7 Scattering theory and relativistic theory of particle interactions

In this chapter we lay foundations for the first approach to formulating relativistic quantum field theories. This approach is close in spirit to the view expressed by Steven Weinberg - one of the founders of the Standard Theory of elementary particle interactions - that quantum field theory is merely a convenient machinery allowing to systematically construct amplitudes of particle reactions (comprising together the S-matrix) satisfying a certain set of physically motivated requirements such as Poincaré covariance, unitarity, cluster decomposition and analyticity (which were formulated in the historical development of high energy particle physics quite independently of the field theory principles). This view, while convenient as a starting point for our considerations, seems, however, too restricted. A more balanced one is probably that quantum field theory is just a quantum theory of some physical system. But what this system really is? In other words, what is the "ontology" underlying the quantum field theory? We will see that to some extent the ultimate formalism we will come to know dispenses us of such questions.[1] Nevertheless, in formulating quantum field theory one has to stick to some "ontology". The two obvious possible choices which lead to the quantum field theory as we know it are particles and fields (but one cannot exclude that the true underlying physical system may ultimately prove to be something else). Therefore, in this chapter and in the two following ones (Chapters 8 and 9) our underlying ontology will be particles. Quantum field theory as a quantum theory of a system of fields[2] will be developed in Chapter 11. We decided to present both formulations because this allows to better understand the foundations of quantum field theory and makes also clear similarities and differences between its versions used in particle physics and in condensed matter and solid state physics.

Adopting particles as the basic ontology is natural in condensed matter and solid state physics. Physical systems considered in these areas can certainly be treated as composed of well known particles[3] (although to be able to capture essential properties of some systems one nowadays frequently considers systems consisting of spins at fixed positions or allows particles to only hop from one site to another of a prescribed lattice - these are effective, purely theoretical constructions done at a higher - so to say - emergent in systems

[1] In this sense quantum field theory seems to favour the view, nowadays widespread, if not prevailing among theoreticians, that only the mathematical formalism matters and the ontology is largely irrelevant; this was most probably the attitude to physical theories of Dirac, but certainly not the one of Bohr!

[2] It is of course possible to formulate quantum field theory as a quantum theory of a mixed system consisting of fields and particles. (This was the approach adopted in Section 3.8 in which quantum theory of radiation was presented as a prototype quantum field theory). In fact, it seems that this may be the most natural point of view on the physical system underlying the quantum field theory: fermionic fields are Grassman algebra valued mappings which hardly, if at all, can be ascribed any physical reality - because of this fermions most probably should be considered true particles. Bosons, in contrast, are most naturally interpreted as quantum excitations of continuous fields.

[3] That is, the question what these particles are made of and why they have properties they have - masses, spins - is entirely irerlevant for problems which are of interest in these areas of physics.

of ordinary particle, level) properties of which - masses, spins, charges, (long distance) interactions, etc. - are well known. The Hilbert space $\mathcal{H}$ of a theory constructed adopting this ontology is naturally a multiparticle space of the same kind as the spaces built in Chapter 5, possessing the vector $|\text{void}\rangle$ from which other vectors can be obtained by the action of an arbitrary number (which can also be infinite) of creation operators corresponding to the kinds of particles which are "put in" into the system (as its fundamental constituents); the theory is constructed by adding to the free hamiltonian H_0 an interaction operator V_{int} acting in $\mathcal{H}$. The resulting quantum mechanics of a many particle system with the Hamiltonian similar to the ones considered in Chapter 5 *is* a model of (nonrelativistic) quantum field theory and properties of excitations of the resulting system, interpreted in terms of quasi-particles, are in general very different than properties of the "fundamental" particles "put" in the system - see the disscussion in Section 5.7.

In the approach developed in this and in the two following chapters, relativistic field theories will be formulated in the similar spirit, as quantum theories of interacting relativistic particles. Therefore the starting point will be a relativistic theory of *free* particles of a finite number of definite kinds, a, b, ..., constructed on the basis of the second quantization formalism of Section 6.5. The "arena" of the latter theory is the big multiparticle Hilbert space $\mathcal{H}$ which is a direct sum of multiple tensor products of single-particle Hilbert spaces $\mathcal{H}_a^{(1)}$, $\mathcal{H}_b^{(1)}$, ..., of several types of particles and of the one dimensional $\mathcal{H}^{(0)}$ as in Section 5.1. The big Hilbert space is therefore spanned by the vector $|\text{void}\rangle$ (which spans $\mathcal{H}^{(0)}$) and all possible multi-particle state-vectors

$$|(\mathbf{p}_1\sigma_1, \mathbf{p}_2\sigma_2, \ldots, \mathbf{p}_N\sigma_N)_0\rangle\,, \tag{7.1}$$

constructed as (appropriately symmetrized/antisymmetrized) tensor products of one-particle state-vectors.[4] In the continuum (i.e. in the infinite space volume) the vectors (7.1) are normalized so that (somewhat symbolically)

$$\begin{aligned}\langle(\mathbf{p}'_N\sigma'_N, \mathbf{p}'_{N-1}\sigma'_{N-1}, \ldots, \mathbf{p}'_1\sigma'_1)_0|(\mathbf{p}_1\sigma_1, \mathbf{p}_2\sigma_2, \ldots, \mathbf{p}_M\sigma_M)_0\rangle \\ = \delta_{NM}\sum_P \zeta^P \delta_\Gamma(\mathbf{p}'_1 - \mathbf{p}_{P(1)}) \ldots \delta_\Gamma(\mathbf{p}'_N - \mathbf{p}_{P(N)})\,,\end{aligned} \tag{7.2}$$

where $\zeta = \pm 1$ depending on whether particles are bosons or fermions and the (anti)symmetrization is understood to be done only within the groups of identical particles. The sum in (7.2) is over permutations within groups of labels corresponding to identical particles and $(-1)^P$ is the sign of the permutation of fermionic labels in a given permutation P. The

[4] If the vector (7.1) represents N_a particles of type a, N_b particles of type b, etc. ($N_a + N_b + \ldots = N$), different groups of labels, e.g. $(\mathbf{p}_i\sigma_i, \ldots, \mathbf{p}_{i+N_a}\sigma_{i+N_a})$ correspond to different types of particles but we do not introduce any additional index to distinguish which labels correspond to which type of particles. Only labels corresponding to identical particles are symmetrized or antisymmetrized as described in Chapter 5. Basis N-particle states constructed as appropriate linear combinations of the states (7.1) are also in use (see Section 6.4 for examples of such alternative bases of the $\mathcal{H}^{(2)}$ subspace). The subscript "0" is used to distinguish these vectors from the *in* and *out* vectors which will also be be labeled by listing the momenta and spin variables of the particles they represent.

symbol $\delta_\Gamma(\mathbf{p}'-\mathbf{p})$ which here is assumed to include also the Kronecker delta of the spin variables σ, depends on the normalization of the one-particle states; with the one usually adopted in nonrelativistic applications $\delta_\Gamma(\mathbf{p}'-\mathbf{p}) = (2\pi)^3\delta_{\sigma'\sigma}\delta^{(3)}(\mathbf{p}'-\mathbf{p})$; in relativistic theories more convenient is the normalization such that $\delta_\Gamma(\mathbf{p}'-\mathbf{p}) = (2\pi)^3 2E_{\mathbf{p}}\delta_{\sigma'\sigma}\delta^{(3)}(\mathbf{p}'-\mathbf{p})$, corresponding to $2E_{\mathbf{p}}$ particles in the unit volume (see Section 10.2). Because in general considerations we will be not interested in the detailed particle compositon of the multi-particle state-vectors, it is practical to introduce a compact notation, in which $|\alpha_0\rangle$ stands for state-vectors of the form (7.1) and the scalar product (7.2) is concisely written as

$$\langle\beta_0|\alpha_0\rangle = \delta(\beta-\alpha) \equiv \delta_{\beta\alpha}\,. \tag{7.3}$$

The completeness relation

$$\hat{1} = |\text{void}\rangle\langle\text{void}| + \sum_{N=1}^{\infty}\left(\sum_{N_1}\sum_{N_2}\ldots\right)\delta_{N,(N_1+N_2+\ldots)}\,\frac{1}{N_1!N_2!\ldots} \tag{7.4}$$
$$\sum_{\sigma_1,\ldots,\sigma_N}\int\prod_{i=1}^{N}d\Gamma_{\mathbf{p}_i}|(\mathbf{p}_1\sigma_1,\ldots,\mathbf{p}_N\sigma_N)_0\rangle\langle(\mathbf{p}_N\sigma_N,\ldots,\mathbf{p}_1\sigma_1)_0|\,,$$

in which the summation is over different numbers N_i of distinct types of particles, will be then compactly written as

$$\hat{1} = \int d\alpha\,|\alpha_0\rangle\langle\alpha_0|\,, \qquad \text{i.e.} \qquad |\Psi\rangle = \int d\alpha\,|\alpha_0\rangle\langle\alpha_0|\Psi\rangle\,. \tag{7.5}$$

where $|\Psi\rangle$ is any vector of the Hilbert space $\mathcal{H}$.

The vectors (7.1) are the eigenvenctors of the free Hamiltonian H_0 which is taken to be a sum $H_0 = H_0^a + H_0^b + \ldots$ of terms (6.113) with the energies $E_a(\mathbf{p}) = \sqrt{\mathbf{p}^2+m_a^2}$, $E_b(\mathbf{p}) = \sqrt{\mathbf{p}^2+m_b^2}$, ... - hence the subscript 0 in $|\alpha_0\rangle$. In the continuum (infinite volume V of the space) the only normalizable eigenvector of H_0 is the vector $|\text{void}\rangle$; the other eigenvectors $|\alpha_0\rangle$ of H_0 are non-normalizable.[5] Because the vector $|\text{void}\rangle$ is also the lowest energy H_0 eigenvector, it will be denoted $|\Omega_0\rangle$ (in relativistic theories the numbers of particles will not be conserved by the interaction, so there is no point to consider, as in nonrelativistic theories, separate H_0 eigenvectors $|\Omega_0^{(N_a,N_b\ldots)}\rangle$ in each $\mathcal{H}^{(N_a,N_b\ldots)}$ subspace). As explained in Section 5.1, even in the finite volume V, when allowed particle momenta form a discrete set (as a result of imposing periodic boundary conditions) and all state-vectors are normalizable, the Hilbert space is not separable - the set of vectors $|\alpha_0\rangle$ which span the big Hilbert space $\mathcal{H}$ is not countable.[6] The separable subspace spanned in the

[5]Non-normalizable state-vectors $|\alpha_0\rangle$, called generalized vectors, are in this respect similar to the plane waves $\psi_{\mathbf{p}} = e^{i\mathbf{p}\cdot\mathbf{x}}$ of ordinary nonrelativistic Quantum Mechanics of a single particle which are generalized (non-normalizable) eigenvectors of the $H_0 = \hat{\mathbf{P}}^2/2m$ and $\hat{\mathbf{P}}$ operators.

[6]This follows from the mathematical facts that for integer M and N both limits

$$\lim_{N\to\infty}\lim_{M\to\infty}M^N \qquad \text{and} \qquad \lim_{N\to\infty}2^N\,,$$

big Hilbert space by the vectors obtained by acting on $|\Omega_0\rangle = |\text{void}\rangle$ with an arbitrary but *finite* number of the creation operators forms the most natural (but not the only one which can be selected) *Fock space*.

The relativistic character of the theory of free particles constructed in Section 6.5, is ensured by the relativistic form of the energies $E_a(\mathbf{p})$, $E_b(\mathbf{p})$, ..., entering the free Hamiltonians H_0^a, H_0^b, ... of the form (6.113) and the possibility of constructing (as bilinear combinations of the creation and annihilation operators of the "fundamental" particles "put in" into the system) the remaining Poincaré group generators $\mathbf{J}_0$, $\mathbf{P}_0$, and $\mathbf{K}_0$, acting in $\mathcal{H}$ satisfying together with H_0 the rules (6.21). Of course the manifestly relativistic character of the dynamics is lost when the theory of free particles is considered not in the continuum; nevertheless, considering the the system of particles as enclosed in a finite volume V is necessary for example to consider thermodynamical properties of a gas of free relativistic (in the sense of their energy-momentum relation) particles.

