6 The Poincaré group and state-vectors representing particles

In this chapter we make the first step towards formulating quantum relativistic theories of interacting particles. We first recall properties of the Poincaré group symmetry generators which should act in the Hilbert space of any relativistic quantum theory. Next, assuming a theory which is relativistic in this sense is given we (following Wigner) single out a class of Hamiltonian and the three-momentum operator joint generalized (i.e. non-normalizable) eigenvectors which, if exist in the theory Hilbert space, should be identified with vectors representing states representing single stable particles and investigate their transformations under Lorentz and discrete (space reflection and time reversal) transformations stressing the essential difference between properties of states of massless and massive particles. Particles are therefore identified with special types of irreducible representations of the Poincaré group; this definition of particles is very convenient, as the problem which particles are truly "elementary", and which are not, is irrelevant for it.

If the dynamics set by a given Hamiltonian is that of free particles (the Hamiltonian has no interaction term) multiparticle state-vectors constructed as tensor products (as in Chapter 5) of one-particle states are also its (generalized) eigenvectors (this property can be taken for the definition of the free Hamiltonian). Some details of the kinematical characterization of two-particle states constructed in this way are recalled in Section 6.4. (The properties of the one-particle state-vectors remain the same whether the Hamiltonian is free or not.) In typical scattering experiments particles which are prepared before the collission and detected afterwards behave, long before and long after the reaction, as free and the full Hamiltonian of a system of interacting particles should possess (generalized) eigenvectors which correspond to such situations. The sense in which such eigenvectors can be related to multiparticle eigenvectors of an appropriate free Hamiltonian will be elucidated in Chapter 7.

6.1 The Poincaré group

Einstein's principle of relativity singles out a class of inertial frames which are all equivalent to each other: anyone of them can be chosen for a reference frame for physical phenomena and, as states the first Einstein's postulate, the formal mathematical form of the physical laws is the same in all of them. The second Einstein's postulate is the equality of the speed of light measured in all these systems. The two postulates form together the basis of the special theory of relativity.

If $x^{\mu} = (ct, x, y, z)$ are the coordinates of an event in the inertial frame $\mathcal{O}$ and $x^{\mu\prime} = (ct', x', y', z')$ are the coordinates of the same event in another inertial frame $\mathcal{O}'$ then the equality of the speed of light measured in $\mathcal{O}$ and $\mathcal{O}'$ is ensured[1] if

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = g_{\mu\nu} dx^{\mu\prime} dx^{\nu\prime} = ds'^2\,, \tag{6.1}$$

where $g_{\mu\nu} = \mathrm{diag}(+1, -1, -1, -1)$. This can be translated into the condition

$$g_{\mu\nu} \frac{\partial x^{\mu\prime}}{\partial x^\lambda} \frac{\partial x^{\nu\prime}}{\partial x^\kappa} = g_{\lambda\kappa}\,, \tag{6.2}$$

the most general solution of which is

$$x^{\mu\prime} = \Lambda^\mu{}_\nu x^\nu - a^\mu\,. \tag{6.3}$$

$\Lambda^\mu{}_\nu$ is here a constant matrix satisfying the relation

$$g_{\mu\nu} \Lambda^\mu{}_\lambda \Lambda^\nu{}_\kappa = g_{\lambda\kappa}\,, \qquad \text{that is,} \qquad \Lambda^T \cdot g \cdot \Lambda = g\,, \tag{6.4}$$

and a^μ is an arbitrary constant four-vector.

The transformations (6.3) form the Poincaré group (in the usual sense: to each element there is the inverse one, the composition of any two transformations is another transformation, etc.). Its elements are the transformations $S(\Lambda,\, a)$. Their composition law reads

$$S(\Lambda_2,\, a_2) \cdot S(\Lambda_1,\, a_1) = S(\Lambda_2 {\cdot} \Lambda_1,\, a_2 + \Lambda_2 {\cdot} a_1)\,. \tag{6.5}$$

The inverse transformation is

$$S^{-1}(\Lambda,\, a) = S(\Lambda^{-1},\; -\Lambda^{-1} {\cdot} a)\,. \tag{6.6}$$

The Poincaré group is a semisimple product of the Lorentz group L (transformations represented by Λ) and of the Abelian group of translations. The Lorentz group is, similarly as the Poincaré one, a Lie group which can be identified with a differentiable manifold. It consists of four disconnected components:

$$L = L_+^\uparrow \cup\ L_-^\uparrow \cup\ L_+^\downarrow \cup\ L_-^\downarrow\,. \tag{6.7}$$

From (6.4) it follows that $(\det\Lambda)^2 = 1$ and $(\Lambda^0{}_0)^2 \geq 1$. The proper ortochronous part $L_+^\uparrow$ of the Lorentz group L consists of matrices Λ having the determinant equal $+1$ and $\Lambda^0{}_0 \geq 1$. Since by a continuous change of parameters of Λ one cannot alter the sign of the determinant, nor arrive at $\Lambda^0{}_0 \leq -1$ starting from $\Lambda^0{}_0 \geq 1$, it is clear that the other components $L_-^\uparrow$, $L_+^\downarrow$ and $L_-^\downarrow$ of the Lorentz group (corresponding respectively to $\det\Lambda = -1$ and $\Lambda^0{}_0 \geq 1$, $\det\Lambda = +1$ and $\Lambda^0{}_0 \leq -1$, $\det\Lambda = -1$ and and $\Lambda^0{}_0 \leq -1$) must

[1] Since the light propagation is specified by the condition $ds^2 = 0$, the equality of the speed of light measured in both systems alone imposes in fact only a weaker condition, namely $ds'^2 = f(x) ds^2$, which leads to the conformal group.

be disconnected (as manifolds). Any Lorentz transformation not belonging to the proper ortochronous component $L_+^\uparrow$ can be obtained by composing a proper ortochronous matrix Λ ($\det\Lambda = +1$ and $\Lambda^0{}_0 \geq 1$) with one of the matrices P, T or $P{\cdot}T$ where

$$P^\mu{}_\nu = \mathrm{diag}(+1, -1, -1, -1)\,, \qquad T^\mu{}_\nu = \mathrm{diag}(-1, +1, +1, +1)\,. \tag{6.8}$$

Taking the passive view, if a state of a system seen by an (inertial) observer $\mathcal{O}$ is represented in a relativistic theory Hilbert space by the vector $|\Psi\rangle$, the state of the same system as seen by another observer $\mathcal{O}'$ whose reference frame is related to the one of $\mathcal{O}$ by the transformation $S(\Lambda, a)$ is represented by the vector

$$|\Psi'\rangle = U(\Lambda, a)|\Psi\rangle\,. \tag{6.9}$$

The operators[2] $U(\Lambda, a)$ form a representation of the Poincaré group (or, more precisely, of its universal covering group, if one wants to avoid projective representations) in the Hilbert space and satisfy

$$U(\Lambda_2, a_2)U(\Lambda_1, a_1) = U(\Lambda_2{\cdot}\Lambda_1, a_2 + \Lambda_2{\cdot}a_1)\,. \tag{6.10}$$

The algebra of the generators of the Poincaré group can be found by considering infinitesimal transformations

$$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu - \omega^\mu{}_\nu\,, \qquad a^\mu = \epsilon^\mu\,. \tag{6.11}$$

From the condition (6.4) it follows that $\omega_{\mu\nu} \equiv g_{\mu\lambda}\omega^\lambda{}_\nu$ is an antisymmetric 4×4 matrix

$$\omega_{\mu\nu} = -\omega_{\nu\mu}\,. \tag{6.12}$$

This shows that the Poincaré group has $6 + 4 = 10$ independent (real) parameters. As in Section 4.1, operators U corresponding to infinitesimal transformations can always be written in the form

$$U(I - \omega, \epsilon) \approx \hat{1} + \frac{i}{2}\,\omega_{\mu\nu}J^{\mu\nu} - i\epsilon_\mu P^\mu, \tag{6.13}$$

where the 10 generators $J^{\mu\nu}$ are Hermitian[3] operators: $(J^{\mu\nu})^\dagger = J^{\mu\nu} = -J^{\nu\mu}$, $P^{\mu\dagger} = P^\mu$.

In order to find the algebra of these generators, that is the structure constants of the Poincaré group, instead of using the method described in Section 4.2, it is easier to

[2]Since any proper ortochronous Lorentz transformation $\Lambda \in L_+^\uparrow$ can be continuously deformed to the identity transformation which in the Hilbert space is represented by the unit operator, all $U(\Lambda, a)$ corresponding to transformations Λ belonging to $L_+^\uparrow$ must be unitary and linear. We will see that the Hilbert space symmetry operator corresponding to P must also be unitary and linear, but the one corresponding to T must be antiunitary and antilinear.

[3]Since the Lorentz group is not compact, it has also non-unitary representations. However, $U(\Lambda, a)$ have to be symmetry operators which preserve probabilities. According to the Wigner theorem (Section 4.1) they must, therefore, be either unitary or antiunitary.

proceed as follows (the same method has been adopted in Secion 4.3). By writing the product $U(\Lambda,\, a)U(I-\omega,\, \epsilon)U^{-1}(\Lambda,\, a)$ on one side in the form

$$U(\Lambda,\, a)\left(\hat{1}+\frac{i}{2}\,\omega_{\mu\nu}J^{\mu\nu}-i\epsilon_\mu P^\mu\right)U^{-1}(\Lambda,\, a)\,, \tag{6.14}$$

and on the other side (using the group composition rules (6.5) and (6.6)) as

$$\begin{aligned} &U(I-\Lambda{\cdot}\omega{\cdot}\Lambda^{-1},\ \Lambda{\cdot}\epsilon+\Lambda{\cdot}\omega{\cdot}\Lambda^{-1}{\cdot}a) \\ &\qquad\approx \hat{1}+\frac{i}{2}\left(\Lambda{\cdot}\omega{\cdot}\Lambda^{-1}\right)_{\mu\nu}J^{\mu\nu}-i\left(\Lambda{\cdot}\epsilon+\Lambda{\cdot}\omega{\cdot}\Lambda^{-1}{\cdot}a\right)_\mu P^\mu, \end{aligned} \tag{6.15}$$

and equating the coefficients of $\omega_{\mu\nu}$ and ϵ_μ on both sides we find[4]

$$U(\Lambda,\, a)P^\mu U^{-1}(\Lambda,\, a)=(\Lambda^{-1})^\mu{}_\lambda P^\lambda\,, \tag{6.16}$$

$$U(\Lambda,\, a)J^{\mu\nu}U^{-1}(\Lambda,\, a)=(\Lambda^{-1})^\mu{}_\lambda(\Lambda^{-1})^\nu{}_\kappa\left(J^{\lambda\kappa}+a^\lambda P^\kappa-a^\kappa P^\lambda\right), \tag{6.17}$$

where we have used the relation (which follows from (6.4))

$$\Lambda_\nu{}^\mu\equiv g_{\nu\lambda}\,\Lambda^\lambda{}_\kappa\, g^{\kappa\mu}=(\Lambda^{-1})^\mu{}_\nu\,. \tag{6.18}$$

The results (6.16), (6.17) show that the four operators P^μ transform as a four-vector whereas the six operators $J^{\mu\nu}$ transform inhomogeneously; the inhomogeneity allows to identify them (or at least their spatial components J^{kl}) with the angular momentum operators. Setting in the formulae (6.16), (6.17) $\Lambda=I-\omega$ and $a=\epsilon$ and writing $U(\Lambda,\, a)$ as in (6.13) we arrive at the commutation rules satisfied by the generators of the Poincaré group:

$$\begin{aligned} &[P^\mu,\ P^\nu]=0\,,\\ &\left[J^{\mu\nu},\ P^\lambda\right]=i\left(P^\mu g^{\nu\lambda}-P^\nu g^{\mu\lambda}\right),\\ &\left[J^{\kappa\lambda},\ J^{\mu\nu}\right]=i\left(J^{\kappa\nu}g^{\lambda\mu}-J^{\kappa\mu}g^{\lambda\nu}-J^{\lambda\nu}g^{\kappa\mu}+J^{\lambda\mu}g^{\kappa\nu}\right). \end{aligned} \tag{6.19}$$

It can be shown, that possible central charges, which could appear in these commutation rules can all be removed by redefining the generators P^μ and $J^{\mu\nu}$. Furthermore, all phase factors (discussed in Section 4.2) which could arise in the composition rule (6.10) of the symmetry operators for topological reasons (see below) are removed by taking for the symmetry group the $SL(2,C)$ group - the universal covering of the Lorentz group in four space-time dimensions.[5]

To distinguish the generators of rotations and of Lorentz boosts it is convenient to introduce the three-dimensional notation

$$J^i\equiv\frac{1}{2}\epsilon^{ijk}J^{jk}\,,\qquad K^i\equiv J^{0i}\,,\qquad H\equiv P^0\,. \tag{6.20}$$

[4]The relations for $U^{-1}P^\mu U$ and $U^{-1}J^{\mu\nu}U$ can be obtained either directly, by repeating the steps, or by substituting here $\Lambda\to\Lambda^{-1}$, $a\to-\Lambda^{-1}\cdot a$.

[5]In general, universal coverings of $SO(1,D)$ and $O(1,D)$ are the $Spin(1,D)$ and $Pin(1,D)$ groups, respectively. However $Spin(1,3)$ is isomorhic to $SL(2,C)$.

