xi^s\\
&= \frac{1}{2}\,g\,g^{ik}\,g^{jl}\,g^{mp}\,\epsilon_{klm}\,\epsilon_{prs}\,d\xi^r \wedge d\xi^s\\
&= \frac{1}{2}\,g\,g^{-1}\,\epsilon^{ijp}\,\epsilon_{prs}\,d\xi^r \wedge d\xi^s\\
&= \frac{1}{2}\left(\delta^i{}_r\delta^j{}_s - \delta^i{}_s\delta^j{}_r\right)d\xi^r \wedge d\xi^s\\
&= \frac{1}{2}\left(d\xi^i \wedge d\xi^j - d\xi^j \wedge d\xi^i\right) = d\xi^i \wedge d\xi^j\,.
\end{aligned}$$

4. Divergence referred to the canonical volume form $\sqrt{g}\,d\xi^1\wedge d\xi^2\wedge d\xi^3 \equiv \hat{f}^1\wedge\hat{f}^2\wedge\hat{f}^3$ is what in General Relativity is called the covariant divergence:

$$V^k{}_{;\,k}\,\sqrt{g}\,d\xi^1 \wedge d\xi^2 \wedge d\xi^3 \equiv \left(\partial_k V^k + \Gamma^k{}_{kj}\,V^j\right)\sqrt{g}\,d\xi^1 \wedge d\xi^2 \wedge d\xi^3\,.$$

We recall the definition of the Christoffel symbols (Krzysztofelki po naszemu) $\Gamma^i{}_{kj}$ in terms of the metric tensor:

$$\Gamma^i{}_{kj} = \frac{1}{2}\,g^{il}\left(\partial_k g_{lj} + \partial_j g_{lk} - \partial_l g_{kj}\right).$$

Hence,

$$
\begin{aligned}
\Gamma^k{}_{kj} &= \frac{1}{2}\, g^{kl} \left(\partial_k g_{lj} + \partial_j g_{lk} - \partial_l g_{kj}\right) = \frac{1}{2}\, g^{kl} \partial_j g_{lk} \\
&= \frac{1}{2}\, \mathrm{tr}\left(g^{-1}\partial_j g\right) = \frac{1}{2}\, \partial_j \ln(g) = \partial_j \ln(\sqrt{g}) \\
&= \frac{1}{\sqrt{g}}\, \frac{\partial}{\partial \xi^j}\, \sqrt{g}\,.
\end{aligned}
$$

This should be compared to $d(*\hat{V})$:

$$
\begin{aligned}
d(*\hat{V}) &= \frac{\partial}{\partial \xi^k} \left(\sqrt{g}\, V^k\right) d\xi^1 \wedge d\xi^2 \wedge d\xi^3 \\
&= \left(\sqrt{g}\, \partial_k V^k + V^k\, \partial_k \sqrt{g}\right) d\xi^1 \wedge d\xi^2 \wedge d\xi^3 \\
&= \left(\partial_k V^k + V^k\, \frac{1}{\sqrt{g}}\, \partial_k \sqrt{g}\right) \sqrt{g}\, d\xi^1 \wedge d\xi^2 \wedge d\xi^3 \\
&= \left(\partial_k V^k + \Gamma^i{}_{ik} V^k\right) \sqrt{g}\, d\xi^1 \wedge d\xi^2 \wedge d\xi^3\,.
\end{aligned}
$$

5. Yet one more useful formula is

$$
\frac{\partial}{\partial \xi^i}\, \ln \det M(\xi) = \mathrm{tr}\left(\frac{\partial M}{\partial \xi^i}\, M^{-1}\right),
$$

for a matrix M depending on ξ^i.
Demonstration: Let us compute a variation of the left hand side for a small variation of the matrix M:

$$
\begin{aligned}
\delta \ln \det M &= \ln \det(M + \delta M) - \ln \det M \\
&= \ln\left(\det(M + \delta M)\, \det{}^{-1} M\right) \\
&= \ln\left(\det(M + \delta M)\, \det(M^{-1})\right) \\
&= \ln\left(\det[(M + \delta M)\cdot M^{-1}]\right) \\
&= \ln\left(\det[I + (\delta M)\cdot M^{-1}]\right) \\
&= \mathrm{tr}[\ln[I + (\delta M)\cdot M^{-1}] = \mathrm{tr}[(\delta M)\cdot M^{-1}]\,.
\end{aligned}
$$

For the partial derivative with respect to ξ^i we take $\delta M = M(\ldots, \xi^i + h^i, \ldots) - M(\ldots, \xi^i, \ldots)$, divide by h^i and take the limit $h^i \to 0$. This proves the formula.