The theory of interacting relativistic particles (which in this approach *is* the quantum field theory) is constructed by adding to the free Hamiltonian H_0 an interaction operator V_{int} acting in the big Hilbert space $\mathcal{H}$ spanned by the vector $|\text{void}\rangle = |\Omega_0\rangle$ and all vectors (7.1). Whether the resulting theory of the "fundamental" particles "put in" into the system and now allowed to interact with each other is still relativistic, that is, whether it is possible to construct in $\mathcal{H}$ new Poincaré group generators $\mathbf{J}$, $\mathbf{P}$, and $\mathbf{K}$ which together with $H = H_0 + V_{\text{int}}$ would satisfy the commutation rules (6.21) and additional physical requirements which will be formulated below depends, of course, on the form of V_{int}.

Assuming that the theory obtained by replacing H_0 by $H = H_0 + V_{\text{int}}$, is still a theory of particles (that is, assuming that the Hamiltonian $H = H_0 + V_{\text{int}}$ has still eigenvectors which can be interpreted as representing some kinds of particles - see below), the question what interactions V_{int} allow for constructing the Poincaré group generators requires formulating the theory in the continuum and is, for this reason, most conveniently investigated within the framework of the scattering theory:[7] S-matrices characterizing interactions of relativistic particles, that is the set of amplitudes allowing to compute probabilities

(relevant for counting bosonic and fermionic basis states) are equal to the power of the continuum. It is precisely the nonseparability of the Hilbert space $\mathcal{H}$ which is at the origin of the mentioned insensitivity of the ultimate formalism to the "ontology" underlying the theory.

[7]Although from the fundamental perspective it should be regarded as matter of pure calculational convenience that quantum field theory (or, more generally, any quantum theory) is formulated in the infinite space volume - there is a strong conviction that essential physics of considered systems must be the same, whether they are considered in the infinite space or as confined to a (large) finite volume and that in the latter case their measurable characteristics (if properly defined) do not depend on the size (if it is sufficiently large) of this volume that is, tend to well defined limits as $V \to \infty$ - the proper formulation of the scattering theory requires considering the theory in the infinite space. The point is that in the finite volume, when all eigenvectors of the Hamiltonian are normalizable, scattering processes cannot be sharply distinguished from the general time evolution of the system: all reactions would occur multiply as time goes and it would not make sense to appeal to the infinite time limits in order to define measurable quantities characterizing what in the real world is observed as scattering processes. Therefore, to meaningfully define state-vectors representing scattering reactions the infinite volume is crucial.

(rates) of particle reactions, should transform in a well defined way when the reference frame is changed - they should be Lorentz covariant. Therefore in this chapter we will first formulate the scattering theory which in its general form applies to the ordinary non-relativistic as well as to relativistic quantum mechanics. In fact, despite some important differences between its simplest version - the theory of scattering by an external potential based on the nonrelativistic quantum mechanics of a single particle and the scattering theory applied to the relativistic quantum mechanics of particles (that is quantum field theory) developed here,[8] keeping in mind the former is helpful in understanding also the latter one. Thus we will first derive general formulae, applicable in relativistic and in nonrelativistic theories, expressing the S-matrix elements in different ways useful in discussing its various aspects and will work out various approximate and iterative ways of computing them in addition to the basic one based on the Dyson expnasion and the Wick theorem. This one will be illustrated here on the example of the elastic scattering of nonrelativistic particles which will serve us to derive the result used in Section 5.5.

We will then investigate in detail the requirement of Lorentz covariance of the S-matrix and will formulate sufficient conditions under which the Hamiltonian $H = H_0 + V_{\rm int}$ leads to a Lorentz covariant S-matrix. It will be seen that if the S-matrix is Poincaré covariant (which is the case if $\mathbf{J}_0$, $\mathbf{P}_0$, and $\mathbf{K}_0$ commute with $V_{\rm int}$) it is also possible to construct the generators $\mathbf{J}$, $\mathbf{P}$, and $\mathbf{K}$ having the required properties. As will turn out, these sufficient conditions are not satisfied in some theories of physical interest and the ultimate Poincaré covariance of their S-matrices must be ensured by additional special features of these theories; nevertheless, the conditions formulated here constitute a useful reference point for further constructions. Finally, we will discuss in some details general properties of S-matrices in relativistic theories such as unitarity, partial wave expansion and its various possible symmetries.

Of course the fact that the relativistic character of the constructed theory is investigated by appealing to the infinite volume limit and the scattering theory does not mean that the S-matrix exhaust all the physically interesting information which can be obtained from it! Once it is formulated as a relativistic theory, various other properties of the underlying system, like for example its thermal properties which require keeping the volume finite, can be investigated by various other methods and means of general quantum mechanics (e.g. by the Rayleigh-Schrödinger stationary perturbative expansion).

7.1 Time evolution, the S-matrix and the S_0 operator

In developing the scattering theory within the quantum mechanics of interacting particles formulated along the lines sketched above we make the following important but physically motivated assumptions. Firstly, we assume that the Hamiltonian $H = H_0 + V_{\rm int}$ is still a

[8] There exist, of course, intermediate level theories based on nonrelativistic quantum mechanics of many particles which, similarly to the relativistic quantum field theories allow to consider multichannel scattering process.

Hamiltonian of a system of particles, by which we mean that it possesses particle-like generalized eigenvectors, which in the sense which will be made precise below have properties similar to the multi-particle generalized eigenvectors (7.1) of a free Hamiltonian $\tilde{H}_0$, not necessarily identical with H_0 we start with, that is of a Hamiltonian $\tilde{H}_0$ which is a sum of (a finite number of) terms $\tilde{H}_0^{\tilde{a}}$, $\tilde{H}_0^{\tilde{b}}, \ldots$ of the form (6.113) with some relativistic energies $E_{\tilde{a}}(\mathbf{p})$, $E_{\tilde{b}}(\mathbf{p}), \ldots$ and with the original operators $a_\sigma^\dagger(\mathbf{p})$, $a_\sigma(\mathbf{p})$ replaced by some other operators $\tilde{a}_\sigma^\dagger(\mathbf{p})$, $\tilde{a}_\sigma(\mathbf{p})$ (constructed out of the original ones by means of some sort of a - perhaps very complicated compared to the one used in Section 5.5 - Bogolyubov transformation) satisfying analogous commutation relations.[9] We assume that all operators $\tilde{a}_\sigma(\mathbf{p})$ annihilate some normalizable (in the continuum) vector $|\tilde{\Omega}_0\rangle$ which is the ground state-vector of $\tilde{H}_0$ and that the $\tilde{H}_0$ generalized eigenvectors $|\tilde{\alpha}_0\rangle$ obtained by acting on $|\tilde{\Omega}_0\rangle$ with the operators $\tilde{a}_\sigma^\dagger(\mathbf{p})$ also span the original Hilbert space. In a relativistic theory this in particular means that $H = H_0 + V_{\rm int}$ possesses, among others, generalized (in the infinite space) eigenvectors which with respect to the transformations generated by the operators H, $\mathbf{P}$, $\mathbf{J}$ and $\mathbf{K}$ satisfying the rules (6.21) transform in the same way as do the discussed in Chapter 6 one-particle states and that the one-particle eigenvectors of $\tilde{H}_0$ have precisely the same properties (with respect to transformations generated by $\tilde{H}_0$, $\tilde{\mathbf{P}}_0$, $\tilde{\mathbf{J}}_0$ and $\tilde{\mathbf{K}}_0$). Furthermore we will assume that, similarly to H_0 (and to $\tilde{H}_0$), the Hamiltonian $H = H_0 + V_{\rm int}$ has (in the infinite space volume) only a single (at least in the Fock space built on the vacuum vector $|\tilde{\Omega}_0\rangle$) normalizable ground-state eigenvector of H denoted $|\Omega\rangle$ and called the vacuum, that the particle-like non-normalizable eigenvectors of H, which will be introduced in Section 7.3, together with $|\Omega\rangle$ span the whole Hilbert space (or at least the Fock space built on the vacuum vector $|\tilde{\Omega}_0\rangle$) and, finally, that the spectra of the Hamiltonians $H = H_0 + V_{\rm int}$ and of $\tilde{H}_0$ are identical.[10] We therefore postulate that there is a *strict one-to-one correspondence* between all eigenvectors of $H = H_0 + V_{\rm int}$ and the eigenvectors of $\tilde{H}_0$ and that energies of the corresponding eigenvectors (with respect to the respective Hamiltonians, H and $\tilde{H}_0$) are equal. The physical motivation for these assumptions is that if $H = H_0 + V_{\rm int}$ is the Hamiltonian of a system of particles, its non-normalizable (in the continuum) eigenvectors should all represent (as is the case in ordinary nonrelativistic quantum mechanical scattering on a fixed potential which does not admit bound states) collision-type processes in which long before and long after the reaction particles look as (mutually) noninteracting. Therefore, it should be possible to associate with a given scattering process the Hilbert space (Heisenberg picture) state-vectors which, in a well defined way, correspond, as far as their transformation properties and energies are concerned, to eigenvectors of some $\tilde{H}_0$.

The assumptions formulated above could of course be checked if the theory could be

[9] It is therefore clear that at least formally, the Poincaré group generators $\tilde{\mathbf{P}}_0$, $\tilde{\mathbf{J}}_0$ and $\tilde{\mathbf{K}}_0$ satisfying together with $\tilde{H}_0$ the commutation relations (6.21) can also be built as operators bilinear in the creation and annihilation operators $\tilde{a}_\sigma^\dagger(\mathbf{p})$, $\tilde{a}_\sigma(\mathbf{p})$.

[10] This is not always true in the scattering theory based on (nonrelativistic) quantum mechanics of a single particle in which the potential $V_{\rm int}$ added to H_0 can lead to the existence of normalizable eigenvectors of H (i.e. bound states) but seems to be quite a natural assumption in a many-particle quantum theory formulated in the infinite space.

solved exactly. Unfortunately, in most cases one has to rely on some sort of approximations which usually hinge on the second assumption we are going to make. A method with the help of which the true spectrum of H could, at least in principle, be investigated will be outlined in Chapter 13.

In their abstract form the assumptions formulated above do not allow to go too far[11] within the approach to the quantum field theory developped in this and in the two following chapters. Therefore, in order to construct theories in which practical calculations (based on a systematic perturbative expansion) can be performed we will make a "technical", simplifying assumption that $V_{\rm int}$ is "small" in the sense that the spectrum of $H = H_0 + V_{\rm int}$ is the same as the spectrum of H_0, i.e. the full H eigenvectors have the properties of the eigenvectors $|\alpha_0\rangle$ of H_0. In other words, we will assume that $\tilde{H}_0 = H_0$ and that the strict one to-one correspondence holds between the the H_0 and H eigenvectors (including the equality of the corresponding eigenvalues) which form alternative bases of the same Fock space. In the considerations of this chapter, however, although we will use the notation H_0, and $|\alpha_0\rangle$, one can treat them as $\tilde{H}_0$ and $|\tilde{\alpha}_0\rangle$.

It is important to realize that these assumptions are neither a priori obvious, nor are they always fulfilled. It could happen that $H = H_0 + V_{\rm int}$ does not possess particle-like eigenstates at all (or not all of its eigenvectors can be interpreted as representing states of particles). This is indeed so in conformal field theory models or theories of "unparticles" discussed in the literature,[12] so that there are theories to which even the general, seemingly well motivated assuption does not apply. Furthermore, even all $H = H_0 + V_{\rm int}$ eigenvectors represent states of particles, they can be in one-to-one corresponcence with eigenvectors of a free-particle Hamiltonian $\tilde{H}_0$ which is very different from H_0 used to build H. The most prominent example of such a theory is Quantum Chromodynamics (QCD) - the theory of strong interactions in which the H_0 eigenvectors represent states of free spin $\frac{1}{2}$ coloured (i.e. transforming nontrivially under the action of the colour $SU(3)_c$ symmetry group) quarks, antiquarks and spin 1, massless coloured gluons, whereas the true H (and, therefore, also $\tilde{H}_0$) eigenvectors represent colourless, i.e. $SU(3)_c$ singlets, baryons, antibaryons and mesons. The approach exploiting the "technical" assumption was largely shaped by the historical developement of quantum electrodynamics of electrons, positrons and photons, which as a quantum field theory is very special in that the interaction between charged particles and photons is quite weak and, moreover, all particles of this theory are absolutely stable.[13] The unified theory of weak and electro-

[11] At least in fully relativistic theories - there are simplified nonrelativistic models, like e.g. the Lee model, in which the Hamiltonian $\tilde{H}_0$ can be explicitly constructed.