In this notation the commutation relations (6.19) take the form

$$\begin{aligned}
&\left[J^i,\ J^j\right] = i\epsilon^{ijk}J^k\,, && \left[K^i,\ K^j\right] = -i\epsilon^{ijk}J^k\,,\\
&\left[J^i,\ K^j\right] = i\epsilon^{ijk}K^k\,, && \left[K^i,\ P^j\right] = -i\delta^{ij}H\,,\\
&\left[J^i,\ P^j\right] = i\epsilon^{ijk}P^k\,, && \left[K^i,\ H\right] = -iP^i\,,\\
&\left[J^i,\ H\right] = 0\,, && \left[P^i,\ P^j\right] = 0\,,\\
& && \left[P^i,\ H\right] = 0\,.
\end{aligned} \tag{6.21}$$

The first of these relations allows to identify J^i with the total angular momentum operators, which generate rotations. The operators K^i generate boosts. P^i and H are the total momentum operator and the Hamiltonian. The crucial difference with respect to the commutation rules (4.48) satisfied by the Galileo group generators are the commutators $[K^i, P^j]$ and $[K^i, H]$: they are nonzero here and this implies that either $\mathbf{P}$ or $\mathbf{K}$ (or both) must be modified when the form of the Hamiltonian is changed.

6.2 The little group and one-particle states

We now assume a relativistic quantum theory is given, that is the operators $\mathbf{J}$, $\mathbf{K}$, $\mathbf{P}$ and H, satisfying the rules (6.21) are realized in some Hilbert space $\mathcal{H}$, and use their properties (6.21) to identify state-vectors which will represent stable particles. Such vectors should exist in the Hilbert space of any theory which is a theory of free or of interacting particles because by definition a single stable particle cannot disappear nor transform spontaneously into another particle or into several other particles (dacay).

Out of the generators of the Poincaré group one can construct two operators:

$$P^2 \equiv P_\mu P^\mu\,, \qquad \text{and} \qquad W^2 \equiv W_\mu W^\mu\,, \tag{6.22}$$

which commute with all the generators P^μ and $J^{\mu\nu}$. Here

$$W^\mu = -\frac{1}{2}\,\epsilon^{\mu\nu\lambda\rho}J_{\nu\lambda}P_\rho\,, \tag{6.23}$$

is the Pauli-Lubański vector (we use $\epsilon^{0123} = -\epsilon_{0123} = 1$). Eigenvalues of $P^\mu P_\mu$ and $W^\mu W_\mu$ label, therefore, irreducible representations of the Poincaré group. Below we consider a subspace of the full Hilbert space on which $P^\mu P_\mu$ and $W^\mu W_\mu$ have fixed eigenvalues. The eigenvalues m^2 of $P^\mu P_\mu$ can a priori be any real numbers. Physically, however, one is interested only in $m^2 \geq 0$, that is, we assume the theory we consider does not predict the existence of tachyons (in other words, that in the Hilbert space of the considered theory there are no $P^\mu P_\mu$ eigenvectors corresponding to $m^2 < 0$; if the Hamiltonian is not one of free particles this is, of course, a dynamical question). Possible eigenvalues of $W^\mu W_\mu$ will be determined below.

In agreement with the physical experience, vectors representing states of single particles should be defined as generalized, i.e. non-normalizable eigenvectors of the operators P^μ (i.e. as eigenvectors of H and $\mathbf{P}$):

$$P^\mu|\mathbf{p},\sigma\rangle = p^\mu|\mathbf{p},\sigma\rangle\,, \qquad \text{so that} \qquad U(1,a)|\mathbf{p},\sigma\rangle = e^{-ia_\mu p^\mu}|\mathbf{p},\sigma\rangle\,. \tag{6.24}$$

The symbol σ labels here different vectors having the same P^μ eigenvalue $p^\mu = (E(\mathbf{p}),\, \mathbf{p})$ with $E(\mathbf{p}) = \sqrt{\mathbf{p}^2 + m^2}$ (we assume, there are no states with the P^μ eigenvalue $p^\mu = (-E(\mathbf{p}),\, \mathbf{p})$). Again, in agreement with the physical notion of a particle, labels σ labeling state-vectors supposed to represent single particles assume by definition only *discrete* values. (Vectors representing e.g. states of several particles can also be chosen as eigenvectors of P^μ - see Section 6.4 - but in this case the counterpart of the label σ takes on continuous values characterizing the relative motion of the particles; note however, that according to the definition adopted an atom in the ground state *is* a particle despite the fact that electron(s) can be knocked out of it!) In addition, one requires that in a given theory the number of different eigenvalues of the operator $P_\mu P^\mu$ on states representing single particles is finite (the theory predicts existence of only a finite number of different particles). Using the transformation properties (6.16) of the four-momentum operator one can write ($U(\Lambda) = U(\Lambda, 0)$)

$$\begin{aligned} P^\mu U(\Lambda)|\mathbf{p},\sigma\rangle &= U(\Lambda)\, U^{-1}(\Lambda) P^\mu U(\Lambda)|\mathbf{p},\sigma\rangle \\ &= U(\Lambda)\, \Lambda^\mu{}_\nu P^\nu|\mathbf{p},\sigma\rangle = \Lambda^\mu{}_\nu p^\nu U(\Lambda)|\mathbf{p},\sigma\rangle\,. \end{aligned} \tag{6.25}$$

This shows that $U(\Lambda)|\mathbf{p},\sigma\rangle$ is the eigenvector of P^μ with the eigenvalue $p^\mu_\Lambda \equiv \Lambda^\mu{}_\nu p^\nu$. Hence, it can be written as a general superposition of such state-vectors:

$$U(\Lambda)|\mathbf{p},\sigma\rangle = \sum_{\sigma'} |\mathbf{p}_\Lambda,\sigma'\rangle\, C_{\sigma'\sigma}(\Lambda, p)\,. \tag{6.26}$$

By appropriately choosing the basis in the Hilbert space, one can always make the matrix $C_{\sigma'\sigma}$ block diagonal, i.e. decompose the general representation of the Lorentz group acting on one-particle states into irreducible representations. Stable particles are then identified with irreducible representations of the Lorentz group. Of course, different particles, like e.g. e^-, e^+, can correspond to isomorphic representations of the Lorentz group (they are distinguished by a charge operator which corresponds to internal symmetries; instead the representations corresponding to μ^- and e^- differ by the values of the Poincaré group Casimir operator $P^2 = P^\mu P_\mu$).

To investigate possible forms of irreducible matrices $C_{\sigma'\sigma}$ and possible eigenvalues of the second Racah operator $W^\mu W_\mu$, one has to give a meaning to the label σ. To this end, we notice that all components of W^μ commute with all components of the P^μ operator and, therefore, one their linear combination (W^μ components do not commute with one another) can be diagonalized simultaneously with the P^μ operators and used for this purpose.[6] One possibility is simply to choose the operator

$$W^0 = \mathbf{J}\cdot\mathbf{P}\,, \tag{6.27}$$

[6] Normally - see Chapter 4 - one uses for this purposes commuting linear combinations of the symmetry

the eigenvalues of which will be written as $|\mathbf{p}|\lambda$, and to identify σ with the *helicity* λ - the projection of the total angular momentum onto the particle's three-momentum. W^0, that is, the helicity quantum number λ, will be used to naturally label states of massless particles. While helicity is physically most useful also as a quantum number of massive particles (it is helicity which is usually most easily accessible to experimental determination), the definition of eigenvectors of the W^0 operator has some subtleties which makes its use a bit complicated, especially in the case of massive particles. Therefore, to label states of a particle of mass $m \neq 0$ one uses the operator $-s^\mu_p W_\mu/m$ with some four-vector s^μ_p the precise form of which will be specified below.

We begin by noting that[7] the only functions of p^μ invariant with respect to proper ortochronous Lorentz transformations are functions of p^2 and, when $p^2 \geq 0$, also of $\text{sgn}(p^0)$. For all four-momenta p^μ with a given p^2 value (the $P_\mu P^\mu$ eigenvalue - the particle mass squared) and $\text{sgn}(p^0)$, if $p^2 > 0$, we choose a *standard* four-momentum k^μ and a *standard* proper ortochronous Lorentz transformation, call it L_p, which transforms the standard four-momentum k^μ into a given four-momentum p^μ:

$$(L_p)^\mu{}_\nu k^\nu = p^\mu\,. \tag{6.28}$$

Next, for arbitrary four-momenta $p^\mu = (\pm E(\mathbf{p}), \mathbf{p})$, where $E(\mathbf{p}) = \sqrt{\mathbf{p}^2 + m^2}$ (with the chosen $p^2 = m^2$ and $\text{sgn}(p^0)$, if $p^2 > 0$) we *define* the state-vectors $|\mathbf{p}, \sigma\rangle$ by the formula

$$|\mathbf{p}, \sigma\rangle = \mathcal{N}_p\, U(L_p)|\mathbf{k}, \sigma\rangle\,, \tag{6.29}$$

in which $\mathcal{N}_p$ is a normalization factor which will be fixed below. Thus, it is sufficient to give the label σ a meaning in the frame in which the four-momentum of the particle is the standard four-momentum k^μ. The standard four-momentum k^μ of massive particles will correspond to the particle at rest: $k^0 = m$, $\mathbf{k} = \mathbf{0}$. On the subspace corresponding to the zero $\mathbf{P}$ eigenvalue the operators J^k and P^i effectively commute; hence J^z, equal in this frame to W^3/m (or any other projection $\hat{\mathbf{s}}\cdot\mathbf{J} = \hat{\mathbf{s}}\cdot\mathbf{W}/m$ with an arbitrarily directed unit three-vector $\hat{\mathbf{s}}$), can be used to define σ. As massless particles cannot be at rest, their

group generators (forming the Cartan subalgebra of the algebra of the symmetry generators). In the case at hands, no linear combination of $J^{\mu\nu}$ commutes with all components of P^μ, but nothing prevents one from using a combination (even dependent on the P^μ's eigenvalues) of W^μ's which are nonlinear combinations of $J^{\mu\nu}$'s and P^μ's. The situation here is different than in nonrelativistic quantum mechanics, in which the spin operators can be separated from the total angular momentum operators and used to give a meaning to the label σ.

[7] Indeed, taking the boost along the three-momentum (an arbitrary transformation belonging to $L^\uparrow_+$ can always be composed out of rotations and a boost along the three-momentum)

$$\text{sgn}(p^{0\prime}) = \text{sgn}(\Lambda^0{}_0\, p^0)\,\text{sgn}\!\left(1 + \frac{\Lambda^0{}_i\, p^i}{\Lambda^0{}_0\, p^0}\right).$$

Since $\Lambda^0{}_0 \geq 1$ (orthochronous transformations) and since $\det\Lambda = 1$ implies that $\Lambda^0{}_0 > |\Lambda^0{}_z|$, while $p^2 \geq 0$ ensures that $|p^0| > |p^z|$, it follows that if $p^2 > 0$, the sign of the second bracket is positive and, therefore, $\text{sgn}(p^{0\prime}) = \text{sgn}(p^0)$.

standard four-momentum will be taken in the form $k^\mu = (\kappa, 0, 0, \kappa)$ with $\kappa > 0$. In this case the label σ will be given a meaning in terms of the eigenvalues of the operator W^0.

Adopting the definition (6.29), and considering an arbitrary transformation Λ we can write ($\mathbf{k} = \mathbf{0}$ if $|\mathbf{p}, \sigma\rangle$ represents a state of a massive particle and $\mathbf{k} = (0, 0, \kappa)$, if it represents a state of massless particle; in the latter case σ will be later replaced by λ)

$$\begin{aligned} U(\Lambda)|\mathbf{p}, \sigma\rangle &= U(\Lambda)\,\mathcal{N}_p\,U(L_p)|\mathbf{k}, \sigma\rangle \\ &= \mathcal{N}_p\,U(L_{\Lambda\cdot p})\,U(L^{-1}_{\Lambda\cdot p}\cdot\Lambda\cdot L_p)|\mathbf{k}, \sigma\rangle\,, \end{aligned} \tag{6.30}$$

where we have multiplied from the left by $\hat{1} = U(L_{\Lambda\cdot p})U^{-1}(L_{\Lambda\cdot p})$ and used the group composition properties[8] of the operators U. From the definition of the transformation L_p it follows that the matrix[9]

$$W^\mu{}_\nu = \left(L^{-1}_{\Lambda\cdot p}\cdot\Lambda\cdot L_p\right)^\mu{}_\nu\,, \tag{6.31}$$

is an element of the so-called *little group* (called also the *stability group*) of the standard four-momentum k^μ, that is, it is an element of the subgroup of the (proper orthochronous) Lorentz group consisting of transformations which do not change the standard four-momentum: $W^\mu{}_\nu k^\nu = k^\mu$. Thus, in agreement with (6.26) for any such W

$$U(W)|\mathbf{k}, \sigma\rangle = \sum_{\sigma'} |\mathbf{k}, \sigma'\rangle\, D_{\sigma'\sigma}(W)\,, \tag{6.32}$$

symbols $D_{\sigma'\sigma}$ have been used in place of $C_{\sigma'\sigma}$. The matrices $D_{\sigma'\sigma}$ form a representation of the little group in the standard sense:

$$\begin{aligned} \sum_{\sigma'} |\mathbf{k}, \sigma'\rangle\, D_{\sigma'\sigma}(W_2\cdot W_1) &= U(W_2\cdot W_1)|\mathbf{k}, \sigma\rangle = U(W_2)\,U(W_1)|\mathbf{k}, \sigma\rangle \\ &= U(W_2)\sum_{\sigma''} |\mathbf{k}, \sigma''\rangle\, D_{\sigma''\sigma}(W_1) = \sum_{\sigma'}\sum_{\sigma''} |\mathbf{k}, \sigma'\rangle D_{\sigma'\sigma''}(W_2) D_{\sigma''\sigma}(W_1) \\ &= \sum_{\sigma'} |\mathbf{k}, \sigma'\rangle\, [D(W_2)\cdot D(W_1)]_{\sigma'\sigma}\,. \end{aligned}$$

The importance of the matrices $D_{\sigma'\sigma}$ stems from the fact that any Lorentz transformation $S(\Lambda, 0)$ of a one-particle state-vector $|\mathbf{p}, \sigma\rangle$, obtained by acting on it with the corresponding operator $U(\Lambda)$, can be expressed through the matrix $D_{\sigma'\sigma}(W)$ corresponding to Λ and acting on the σ label. Indeed, combining the formulae (6.30) - (6.32) and (6.29) one obtains

$$U(\Lambda)|\mathbf{p}, \sigma\rangle = (\mathcal{N}_p/\mathcal{N}_{\Lambda\cdot p})\sum_{\sigma'} |\mathbf{p}_\Lambda, \sigma'\rangle D_{\sigma'\sigma}(W(\Lambda, p))\,. \tag{6.33}$$

[8]Since we deal with the universal covering of the Poincaré group, no phase factors enter the game here.