[12] Eigenvectors of free Hamiltonians H_0 of such theories represent massless particles.

[13] Positronium - a bound state of electron and positron - is unstable. However already electrodynamics of electrons and muons (which are stable in the absence of weak interactions) does not fully fit into the assumed scheme (although the interaction is still weak): the bound state of electron and antimuon (or of positron and muon) is stable and therefore the nonnormalizable eivenvectors of the full Hamiltonian of such electrodynamics should correspond, strictly speaking, to the eigenvectors of $\tilde{H}_0$ which is the free Hamiltonian of electrons positrons, muons, antimuons, and of $e^-\mu^+$ and $e^+\mu^-$ bound states treated as elementary particles.

magnetic interactions (of which quantum electrodynamics is only a part) is also weakly coupled but certainly violates the assumption that there is a strict one-to-one correspondence between the particle-like eigenvectors of H and of H_0: $W^{\pm}$ bosons of spin 1, muons, taons are "put in" into the theory as particles and have the corresponding eigenvectors of H_0 but not being absolutely stable they have no, strictly speaking, their counterparts among the particle-like eigenvectors of H.

Thus, the assumptions adopted in the approach to quantum field theory based on relativistic quantum mechanics of particles, which is developed in Chapters 7-9 can be satisfied only in a very special (rather narrow) class of theories and require in addition a judicious construction of the interaction operator $V_{\rm int}$. This will become clear in Section 9.7, where it will turn out that observance of these assumptions (by appropriately adjusting $V_{\rm int}$) is crucial for avoiding some type of ill defined contributions to the transition amplitudes (S-matrix elements) that would otherwise occur in perturbative calculations.

With the two assumptions clearly spelled out as above it becomes possible to formulate the scattering theory based on relativistic Quantum Mechanics of particles in the Fock space spanned by the H_0 eigenvectors. To the proper Hilbert space which we will consider initially belong all possible *normalizable* state-vectors $|\Psi\rangle$ that can be constructed out of the Fock space of generalized H_0 eigenvectors as

$$|\Psi\rangle = \int d\alpha \, |\alpha_0\rangle\langle\alpha_0|\Psi\rangle \equiv \int d\alpha \, |\alpha_0\rangle \, \psi(\alpha) \, , \tag{7.6}$$

with integrable profiles $\psi(\alpha)$: $\int d\alpha |\psi(\alpha)|^2 = 1$. One can then consider such states prepared at $t = 0$ and their time evolution generated either by H or H_0. Guided by the physical intuition and in line with the general framework adopted, we assume that the Schrödinger picture counterparts (we set $\hbar = c = 1$)

$$|\Psi(t)\rangle = e^{-iHt}|\Psi\rangle \equiv U(t,0)|\Psi\rangle \, , \tag{7.7}$$

(in the notation of Chapter 1) of normalizable Heisenberg picture state-vectors $|\Psi\rangle$ which represent reactions between particles converge as $t \to \mp\infty$ (in the sense of convergence in the Hilbert space of sequences of vectors) to some state-vectors[14]

$$|\Psi_{\rm as}^{\rm in/out}(t)\rangle = e^{-iH_0 t}|\Psi_{\rm as}^{\rm in/out}\rangle \equiv U_0(t,0)|\Psi_{\rm as}^{\rm in/out}\rangle \, , \tag{7.8}$$

because in experiments one prepares states representing particles which before the collision are well localized and separated in space and are therefore from the practical point of view non-interacting with each other; likewise, long after the collision particles are again well separated and again look as mutually non-interacting. Thus, any state $|\Psi\rangle$ representing a scattering process can be written (employing the notation of Chapter 1) as

$$|\Psi\rangle = \lim_{t\to\mp\infty} U^{\dagger}(t,0)U_0(t,0)|\Psi_{\rm as}^{\rm in/out}\rangle = U_I^{\dagger}(t,0)|\Psi_{\rm as}^{\rm in/out}\rangle \, . \tag{7.9}$$

[14]The states $|\Psi_{\rm as}^{\rm in/out}\rangle$ considered here should not be identified with the *in* and *out* states analogous to the ones introduced in Section 1.3; the states playing the roles of the *in* and *out* states in the present context will be defined in Section 7.3.

As any smooth, normalized superposition (7.6) of the H_0 eigenvectors $|\alpha_0\rangle$ should represent a possible initial or a possible final state of particles which will participar of have participated in some reaction, one assumes that on the whole proper Hilbert space the operators

$$\Omega(t) \equiv e^{iHt}e^{-iH_0t} = U_I^\dagger(t,0)\,, \tag{7.10}$$

do have the limits[15]

$$\lim_{t\to\mp\infty}\Omega(t) = \Omega(\mp\infty) \equiv \Omega_\pm\,, \tag{7.11}$$

on any normalizable smooth superposition of the $|\alpha_0\rangle$ vectors. $\Omega_\pm$ are called Møller operators. Since

$$\frac{d}{dt}\,\Omega(t) = \frac{d}{dt}\left(e^{iHt}e^{-iH_0t}\right) = i\,U^\dagger(t,0)V_{\rm int}U_0(t,0) \equiv i\,U_I^\dagger(t,0)V_{\rm int}^I(t)\,,$$

and since $\Omega(0) = \hat{1}$, the operator $\Omega(t)$ can alternatively be defined by the integral relation

$$\Omega(t) = \hat{1} + i\int_0^t dt'\,U^\dagger(t',0)V_{\rm int}U_0(t',0)\,. \tag{7.12}$$

Furthermore, as the operators $\Omega(t)$ are unitary for any fixed t, that is satisfy $\Omega^\dagger(t)\Omega(t) = \hat{1}$ (and also $\Omega(t)\,\Omega^\dagger(t) = \hat{1}$) the Møller operators $\Omega_\pm$ are at least isometric, which means that similarly to $\Omega(t)$ they are defined on the whole Hilbert space $\mathcal{H}$ and preserve the norm: $(\Omega_\pm\Psi|\Omega_\pm\Psi) = (\Psi|\Psi)$ and, therefore, also the scalar products of normalizable states:

$$(\Omega_\pm\Phi|\Omega_\pm\Psi) = (\Phi|\Psi)\,, \tag{7.13}$$

that is they satisfy the relations

$$\Omega_+^\dagger\Omega_+ = \Omega_-^\dagger\Omega_- = \hat{1}\,,$$

(but, in general, not necessarily the relations $\Omega_+\Omega_+^\dagger = \Omega_-\Omega_-^\dagger = \hat{1}$). In relativistic quantum mechanics of particles (i.e. in QFT) one assumes that[16] $\Omega_\pm\mathcal{H} = \mathcal{H}$ (and not $\Omega_\pm\mathcal{H} \subset \mathcal{H}$), that is that any $\mathcal{H}$ space state-vector can be represented as the image of the action of Ω_+ and Ω_- on some states $|\Psi_{\rm as}^{\rm in/out}\rangle$:[17]

$$|\Psi\rangle = \Omega_+|\Psi_{\rm as}^{\rm in}\rangle = \Omega_-|\Psi_{\rm as}^{\rm out}\rangle\,. \tag{7.14}$$

[15]Since $\Omega_\pm$ clearly correspond to the $\varepsilon \to 0$ limits of the operators $U_I^\varepsilon(0,\mp\infty)$ considered in Section 1.2, this amounts to assuming that these regularized interaction picture evolution operators do have finite $\varepsilon \to 0$ limits on all smooth, normalized superpositions of the H_0 eigenvectors. In particular, one assumes here that the operators $U_I^\varepsilon(0,\mp\infty)$ acting on the ground-state eigenvector $|\Omega_0\rangle$ give in the limit $\varepsilon \to 0$ directly the state-vectors $|\Omega_\pm\rangle$, which are normalized lowest energy H eigenvectors and can differ one from another only by a phase factor (recall that in the Gell-Mann - Low construction one obtains the same eigenvector of H, whether one considers the $t \to -\infty$ or $t \to \infty$ limit). As disscussed, and as will be seen, this can be true only if the interaction $V_{\rm int}$ is judiciously adjusted.

[16]This is not necessarily true in nonrelativistic Quantum Mechanics of a single particle. See Appendix E.

[17]Again, this is in line with the intuition that if there exists a stable bound state, it can be prepared in the far past and registered in the far future.

(This implies that the relations $\Omega_+\Omega_+^\dagger = \Omega_-\Omega_-^\dagger = \hat{1}$ also do hold.)

Already at this point one can introduce the S_0 operator. As usually in a quantum theory, one is interested in scalar products $S_{\Phi\Psi} \equiv \langle\Phi|\Psi\rangle$ of normalized states. Expressing $|\Psi\rangle$ as the Ω_+ image of the appropriate $|\Psi_{\rm as}^{\rm in}\rangle$ and $|\Phi\rangle$ as the Ω_- image of $|\Phi_{\rm as}^{\rm out}\rangle$ one gets

$$S_{\Phi\Psi} \equiv \langle\Phi|\Psi\rangle = \langle\Phi_{\rm as}^{\rm out}|S_0|\Psi_{\rm as}^{\rm in}\rangle\,, \tag{7.15}$$

where the S_0 operator is defined as the product (compare the formula (1.72))

$$S_0 \equiv \Omega_-^\dagger\Omega_+\,. \tag{7.16}$$

It maps the asymptotic "incoming" states onto the corresponding "outgoing" ones: $|\Psi_{\rm as}^{\rm out}\rangle = S_0|\Psi_{\rm as}^{\rm in}\rangle$. The scalar product $S_{\Phi\Psi}$ - which is equal to the appropriate matrix elements of S_0 - has the natural interpretation of the probability amplitude of finding the system in the state $|\Phi\rangle$, which, if evolved in time,[18] would become in the far future indistinguishable from an appropriately (i.e. with H_0) evolved state $|\Phi_{\rm as}^{\rm out}\rangle$ which has direct interpretation in terms of noninteracting (and spatially separated in the far future) particles, if it is prepared as the state $|\Psi\rangle$ which, if evolved in time, has in the far past a similar free-particle interpretation, being indistinguishable from the evolved state $|\Psi_{\rm as}^{\rm in}\rangle$. The scalar products $S_{\Phi\Psi}$ thus contain answers to a prevailing amount of experimentally accessible questions which usually are formulated in the form "what is the probability that the detectors will register a given free-particle state produced as a result of an interaction of particles which long before interaction were prepared (in the accelerator) as another free-particles state?"

It is also convenient to introduce an alternative notation (corresponding to a slightly different labeling of states) and to call $|\Psi_+\rangle$ and $|\Psi_-\rangle$ the two different states which are the images of the same state $|\Psi\rangle$ under Ω_+ and Ω_-, respectively. Thus, in this notation $|\Psi_\pm\rangle = \Omega_\pm|\Psi\rangle$ which means that

$$\lim_{t\to\mp\infty} U(t,0)|\Psi_\pm\rangle = \lim_{t\to\mp\infty} U_0(t,0)|\Psi\rangle\,. \tag{7.17}$$

Scalar products (7.15) can be now written as $\langle\Phi_-|\Psi_+\rangle = \langle\Phi|S_0|\Psi\rangle$ and since

$$|\Psi_+\rangle = \Omega_+\Omega_-^\dagger|\Psi_-\rangle\,, \qquad |\Psi_-\rangle = \Omega_-\Omega_+^\dagger|\Psi_+\rangle\,, \tag{7.18}$$

they can also be expressed as the matrix elements

$$S_{\Phi\Psi} \equiv \langle\Phi_-|\Psi_+\rangle = \langle\Phi_-|S|\Psi_-\rangle = \langle\Phi_+|S|\Psi_+\rangle\,, \tag{7.19}$$

of the S operator

$$S \equiv \Omega_+\Omega_-^\dagger\,. \tag{7.20}$$

[18]Notice that the states are always identified at $t = 0$; that is we implicitly work in the Heisenberg picture (see Section 1.1) which in a relativistic theory allows to keep its Poincaré covariance as manifest as it is possible.