[9]Do not confuse it with the operator W^μ!

	p^2	$\mathrm{sgn}(p^0)$	standard four − vector k^μ	little group
i)	$m^2 > 0$	$p^0 > 0$	$(m, 0, 0, 0)$	$SO(3)$
ii)	$m^2 > 0$	$p^0 < 0$	$(-m, 0, 0, 0)$	$SO(3)$
iii)	$m^2 = 0$	$p^0 > 0$	$(\kappa, 0, 0, \kappa)$	$E(2)$
iv)	$m^2 = 0$	$p^0 < 0$	$(-\kappa, 0, 0, \kappa)$	$E(2)$
v)	$-\kappa^2 < 0$		$(0, 0, 0, \kappa)$	$SO(1, 2)$
vi)	$p^\mu = 0$		$(0, 0, 0, 0)$	$SO(1, 3)$

Table 1: Possible little groups

Therefore the problem of classifying and finding the matrices $C_{\sigma'\sigma}$ reduces to the simpler problem of finding and classifying the matrices $D_{\sigma'\sigma}$. (As the little group of the standard four-momentum k^μ of massive particles is the $SO(3)$ group, the matrices $D(W)$ appearing in the transformation rule (6.32) of state-vectors representing such particles will be just the D-matrices introduced in Section 4.4). To prevent appearances of the awkward factors $\mathcal{N}_p$ we will use the relativistic normalization of the one-particle states:

$$\langle \mathbf{p}', \sigma' | \mathbf{p}, \sigma \rangle = (2\pi)^3 2E_{\mathbf{p}} \delta_{\sigma'\sigma} \delta^{(3)}(\mathbf{p}' - \mathbf{p}) \equiv \delta_{\sigma'\sigma} \delta^{(3)}_\Gamma(\mathbf{p}' - \mathbf{p}) \,. \tag{6.34}$$

This corresponds to the relativistically invariant measure

$$\int \frac{d^3\mathbf{p}}{(2\pi)^3 2p^0} = \int \frac{d^3\mathbf{p}_\Lambda}{(2\pi)^3 2p^0_\Lambda} \,. \tag{6.35}$$

With this normalization $\mathcal{N}_{\Lambda\cdot p} = \mathcal{N}_p$ and we will simply set $\mathcal{N}_p = 1$. Finally, we notice that from the relation $U(\Lambda, a) = U(\Lambda, 0)\, U(I,\, \Lambda^{-1}\cdot a)$ - cf. (6.5) - $p \cdot (\Lambda^{-1}\cdot a) = p_\Lambda \cdot a$ and from (6.24) it follows that

$$U(\Lambda, a)|\mathbf{p}, \sigma\rangle = e^{-ia\cdot p_\Lambda} \sum_{\sigma'} |\mathbf{p}_\Lambda, \sigma'\rangle\, D_{\sigma'\sigma}(W(\Lambda, p)) \,. \tag{6.36}$$

Table 1 lists all possible (in four-dimensional space-time) little groups. $E(2)$ is the group of movements of a two-dimensional plane which consists of rotations (around the axis perpendicular to the plane) and translations. We assume that only the possibilities labeled i), iii) and vi) are realized in relativistic quantum theories which are theories of (interacting) particles, that is that in the Hilbert spaces of such theories (or at least in their exploited separable subspaces) there are no vectors corresponding to the remaining three possibilites. The case vi) corresponds to the vacuum state, which is the lowest energy eigenvectors of the system's Hamiltonian H. Its stability group is the whole Lorentz group, that is, it is preserved by all the Poincaré transformations:[10]

$$U(\Lambda,\, a)|\Omega\rangle = |\Omega\rangle\,. \tag{6.37}$$

Although the existence in the system's full Hilbert space of more than one such vacuum vector $|\Omega\rangle$ is not excluded,[11] one assumes that (in the limit of infinite space volume) in the separable subspace of the full Hilbert space one is working in there is only one such state-vector which is normalizable (and mormalized $\langle\Omega|\Omega\rangle = 1$). We will now consider the remaining two cases separately.

Massive particles ($m^2 > 0$, $p^0 > 0$)

The little group of the standard four-vector k^μ is in this case the rotation group $SO(3)$ generated by the three operators J^i (their action obviously does not change the standard four-momentum $(m, \mathbf{0})$, and which on the subspace corresponding to zero $\mathbf{P}$ eigenvalue effectively commute with all P^μ's). The algebra of these operators can be solved as in the nonrelativistic case. Its unitary irreducible representations are known to be labeled by the spin quantum number $s = 0, \frac{1}{2}, 1, \dots$, with $s(s+1)$ being the eigenvalue of $\mathbf{J}^2$, and have dimensions $2s+1$. Since on the vectors $|\mathbf{0}, \sigma\rangle$ the Lorentz invariant operator $W^\mu W_\mu$ reduces to $-m^2\mathbf{J}^2$, it follows that on a representation of the Poincaré group furnished by state-vectors of a single massive particle of mass m the eigenvalue of $W^\mu W_\mu$ is equal $-m^2 s(s+1)$ with s being the particle's spin.

A Lorentz transformation $\Lambda^\mu{}_\nu = W^\mu{}_\nu$ belonging to the little group of the standard four-vector k^μ of a massive particle has the form

$$W^\mu{}_\nu = \begin{pmatrix} 1 & \\ & R \\ & \end{pmatrix}, \tag{6.38}$$

where R is a 3×3 $SO(3)$ matrix. $D^{(s)}_{\sigma'\sigma}$ matrices corresponding to infinitesimal transformations $R^i{}_j = \delta^i{}_j - \omega^i{}_j$ must have the form (cf. (6.13))

$$D^{(s)}_{\sigma'\sigma} = \delta_{\sigma'\sigma} + \frac{i}{2}\,\omega_{kl}\left(J^{kl}_{(s)}\right)_{\sigma'\sigma}\,. \tag{6.39}$$

The explicit form of the matrix generators $J^{kl}_{(s)}$ (matrix elements of the generators $J^{kl} = \epsilon^{kli} J^i$) is obtained by solving algebraically the first of the commutation relations (6.21).

[10] One assumes here that the lowest H eigenvalue - the ground state energy of the system - is zero. This assumption can be relaxed: one can admit a nonzero vacuum energy in some Lorentz frame at the price of having a nonzero vacuum state total three-momentum in other frames; as long as gravity is not taken into account one can argue that this part of the total three-momentum of the system and of the total system's energy is unobservable.

[11] Supersymmetric gauge theories typically have multiple vacua. Also ordinary gauge theories (Chapter 20) posses many vacua distinguished by a topological invariant. In the "thermodynamic limit" all these vacua belong, as assumed, to different separable subspaces.

This is described in standard textbooks of quantum mechanics. Choosing for the basis of the $\mathbf{k} = \mathbf{0}$ subspace the vectors $|\mathbf{0}, \sigma\rangle$ such that $J^z|\mathbf{0}, \sigma\rangle = \sigma|\mathbf{0}, \sigma\rangle$, the result (quoted also in Section 4.4) is

$$\begin{aligned} \left(J^{12}_{(s)}\right)_{\sigma\sigma'} &\equiv \left(J^{z}_{(s)}\right)_{\sigma\sigma'} = \sigma\, \delta_{\sigma\sigma'}\,, \\ \left(J^{23}_{(s)} \pm iJ^{31}_{(s)}\right)_{\sigma\sigma'} &\equiv \left(J^{x}_{(s)} \pm iJ^{y}_{(s)}\right)_{\sigma\sigma'} = \delta_{\sigma,\sigma'\pm1}\sqrt{(s\mp\sigma')(s\pm\sigma'+1)}\,. \end{aligned} \tag{6.40}$$

If the rotation R in the transformation W (6.38) is parametrized by three Euler angles and represented in the form $e^{-i\alpha\mathcal{J}^z_{\rm vec}}\cdot e^{-i\beta\mathcal{J}^y_{\rm vec}}\cdot e^{-i\gamma\mathcal{J}^z_{\rm vec}}$ (see Section 4.4), the corresponding matrix in (6.36) is just the matrix $D^{(s)}_{\sigma'\sigma}(\alpha,\beta,\gamma)$ defined in (4.89).

Out of the state-vectors $|\mathbf{0}, \sigma\rangle$ it is easy to construct the vectors $|\mathbf{0}, \sigma_s\rangle$ such that $\hat{\mathbf{s}}\cdot\mathbf{J}|\mathbf{0}, \sigma_s\rangle = \sigma_s|\mathbf{0}, \sigma_s\rangle$, where $\hat{\mathbf{s}}$ is an arbitrary vector of unit length (the state-vectors $|\mathbf{0}, \sigma\rangle$ correspond to $\hat{\mathbf{s}} = \hat{\mathbf{z}}$). The vectors $|\mathbf{p}, \sigma_s\rangle$ with $\mathbf{p} \neq 0$ are then given by

$$|\mathbf{p}, \sigma_s\rangle = U(L_p)|\mathbf{0}, \sigma_s\rangle\,, \tag{6.41}$$

and can be shown to be the eigenvectors

$$-\frac{s^\mu_p\, W_\mu}{m}\,|\mathbf{p}, \sigma_s\rangle = \sigma_s|\mathbf{p}, \sigma_s\rangle\,, \tag{6.42}$$

of the operator $s^\mu_p\, W_\mu$ in which

$$s^\mu_p = (L_p)^\mu{}_\nu\, s^\nu_{\rm rest}\,, \qquad\qquad s^\nu_{\rm rest} = (0, \hat{\mathbf{s}})\,. \tag{6.43}$$

One concludes therefore, that the vectors $|\mathbf{p}, \sigma_s\rangle$ defined by (6.41), which form an irreducible representation (labeled by $P^\mu P_\mu = m^2$ and $W^\mu W_\mu = -m^2 s(s+1)$) of the Poincaré group, are the eigenvectors of the $\mathbf{P}$ and $-s^\mu_p\, W_\mu/m$ operators with the eigenvalues $\mathbf{p}$ and σ_s, respectively.

Under an arbitrary Lorentz transformation Λ the state-vectors $|\mathbf{p}, \sigma\rangle$ (labeled by the J^z eigenvalue σ in the particle's rest frame) transform according to the rule

$$U(\Lambda)|\mathbf{p}, \sigma\rangle = \sum_{\sigma'} |\mathbf{p}_\Lambda, \sigma'\rangle\, D^{(s)}_{\sigma'\sigma}(W(\Lambda, p))\,, \tag{6.44}$$

with the matrices $D^{(s)}_{\sigma'\sigma}(R)$ generated by (6.40). The whole task is, therefore, to find the element $W(\Lambda, p)$ of the Lorentz group which must be a rotation

$$W = L^{-1}_{\Lambda\cdot p}\cdot\Lambda\cdot L_p = R\,. \tag{6.45}$$

To this end, one has to specify first the standard transformation L_p. We choose the following one:

$$\begin{aligned} (L_p)^0{}_0 &= \gamma\,, & (L_p)^i{}_j &= \delta^i{}_j - (\gamma-1)\frac{p^i p_j}{\mathbf{p}^2}\,, \\ (L_p)^i{}_0 &= \frac{p^i}{|\mathbf{p}|}\sqrt{\gamma^2-1}\,, & (L_p)^0{}_j &= -\frac{p_j}{|\mathbf{p}|}\sqrt{\gamma^2-1}\,, \end{aligned} \tag{6.46}$$

with

$$\gamma \equiv \sqrt{1+\frac{\mathbf{p}^2}{m^2}} = \frac{E(\mathbf{p})}{m} = \frac{1}{\sqrt{1-\mathbf{v}^2}}\,. \tag{6.47}$$

It is easy to see that

$$(L_p)^0{}_0\, m = \gamma\, m = E\,, \qquad (L_p)^i{}_0\, m = \frac{p^i}{|\mathbf{p}|}\sqrt{\frac{\mathbf{p}^2}{m^2}}\, m = p^i\,, \tag{6.48}$$

i.e. $p^\mu = (L_p)^\mu{}_\nu k^\nu$, as required. It can be also shown that L_p given above, which in fact is just the boost in the direction opposite to $\mathbf{p}$, is the composition

$$L_p = R_{\hat{\mathbf{z}}}(\hat{\mathbf{p}}) \cdot B_z(|\mathbf{p}|) \cdot R^{-1}_{\hat{\mathbf{z}}}(\hat{\mathbf{p}})\,, \tag{6.49}$$

where $R_{\hat{\mathbf{z}}}(\hat{\mathbf{p}})$ is the rotation making a vector pointing in the $\hat{\mathbf{p}}$ direction out of a vector pointing in the $\hat{\mathbf{z}}$ direction and $B_z(|\mathbf{p}|)$ is the boost along the z axis changing the four-vector $(m, 0, 0, 0)$ into $(E(\mathbf{p}), 0, 0, |\mathbf{p}|)$. In turn, the rotation $R_{\hat{\mathbf{z}}}(\hat{\mathbf{p}})$ can be obtained as a composition of two successive rotations (see Appendix D)

$$R_{\hat{\mathbf{z}}}(\hat{\mathbf{p}}) = e^{-i\phi_{\mathbf{p}}\mathcal{J}^z_{\rm vec}} \cdot e^{-i\theta_{\mathbf{p}}\mathcal{J}^y_{\rm vec}}\,. \tag{6.50}$$

The matrices $\mathcal{J}^i_{\rm vec}$ are the Lorentz group generators in the vector representation (D.3). In the Hilbert space the rotation (6.50) acts through the operator

$$U(R_{\hat{\mathbf{z}}}(\hat{\mathbf{p}})) = e^{-i\phi_{\mathbf{p}}J^z} e^{-i\theta_{\mathbf{p}}J^y}\,, \tag{6.51}$$

in which $\theta_{\mathbf{p}}$ and $\phi_{\mathbf{p}}$ are the angles which characterize the direction of the unit vector $\hat{\mathbf{p}}$:

$$\hat{\mathbf{p}} = (\sin\theta_{\mathbf{p}}\cos\phi_{\mathbf{p}},\ \ \sin\theta_{\mathbf{p}}\sin\phi_{\mathbf{p}},\ \ \cos\theta_{\mathbf{p}})\,. \tag{6.52}$$

Their ranges $0 \le \theta_{\mathbf{p}} \le \pi$, $0 \le \phi_{\mathbf{p}} \le 2\pi$ have to be specified explicitly because changing $\phi_{\mathbf{p}}$ to $\phi_{\mathbf{p}} + 2\pi$ would give the same rotation (6.50) but the corresponding Hilbert space operator $U(R_{\hat{\mathbf{z}}}(\hat{\mathbf{p}}))$ would give the opposite sign when acting on states of half-integer spin particles.[12] Notice, that the rotation $R_{\hat{\mathbf{z}}}(\hat{\mathbf{p}})$ could be defined with an additional factor $e^{+i\phi_{\mathbf{p}}\mathcal{J}^z_{\rm vec}}$ on its right extreme in agreement with the frequently used convention (see Appendix D). This extra factor could even depend on an angle different than $\phi_{\mathbf{p}}$ as it has no effect when acting on the standard vector. From L_p and $U(L_p)$ such an extra factor would simply drop out, because being a rotation in the xy plane it would commute with the boost $B_z(|\mathbf{p}|)$ which acts in the tz "plane".

An important property of the adopted definition of the standard transformation L_p is that whenever the transformation Λ is itself an ordinary three-dimensional rotation R,

[12]In other words, giving explicitly these angles amounts to specifying which one of the two $SL(2, C)$ transformations, which correspond to the same element of the Lorentz group, is chosen to define the standard transformation L_p.

the corresponding little group element $W(\Lambda, p) = W(R, p) = R$ (in fact, this is the reason for which L_p in (6.49) is defined with the factor $R^{-1}_{\hat{\mathbf{z}}}(\hat{\mathbf{p}})$ on its right extreme). To prove this, let us write $W(\Lambda, p)$ explicitly[13]

$$\begin{aligned} W(R,p) &= L^{-1}_{R\cdot p} \cdot R \cdot L_p \\ &= R_{\hat{\mathbf{z}}}(R\cdot\hat{\mathbf{p}}) \cdot B^{-1}_z(|R\cdot\mathbf{p}|) \cdot R^{-1}_{\hat{\mathbf{z}}}(R\cdot\hat{\mathbf{p}}) \cdot R \cdot R_{\hat{\mathbf{z}}}(\hat{\mathbf{p}}) \cdot B_z(|\mathbf{p}|) \cdot R^{-1}_{\hat{\mathbf{z}}}(\hat{\mathbf{p}})\,. \end{aligned}$$

The three successive transformations (rotations): $R^{-1}_{\hat{\mathbf{z}}}(R\cdot\hat{\mathbf{p}}) \cdot R \cdot R_{\hat{\mathbf{z}}}(\hat{\mathbf{p}})$ transform first a vector pointing in the z direction into a vector pointing in the direction $\hat{\mathbf{p}}$, then rotate it by R and finally produce again a vector parallel to the z axis; hence together they must simply be a rotation in the xy plane; such a rotation commutes with $B_z(|\mathbf{p}|)$ which has the matrix structure

$$B_z(|\mathbf{p}|) = \begin{pmatrix} \gamma & & & \sqrt{\gamma^2-1} \\ & 1 & 0 & \\ & 0 & 1 & \\ \sqrt{\gamma^2-1} & & & \gamma \end{pmatrix}. \tag{6.53}$$

Since $B^{-1}_z(|R\cdot\mathbf{p}|) = B^{-1}_z(|\mathbf{p}|)$ - by definition rotations R do not change the length of a three-vector - we indeed get $W(R,p) = R$. Thus, owing to the choice of L_p, state-vectors of massive particles (as far as their three-momentum $\mathbf{p}$ and spin σ labels are concerned) behave under rotations as in nonrelativistic quantum mechanics.

Massless particles ($m^2 = 0$, $p^0 > 0$)

The little group of the standard vector $k^\mu = (\kappa, 0, 0, \kappa)$ - without loss of generality we can set $\kappa = 1$ - is the group $E(2)$ the structure of which is most easily unraveled by using the following trick. Let us, in addition to k^μ, consider also the unit time-like vector $t^\mu = (1, 0, 0, 0)$. By definition, the little group elements W satisfy $W^\mu{}_\nu k^\nu = k^\mu$. From this and from the basic property of $W^\mu{}_\nu$'s as Lorentz transformations (the preservation of the scalar products of four-vectors) it follows that

$$\begin{aligned} &(W\cdot t)_\mu (W\cdot t)^\mu = t_\mu t^\mu = 1\,, \\ &(W\cdot t)_\mu k^\mu = (W\cdot t)_\mu (W\cdot k)^\mu = t_\mu k^\mu = 1\,. \end{aligned}$$

The second equality implies that the most general structure of the vector $(W\cdot t)^\mu$ reads:

$$(W\cdot t)^\mu = (1+\zeta,\, \alpha,\, \beta,\, \zeta)\,, \tag{6.54}$$

and the first one imposes the condition $\zeta = (\alpha^2 + \beta^2)/2$. The result of the action of $W^\mu{}_\nu$ on t^ν is therefore the same as of the Lorentz transformation S of the form (it is

[13] In principle R is a 4×4 Lorentz transformation matrix; nevertheless we use here the notation $R\cdot\mathbf{p}$ to denote corresponding rotation of the three-vector $\mathbf{p}$.

straightforward to check that $S^T \cdot g \cdot S = g$)

$$S^{\mu}{}_{\nu} = \begin{pmatrix} 1+\zeta & \alpha & \beta & -\zeta \\ \alpha & 1 & 0 & -\alpha \\ \beta & 0 & 1 & -\beta \\ \zeta & \alpha & \beta & 1-\zeta \end{pmatrix} . \tag{6.55}$$

This does not mean that W is identical with S, but only that $S^{-1}(\alpha,\beta) \cdot W \cdot t = t$, that is, that $S^{-1}(\alpha,\beta) \cdot W$ is a rotation. Moreover, since $W \cdot k = k$ and $S \cdot k = k$, the product $S^{-1}(\alpha,\beta) \cdot W$ preserves also k^{μ}, and, therefore, can only be a rotation by some angle θ around the z axis. Thus, the most general element of the $E(2)$ group has the form

$$W(\theta,\alpha,\beta) = S(\alpha,\beta) \cdot R_z(\theta) . \tag{6.56}$$

The transformations corresponding to $\theta = 0$ and those corresponding to $\alpha = \beta = 0$ form two Abelian subgroups of $E(2)$:

$$\begin{aligned} \theta = 0 : \qquad & S(\alpha_2,\beta_2) \cdot S(\alpha_1,\beta_1) = S(\alpha_2+\alpha_1, \beta_2+\beta_1) , \\ \alpha = \beta = 0 : \qquad & R_z(\theta_2) \cdot R_z(\theta_1) = R_z(\theta_2+\theta_1) . \end{aligned}$$

Moreover, as can be explicitly checked,

$$R_z(\theta) \cdot S(\alpha,\beta) \cdot R_z^{-1}(\theta) = S(\alpha\cos\theta + \beta\sin\theta, -\alpha\sin\theta + \beta\cos\theta) , \tag{6.57}$$

which shows that the $\theta = 0$ subgroup is an invariant subgroup of $E(2)$. It is easy to see that $W(\theta,\alpha,\beta)$ do indeed form the group of movements of a two dimensional plane, wchich translate its points by the vector (α,β) and rotate them in the plane around an axis perpendicular to it by the angle θ.

We now find representations of the algebra of the matrices $W(\theta,\alpha,\beta)$ on vectors representing physical states of a massless particle. Since $W(\theta,\alpha,\beta)$ is a Lorentz transformation it is possible to write its infinitesimal form as

$$W^{\mu}{}_{\nu}(\theta,\alpha,\beta) = \delta^{\mu}{}_{\nu} - \omega^{\mu}{}_{\nu} . \tag{6.58}$$

By comparing this with the form (6.55) of $S^{\mu}{}_{\nu}$ and with $R_z(\theta)$ we find

$$\omega^{\mu}{}_{\nu} = \begin{pmatrix} 0 & -\alpha & -\beta & 0 \\ -\alpha & 0 & -\theta & \alpha \\ -\beta & \theta & 0 & \beta \\ 0 & -\alpha & -\beta & 0 \end{pmatrix} , \tag{6.59}$$

or

$$\omega_{\mu\nu} = \begin{pmatrix} 0 & -\alpha & -\beta & 0 \\ \alpha & 0 & \theta & -\alpha \\ \beta & -\theta & 0 & -\beta \\ 0 & \alpha & \beta & 0 \end{pmatrix} . \tag{6.60}$$

It follows, that the infinitesimal form of the corresponding symmetry operator $U(W(\theta, \alpha, \beta))$ acting in the Hilbert space reads

$$U(W(\theta, \alpha, \beta)) \equiv U(S(\alpha, \beta))\, U(R_z(\theta)) \approx \hat{1} + \frac{i}{2}\omega_{\mu\nu} J^{\mu\nu} = \hat{1} - i(\alpha A + \beta B - \theta J^z)\,, \quad (6.61)$$

where in addition to $J^z = J^{12}$ we have defined the following two combinations of the generators:

$$A \equiv J^{01} + J^{13} = K^x - J^y\,, \qquad B \equiv J^{02} + J^{23} = K^y + J^x\,. \quad (6.62)$$

These three operators, which in the Hilbert space generate transformations belonging to the stability group of the standard four-vector $k^\mu = (\kappa, 0, 0, \kappa)$, satisfy the following commutation relations:

$$[J^z,\ A] = iB\,, \qquad [J^z,\ B] = -iA\,, \qquad [A,\ B] = 0\,, \quad (6.63)$$

which follow directly from the commutation rules (6.21). To solve this algebra in the subspace of state-vectors $|\mathbf{k}\rangle$ such that $P^\mu|\mathbf{k}\rangle = k^\mu|\mathbf{k}\rangle$ (as was done in the case of massive particles), one can work in the basis formed by common eigenvectors of the commuting operators A and B:

$$A\,|\mathbf{k}, a, b\rangle = a\,|\mathbf{k}, a, b\rangle\,, \qquad B\,|\mathbf{k}, a, b\rangle = b\,|\mathbf{k}, a, b\rangle\,. \quad (6.64)$$

However, using the same method, by which the transformation properties of the Poincaré group generators P^μ and $J^{\mu\nu}$ have been established in Section 6.1, it can be shown that

$$\begin{aligned} U(R_z(\theta))\, A\, U^{-1}(R_z(\theta)) &= A\cos\theta - B\sin\theta\,, \\ U(R_z(\theta))\, B\, U^{-1}(R_z(\theta)) &= A\sin\theta + B\cos\theta\,. \end{aligned}$$

Consequently, if $|\mathbf{k}, a, b\rangle$ is an eigenvector of A and B (with the eigenvalues a and b, respectively), the same property have also all vectors of the form

$$|\mathbf{k}, a, b\rangle_\theta \equiv U^{-1}(R_z(\theta))\, |\mathbf{k}, a, b\rangle\,, \quad (6.65)$$

with and arbitrary angle θ, which have the A and B eigenvalues equal $a\cos\theta - b\sin\theta$ and $a\sin\theta + b\cos\theta$, respectively. Thus, the spectra of the operators A and B are continuous, contrary to what is observed: massless particles do not carry any continuous quantum numbers. One is, therefore, forced to declare, that only the A and B eigenvectors with the eigenvalues $a = 0$, $b = 0$ represent physical states of massless particles. It will become clear later, that the subgroup of the little group parametrized by α and β is related to gauge invariance which is an inevitable feature of a consistent (quantum) theory of massless particles of spin grater than 1/2.