The S operator (which has been used in Section 1.3 - c.f. the formula (1.73)) which is different from the S_0 one[19] will be of little use in the approach developed in Chapters 7-9 (it becomes of relevant only in the approach based on Green's functions, when the structure of the asymptotic states is reconstructed from the poles of these functions). However the notation $|\Psi_\pm\rangle$ will be useful.

Under the assumptions underlying the considerations of this section the operators $H = H_0 + V_{\rm int}$ and H_0 satisfy the important *intertwining relation*

$$H\,\Omega_\pm = \Omega_\pm H_0\,, \tag{7.21}$$

which in particular implies[20] that $\Omega_\pm^\dagger H \Omega_\pm = H_0$. Indeed,

$$e^{iHt}\,\Omega_\pm = e^{iHt} \lim_{\tau\to\mp\infty} \left(e^{iH\tau}e^{-iH_0\tau}\right) = \lim_{\tau\to\mp\infty} \left(e^{iH(\tau+t)}e^{-iH_0(\tau+t)}\right) e^{iH_0t} = \Omega_\pm\, e^{iH_0t}\,.$$

Differentiating this equality with respect to t at $t = 0$ yields the relation (7.21). This result should be compared with the more rigorously derived formula (1.29); this again shows that (7.21) can hold only if the interaction $V_{\rm int}$ is very special. The intertwining relations (7.21) mean, in particular, that

$$\langle\Psi|H|\Psi\rangle = \langle\Psi^{\rm in}_{\rm as}|H_0|\Psi^{\rm in}_{\rm as}\rangle = \langle\Psi^{\rm out}_{\rm as}|H_0|\Psi^{\rm out}_{\rm as}\rangle\,.$$

Furthermore, exploiting (7.21) it is easy to see that

$$[S_0,\ H_0] = 0\,. \tag{7.22}$$

Indeed:[21]

$$S_0H_0 = \Omega_-^\dagger\Omega_+H_0 = \Omega_-^\dagger H\,\Omega_+ = \Omega_-^\dagger H\,\Omega_-\Omega_-^\dagger\Omega_+ = H_0\,\Omega_-^\dagger\Omega_+ = H_0S_0\,. \tag{7.23}$$

Since any normalizable state-vector $|\Psi\rangle$ can be written as a superposition of non-normalizable generalized (i.e. not belonging to the proper Hilbert space) H_0 eigenvectors $|\alpha_0\rangle$, one can write

$$\langle\beta_0|\Psi^{\rm out}_{\rm as}\rangle = \langle\beta_0|S_0|\Psi^{\rm in}_{\rm as}\rangle = \int d\alpha\,\langle\beta_0|S_0|\alpha_0\rangle\langle\alpha_0|\Psi^{\rm in}_{\rm as}\rangle\,.$$

[19]The difference between the S_0 and S operators is particularly sharp in the nonrelativistic potential scattering theory, if H possesses bound states: while S_0 acts nontrivially on the whole Hilbert space $\mathcal{H}$, S annihilates the whole subspace $\mathcal{H}_{\rm bound}$ - see Appendix E.

[20]If the operators $\Omega_\pm$ are truly unitary, these relations imply that the spectra of H and H_0 are identical (which is one of our assumptions adopted here). This shows that in the case of ordinary potential scattering $\Omega_\pm$ cannot be unitary if H has bound states (normalizable eigenvectors) because the spectrum of H_0 is continuous. In such a case $\Omega_\pm$ are only isometric operators.

[21]We assume here that the Møller operators are unitary; see (E.2) in Appendix E for a justification in the case they are only isometric.

One is thus led to consider the matrix elements $S_{\beta\alpha} \equiv \langle\beta_0|S_0|\alpha_0\rangle$. From (7.22) it follows that

$$0 = \langle\beta_0|[H_0,\ S_0]|\alpha_0\rangle = (E_\beta - E_\alpha)\langle\beta_0|S_0|\alpha_0\rangle\,,$$

which shows that $\langle\beta_0|S_0|\alpha_0\rangle \propto \delta(E_\beta - E_\alpha)$ (because $x\delta(x) = 0$). Furthermore, because for $V_{\rm int} = 0$ the S_0 operator reduces to the unit operator, it is convenient to write

$$S_0 = \hat{1} - iT_0\,, \tag{7.24}$$

thereby introducing the reaction operator T_0. Thus

$$S_{\beta\alpha} \equiv \langle\beta_0|S_0|\alpha_0\rangle = \delta_{\alpha\beta} - 2\pi i\,\delta(E_\beta - E_\alpha)\,t_{\beta\alpha}(E_\alpha)\,, \tag{7.25}$$

where $2\pi\delta(E_\beta - E_\alpha)\,t_{\beta\alpha}(E_\alpha) = \langle\beta_0|T_0|\alpha_0\rangle$. As will be shown in Chapter 10, it is precisely the quantity $t_{\beta\alpha}(E_\alpha)$ which is needed to compute the rate of the process $\alpha \to \beta$. In the case of the nonrelativistic potential scattering the quantity $t_{\beta\alpha} \equiv t(\mathbf{p}', \mathbf{p})$ is directly related to the standard scattering amplitude $f(\theta)$ - see Appendix E. All measurable characteristics of scattering processes predicted by a given theory defined by the Hamiltonian $H = H_0 + V_{\rm int}$ can be extracted from the matrix elements $S_{\beta\alpha}$ of the corresponding S_0 (or T_0) operator of this theory. One useful representation for this operator will be derived directly from the differential equation satisfied by the (interaction picture) evolution operator introduced in Section 1.1:

$$U_I(t_2, t_1) = e^{iH_0t_2}e^{-iH(t_2-t_1)}e^{-iH_0t_1} = \Omega_-^\dagger(t_2)\Omega_+(t_1)\,, \tag{7.26}$$

of which S_0 is the double limit:

$$S_0 = \Omega_-^\dagger\Omega_+ = \lim_{t_2\to+\infty}\lim_{t_1\to-\infty} U_I(t_2, t_1)\,. \tag{7.27}$$

It will be therefore possible to evaluate S-matrix elements using the Dyson expansion of Section 5.8 (in conjunction with the Wick theorem of Section 5.9). Before exploiting this representation of the S_0 operator one has, however, to introduce the non-normalizable (generalized) H eigenvectors and the rezolvent operators which allow to relate these to the H_0 eigenvectors $|\alpha_0\rangle$.

7.2 Rezolvents and the T operator representation of the S-matrix

A very important role in the formal scattering theory is played by the *rezolvent* operators

$$G(z) \equiv (z - H)^{-1}\,, \qquad \text{and} \qquad G_0(z) \equiv (z - H_0)^{-1}\,. \tag{7.28}$$

Their matrix elements between normalizable states are analytic functions on the complex z plane except for isolated poles corresponding to normalizable H (H_0) eigenstates and a branch cut along the continuous part of the H (H_0) spectrum.

Substituting for A and B in the obvious operator identity

$$\frac{1}{A} - \frac{1}{B} = \frac{1}{B}\,(B - A)\,\frac{1}{A}\,,$$

the operators $z - H$ and $z - H_0$ (the operators $z - H_0$ and $z - H$) one obtains two relations

$$\begin{aligned} G(z) &= G_0(z) + G_0(z)\,V_{\rm int}\,G(z)\,, \\ G(z) &= G_0(z) + G(z)\,V_{\rm int}\,G_0(z)\,, \end{aligned} \tag{7.29}$$

It is also easy to see that because $H = H^\dagger$ ($H_0 = H_0^\dagger$),

$$G(z^*) = [G(z)]^\dagger\,, \qquad G_0(z^*) = [G_0(z)]^\dagger\,. \tag{7.30}$$

Matrix elements of the resolvent operator $G_0(z)$ between the non-normalizable H_0 eigenvectors are explicitly given by

$$\langle \beta_0 | G_0(z) | \alpha_0 \rangle = \delta_{\beta\alpha}\,\frac{1}{z - E_\alpha}\,. \tag{7.31}$$

Another very important operator is the $T(z)$ operator defined as

$$T(z) \equiv V_{\rm int} + V_{\rm int}\,G(z)\,V_{\rm int}\,. \tag{7.32}$$

It has the same analytic properties as $G(z)$ and satisfies the following relations

$$\begin{aligned} G_0(z)\,T(z) &= G(z)\,V_{\rm int}\,, \\ T(z)\,G_0(z) &= V_{\rm int}\,G(z)\,, \end{aligned} \tag{7.33}$$

which readily follow from the relations (7.29). They allow to express $G(z)$ through $T(z)$: replacing in (7.29) $V_{\rm int}\,G(z)$ (or $G(z)\,V_{\rm int}$) using (7.33) one gets

$$G(z) = G_0(z) + G_0(z)\,T(z)\,G_0(z)\,. \tag{7.34}$$

Using (7.33) in (7.32) leads instead to

$$T(z) = V_{\rm int} + V_{\rm int}\,G_0(z)\,T(z)\,, \tag{7.35}$$

which is known as the Lippman-Schwinger equation for $T(z)$. Iterating it yields the series

$$T(z) = V_{\rm int} + V_{\rm int}\,G_0(z)\,V_{\rm int} + V_{\rm int}\,G_0(z)\,V_{\rm int}\,G_0(z)\,V_{\rm int} + \ldots \tag{7.36}$$

Matrix elements of the S_0 operator between generalized H_0 eigenstates can be expressed through the operator $T(z)$. To this end, instead of representing S_0 as the double limit of the $U_I(t_2, t_1)$ operator, as in (7.27), it is written as the single limit

$$S_0 = \Omega_-^\dagger\,\Omega_+ = \lim_{\tau\to\infty} U_I(\tau, -\tau) = \lim_{\tau\to\infty} e^{iH_0\tau} e^{-2iH\tau} e^{iH_0\tau}\,.$$

Differentiating with respect to τ the operator $U_I(\tau,-\tau)$ one obtains the differential equation satisfied by it, which, together with the obvious boundary condition at $\tau=0$, allows to write for this operator an integral expression, similar to (1.6). Using it, matrix element of the S_0 operator between normalizable states can be written as

$$\langle\Phi|S_0|\Psi\rangle = \langle\Phi|\Psi\rangle - i\int_0^\infty dt\, e^{-\varepsilon t}\langle\Phi|e^{iH_0t}\,V_{\rm int}\,e^{-2iHt}e^{iH_0t} + e^{iH_0t}e^{-2iHt}\,V_{\rm int}\,e^{iH_0t}|\Psi\rangle\,.$$

The factor $e^{-\varepsilon t}$ is not necessary when the matrix element is taken between two normalizable states, but when introduced,[22] it allows to replace $|\Psi\rangle$ and $|\Phi\rangle$ by the generalized H_0 eigenvectors $|\alpha_0\rangle$ and $|\beta_0\rangle$:

$$\begin{aligned}\langle\beta_0|S_0|\alpha_0\rangle &= \delta_{\beta\alpha} - i\int_0^\infty dt\,\langle\beta_0|V_{\rm int}\,e^{i(E_\beta+E_\alpha-2H+i0)t} + e^{i(E_\beta+E_\alpha-2H+i0)t}\,V_{\rm int}|\alpha_0\rangle\\ &= \delta_{\beta\alpha} + \frac{1}{2}\langle\beta_0|V_{\rm int}\,G\Big(\frac{E_\beta+E_\alpha}{2}+i0\Big) + G\Big(\frac{E_\beta+E_\alpha}{2}+i0\Big)V_{\rm int}|\alpha_0\rangle\,.\end{aligned}$$