In the subspace $a = b = 0$ of the Hilbert space the three generators, A, B, and J^z effectively commute, and physical state-vectors can be labeled by their J^z eigenvalues[14]

$$J^z|\mathbf{k}, \lambda\rangle = \lambda|\mathbf{k}, \lambda\rangle\,. \quad (6.66)$$

[14]Instead of σ we now use the symbol λ.

λ is just the helicity because J^z is simply the projection of the total angular momentum onto the direction of the standard three-momentum $\mathbf{k} = (0, 0, \kappa)$. Therefore[15]

$$U(W(\theta, \alpha, \beta))\,|\mathbf{k}, \lambda\rangle = e^{-i(\alpha A + \beta B)} e^{i\theta J^z} |\mathbf{k}, \lambda\rangle = e^{i\theta\lambda} |\mathbf{k}, \lambda\rangle\,, \tag{6.67}$$

which means that the matrices $D_{\lambda'\lambda}(W)$ appearing in the transformation rule (6.32) of state-vectors representing massless particles have the simple form

$$D_{\lambda'\lambda}(W(\Lambda, p)) = e^{i\theta\lambda}\,\delta_{\lambda'\lambda}\,. \tag{6.68}$$

Thus, the action of an arbitrary Poincaré transformation $S(\Lambda, a)$ on a state-vector $|\mathbf{p}, \lambda\rangle$ of a single massless particle reads (cf. the formula (6.36))

$$U(\Lambda, a)|\mathbf{p}, \lambda\rangle = e^{-ia\cdot p_\Lambda}\, e^{i\theta\lambda}\, |\mathbf{p}_\Lambda, \lambda\rangle\,. \tag{6.69}$$

The angle θ in this formula has to be found by forming the little group matrix $W(\Lambda, p)$ and decomposing it according to (6.56):

$$W(\Lambda, p) = L^{-1}_{\Lambda\cdot p} \cdot \Lambda \cdot L_p = S(\alpha(\Lambda, p), \beta(\Lambda, p)) \cdot R_z(\theta(\Lambda, p))\,. \tag{6.70}$$

To use this formula one has to specify the standard Lorentz transformation L_p which transforms the standard four-momentum k^μ into a given four-vector p^μ: $(L_p)^\mu{}_\nu k^\nu = p^\mu$. In the case of massless particles one takes

$$L_p = R_{\hat{\mathbf{z}}}(\hat{\mathbf{p}}) \cdot B_z(|\mathbf{p}|/\kappa)\,, \tag{6.71}$$

(for $k^\mu = (\kappa, 0, 0, \kappa)$), where the action on the Hilbert space states of the rotation $R_{\hat{\mathbf{z}}}(\hat{\mathbf{p}})$ defined by (6.50) is specified in (6.51), and the Lorentz boost along the z axis reads[16]

$$B_z(u) = \begin{pmatrix} \frac{u^2+1}{2u} & 0 & 0 & \frac{u^2-1}{2u} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{u^2-1}{2u} & 0 & 0 & \frac{u^2+1}{2u} \end{pmatrix}, \tag{6.72}$$

with $u = \sqrt{(1+v)/(1-v)}$, that is, $\gamma = (u^2+1)/2u$ and $v\gamma = (u^2-1)/2u$. Of course, with L_p given by (6.71) the factor S in (6.70) equals unity, if $\Lambda = R$.

Up to now we have not encountered any restriction on the values the helicity λ of a massless particle can assume. In fact, no such a restriction can be derived algebraically.[17] There is however, a topological argument which shows that λ can assume only integer or half-integer values. The Lorentz group $SO(1, 3)$ is doubly connected, its universal covering

[15] Since $[A, B] = 0$, the form $U(S(\alpha, \beta)) = e^{-i(\alpha A + \beta B)}$ follows from (4.27).

[16] The form of $B_z(u)$, which is very different from the boost appearing in the standard transformation L_p (6.49) for *massive* particles and misleadingly denoted also by $B_z(\cdot)$, follows from the requirement that $|\mathbf{p}| = \gamma(1+v)\kappa$ when one goes over to the frame moving along the z axis with the velocity $-v$. Thus, $u \equiv |\mathbf{p}|/\kappa = \sqrt{(1+v)/(1-v)}$, that is, $v = (u^2-1)/(u^2+1)$ and $\gamma v = (u^2-1)/2u$.

[17] See "The Feynman Lectures on Physics", vol. III, a footnote in Section 17.4.

group being the $SL(2,C)$ group of all complex matrices of unit determinant. This can be shown in a way similar to the one which was used (in Section 4.2) to show that $SU(2)$ is the twofold covering of $SO(3)$. Consider a four-vector V^μ and the four 2×2 matrices σ^μ:

$$\sigma^\mu = (1, \boldsymbol{\sigma})\,. \tag{6.73}$$

It is easy to check, that $\det(V_\mu\sigma^\mu) = V^\mu V_\mu$. If a matrix M belongs to $SL(2,C)$,

$$\det\big(M\cdot V_\mu\sigma^\mu\cdot M^\dagger\big) = \det(V_\mu\sigma^\mu)\,, \tag{6.74}$$

because (by the definition of the $SL(2,C)$ group) $\det(M) = \det(M^\dagger) = 1$. Since

$$M\cdot V_\mu\,\sigma^\mu\cdot M^\dagger = V'_\mu\,\sigma^\mu\,, \tag{6.75}$$

it follows that to each M corresponds a Lorentz transformation $\Lambda(M)$ transforming V_μ into V'_μ. Moreover, it is clear that $\Lambda(M) = \Lambda(-M)$. This shows that $SO(1,3) = SL(2,C)/\mathbb{Z}_2$, where $\mathbb{Z}_2$ is the invariant subgroup of $SL(2,C)$ consisting of the matrices I and $-I$.

To investigate the topology of the $SL(2,C)$ group one can use the fact that any matrix M belonging to it can be written as $M = U\cdot e^H$, where U is a unitary matrix and H is a Hermitian matrix (this is called the *polar decomposition*). Since $\det(U\cdot U^\dagger) = 1$, and $\det(e^H) = e^{\mathrm{tr}H}$ is real and positive, $\det(M) = 1$ requires $\det(U) = 1$ and $\mathrm{tr}H = 0$. The topology of the $SL(2,C)$ group is therefore the product topology of the set of unitary unimodular 2×2 matrices U and the set of the Hermitian traceless 2×2 matrices H. Any such matrix can be written as a linear combination of the three Pauli matrices with arbitrary real coefficients. On the other hand, any unitary unimodular 2×2 matrix can be parametrized by four real parameters a, b, c, d:

$$U = \begin{pmatrix} a+ib & c+id \\ -c+id & a-ib \end{pmatrix}, \tag{6.76}$$

which are subjected to the constraint $a^2+b^2+c^2+d^2 = 1$ (so that $\det U = 1$). Therefore, the topology of $SL(2,C)$ is that of the Cartesian product $S_3\times\mathbb{R}^3$. $SL(2,C)$ is therefore simply connected - any path connecting two its points can be continuously deformed to another such a path. The fact that $SO(1,3) = SL(2,C)/\mathbb{Z}_2$ makes $SO(1,3)$ doubly connected: taking the quotient with respect to $\mathbb{Z}_2$ amounts to identifying U and $-U$ in the polar decomposition[18] therefore a path that makes a jump (or an odd number of jumps) from U to $-U$ cannot be continuously deformed to a path connecting the same two group points which does not make any jump (or makes an even number of jumps); however, a path which makes two jumps can be continuously deformed to a path without jumps (more generally, paths with the same number of jumps modulo 2 are deformable to each other).

[18] This follows from the fact that if $H = x\sigma^x + y\sigma^y + z\sigma^z$ with real x, y and z, $\mathrm{tr}e^H = 2\cosh r > 0$ where $r = \sqrt{x^2+y^2+z^2}$, while $\mathrm{tr}(-I) = -2$.

Let us take two paths $\Lambda_1(\xi_1)$ and $\Lambda_2(\xi_2)$ in the Lorentz group manifold, where ξ_1 and ξ_2 are two parameters, $0 \leq \xi \leq 1$, such that $\Lambda_i(0) = I$ and $\Lambda_i(1) = \Lambda_i$ and consider the following composition of the corresponding symmetry operators acting in the Hilbert space

$$U^{-1}(\Lambda_2(\xi_2)\cdot\Lambda_1(\xi_1))\, U(\Lambda_2(\xi_2))\, U(\Lambda_1(\xi_1))\,. \tag{6.77}$$

When ξ_1 and ξ_2 vary from 0 to 1, the product $U(\Lambda_2(\xi_2))\, U(\Lambda_1(\xi_1))$ on the right can be viewed as the operator corresponding to $\Lambda_2\cdot\Lambda_1$ but defined along a path going first from I to Λ_1 and only then from Λ_1 to $\Lambda_2\cdot\Lambda_1$. Because $SO(1,3)$ is, as argued, doubly connected, this may not be the same operator as the one corresponding to $\Lambda_2\cdot\Lambda_1$ which is obtained by integrating the differential equation (4.31) along the standard path going directly from I to $\Lambda_2\cdot\Lambda_1$. Therefore, although in the $SO(1,3)$ group itself always

$$(\Lambda_2(\xi_2)\cdot\Lambda_1(\xi_1))^{-1}\cdot\Lambda_2(\xi_2)\cdot\Lambda_1(\xi_1) = I\,, \tag{6.78}$$

the operator (6.77) may differ from the unit operator if the path $I \to \Lambda_1 \to \Lambda_2\cdot\Lambda_1$ and the standard path $I \to \Lambda_2\cdot\Lambda_1$ are not continuously deformable to each other. However,

$$\left[U^{-1}(\Lambda_2(\xi_2)\cdot\Lambda_1(\xi_1))\, U(\Lambda_2(\xi_2))\, U(\Lambda_1(\xi_1))\right]^2 = \hat{1}\,, \tag{6.79}$$

because the path $I \to \Lambda_1 \to \Lambda_2\cdot\Lambda_1 \to I \to \Lambda_1 \to \Lambda_2\cdot\Lambda_1 \to I$ is always continuously contractible to the point I. Hence,

$$U(\Lambda_2(\xi_2))\, U(\Lambda_1(\xi_1)) = \pm U(\Lambda_2(\xi_2)\cdot\Lambda_1(\xi_1))\,, \tag{6.80}$$

that is the actions of the operators $U(\Lambda_2(\xi_2))\, U(\Lambda_1(\xi_1))$ and of $U(\Lambda_2(\xi_2)\cdot\Lambda_1(\xi_1))$ on state-vectors may differ only by the sign. To see that this implies a restriction on possible values of the helicity λ of massless particles it suffices to consider $\Lambda_1 = R_z(\pi)$ and $\Lambda_2 = R_z(\pi)$. Obviously, $\Lambda_2\cdot\Lambda_1 = I$, but

$$U(R_z(\pi))\, U(R_z(\pi)) = \pm U(I) = \pm\hat{1}\,. \tag{6.81}$$

Since the action of U on one-particle state-vectors of massless particles is realized through the matrices $D_{\lambda'\lambda}$ of the form (6.68), it is clear that this means that the helicity λ can assume only integer or half-integer values.

Because the matrices $D_{\lambda'\lambda}(W(\Lambda, p))$ are diagonal, helicity of a massless particle is a Lorentz invariant quantity. That is, a massless particle having helicity λ in one reference frame $\mathcal{O}$, has the same helicity λ in any other reference frame[19] $\mathcal{O}'$ (only the phase of its state-vector can be altered due to the strict specification of the angles defining the directions $\mathbf{p}$ and $\mathbf{p}_\Lambda$). It follows, that from the point of view of the proper ortochronous Lorentz transformations there could exist only photons of, say, helicity -1. One could say

[19] Helicity of a massive particle - the corresponding one-particle states are defined by (6.99) - can change as a result of a boost reversing the direction of the particle's three-momentum.

that helicity -1 photons and helicity $+1$ photons are different kinds of particles. We will see however, that the parity transformation - if parity is to be a symmetry of at least some interactions - requires the existence of photons of both helicities and this justifies treating them as two internal states of the same particle. Parity is not a symmetry of the weak interactions, and it became customary in the past (but in fact purely conventionally!) to treat massless neutrinos of helicity $-1/2$ as different particles from the helicity $+1/2$ neutrinos (which were called antineutrinos). The physics of *massless* neutrinos would not change, however, if one had declared that these were two internal states of the same particle: if neutrinos were strictly massless there would be no way to experimentally distinguish between the so-called Majorana neutrinos (which are their own antiparticles and have both helicity states) and the Weyl neutrinos (which have only one helicity state and are different from their antiparticles which have opposite helicity). In the years that have passed, however, evidence has been accumulated that the neutrinos are *not* massless, and therefore, the issue whether neutrinos are massive Majorana particles or massive Dirac particles (that is neutrinos and antineutrinos both having two spin projections) is no longer an academic one...