Using the operator identities (7.33) one can replace here the operators $G(z)$ by the $G_0(z)$ ones which can act directly on the states $|\alpha_0\rangle$ and $\langle\beta_0|$. The second term can be then cast in the form

$$\left\{\frac{1}{E_\beta-E_\alpha+i0}+\frac{1}{E_\alpha-E_\beta+i0}\right\}\langle\beta_0|T\Big(\frac{E_\beta+E_\alpha}{2}+i0\Big)|\alpha_0\rangle\,,$$

which, upon using the Sochocki formula, leads to

$$\langle\beta_0|S_0|\alpha_0\rangle = \delta_{\beta\alpha} - 2\pi i\,\delta(E_\beta-E_\alpha)\,\langle\beta_0|T(E_\alpha+i0)|\alpha_0\rangle\,. \tag{7.37}$$

This shows that the matrix element $t_{\beta\alpha}$ of the T_0 operator introduced in (7.25) is given by the special limit $z\to E_\alpha+i0$ of the general matrix element of the $T(z)$ operator. Combining this with the truncated to its first term iterative solution (7.36) of the Lippmann-Schwinger equation (7.35) for $T(z)$, one immediately obtains the formula known as the *Born approximation*

$$t_{\beta\alpha}(E_\alpha) \approx \langle\beta_0|V_{\rm int}|\alpha_0\rangle\,. \tag{7.38}$$

If it is known how to compute the action of $V_{\rm int}$ on free particle states (which is precisely the case, when $V_{\rm int}$ is expressed in terms of the creation and annihilation operators of free particles) this formula provides the simplest working approximation to amplitudes of particle reactions.[23]

[22] It is usually (incorrectly) introduced from the beginning as a factor ensuring "adiabatic" switching on and off the interaction - something which certainly does not happen in Nature! Notice also that if $V_{\rm int}$ was defined with such a factor, the evolution operator $U(t,t_0)$ corresponding to the Hamiltonian H (explicitly time dependent then) would have to have the form (1.9) instead of $e^{-iH(t-t_0)}$.

[23] However, frequently in relativistic theories of interacting particles $t_{\beta\alpha}(E_\alpha)=0$ in this approximation. In Quantum Field Theory the name "Born approximation" is sometimes also used to denote what otherwise is called the tree-level approximation (see Chapter 9); it coincides with (7.38) only for very special interactions $V_{\rm int}$.

7.3 *In* and *out* state-vectors

One can now define the *in* and *out* generalized state-vectors $|\alpha_\pm\rangle$ by the formula

$$|\alpha_\pm\rangle \equiv \Omega_\pm|\alpha_0\rangle \equiv \lim_{t\to\mp\infty} e^{iHt}e^{-iH_0t}|\alpha_0\rangle\,. \tag{7.39}$$

Owing to the intertwining relations (7.21), $|\alpha_\pm\rangle$ turn out to be just the generalized (non-normalizable) eigenvectors of the Hamiltonian $H = H_0 + V_{\rm int}$:

$$H|\alpha_\pm\rangle = E_\alpha|\alpha_\pm\rangle\,, \tag{7.40}$$

with the eigenvalue E_α equal to the energy (w.r.t. H_0) of the corresponding $|\alpha_0\rangle$ states. With the assumption that the spectra of H and H_0 are identical, the vectors $|\alpha_0\rangle$, $|\alpha_+\rangle$ and $|\alpha_-\rangle$ related to each other in the same way as are the vectors $|\Psi\rangle$, $|\Psi_+\rangle$ and $|\Psi_-\rangle$ in (7.17) form three equivalent bases (of generalized vectors) of the theory Hilbert space $\mathcal{H}$. From this point of view the S-matrix elements

$$S_{\beta\alpha} = \langle\beta_-|\alpha_+\rangle = \langle\beta_0|S_0|\alpha_0\rangle\,, \tag{7.41}$$

form a collection of numbers, such that

$$|\alpha_+\rangle = \int d\beta\,|\beta_-\rangle\,S_{\beta\alpha}\,, \qquad \langle\beta_-| = \int d\alpha\,\langle\alpha_+|\,S_{\beta\alpha}\,. \tag{7.42}$$

As a matrix connecting two complete sets of orthonormal states (it is just the matrix of the change of bases) $S_{\beta\alpha}$ must be unitary:

$$\int d\beta\,S^*_{\beta\gamma}S_{\beta\alpha} = \int d\beta\,\langle\gamma_+|\beta_-\rangle\langle\beta_-|\alpha_+\rangle = \langle\gamma_+|\alpha_+\rangle = \delta_{\gamma\alpha}\,. \tag{7.43}$$

This reflects also the unitarity of the S_0 operator: $S_0^{-1} = S_0^\dagger$. The state-vectors $|\alpha_+\rangle$ and $|\alpha_-\rangle$ are, in turn, connected by the S operator defined in (7.20):

$$S|\alpha_-\rangle = |\alpha_+\rangle\,, \qquad \text{or} \qquad \langle\beta_+|S = \langle\beta_-|\,, \tag{7.44}$$

so that, in analogy to (7.19),

$$S_{\beta\alpha} = \langle\beta_+|S|\alpha_+\rangle = \langle\beta_-|S|\alpha_-\rangle\,. \tag{7.45}$$

From the practical point of view (7.39) establishes a strict *one-to-one correspondence* between the *in* and *out* eigenvectors of H and the eigenvectors of H_0 on which the formulation of the perturbative calculation of the S-matrix elements will be based. (This strict correspondence will be relaxed only in Chapter 13 where a more flexible, nonperturbative in essence, way of accessing S-matrix elements will be formulated).

Since any normalizable state $|\Psi\rangle$ can be decomposed into the generalized H_0 eigenvectors $|\alpha_0\rangle$: $|\Psi\rangle = \int d\alpha\, |\alpha_0\rangle\, \psi(\alpha)$, from the relation $|\Psi_\pm\rangle = \Omega_\pm|\Psi\rangle$ one gets

$$|\Psi_\pm\rangle = \Omega_\pm \int d\alpha\, |\alpha_0\rangle\, \psi(\alpha) = \int d\alpha\, |\alpha_\pm\rangle\, \psi(\alpha)\,, \tag{7.46}$$

That is, normalizable state-vectors $|\Psi_\pm\rangle$ decompose onto the generalized H eigenvectors $|\alpha_\pm\rangle$ with the same profile $\psi(\alpha)$ as do their $\Omega_\pm^\dagger$ images onto the generalized H_0 eigenvectors $|\alpha_0\rangle$. Moreover, from the fact that the $\Omega_\pm$ operators preserve the scalar product of normalizable states (cf. (7.13)) it follows that

$$\langle\beta_\pm|\alpha_\pm\rangle = \langle\beta_0|\alpha_0\rangle = \delta_{\beta\alpha}\,. \tag{7.47}$$

Since the *in* and *out* state-vectors $|\alpha_+\rangle$ and $|\alpha_-\rangle$ are in the one-to-one correspondence with the free particle vectors $|\alpha_0\rangle$, in addition to the operators $a(\mathbf{k},\sigma)$, $a^\dagger(\mathbf{k},\sigma)$ (which build the states $|\alpha_0\rangle$ out of $|\Omega_0\rangle$) one can define also the *in* and *out* creation and annihilation operators $a_{\rm in}(\mathbf{k},\sigma)$, $a_{\rm in}^\dagger(\mathbf{k},\sigma)$ and $a_{\rm out}(\mathbf{k},\sigma)$, $a_{\rm out}^\dagger(\mathbf{k},\sigma)$ which acting on the corresponding vacua[24] $|\Omega_\pm\rangle = \Omega_\pm|\Omega_0\rangle$ build the *in* and *out* states. These operators satisfy the same commutation relations as do the original operators $a(\mathbf{k},\sigma)$, $a^\dagger(\mathbf{k},\sigma)$ and have the same transformation properties (in the relativistic case with respect to the full Poincaré symmetry group generated by $H = H_0 + V_{\rm int}$, $\mathbf{P}$, $\mathbf{J}$ and $\mathbf{K} = \mathbf{K}_0 + \mathbf{W}$ - see section 7.5) as do the operators creating and annihilating the free-particle states $|\alpha_0\rangle$. From (7.44) it then follows (cf. (1.59)) that

$$S^\dagger a_{\rm in}^\dagger(\mathbf{k},\sigma)S = a_{\rm out}^\dagger(\mathbf{k},\sigma)\,, \qquad S^\dagger a_{\rm in}(\mathbf{k},\sigma)S = a_{\rm out}(\mathbf{k},\sigma)\,. \tag{7.48}$$

Finally, it should be stressed that by themselves the vectors $U(t,0)|\alpha_\pm\rangle = e^{-iE_\alpha t}|\alpha_\pm\rangle$ do not converge to $U_0(t,0)|\alpha_0\rangle = e^{-iE_\alpha t}|\alpha_0\rangle$ in the limits $t \to \mp\infty$. The convergence holds only for normalizable states built as smooth superpositions of such states. Nevertheless, (7.39) stay true in the literal sense.

The operator identities established above allow to derive useful representations for the *in* and *out* states $|\alpha_\pm\rangle$ either in terms of the rezolvent $G(z)$ or in terms of the $T(z)$ operator. To this end we consider first the action of $\Omega_\pm$ on a normalizable state-vector $|\Psi\rangle$. One gets then the scattering states $|\Psi_\pm\rangle$ which, using the formula (7.12) can be written as

$$|\Psi_\pm\rangle = \Omega_\pm|\Psi\rangle = |\Psi\rangle + i\int_0^{\mp\infty} dt'\, e^{-\varepsilon|t'|}\, U^\dagger(t',0)V_{\rm int}U_0(t',0)|\Psi\rangle\,. \tag{7.49}$$

Again the factor $e^{-\varepsilon|t'|}$ is not necessary for convergence when $|\Psi\rangle$ is a normalizable state, but is necessary when $|\Psi\rangle$ is decomposed into generalized H_0 eigenstates $|\alpha_0\rangle$:

$$|\Psi_\pm\rangle = |\Psi\rangle + i\int d\alpha \int_0^{\mp\infty} dt\, e^{-i(E_\alpha - H \pm i\varepsilon)t}\, V_{\rm int}|\alpha_0\rangle\langle\alpha_0|\Psi\rangle$$

[24]Do not confuse the $|\Omega_\pm\rangle$ vacua with the M$\emptyset$ller operators $\Omega_\mp$. The vacua $|\Omega_+\rangle$ and $|\Omega_-\rangle$ of closed systems, i.e. systems, the Hamiltonians H of which do not depend on time, differ only by a phase factor.

$$= |\Psi\rangle + \int d\alpha\, G(E_\alpha \pm i0)\, V_{\rm int}|\alpha_0\rangle\langle\alpha_0|\Psi\rangle\,. \tag{7.50}$$

To obtain the representations of the *in* and *out* states $|\alpha_\pm\rangle$ one rewrites (7.50), decomposing $|\Psi\rangle$ onto the $|\alpha_0\rangle$ states, in the form

$$|\Psi_\pm\rangle = \int d\alpha \left(|\alpha_0\rangle + G(E_\alpha \pm i0)\, V_{\rm int}|\alpha_0\rangle \right) \psi(\alpha)\,.$$

Comparing this with (7.46) one gets the representation

$$|\alpha_\pm\rangle = |\alpha_0\rangle + G(E_\alpha \pm i0)\, V_{\rm int}|\alpha_0\rangle\,. \tag{7.51}$$

Yet another representation can be obtained using the identity

$$T(E_\alpha \pm i0)|\alpha_0\rangle = V_{\rm int}[\hat{1} + G(E_\alpha \pm i0)V_{\rm int}]|\alpha_0\rangle = V_{\rm int}|\alpha_\pm\rangle\,, \tag{7.52}$$

which follows from the definition (7.32) of the $T(z)$ operator and (7.51). This relation, combined with the result (7.37), immediately allows to write the element $t_{\beta\alpha}(E_\alpha)$ in the S_0 matrix element (7.25) as[25]

$$t_{\beta\alpha}(E_\alpha) = \langle\beta_0|T(E_\alpha + i0)|\alpha_0\rangle = \langle\beta_0|V_{\rm int}|\alpha_+\rangle\,. \tag{7.53}$$

The identity (7.52) applied to (7.51) after trading in this formula the product $G(E_\alpha \pm i0)\, V_{\rm int}$ for $G_0(E_\alpha \pm i0)\, T(E_\alpha \pm i0)$ in agreement with (7.33), leads to the Lippmann-Schwinger equation for $|\alpha_\pm\rangle$:

$$|\alpha_\pm\rangle = |\alpha_0\rangle + G_0(E_\alpha \pm i0)\, V_{\rm int}|\alpha_\pm\rangle \equiv |\alpha_0\rangle + \frac{1}{E_\alpha - H_0 \pm i0}\, V_{\rm int}|\alpha_\pm\rangle\,, \tag{7.54}$$

or

$$|\alpha_\pm\rangle = |\alpha_0\rangle + \int d\beta\, |\beta_0\rangle\, \frac{t_{\beta\alpha}(E_\alpha)}{E_\alpha - E_\beta \pm i0}\,. \tag{7.55}$$

Notice that the formula (7.54) agrees with the identification of the $|\alpha_\pm\rangle$ vectors as the eigenvectors of H, if the relation (7.40) is rewritten in the form

$$(E_\alpha - H_0)|\alpha_\pm\rangle = V_{\rm int}|\alpha_\pm\rangle\,.$$

The $\pm i0$ prescription specifies the way of inverting the operator $(E_\alpha - H_0)$ which has $|\alpha_0\rangle$ as its zero eigenvector.[26] Iterating the Lippmann-Schwinger equation (7.54) eg. for $|\alpha_+\rangle$

[25] Similar representation of $t_{\beta\alpha}(E_\alpha)$ in terms of the *out* state is obtained by taking the Hermitian conjugation of $T(E_\alpha - i0)|\beta_0\rangle = V_{\rm int}|\beta_-\rangle$ and using the property $T^\dagger(z) = T(z^*)$. This leads to $t_{\beta\alpha}(E_\alpha) = \langle\beta_-|V_{\rm int}|\alpha_0\rangle$.

[26] Weinberg in his book derives the formula (7.55) directly from this equality. His derivation (quicker than the one given here) suffers, however, from the not fully convincing application of the residue method to the integral over E_β implicit in (7.55): it does not extend to the whole real axis as requires this method, but is restricted to $E_\beta > M_{\rm min} \geq 0$ (energy of the states $|\alpha_0\rangle$ representing particles is never negative).

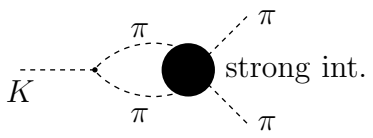

Figure 7.1: Strong interaction induced rescattering of pions produced in the decay of Kaon.

gives the series

$$|\alpha_+\rangle = |\alpha_0\rangle + G_0(E_\alpha + i0)\, V_{\rm int}|\alpha_0\rangle + G_0(E_\alpha + i0)\, V_{\rm int} G_0(E_\alpha + i0)\, V_{\rm int}|\alpha_0\rangle + \dots\,, \tag{7.56}$$

When closed from the left with $\langle\beta_0|V_{\rm int}$, it reproduces the Born series for $t_{\beta\alpha}(E_\alpha) = \langle\beta_0|T(E_\alpha + i0)|\beta_0\rangle$ which can be obtained from (7.36); the latter series, truncated to the first term, gives the Born approximation (7.38).

Another useful approximation can be obtained if the interaction $V_{\rm int}$ consists of two parts: $V_{\rm int} = V_{\rm strong} + V_{\rm weak}$ of which one is "strong" and the other one "weak". One is then interested in accounting for the strong interactions exactly, while the effects of the weak ones can be treated in the simplest approximation. To this end, in addition to the *in* and *out* eigenstates $|\alpha_\pm\rangle$ of the full Hamiltonian $H = H_0 + V_{\rm strong} + V_{\rm weak}$ one defines also the *in* and *out* states with respect to the strong interaction

$$|\beta_\pm^{\rm strong}\rangle = |\beta_0\rangle + \frac{1}{E_\beta - H_0 \pm i0} V_{\rm strong}|\beta_\pm^{\rm strong}\rangle\,, \tag{7.57}$$

so that

$$\langle\beta_0| = \langle\beta_\pm^{\rm strong}| - \langle\beta_\pm^{\rm strong}|V_{\rm strong}\frac{1}{E_\beta - H_0 \mp i0}\,. \tag{7.58}$$

The full matrix $t_{\beta\alpha}(E_\alpha)$ (7.53) can be then written in the form

$$\begin{aligned} t_{\beta\alpha} &= \left[\langle\beta_-^{\rm strong}| - \langle\beta_-^{\rm strong}|V_{\rm strong}\frac{1}{E_\beta - H_0 + i0}\right](V_{\rm strong} + V_{\rm weak})|\alpha_+\rangle \\ &= \langle\beta_-^{\rm strong}|V_{\rm weak}|\alpha_+\rangle + \langle\beta_-^{\rm strong}|V_{\rm strong}|\alpha_0\rangle\,, \end{aligned} \tag{7.59}$$

where the formula (7.54) with $V_{\rm int}$ replaced by $V_{\rm strong} + V_{\rm weak}$ has been used (in the denominator E_β can be replaced by E_α because we need $t_{\beta\alpha}$ for $E_\beta = E_\alpha$) to replace the product $[E_\beta - H_0 + i0]^{-1}(V_{\rm strong} + V_{\rm weak})|\alpha_+\rangle$ by $|\alpha_+\rangle - |\alpha_0\rangle$. This (exact) formula is most useful if the strong interaction cannot induce the $\alpha \to \beta$ transition. The second term, which is just $t_{\beta\alpha}$ in the absence of weak interactions (just set $V_{\rm weak}$ to zero in the formula given in the footnote related to the formula (7.53) to see it!), that is corresponds to all possible transitions $\alpha \to \beta$ induced by $V_{\rm strong}$ alone, is then zero and, moreover, since the effects of $V_{\rm weak}$ are small, one can approximate the full Hamiltonian *in* state $|\alpha_+\rangle$ in the

first term by $|\alpha_+^{\rm strong}\rangle$. The resulting formula $t_{\beta\alpha} \approx \langle\beta_-^{\rm strong}|V_{\rm weak}|\alpha_+^{\rm strong}\rangle$ is used e.g. in nuclear physics to compute rates of nuclear weak beta decays ($|\alpha_+^{\rm strong}\rangle$ and $|\alpha_-^{\rm strong}\rangle$ are then the initial and final nucleon states). Furthermore, using the property (7.42) of the S matrix, this formula can be rewritten as

$$t_{\beta\alpha} = \int d\gamma\, S_{\beta\gamma}^{\rm strong}\, \langle\gamma_+^{\rm strong}|V_{\rm weak}|\alpha_+^{\rm strong}\rangle\,. \tag{7.60}$$

In this form it is used to account for the strong interaction re-scattering effects (shown graphically in figure 7.1) in hadronic weak decays; such effects are crucial for CP violation effects in the Kaon system.

The Born formula (7.38) is the first term of the entire perturbative series which is obtained either by sandwiching the series (7.36) between the states $\langle\beta_0|$ and $|\alpha_0\rangle$ and evaluating it for $z = E_\alpha + i0$, or by using the Lippmann-Schwinger formula (7.55) for $|\alpha_+\rangle$ in the exact expression (7.53) for $t_{\beta\alpha}$:

$$t_{\beta\alpha} \equiv \langle\beta_0|V_{\rm int}|\alpha_+\rangle = V_{\beta\alpha} + \int d\gamma\, \frac{V_{\beta\gamma}t_{\gamma\alpha}(E_\alpha)}{E_\alpha - E_\gamma + i0}\,, \tag{7.61}$$

where $V_{\beta\alpha} \equiv \langle\beta_0|V_{\rm int}|\alpha_0\rangle$. Iterating this equation yields the series:

$$\begin{aligned} t_{\beta\alpha} = V_{\beta\alpha} &+ \int d\gamma\, \frac{V_{\beta\gamma}V_{\gamma\alpha}}{E_\alpha - E_\gamma + i0} \\ &+ \int d\gamma \int d\gamma' \frac{V_{\beta\gamma}V_{\gamma\gamma'}V_{\gamma'\alpha}}{(E_\alpha - E_\gamma + i0)(E_\alpha - E_{\gamma'} + i0)} + \ldots, \end{aligned} \tag{7.62}$$

This is the so-called "old-fashioned" perturbation calculus. While in some situations it is convenient to investigate some specific issues, its main drawback in relativistic theories is the lack of manifest Lorentz covariance.

An alternative, more satisfactory in this respect, approach to perturbative calculation of the S-matrix elements is developed by starting directly from the formulae (7.27) and (7.26). Differentiating the latter with respect to t_2 we get as in section 1.1 the formula (1.23) and, hence,

$$S_0 = U_I(+\infty, -\infty) = T\exp\left(-i\int_{-\infty}^{+\infty} dt\, V_{\rm int}^I(t)\right), \tag{7.63}$$

where the interaction operator in the Dirac picture reads

$$V_{\rm int}^I(t) \equiv e^{iH_0 t}\, V_{\rm int}\, e^{-iH_0 t}\,. \tag{7.64}$$

The formula (7.63) is the basis of the commonly used time-dependent perturbation calculus.

The equivalence of the formula (7.63) and the formulae (7.25) and (7.62) should be clear from the construction (at least at the formal level), but can also be seen directly by making use of the identity

$$\frac{1}{E_\alpha - E_\beta + i0} = \frac{1}{i}\int_0^{+\infty} d\tau\, e^{i(E_\alpha - E_\beta + i0)\tau}\,, \tag{7.65}$$

to represent the energy denominators in (7.62). For example, the first terms in the expansion of (7.63) give

$$\begin{aligned} S_{\beta\alpha} = \langle\beta_0|S_0|\alpha_0\rangle &= \langle\beta_0|1 - i\int_{-\infty}^{+\infty} dt\, V^I_{\rm int}(t) + \ldots|\alpha_0\rangle \\ &= \delta_{\beta\alpha} - i\int_{-\infty}^{+\infty} dt\, e^{-i(E_\alpha - E_\beta)t}\, V_{\beta\alpha} + \ldots \\ &= \delta_{\beta\alpha} - 2\pi i\,\delta(E_\alpha - E_\beta)\, V_{\beta\alpha} + \ldots \\ &\quad + \frac{(-i)^2}{2!}\int_{-\infty}^{+\infty} dt_1 \int_{-\infty}^{+\infty} dt_2\, \langle\beta_0|T\left(V^I_{\rm int}(t_1)V^I_{\rm int}(t_2)\right)|\alpha_0\rangle + \ldots \end{aligned} \tag{7.66}$$

and so on. The advantage of the time-dependent perturbative expansion based on the formula (7.63) lies in the fact that in relativistic theories it allows to keep Lorentz invariance manifest at each stage of the calculations.

7.4 Scattering of nonrelativistic particles

Before discussing scattering of relativistic particles (as a mean to construct theories of relativistic particles interactions) it is instructive to see how the developed formalism applies to the simpler case of the nonrelativistic elastic scattering. Its application to the scattering of a single spinless particle on a fixed potential - the simplest possible case which can be treated in the framework of the ordinary nonrelativistic quantum mechanics - is discussed in Appendix E. Here we apply the S-matrix approach to the problem of the elastic scattering of two nonrelativistic identical particles (fermions or bosons of arbitrary spin). We first recall the usual treatment of this problem in the framework of the two-body Schrödinger equation and then reformulate it using the formalism of second quantization of Chapter 5. The purpose of this is twofold: firstly we want to show how the Dyson expansion (Section 5.8) and the Wick theorem (Section 5.9) provide an efficient and flexible mean to compute scattering cross sections; we also want to establish the relation between elements $t_{\beta\alpha}(E_\alpha)$ (7.24) of the T_0 operator (7.25) to the ordinary scattering amplitude $f(k,\theta)$ known from the Schrödinger equation based approach. Secondly we want to derive the result (5.94) used in the discussion of the ground state of a system of interacting bosons. This will also give the opportunity to go beyond the first nontrivial order of the Dyson expansion and to have a first encounter with the problem of divergences and their treatement (within the relativistic theory this problem will be discussed in Chapter 14).