6.3 Parity and time reversal

For completeness we also investigate the action of the operators

$$\mathcal{P} \equiv U(P, 0) \qquad \text{and} \qquad \mathcal{T} \equiv U(T, 0)\,,$$

on one-particle states. Before we do it, it is necessary to make the following comments. In considering the Hilbert space constructed out of (appropriately symmetrized or antisymmetrized) tensor products of one-particle states one can always assume that the operators $\mathcal{P}$ and $\mathcal{T}$, acting in the way established below on the $|\alpha_0\rangle$ states (defined in Chapter 7), exist (that is, the Hilbert space of free particles can always be appropriately enlarged). The problem whether parity and/or time reversal are are good symmetry operations also in a theory of interacting particles depends then on the structure of interactions, that is on the questions whether $H = H_0 + V_{\rm int}$ commutes with the $\mathcal{P}$ and/or $\mathcal{T}$ operators. In fact, owing to the experiment of C.S. Wu (and that of L. Lederman, R.L. Garwin and M. Weinrich, see Section **??**) it is now known that in the real world $\mathcal{P}$ is not a good symmetry (and this fact has become the cornerstone of the Standard Model of fundamental interactions); similarily neither is $\mathcal{T}$, as was (under the assumption of CPT invariance - see below) revealed by the experiment of J.H. Christenson, J.W. Cronin, V.L. Fitch and R. Turlay (see Chapter 12). Therefore neither parity nor time reversal are, strictly speaking, good symmetries of fundamental particle interactions. However, if the weak interactions are neglected, that is one considers only strong and electromagnetic interactions of elementary particles, they are good symmetries and as such play an important role in our understanding of strong and electromagnetic interactions. Here we assume that parity and time reversal are symmetries of the complete theory.

From the composition rule (6.5) it follows that

$$\begin{aligned}\mathcal{P}\,U(\Lambda,a)\,\mathcal{P}^{-1} &= U(P\cdot\Lambda\cdot P^{-1},\,P\cdot a)\,,\\ \mathcal{T}\,U(\Lambda,a)\,\mathcal{T}^{-1} &= U(T\cdot\Lambda\cdot T^{-1},\,T\cdot a)\,,\end{aligned}\tag{6.82}$$

(the matrices P and T are given in (6.8)). Taking Λ and a to be infinitesimal one finds the relations

$$\begin{aligned}\mathcal{P}\,iJ^{\mu\nu}\,\mathcal{P}^{-1} &= (P^{-1})^{\mu}{}_{\lambda}(P^{-1})^{\nu}{}_{\kappa}iJ^{\lambda\kappa}\,, \qquad \mathcal{P}\,iP^{\mu}\,\mathcal{P}^{-1} = (P^{-1})^{\mu}{}_{\lambda}iP^{\lambda}\,,\\ \mathcal{T}\,iJ^{\mu\nu}\,\mathcal{T}^{-1} &= (T^{-1})^{\mu}{}_{\lambda}(T^{-1})^{\nu}{}_{\kappa}iJ^{\lambda\kappa}\,, \qquad \mathcal{T}\,iP^{\mu}\,\mathcal{T}^{-1} = (T^{-1})^{\mu}{}_{\lambda}iP^{\lambda}\,.\end{aligned}\tag{6.83}$$

From these relations one readily infers that $\mathcal{P}$ has to be unitary, while $\mathcal{T}$ must be antiunitary: antiunitarity of $\mathcal{P}$ would lead to $\mathcal{P}H\mathcal{P}^{-1} = -H$, which in turn would imply that to each H eigenvector $|\Psi\rangle$ of energy E there exists the H eigenvector $\mathcal{P}^{-1}|\Psi\rangle$ of energy $-E$. The spectrum of the Hamiltonian would then not be bounded from below. In contrast, the same reasoning leads to the conclusion that $\mathcal{T}$ must be antiunitary. Writing the generators in the three-dimensional notation one has, therefore:

$$\begin{aligned}\mathcal{P}\,\mathbf{J}\,\mathcal{P}^{-1} &= \mathbf{J}\,, & \mathcal{T}\,\mathbf{J}\,\mathcal{T}^{-1} &= -\mathbf{J}\,,\\ \mathcal{P}\,\mathbf{K}\,\mathcal{P}^{-1} &= -\mathbf{K}\,, & \mathcal{T}\,\mathbf{K}\,\mathcal{T}^{-1} &= \mathbf{K}\,,\\ \mathcal{P}\,\mathbf{P}\,\mathcal{P}^{-1} &= -\mathbf{P}\,, & \mathcal{T}\,\mathbf{P}\,\mathcal{T}^{-1} &= -\mathbf{P}\,.\end{aligned}\tag{6.84}$$

We now consider the action of $\mathcal{P}$ and $\mathcal{T}$ on one-particle states.[20] Again the cases of massive and massless particles have to be treated separately.

P and T transformations of states of massive particles

If $m^2 > 0$, since the state-vector $|\mathbf{0},\sigma\rangle$ ($\mathbf{0}$ represents here the standard four-momentum $k^{\mu} = (m,\mathbf{0})$) is an eigenvector of H, $\mathbf{P}$ and J^z with the eigenvalues m, $\mathbf{0}$ and σ, from the rules (6.84) it follows that

$$\mathcal{P}|\mathbf{0},\sigma\rangle = \eta_{\sigma}|\mathbf{0},\sigma\rangle\,,\tag{6.85}$$

where (because η_{σ} are eigenvalues of the unitarity operator $\mathcal{P}$) $|\eta_{\sigma}| = 1$. In fact, the phase factor η_{σ} cannot depend on σ (as can be easily seen by acting with $\mathcal{P}$ on both sides of the equalities $(J^x \pm iJ^y)|\mathbf{0},\sigma\rangle = \sqrt{\cdots}\,|\mathbf{0},\sigma\pm 1\rangle$). Therefore η is an intrinsic property of a given particle, called its *intrinsic parity*; it is frequently denoted just by P.

Using the explicit form (6.46) of the standard transformation L_p it is easy to see that

$$P\cdot L_p\cdot P^{-1} = L_{P\cdot p}\,, \qquad \text{where} \qquad (P\cdot p)^{\mu} = (E(\mathbf{p}),\ -\mathbf{p})\,.\tag{6.86}$$

This allows to find

$$\mathcal{P}|\mathbf{p},\sigma\rangle = U(P\cdot L_p\cdot P^{-1})\mathcal{P}|\mathbf{0},\sigma\rangle = \eta\,U(L_{P\cdot p})|\mathbf{0},\sigma\rangle = \eta|-\mathbf{p},\sigma\rangle\,,\tag{6.87}$$

[20] The assumption that parity and time-reversal are good symmetries of the theory means that $\mathcal{P}|\Omega\rangle = |\Omega\rangle$, $\mathcal{T}|\Omega\rangle = |\Omega\rangle$.

as could be expected. We will need this result when constructing causal (free) field operators in Chapter 8.

As to the time reversal operator, from the rules (6.84) it follows that $J^z\mathcal{T}|\mathbf{0},\sigma\rangle = -\mathcal{T}J^z|\mathbf{0},\sigma\rangle = -\sigma\mathcal{T}|\mathbf{0},\sigma\rangle$. Therefore

$$\mathcal{T}|\mathbf{0},\sigma\rangle = \zeta_\sigma|\mathbf{0},-\sigma\rangle\,, \tag{6.88}$$

where again $|\zeta_\sigma| = 1$, because of the antiunitarity of $\mathcal{T}$.[21] Using a similar reasoning as in the case of the $\mathcal{P}$ operator, one can show that $\zeta_{\sigma\pm1} = -\zeta_\sigma$, or, in other words,

$$\zeta_\sigma = (-1)^{s-\sigma}\zeta\,. \tag{6.89}$$

Because the operator $\mathcal{T}$ is antiunitary, the phase ζ is unphysical (nothing like "intrinsic time-reversality" exists) and can be removed by redefining the one-particle state-vectors. Indeed, let $|\mathbf{0},\sigma\rangle' \equiv \sqrt{\zeta}|\mathbf{0},\sigma\rangle$. Then (recall: $\mathcal{T}\zeta = \zeta^*\mathcal{T}$!)

$$\begin{aligned}\mathcal{T}|\mathbf{0},\sigma\rangle' = \mathcal{T}\sqrt{\zeta}\,|\mathbf{0},\sigma\rangle &= \sqrt{\zeta^*}\,\mathcal{T}|\mathbf{0},\sigma\rangle = \sqrt{\zeta^*}\,\zeta\,(-1)^{s-\sigma}|\mathbf{0},-\sigma\rangle\\ &= \sqrt{\zeta^*\zeta}\,(-1)^{s-\sigma}|\mathbf{0},-\sigma\rangle' = (-1)^{s-\sigma}|\mathbf{0},-\sigma\rangle'\,.\end{aligned}$$

Furthermore, from the explicit form (6.46) of L_p we get that $T\cdot L_p\cdot T^{-1} = P\cdot L_p\cdot P^{-1} = L_{P\cdot p}$ (because $T^\mu_{\ \nu} = -P^\mu_{\ \nu}$ - cf. (6.8)), so we find

$$\mathcal{T}|\mathbf{p},\sigma\rangle = \zeta(-1)^{s-\sigma}|-\mathbf{p},-\sigma\rangle\,. \tag{6.90}$$

It follows (in agreement with the discussion of Section 4.8) that

$$\mathcal{T}^2|\mathbf{p},\sigma\rangle = \zeta^*(-1)^{s-\sigma}\mathcal{T}|-\mathbf{p},-\sigma\rangle = \zeta^*\zeta(-1)^{s-\sigma}(-1)^{s+\sigma}|\mathbf{p},\sigma\rangle = (-1)^{2s}|\mathbf{p},\sigma\rangle\,,$$

that is, $\mathcal{T}^2$ acting on states of half-integer spin massive particles gives -1. (Compare with the result (4.197) in the nonrelativistic case).

P and T transformations of states of massless particle

In the $m=0$ case, the state-vector $|\mathbf{k},\lambda\rangle$ is the eigenvector of P^μ with the eigenvalue $(\kappa,0,0,\kappa)$ and of J^z with the eigenvalue λ. The operator $\mathcal{P}$ acting on this state produces a state with a non-standard four-momentum, so it is more convenient to consider the action on $|\mathbf{k},\lambda\rangle$ of the operator $U(R_y(\pi))\mathcal{P}$ where $R_y(\pi) = e^{i\pi\mathcal{J}^y_{\rm vec}}$ is the rotation around the y-axis by π. Since $U^{-1}(R_y(\pi))J^zU(R_y(\pi)) = -J^z$ and $J^z\mathcal{P} = \mathcal{P}J^z$ it follows that

$$U(R_y(\pi))\mathcal{P}|\mathbf{k},\lambda\rangle = \eta_\lambda|\mathbf{k},-\lambda\rangle\,. \tag{6.91}$$

[21] This follows from the equality (written in the mathematical notation, see Section 4.1)

$$\zeta^*_{\sigma'}\zeta_\sigma(\Psi_{\mathbf{0},-\sigma'}|\Psi_{\mathbf{0},-\sigma}) = (\mathcal{T}\Psi_{\mathbf{0},\sigma'}|\mathcal{T}\Psi_{\mathbf{0},\sigma}) = (\Psi_{\mathbf{0},\sigma'}|\mathcal{T}^\dagger\mathcal{T}\Psi_{\mathbf{0},\sigma})^* = (\Psi_{\mathbf{0},\sigma'}|\Psi_{\mathbf{0},\sigma})^*\,.$$

and the fact that $(\Psi_{\mathbf{0},-\sigma'}|\Psi_{\mathbf{0},-\sigma}) = (\Psi_{\mathbf{0},\sigma'}|\Psi_{\mathbf{0},\sigma})^* \propto \delta_{\sigma'\sigma}$.

To find the action of $\mathcal{P}$ on the state-vector $|\mathbf{p},\lambda\rangle$ we write

$$\begin{aligned}\mathcal{P}|\mathbf{p},\lambda\rangle &= \mathcal{P}\,U(R_{\hat{\mathbf{z}}}(\hat{\mathbf{p}}))\,U(B_z(|\mathbf{p}|/\kappa))\,|\mathbf{k},\lambda\rangle\\ &= U(R_{\hat{\mathbf{z}}}(\hat{\mathbf{p}}))\,\mathcal{P}\,U(B_z(|\mathbf{p}|/\kappa))\,\mathcal{P}^{-1}U^{-1}(R_y(\pi))\,U(R_y(\pi))\,\mathcal{P}|\mathbf{k},\lambda\rangle\\ &= \eta_\lambda\,U(R_{\hat{\mathbf{z}}}(\hat{\mathbf{p}}))\,U^{-1}(R_y(\pi))\,U(B_z(|\mathbf{p}|/\kappa))\,|\mathbf{k},-\lambda\rangle\,,\end{aligned}$$

where in going to the second line we have used the fact that because of the rule $\mathcal{P}\mathbf{J}\mathcal{P}^{-1}=\mathbf{J}$, the parity operator $\mathcal{P}$ commutes with $U(R_{\hat{\mathbf{z}}}(\hat{\mathbf{p}}))$, and for the last equality, the fact that $R_y(\pi)\cdot P=\mathrm{diag}(1,1,-1,1)$, and therefore its inverse, commutes with $B(|\mathbf{p}|/\kappa)$ given in (6.72) (and so must do the corresponding operators). Now, $R_{\hat{\mathbf{z}}}(\hat{\mathbf{p}})\cdot R_y^{-1}(\pi)$ rotates a vector pointing in the $\hat{\mathbf{z}}$ direction into a vector pointing in the $-\hat{\mathbf{p}}$ direction, but there is a subtlety related to the fact that $U(R_{\hat{\mathbf{z}}}(\hat{\mathbf{p}}))\,U^{-1}(R_y(\pi))$ is not necessarily equal to $U(R_{\hat{\mathbf{z}}}(-\hat{\mathbf{p}}))$. If $\hat{\mathbf{p}}$ is given by (6.52) then according to (6.51)

$$U(R_{\hat{\mathbf{z}}}(-\hat{\mathbf{p}}))=e^{-i(\phi_{\mathbf{p}}\pm\pi)J^z}e^{-i(\pi-\theta_{\mathbf{p}})J^y}\,, \tag{6.92}$$

where the sign $+$ $(-)$ applies if $0<\phi_{\mathbf{p}}<\pi$ $(\pi<\phi_{\mathbf{p}}<2\pi)$ so that $\phi_{\mathbf{p}}\pm\pi$ always remains in the range $(0,2\pi)$ as it should. One can then compute[22]

$$\begin{aligned}&U^{-1}(R_{\hat{\mathbf{z}}}(-\hat{\mathbf{p}}))\,U(R_{\hat{\mathbf{z}}}(\hat{\mathbf{p}}))\,U^{-1}(R_y(\pi))\\ &\quad= e^{+i(\pi-\theta_{\mathbf{p}})J^y}e^{+i(\phi_{\mathbf{p}}\pm\pi)J^z}e^{-i\phi_{\mathbf{p}}J^z}e^{-i\theta_{\mathbf{p}}J^y}e^{-i\pi J^y}\\ &\quad= e^{-i\theta_{\mathbf{p}}J^y}\left(e^{+i\pi J^y}e^{\pm i\pi J^z}e^{-i\pi J^y}\right)e^{-i\theta_{\mathbf{p}}J^y}\\ &\quad= e^{-i\theta_{\mathbf{p}}J^y}\left(e^{\mp i\pi J^z}e^{-i\theta_{\mathbf{p}}J^y}e^{\pm i\pi J^z}\right)e^{\mp i\pi J^z}=e^{\mp i\pi J^z}.\end{aligned}$$