In ordinary nonrelativistic Quantum Mechanics the amplitude of the elastic scattering of two particles of masses m_1 and m_2 is obtained by solving the two-body Schrödinger equation

$$\left(-\frac{\hbar^2}{2m_1}\,\boldsymbol{\nabla}^2_{\mathbf{r}_1} - \frac{\hbar^2}{2m_2}\,\boldsymbol{\nabla}^2_{\mathbf{r}_2} + V_{\rm pot}(\mathbf{r}_1-\mathbf{r}_2)\right)\Psi(\mathbf{r}_1,\mathbf{r}_2) = E\,\Psi(\mathbf{r}_1,\mathbf{r}_2)\,.$$

In the variables $\mathbf{r}=\mathbf{r}_1-\mathbf{r}_2$ and $\mathbf{R}=(m_1\mathbf{r}_1+m_2\mathbf{r}_2)/(m_1+m_2)$ this equation takes the form

$$\left(-\frac{\hbar^2}{2M}\,\boldsymbol{\nabla}^2_{\mathbf{R}} - \frac{\hbar^2}{2m_{\rm red}}\,\boldsymbol{\nabla}^2_{\mathbf{r}} + V_{\rm pot}(\mathbf{r})\right)\Psi(\mathbf{R},\mathbf{r}) = E\,\Psi(\mathbf{R},\mathbf{r})\,,$$

where $M=m_1+m_2$ and $m_{\rm red}=m_1m_2/(m_1+m_2)$ is the reduced mass of the two-particle system. Writing then $\Psi(\mathbf{R},\mathbf{r})=\psi(\mathbf{r})\exp(i\mathbf{P}\cdot\mathbf{R}/\hbar)$ reduces the problem to the one of scattering in an external potential $V_{\rm pot}(\mathbf{r})$ of a single fictitious particle of mass $m_{\rm red}$:

$$\left(-\frac{\hbar^2}{2m_{\rm red}}\,\boldsymbol{\nabla}^2_{\mathbf{r}} + V_{\rm pot}(\mathbf{r})\right)\psi(\mathbf{r}) = E'\psi(\mathbf{r})\,. \tag{7.67}$$

Here $E'=E-\mathbf{P}^2/2M$ is the energy of the two scattering particles in their center of mass system (CMS). In this system, going over to which means just setting $\mathbf{P}=\mathbf{0}$, the momenta of the two particles are $\hbar\mathbf{k}$ and $-\hbar\mathbf{k}$ and the energy $E'\equiv E$ ascribed to the fictitious particle is the total energy of the two colliding particles:

$$E=\frac{1}{2}m_1\mathbf{v}_1^2+\frac{1}{2}m_2\mathbf{v}_2^2=\frac{\hbar^2}{2}\left(m_1\,\frac{\mathbf{k}^2}{m_1^2}+m_2\,\frac{\mathbf{k}^2}{m_2^2}\right)=\frac{\hbar^2\mathbf{k}^2}{2m_{\rm red}}\,.$$

Therefore, the vector $\mathbf{k}$ playing the role of the wave vector of the fictitious particle of mass $m_{\rm red}$ must be identified with the wave vector of one of the scattering particles. The scattering amplitude $f(k,\theta)$, in which $k\equiv|\mathbf{k}|$, is defined in terms of the asymptotic form ($|\mathbf{k}'|=|\mathbf{k}|$, $\mathbf{k}'\cdot\mathbf{k}=\mathbf{k}^2\cos\theta$)

$$\psi^{(+)}_{\mathbf{k}}(\mathbf{r}) = e^{i\mathbf{k}\cdot\mathbf{r}} + \frac{f(\mathbf{k}',\mathbf{k})}{r}\,e^{ikr} \equiv e^{i\mathbf{k}\cdot\mathbf{r}} + \frac{f(k,\theta)}{r}\,e^{ikr}\,, \tag{7.68}$$

of the solution of the Schrödinger equation (7.67) (with $E'=\hbar^2\mathbf{k}^2/2m_{\rm red}$), and the differential scattering cross section $d\sigma/d\Omega$ is simply given by $|f(k,\theta)|^2$.

If the scattering particles are indistinguishable and both have spin s (integer or half-integer) but the potential is spin-independent, the complete wave function of the system can be written as a product $\Psi(\mathbf{r}_1,\mathbf{r}_2)\chi(\sigma_1,\sigma_2)$ in which $\sigma_{1,2}=-s,\ldots,+s$, of the spin part and the space part. According to the principles of Quantum Mechanics, the complete wave function of indistinguishable particles moving in the three-dimensional space must be either totally symmetric $\Psi(\mathbf{r}_1,\mathbf{r}_2)\chi(\sigma_1,\sigma_2)=+\Psi(\mathbf{r}_2,\mathbf{r}_1)\chi(\sigma_2,\sigma_1)$ or totally antisymmetric $\Psi(\mathbf{r}_1,\mathbf{r}_2)\chi(\sigma_1,\sigma_2)=-\Psi(\mathbf{r}_2,\mathbf{r}_1)\chi(\sigma_2,\sigma_1)$, depending on whether s is integer or half-integer

(this is the celebrated spin-statistics connection which will be discussed in Chapter 8). The total spin S of the two particle system is in this situation preserved separately (from the orbital angular momentum) and the spin part $\chi(\sigma_1, \sigma_2)$ of the wave function can be chosen to be symmetric or antisymmetric. The function $\psi(\mathbf{r})$ in the decomposition $\Psi(\mathbf{r}_1, \mathbf{r}_2) = \psi(\mathbf{r}) \exp(i\mathbf{P}\cdot\mathbf{R}/\hbar)$ must then be[27] either even $\psi(-\mathbf{r}) = \psi(\mathbf{r})$ or odd $\psi(-\mathbf{r}) = -\psi(\mathbf{r})$ in order that $\psi(\mathbf{r})\chi(\sigma_1, \sigma_2)$ has the appropriate symmetry corresponding to the spin s of the two indistinguishable particles. The general rule given in the Landau & Lifschitz textbook (Vol. III, par. 137) is that if the total spin S of the two particles is an even number ($S = 0, 2, \ldots$), $\psi(\mathbf{r})$ must be even and the asymptotic solution of the Schrödinger equation must be taken in the form

$$\psi^{(+)}_{\mathbf{k}}(\mathbf{r}) = e^{i\mathbf{k}\cdot\mathbf{r}} + e^{-i\mathbf{k}\cdot\mathbf{r}} + [f(k,\theta) + f(k,\pi-\theta)]\,\frac{e^{ikr}}{r}\,,$$

whereas when the total spin S is an odd number ($S = 1, 3, \ldots$), $\psi(\mathbf{r})$ must be odd and the asymptotic solution of the Schrödinger equation must be constructed as

$$\psi^{(+)}_{\mathbf{k}}(\mathbf{r}) = e^{i\mathbf{k}\cdot\mathbf{r}} - e^{-i\mathbf{k}\cdot\mathbf{r}} + [f(k,\theta) - f(k,\pi-\theta)]\,\frac{e^{ikr}}{r}\,.$$

Indeed, if the two particles have no spin, $s = 0$, then also $S = 0$ (so it is even) and $\psi(\mathbf{r})$ must be even; similarly, if $s = \frac{1}{2}$ but $S = 0$ (again S is even), which means that $\chi(\sigma_1, \sigma_2) = -\chi(\sigma_2, \sigma_1)$ and $\psi(\mathbf{r})$ must again be even etc. Thus the proper scattering amplitude, the modulus squared of which gives the differential cross section, is

$$f(k,\theta) + f(k,\pi-\theta)\,,$$

when $S = 0, 2, \ldots$ and

$$f(k,\theta) - f(k,\pi-\theta)\,,$$

when $S = 1, 2, \ldots$ and the differential scattering cross sections are given either by $|f(k,\theta) + f(k,\pi-\theta)|^2$ or by $|f(k,\theta) - f(k,\pi-\theta)|^2$.

To relate the ordinary scattering amplitude $f(|\mathbf{k}|,\theta)$ to the matrix element of the operator T_0 defined in (7.24) and to the amplitude $\mathcal{A}$ (it is $\mathcal{A}$ that that is naturally obtained in the formalism based on the Dyson expansion), which is obtained from the element $t_{\beta\alpha}$ defined by (7.25) after factoring out from it $(2\pi)^3\delta^{(3)}((\mathbf{P}_\beta - \mathbf{P}_\alpha)/\hbar)$ (see (7.80)) we consider the two-body spin-independent interaction (in the formalism of second quantization)

$$V_{\rm int} = \frac{1}{2}\int d^3\mathbf{x}\int d^3\mathbf{y}\,\psi^\dagger_\alpha(\mathbf{x})\psi^\dagger_\beta(\mathbf{y})\,V_{\rm pot}(|\mathbf{x}-\mathbf{y}|)\,\psi_\beta(\mathbf{y})\psi_\alpha(\mathbf{x})\,, \tag{7.69}$$

with a general two-body translationally and rotationally invariant interaction potential $V_{\rm pot}$ and the field operators constructed as in (5.46). The first nontrivial term of the

[27] The factor $\exp(i\mathbf{P}\cdot\mathbf{R}/\hbar)$ in which in this case $\mathbf{R} = \frac{1}{2}(\mathbf{r}_1 + \mathbf{r}_2)$ is obviously symmetric.

expansion of the formula (7.63) with the initial and final states in which $|\alpha_0\rangle$ and $|\beta_0\rangle$ are the states (eigenstates of H_0) of two free fermions:

$$|\alpha_0\rangle = a^\dagger_{\sigma_2}(\mathbf{k}_2)\, a^\dagger_{\sigma_1}(\mathbf{k}_1)|0\rangle \equiv a^\dagger_2 a^\dagger_1|0\rangle\,,$$
$$|\beta_0\rangle = a^\dagger_{\sigma'_2}(\mathbf{k}'_2)\, a^\dagger_{\sigma'_1}(\mathbf{k}'_1)|0\rangle \equiv a^\dagger_{2'} a^\dagger_{1'}|0\rangle\,, \tag{7.70}$$

then reads[28]

$$S_{\beta\alpha} = \delta_{\beta\alpha} - \frac{i}{2\hbar}\int d^3\mathbf{x}\int d^3\mathbf{y}\int dt\, V_{\rm pot}(\mathbf{x}-\mathbf{y})\langle a_{1'}a_{2'}\,\mathrm{T}[\psi^{\dagger I}_\sigma(t,\mathbf{x})\psi^{\dagger I}_{\bar\sigma}(t,\mathbf{y})\psi^I_{\bar\sigma}(t,\mathbf{y})\psi^I_\sigma(t,\mathbf{x})]a^\dagger_2 a^\dagger_1\rangle,$$

where $\langle\ldots\rangle$ stands for the expectation value of $\ldots$ in the state $|\mathrm{void}\rangle$. As the chronological product is irrelevant here (all operators under it are taken at the same instant), the matrix element can be easily worked out, after moving all the c-number factors and integrals involved in the field operators outside the brackets, by just (anti)commuting the creation operators to the left and the annihilation operators to the right, so that they ultimately act on the $|\mathrm{void}\rangle$ vectors giving zeroes. These operations produce Dirac delta functions depending on the wave vectors and the Kronecker deltas in the spin labels. Furthermore, representing the potential $V_{\rm pot}(\mathbf{x}-\mathbf{y})$ as a Fourier transform as in (**??**) allows to explicitly perform the integrals over the time t and the positions $\mathbf{x}$ and $\mathbf{y}$. This produces two additional three-dimensional Dirac delta functions depending on the wave vectors and one delta function expressing the conservation of the linear combination of the frequencies $\omega_{\mathbf{k}}$ (originating from the field operators). Finally using the Dirac deltas all integrals over the wave vectors originating from the field operators and from the Fourier transform of $V_{\rm pot}(\mathbf{x}-\mathbf{y})$ can be eliminated and one obtains[29]

$$S_{\beta\alpha} = \delta_{\beta\alpha} - \frac{i}{\hbar}\,(2\pi)^4\delta(\omega_{\mathbf{k}'_1}+\omega_{\mathbf{k}'_2}-\omega_{\mathbf{k}_1}-\omega_{\mathbf{k}_2})\,\delta^{(3)}(\mathbf{k}'_1+\mathbf{k}'_2-\mathbf{k}_1-\mathbf{k}_2)\,\mathcal{A}_{\beta\alpha}\,, \tag{7.71}$$

with[30]

$$\mathcal{A}_{\beta\alpha} = \delta_{\sigma_1\sigma'_1}\delta_{\sigma_2\sigma'_2}\,\tilde V_{\rm pot}(|\mathbf{k}'_1-\mathbf{k}_1|) \pm \delta_{\sigma_1\sigma'_2}\delta_{\sigma_2\sigma'_2}\,\tilde V_{\rm pot}(|\mathbf{k}'_2-\mathbf{k}_1|)\,, \tag{7.72}$$