Hence,[23]

$$U(R_{\hat{\mathbf{z}}}(\hat{\mathbf{p}}))\,U^{-1}(R_y(\pi))=U(R_{\hat{\mathbf{z}}}(-\hat{\mathbf{p}}))\,e^{\mp i\pi J^z}, \tag{6.93}$$

and finally (using again the fact that rotations around the z axis commute with boosts along the z axis) one gets

$$\mathcal{P}|\mathbf{p},\lambda\rangle=\eta_\lambda\,e^{\pm i\pi\lambda}|-\mathbf{p},-\lambda\rangle\,. \tag{6.94}$$

Setting $\eta_\lambda=\eta e^{\mp i(\lambda+s)\pi}$ ensures that the state-vectors representing single massless particles of definite helicity (the so-called helicity state-vectors) transform under parity in the same way as do the state-vectors (defined in (6.99)) of massive particles of definite helicity. Thus, finally[24]

$$\mathcal{P}|\mathbf{p},\lambda\rangle=\eta\,e^{\mp i\pi s}|-\mathbf{p},-\lambda\rangle\,, \tag{6.95}$$

[22] Using similar manipulations one can check that for massive particles indeed $\mathcal{P}U(L_p)\mathcal{P}^{-1}=U(L_{P\cdot p})$ (without any extra phase factors), thus better justifying the formula (6.87).

[23] Were the rotation (6.50) defined with an extra factor $e^{i\tilde{\phi}_{\mathbf{p}}J^z}$ on its right extreme, we would get $U(R_{\hat{\mathbf{z}}}(\hat{\mathbf{p}}))\,U^{-1}(R_y(\pi))=U(R_{\hat{\mathbf{z}}}(-\hat{\mathbf{p}}))\,e^{-2i(\tilde{\phi}_{\mathbf{p}}\pm\pi)J^z}$.

[24] Superficially (6.95) may look like no change of helicity, but one has to remember that λ acquires its proper meaning only in the frame in which the four-momentum of the particle is standard. In that frame $-\lambda$ means that the helicity *is* reversed by the parity operation.

which shows that the phase factor appearing in the parity transformation rule of the helicity state-vectors of massless (and massive) half-integer spin particles is discontinuous (e.g. for $s = \frac{1}{2}$ the phase factor is $-i$ or $+i$, depending on whether $p^y > 0$ or $p^y < 0$). From (6.95) it follows, that a theory of massless particles can be invariant under parity transformations only if particles of both opposite helicities exist. For example, quantum electrodynamics is parity invariant because photons come with two helicities.

To find the action of $\mathcal{T}$ on state-vectors of massless particles with an arbitrary four-momentum p^μ we again consider

$$U(R_y(\pi))\mathcal{T}|\mathbf{k}, \lambda\rangle = \zeta_\lambda|\mathbf{k}, \lambda\rangle\,, \tag{6.96}$$

which is justified in the same way as (6.91). Then

$$\begin{aligned}\mathcal{T}|\mathbf{p}, \lambda\rangle &= \mathcal{T}U(R_{\hat{\mathbf{z}}}(\hat{\mathbf{p}})\cdot B(|\mathbf{p}|/\kappa))\,|\mathbf{k}, \lambda\rangle\\ &= U(T\cdot R_{\hat{\mathbf{z}}}(\hat{\mathbf{p}})\cdot B(|\mathbf{p}|/\kappa))\,\mathcal{T}^{-1}U^{-1}(R_y(\pi))\,U(R_y(\pi))\mathcal{T}|\mathbf{k}, \lambda\rangle\\ &= \zeta_\lambda\,U(R_{\hat{\mathbf{z}}}(\hat{\mathbf{p}}))\,U\big(R_y^{-1}(\pi)\big)\,U(B(|\mathbf{p}|/\kappa))\,|\mathbf{k}, \lambda\rangle\\ &= \zeta_\lambda\,e^{\mp i\pi\lambda}|-\mathbf{p}, \lambda\rangle\,,\end{aligned}$$

where we have again used first the fact that T ($\mathcal{T}$) commutes with $R_{\hat{\mathbf{z}}}(\hat{\mathbf{p}})$ (with $U(R_{\hat{\mathbf{z}}}(\hat{\mathbf{p}}))$ - because it is antilinear!) and next, that $R_y(\pi)\cdot T = \mathrm{diag}(-1, 1, 1, -1)$ commutes with $B(|\mathbf{p}|/\kappa)$ given in (6.72) (and so must do the operators). Finally we have used the result (6.93). Continuity of the transformation rule of the massive particle helicity states for $m \to 0$ can be ensured by setting $\zeta_\lambda = \zeta$. Thus, finally

$$\mathcal{T}|\mathbf{p}, \lambda\rangle = \zeta\,e^{\mp i\pi\lambda}|-\mathbf{p}, \lambda\rangle\,.$$

In Chapter 8 it will become clear, that to build particle interactions preserving some quantum numbers, like e.g. the electric charge, each particle carrying such a charge should be accompanied by its antiparticle of the same mass and spin but carrying the opposite charge. In the Hilbert space constructed as a tensor product of one-particle state-vectors (i.e. spanned by the $|\alpha_0\rangle$ states) one can then introduce a unitary charge conjugation operator $\mathcal{C}$ the action of which is defined by

$$\begin{aligned}\mathcal{C}|\mathrm{particle}(\mathbf{p}, \sigma)\rangle &= \xi|\mathrm{antiparticle}(\mathbf{p}, \sigma)\rangle\,,\\ \mathcal{C}|\mathrm{antiparticle}(\mathbf{p}, \sigma)\rangle &= \xi^c|\mathrm{particle}(\mathbf{p}, \sigma)\rangle\,,\end{aligned} \tag{6.97}$$

(similar formulae apply also to the helicity state-vectors $|\mathbf{p}, \lambda\rangle$). The phase factors ξ and ξ^c are the charge conjugation parities (denoted also C) of the particle and antiparticle, respectively. If a particle is its own antiparticle (does not carry any conserved charge), $\xi = \xi^c$. In general, unitarity of $\mathcal{C}$ implies $\xi^c = \xi^*$. The problem whether charge conjugation is a symmetry of the theory of interacting particles reduces to the question whether the operator $\mathcal{C}$ commutes with $H = H_0 + V_{\mathrm{int}}$. If it does, (6.97) applies to states representing true paricles predicted by the given theory.

While in various quantum (field) theories of interacting particles the separate discrete transformations P, C and T may be broken by interactions (all of them are broken in the real world - see Chapter 12), in which case the corresponding operators $\mathcal{P}$, $\mathcal{C}$ and $\mathcal{T}$ properly acting on the *in* and *out* states strictly speaking do not exist, the combined CPT transformation always remains a good symmetry if the theory is covariant with respect to the proper ortochronous part of the Poincaré group and local (this notion will become more clear only later). Each one-particle state must have therefore its CPT counterpart. This means that antiparticles can always be defined by the action of the $\mathcal{CPT}$ operator

$$\mathcal{CPT}|\text{particle}(\mathbf{p},\sigma)\rangle = \zeta\eta\xi(-1)^{s-\sigma}|\text{antiparticle}(\mathbf{p},-\sigma)\rangle\,, \tag{6.98}$$

There can also be CPT self-conjugate particles. A newly discovered (in the years 2012/2013) Higgs boson h^0 is an example of such a particle of spin 0; existence of elementary CPT self-conjugate spin $\frac{1}{2}$ fermions, commonly called *Majorana* particles, has not been yet established experimentally (perhaps neutrinos have this nature).

6.4 State vectors representing two particles

Here we briefly discuss two-particle state-vectors in the representation of the angular momentum. This representation allows to analyze the angular momentum content of the final states of various reactions with two particles in the initial and final states. It will also be necessary to derive some consequences of unitarity of the scattering matrix (Section 7.6).

State-vectors representing two *noninteracting* particles[25] (i.e. state-vectors which are two-particle generalized eigenvectors of a free Hamiltonian H_0) are most simply characterized by two three-momenta $\mathbf{p}_1$ and $\mathbf{p}_2$ of these particles and their spin projections σ_1 and σ_2 or, usually more conveniently, by their helicities λ_1 and λ_2. Helicities (as we have seen) are the natural spin labels for states of massless particles. The helicity one-particle state-vectors of massive particles, which are generalized eigenvectors of the operator $W^0 = \mathbf{J}\cdot\hat{\mathbf{P}}$ (6.23), are defined by the formula

$$|\mathbf{p},\lambda\rangle = U(R_{\hat{\mathbf{z}}}(\hat{\mathbf{p}}))\,U(B_z(|\mathbf{p}|))|\mathbf{0},\lambda\rangle\,, \tag{6.99}$$

in which $J^z|\mathbf{0},\lambda\rangle = \lambda|\mathbf{0},\lambda\rangle$ (compare this formula with (6.71); note however, that the boost in (6.99) is the one defined in (6.53), whereas the boost in (6.71) is the one given by (6.72)). The two-particle state-vectors $|\mathbf{p}_1,\lambda_1,\mathbf{p}_2,\lambda_2\rangle$ constructed simply as $|\mathbf{p}_1,\lambda_1\rangle\otimes|\mathbf{p}_2,\lambda_2\rangle$ if the two particles of spin s are distinct, and as (see Chapter 5)

$$|\mathbf{p}_1,\lambda_1,\mathbf{p}_2,\lambda_2\rangle = \frac{1}{\sqrt{2}}\left(|\mathbf{p}_1,\lambda_1\rangle\otimes|\mathbf{p}_2,\lambda_2\rangle + (-1)^{2s}|\mathbf{p}_2,\lambda_2\rangle\otimes|\mathbf{p}_1,\lambda_1\rangle\right), \tag{6.100}$$

[25]It will be seen in Section **??**hat properties state-vectors representing two (or more) interacting particles with respect to the Poincaré group transformations are the same as the properties of the corresponding state-vectors of free particles. Hence the results of this section carry over also to this case.

if they are identical (indistinguishable), are normalized conventionally:

$$\langle\mathbf{p}_1', \lambda_1', \mathbf{p}_2', \lambda_2'|\mathbf{p}_1, \lambda_1, \mathbf{p}_2, \lambda_2\rangle = \delta_\Gamma^{(3)}(\mathbf{p}_1' - \mathbf{p}_1)\delta_\Gamma^{(3)}(\mathbf{p}_2' - \mathbf{p}_2)\delta_{\lambda_1'\lambda_1}\delta_{\lambda_2'\lambda_2} \pm \text{perm.} \quad (6.101)$$

where the permutation term is present if the two particles are identical and the symbol $\delta_\Gamma^{(3)}$ is defined in (6.34). In the case of massive particles it is of course also possible to form the basis out of tensor products of the $|\mathbf{p}, \sigma\rangle$ states defined by (6.29).