[28] $V^I_{\rm int}(t)$ defined in (7.64) is obtained by simply inserting in (7.69) the field operators taken in the interaction picture (instead of the Schrödinger picture ones) which amounts to replacing in the formulae (5.46) $\pm i\mathbf{k}\cdot\mathbf{x}$ by $\mp i(\omega_{\mathbf{k}}t - \mathbf{k}\cdot\mathbf{x})$ where $\omega_{\mathbf{k}} = \hbar\mathbf{k}^2/2m$.

[29] It is instructive to check the dimensions: since the wave vectors have dimension $[L]^{-1}$,

$$\delta_{\alpha\beta} = (2\pi)^6[\delta_{\sigma'_1\sigma_1}\delta_{\sigma'_2\sigma_2}\delta^{(3)}(\mathbf{k}'_1-\mathbf{k}_1)\delta^{(3)}(\mathbf{k}'_2-\mathbf{k}_2) \pm \delta_{\sigma'_1\sigma_2}\delta_{\sigma'_2\sigma_1}\delta^{(3)}(\mathbf{k}'_1-\mathbf{k}_2)\delta^{(3)}(\mathbf{k}'_2-\mathbf{k}_1)]\,,$$

has dimension $[L]^6$ and the second term has dimension (the dimension of ω is $[T]^{-1}$) $\hbar^{-1}[T][L]^3[L]^2[L]\times$energy $= (\hbar c)^{-1}c[T][L]^3[L]^2[L]\times$energy which is also $[L]^6$ because the dimension of $\hbar c$ is energy$\times[L]$.

[30] If the potential is of the Yukawa form $V_{\rm pot}(\mathbf{x}) = \frac{g^2}{|\mathbf{x}|}\,e^{-M_\phi c|\mathbf{x}|/\hbar}$ with the coupling constant g (g^2 has the physical dimension of energy times length and M_ϕ has the mass dimension - such a potential arises as a low energy limit of a relativistic interation mediated by a boson of mass M_ϕ) then

$$\mathcal{A}_{\beta\alpha} = 4\pi\, g^2\left[\frac{\delta_{\sigma'_1\sigma_1}\delta_{\sigma'_2\sigma_2}}{(\mathbf{k}'_1-\mathbf{k}_1)^2 + M^2_\phi c^2/\hbar^2} \pm \frac{\delta_{\sigma'_1\sigma_2}\delta_{\sigma'_2\sigma_1}}{(\mathbf{k}'_2-\mathbf{k}_1)^2 + M^2_\phi c^2/\hbar^2}\right].$$

where the + sign applies to bosons and the − sign to fermions. Using the CMS kinematics (in the four-vector notation):

$$\begin{aligned}
k_1^\mu &= (E,\, 0,\, 0,\, |\mathbf{k}|)\,, \\
k_2^\mu &= (E,\, 0,\, 0,\, -|\mathbf{k}|)\,, \\
(k_1')^\mu &= (E,\, 0,\, |\mathbf{k}|s_\theta,\, |\mathbf{k}|c_\theta)\,, \\
(k_2')^\mu &= (E,\, 0,\, -|\mathbf{k}|s_\theta,\, -|\mathbf{k}|c_\theta)\,,
\end{aligned}$$

one gets $|\mathbf{k}_1 - \mathbf{k}_1'| = 2|\mathbf{k}|\sin(\theta/2)$, $|\mathbf{k}_1 - \mathbf{k}_2'| = 2|\mathbf{k}|\cos(\theta/2) = 2|\mathbf{k}|\sin((\pi-\theta)/2)$, so that

$$\mathcal{A}_{\beta\alpha} = \delta_{\sigma_1'\sigma_1}\delta_{\sigma_2'\sigma_2}\,\tilde{V}_{\rm pot}(|\mathbf{k}|\sin(\theta/2)) \pm \delta_{\sigma_1'\sigma_2}\delta_{\sigma_2'\sigma_1}\tilde{V}_{\rm pot}(|\mathbf{k}|\sin((\pi-\theta/2))\,.$$

The two terms of $\mathcal{A}_{\beta\alpha}$ must be therefore proportional to the first terms in the expansions of the nonrelativistic amplitudes $f(k,\theta)$ and $f(k,\pi-\theta)$.

One can now check the quoted rules of Landau & Lifschitz for $f(k,\theta) \pm f(k,\pi-\theta)$. If the (identical) scattering particles are spinless bosons, the amplitude $\mathcal{A}_{\beta\alpha}$ (7.72) comes with the plus sign between its two terms and the rule is obviously satisfied (the total spin $S=0$). Consider now the scattering of spin $s=1/2$ fermions. How it happens that when the total spin S is even (that is $S=0$), the pieces with θ and $\pi-\theta$ combine with the plus sign (despite the minus sign between the two terms in (7.72))? Denote $S_\theta \equiv \tilde{V}_{\rm pot}(|\mathbf{k}|\sin(\theta/2))$ and $C_\theta \equiv \tilde{V}_{\rm pot}(|\mathbf{k}|\cos(\theta/2))$ and consider the scattering amplitudes with different spin configurations. One finds:

$$\begin{aligned}
\mathcal{A}(\uparrow\uparrow \;\longrightarrow\; \uparrow\uparrow) &= S_\theta - C_\theta\,, \\
\mathcal{A}(\downarrow\downarrow \;\longrightarrow\; \downarrow\downarrow) &= S_\theta - C_\theta\,.
\end{aligned}$$

In these two cases both terms contribute because all the spin dependent Kronecker deltas in (7.72) are nonzero. In contrast,

$$\begin{aligned}
\mathcal{A}(\uparrow\downarrow \;\longrightarrow\; \uparrow\downarrow) &= S_\theta\,, \\
\mathcal{A}(\downarrow\uparrow \;\longrightarrow\; \downarrow\uparrow) &= S_\theta\,, \\
\mathcal{A}(\uparrow\downarrow \;\longrightarrow\; \downarrow\uparrow) &= -C_\theta\,, \\
\mathcal{A}(\downarrow\uparrow \;\longrightarrow\; \uparrow\downarrow) &= -C_\theta\,,
\end{aligned}$$

because when the initial (and therefore also final) spins are opposite, only one of the two terms contributes. Now, if the two initial fermions are in the $S=0$ total spin state the final ones must be in the same spin state too, because spin is separately (independently of the orbital angular momentum) preserved and to obtain the corresponding scattering amplitude one must combine the four amplitudes as follows:

$$\begin{aligned}
\mathcal{A}\left(\frac{\uparrow\downarrow - \downarrow\uparrow}{\sqrt{2}} \longrightarrow \frac{\uparrow\downarrow - \downarrow\uparrow}{\sqrt{2}}\right) = \frac{1}{2}\,[&\mathcal{A}(\uparrow\downarrow\longrightarrow\uparrow\downarrow) - \mathcal{A}(\uparrow\downarrow\longrightarrow\downarrow\uparrow) \\
&-\mathcal{A}(\downarrow\uparrow\longrightarrow\uparrow\downarrow) + \mathcal{A}(\downarrow\uparrow\longrightarrow\downarrow\uparrow)] = S_\theta + C_\theta\,.
\end{aligned}$$

And indeed the quoted rule is obeyed. Instead, if the two fermions were in the $S = 1$ total spin state with $S^z = 0$, the amplitude would be

$$\mathcal{A}\left(\frac{\uparrow\downarrow + \downarrow\uparrow}{\sqrt{2}} \longrightarrow \frac{\uparrow\downarrow + \downarrow\uparrow}{\sqrt{2}}\right) = \frac{1}{2}\left[\mathcal{A}(\uparrow\downarrow\longrightarrow\uparrow\downarrow) + \mathcal{A}(\uparrow\downarrow\longrightarrow\downarrow\uparrow)\right.$$
$$\left. + \mathcal{A}(\downarrow\uparrow\longrightarrow\uparrow\downarrow) + \mathcal{A}(\downarrow\uparrow\longrightarrow\downarrow\uparrow)\right] = S_\theta - C_\theta\,,$$

again in agreement with the general rule. In the similar manner one can check the working of these rules in the scattering of say two identical spin 1 bosons.

Finally using the considered approximation to the complete amplitude, it is straightforward to establish the (generally valid) relation between the amplitude $\mathcal{A}$ in (7.71) and the ordinary scattering amplitude $f(k,\theta)$. Taking opposite directions of spins of the two scattering particles, one obtains

$$\mathcal{A} = \tilde{V}_{\rm pot}(|\mathbf{k}'_1 - \mathbf{k}|)\,.$$

The corresponding scattering amplitude $f(k,\theta)$ (without the spin factors) defined by the asymptotic form (7.68) of the solution of the Schrödinger equation (7.67) can be computed using the Born approximation which gives

$$f_{\rm Born}(k,\theta) = -\frac{m_{\rm red}}{2\pi\hbar^2}\int d^3\mathbf{r}\, e^{-i(\mathbf{k}'-\mathbf{k})\cdot\mathbf{r}}\, V_{\rm pot}(\mathbf{r}) = -\frac{m_{\rm red}}{2\pi\hbar^2}\,\tilde{V}_{\rm pot}(|\mathbf{k}'_1 - \mathbf{k}|)\,.$$

Comparing the two amplitudes one finds that the rule is

$$f(k,\theta) = -\frac{m_{\rm red}}{2\pi\hbar^2}\,\mathcal{A}(k,\theta)\,. \tag{7.73}$$

If the two scattering particles are identical and both have mass m, then $m_{\rm red} = m/2$. It should be clear that higher orders of the Dyson expansion of the formula (7.63) will yield higher order terms of the Born expansion of the scattering amplitude. The great advantage of the formalism based on the second quantization formalism is that symmetry requirements are automatically fulfilled (they are encoded in the properties of the field operators). One is also not bound to the center of mass frame - particles in the initial state (7.70) can have arbitrary momenta; if the scattering particles are different, the reduced mass $m_{\rm red}$ will come out automatically from the kinematics of the process. Moreover one can easily consider also spin dependent interactions (the arguments bases on separate spin conservation are then invalid and the analysis in the language of the wave functions becomes difficult).

In general, the elastic scattering amplitude $f(k,\theta)$ defined by (7.68) can be expressed through the partial wave shifts as

$$f(k,\theta) = \frac{1}{k}\sum_{\ell=0}^{\infty}(2\ell+1)\, e^{i\delta_\ell(k)}\sin\delta_\ell(k)\, P_\ell(\cos\theta)$$
$$= \sum_{\ell=0}^{\infty}(2\ell+1)\,\frac{1}{k\cot\delta_\ell - ik}\,P_\ell(\cos\theta)\,,$$