An alternative basis of the two-particle subspace of the Hilbert space is provided by the vectors characterized by the total momentum $\mathbf{P} = \mathbf{p}_1 + \mathbf{p}_2$ and the momentum $\mathbf{p}$ of, say, the first of the two particles *in their center of mass frame*

$$|\mathbf{P}, \mathbf{p}, \lambda_1, \lambda_2\rangle \equiv |\mathbf{P}, \sqrt{s}, \hat{\mathbf{p}}, \lambda_1, \lambda_2\rangle \,, \quad (6.102)$$

where the Mandelstam variable $\sqrt{s} \equiv \sqrt{\mathbf{p}^2 + m_1^2} + \sqrt{\mathbf{p}^2 + m_2^2}$ is the total energy of the two-particle system in its center of mass frame. In other words, it is the *invariant mass* of the two particle system.[26] The direction $\hat{\mathbf{p}}$ can of course be specified by giving the two angles $\vartheta_\mathbf{p}$ and $\varphi_\mathbf{p}$ (defined in the center of mass frame). The Mandelstam variable $\sqrt{s}$ can also be traded for the total energy $P^0 = \sqrt{s + \mathbf{P}^2}$ in the Laboratory frame (i.e. the frame in which the total three-momentum is $\mathbf{P}$). The state-vectors (6.102) are defined by the prescription

$$|\mathbf{P}, \sqrt{s}, \hat{\mathbf{p}}, \lambda_1, \lambda_2\rangle = U(R_{\hat{\mathbf{z}}}(\hat{\mathbf{P}}))\, U(B_z(|\mathbf{P}|))|\mathbf{0}, \sqrt{s}, \hat{\mathbf{p}}, \lambda_1, \lambda_2\rangle \,, \quad (6.103)$$

i.e. by a specific Lorentz transformation of the center of mass states $|\mathbf{0}, \sqrt{s}, \hat{\mathbf{p}}, \lambda_1, \lambda_2\rangle \equiv |\mathbf{0}, \mathbf{p}, \lambda_1, \lambda_2\rangle$, which in turn are constructed as the tensor product[27] (symmetrized or antisymmetrized as in (6.100), if the two particles are identical) of the two helicity state-vectors: $|\mathbf{p}, \lambda_1\rangle = U(R_{\hat{\mathbf{z}}}(\hat{\mathbf{p}}))|\hat{\mathbf{z}}|\mathbf{p}|, \lambda_1\rangle$ and the state-vector[28] $e^{-i\pi s_2}U(R_{\hat{\mathbf{z}}}(\hat{\mathbf{p}}))|-\hat{\mathbf{z}}|\mathbf{p}|, \lambda_2\rangle$. In this way the helicity labels λ_1, λ_2 *are defined in the center of mass frame*. For this reason the state-vectors $|\mathbf{P}, \mathbf{p}, \lambda_1, \lambda_2\rangle \equiv |\mathbf{P}, \sqrt{s}, \hat{\mathbf{p}}, \lambda_1, \lambda_2\rangle$ are related to the state-vectors $|\mathbf{p}_1, \lambda_1, \mathbf{p}_2, \lambda_2\rangle$ (which have their helicity labels defined in the Laboratory frame) by

$$|\mathbf{P}, \mathbf{p}, \lambda_1, \lambda_2\rangle = e^{-i\pi s_2}\sum_{\lambda_1'}\sum_{\lambda_2'}|\mathbf{p}_1, \lambda_1', \mathbf{p}_2, \lambda_2'\rangle\, D^{(s_1)}_{\lambda_1'\lambda_1}(R_1)\, D^{(s_2)}_{\lambda_2'\lambda_2}(R_2)\,, \quad (6.104)$$

where the respective little group rotations R_1 and R_2 result from the action as in (6.36) of the Lorentz transformation $U(\Lambda) = U(R_{\hat{\mathbf{z}}}(\hat{\mathbf{P}}))U(B_z(|\mathbf{P}|))$ on the one-particle states $|\mathbf{p}, \lambda_1\rangle$

[26] Since these are states of free particles there is no question about their interaction energy (the so called *mass defect*).

[27] The action of the symmetry operators like $U(R_{\hat{\mathbf{z}}}(\hat{\mathbf{P}}))$ or $U(B_z(|\mathbf{P}|))$ on such tensor products of state-vectors of single particles follows the rule (5.53).

[28] Due to the extra phase factor the state-vector $e^{-i\pi s_2}U(R_{\hat{\mathbf{z}}}(\hat{\mathbf{p}}))|-\hat{\mathbf{z}}|\mathbf{p}|, \lambda_2\rangle$ is, if the particles are massive, identical with the state-vector $U(B_{-z}(|\mathbf{p}|))|\mathbf{0}, -\lambda_2\rangle$; in this way the two particles represented by the state-vector $|\mathbf{0}, \sqrt{s}, \hat{\mathbf{p}}, \lambda_1, \lambda_2\rangle$ are treated symmetrically, what in turn leads to a simple time reversal transformation rule for these state-vectors. Notice also that in the case of half-integer spin particles $U(R_{\hat{\mathbf{z}}}(\hat{\mathbf{p}}))|-\hat{\mathbf{z}}|\mathbf{p}|, \lambda_2\rangle$ differs, when $\varphi_\mathbf{p} > \pi$, from $U(R_{\hat{\mathbf{z}}}(-\hat{\mathbf{p}}))|\hat{\mathbf{z}}|\mathbf{p}|, \lambda_2\rangle$.

and $U(R_{\hat{\mathbf{z}}}(\hat{\mathbf{p}}))|{-\hat{\mathbf{z}}}|\mathbf{p}|,\lambda_2\rangle \propto |{-\mathbf{p}},\lambda_2\rangle$; if the particle i is massive, the corresponding rotation is $R_i = R^{-1}_{\hat{\mathbf{z}}}(\hat{\mathbf{p}}^\Lambda)\cdot W(\Lambda,p_i)\cdot R_{\hat{\mathbf{z}}}(\hat{\mathbf{p}})$ where $W(\Lambda,p_i)$ is given by the composition (6.31); if the particle i is massless, the corresponding matrix $D^{(s_i)}_{\lambda_i'\lambda_i}(R_i)$ is just equal $\delta_{\lambda_i'\lambda_i}e^{i\lambda_i\theta_i}$ with θ_i determined from (6.70).

It can be shown that the scalar product of the two vectors $|\mathbf{P},\mathbf{p},\lambda_1,\lambda_2\rangle$ (6.102) representing two distinct particles is given by

$$\begin{aligned}\langle\mathbf{P}',\mathbf{p}',\lambda_1',\lambda_2'|\mathbf{P},\mathbf{p},\lambda_1,\lambda_2\rangle &= \delta^{(3)}_\Gamma(\mathbf{p}_1'-\mathbf{p}_1)\,\delta^{(3)}_\Gamma(\mathbf{p}_2'-\mathbf{p}_2)\,\delta_{\lambda_1'\lambda_1}\delta_{\lambda_2'\lambda_2}\\ &= (2\pi)^4\delta^{(4)}(P'-P)\,16\pi^2\,\frac{\sqrt{s}}{|\mathbf{p}|}\,\delta^{(2)}(\Omega_{\hat{\mathbf{p}}'}-\Omega_{\hat{\mathbf{p}}})\,\delta_{\lambda_1'\lambda_1}\delta_{\lambda_2'\lambda_2}\,. \quad (6.105)\end{aligned}$$

If the two particles are identical, one has to add to the right hand side in the first line the term $(-1)^{2s}\delta^{(3)}_\Gamma(\mathbf{p}_1'-\mathbf{p}_2)\delta^{(3)}_\Gamma(\mathbf{p}_2'-\mathbf{p}_1)\,\delta_{\lambda_1'\lambda_2}\delta_{\lambda_2'\lambda_1}$ and replace $\delta^{(2)}(\Omega_{\hat{\mathbf{p}}'}-\Omega_{\hat{\mathbf{p}}})\,\delta_{\lambda_1'\lambda_1}\delta_{\lambda_2'\lambda_2}$ by

$$\delta^{(2)}(\Omega_{\hat{\mathbf{p}}'}-\Omega_{\hat{\mathbf{p}}})\,\delta_{\lambda_1'\lambda_1}\delta_{\lambda_2'\lambda_2}+\delta^{(2)}(\Omega_{\hat{\mathbf{p}}'}-\Omega_{-\hat{\mathbf{p}}})\,\delta_{\lambda_1'\lambda_2}\delta_{\lambda_2'\lambda_1}\,,$$

(without any $(-1)^{2s}$ factor!) in the second line.

One next introduces state-vectors of two particles with fixed total angular momentum j in their center of mass frame. They are given (for distinct and identical particles alike) by the formula

$$|\mathbf{P},\sqrt{s},\lambda_1,\lambda_2,j,m_j\rangle = \sqrt{\frac{2j+1}{4\pi}}\int d\Omega_{\hat{\mathbf{p}}}\,|\mathbf{P},\sqrt{s},\hat{\mathbf{p}},\lambda_1,\lambda_2\rangle\,D^{(j)*}_{m_j,\lambda_1-\lambda_2}(\Omega_{\hat{\mathbf{p}}})\,. \qquad (6.106)$$

The matrices $D^{(j)}_{m_j,\lambda_1-\lambda_2}(\Omega_{\hat{\mathbf{p}}})\equiv D^{(j)}_{m_j,\lambda_1-\lambda_2}(\varphi_{\mathbf{p}},\vartheta_{\mathbf{p}},0)$ defined by the formulae (4.89) and (4.90) are explicitly given in B. It can be shown that if the two particles are identical,

$$|\mathbf{P},\sqrt{s},\lambda_2,\lambda_1,j,m_j\rangle = (-1)^j|\mathbf{P},\sqrt{s},\lambda_1,\lambda_2,j,m_j\rangle\,, \qquad (6.107)$$

so that for j odd such state-vectors vanish if $\lambda_1=\lambda_2$. This is a reflection of the usual rule (following from the requirements imposed by the Bose-Einstein or Fermi-Dirac statistics) that two identical bosons in the symmetric spin state cannot be in P, F,... waves whereas two fermions in the symmetric spin state cannot be in S, D,... waves.

The generalized vectors (6.106) are eigenvectors of $P^\mu P_\mu$ with the eigenvalue s and of $W^\mu W_\mu$ with the eigenvalues $-s\,j(j+1)$ so that j is the total angular momentum of the two-particle system and $\sqrt{s}$ its "mass". When $\mathbf{P}=0$ (i.e. when the laboratory frame coincides with the two-particle system's center of mass frame), they are eigenvectors of W^3 with the eigenvalue $\sqrt{s}\,m_j$, (m_j is therefore the total angular momentum projection onto the z axis) while for $\mathbf{P}\neq 0$ - of W^0 with the eigenvalue $|\mathbf{P}|\,m_j$. When $\mathbf{P}\neq 0$, the state-vectors $|\mathbf{P},\sqrt{s},\lambda_1,\lambda_2,j,m_j\rangle$ are therefore the analogs of the helicity state-vectors (6.99) of a massive particle (of mass $\sqrt{s}$); the quantum number j is then called the total spin S of the two-particle system and m_j acquires the interpretation of the system's total helicity (it is denoted by Λ).

The scalar product of two such vectors representing a state of two distinct particles is given by

$$\langle \mathbf{P}', \sqrt{s'}, \lambda_1', \lambda_2', j', m_j' | \mathbf{P}, \sqrt{s}, \lambda_1, \lambda_2, j, m_j \rangle$$
$$= (2\pi)^4 \delta^{(4)}(P' - P)\, 16\pi^2 \frac{\sqrt{s}}{|\mathbf{p}|}\, \delta_{j'j}\, \delta_{m_j' m_j}\, \delta_{\lambda_1' \lambda_1}\, \delta_{\lambda_2' \lambda_2}\,, \tag{6.108}$$

where the relation

$$\int d\Omega_{\hat{\mathbf{p}}}\, D^{(j')}_{m_j',\lambda}(\Omega_{\hat{\mathbf{p}}})\, D^{(j)*}_{m_j,\lambda}(\Omega_{\hat{\mathbf{p}}}) = \frac{4\pi}{2j+1}\, \delta_{j'j}\, \delta_{m_j',m_j}\,, \tag{6.109}$$

has been used. If the two particles are identical, the helicity Kronecker deltas in (6.108) have to be replaced by

$$\delta_{\lambda_1' \lambda_1}\, \delta_{\lambda_2' \lambda_2} + (-1)^j\, \delta_{\lambda_1' \lambda_2}\, \delta_{\lambda_2' \lambda_1}\,,$$

(so that one gets zero for j odd and $\lambda_1 = \lambda_2$). The relation inverse to (6.106), valid for distinct and identical particles alike, reads

$$|\mathbf{P}, \sqrt{s}, \hat{\mathbf{p}}, \lambda_1, \lambda_2\rangle = \sum_j^{\infty} \sum_{m_j=-j}^{+j} \sqrt{\frac{2j+1}{4\pi}}\, |\mathbf{P}, \sqrt{s}, \lambda_1, \lambda_2, j, m_j\rangle D^{(j)}_{m_j,\lambda_1-\lambda_2}(\Omega_{\hat{\mathbf{p}}})\,, \tag{6.110}$$

with the sum over j running over all integer (including zero) or all half-integer values. The formula (6.110) will be used in Section 7.6 to explore consequences of the unitarity of the scattering matrix.

6.5 Summary

We have investigated the transformation properties of one-particle state-vectors which follow from the postulate that the Poincaré group is the group of symmetry transformations. Starting from the one-particle state-vectors constructed in Section 6.2 the big Hilbert space $\mathcal{H} = \oplus_{N=0}^{\infty} \mathcal{H}^{(N)}$ spanned by the (appropriately symmetrized/antisymmetrized) tensor products of one-particle state-vectors can be constructed along the lines outlined in Chapter 5 and assuming that the vector $|\text{void}\rangle$ is invariant with respect to all transformatios forming the Poicaré group: $U(\Lambda, a)|\text{void}\rangle = |\text{void}\rangle$. The corresponding creation and annihilation operators can be then introduced and share the Poincaré transformation properties of the corresponding one-particle states:

$$U(\Lambda, a)\, a_\sigma^\dagger(\mathbf{p})\, U^{-1}(\Lambda, a) = e^{-ip_\Lambda^\mu a_\mu} \sum_{\sigma'} a_{\sigma'}^\dagger(\mathbf{p}_\Lambda) D_{\sigma'\sigma}(W(\Lambda, p))\,,$$
$$U(\Lambda, a)\, a_\sigma(\mathbf{p})\, U^{-1}(\Lambda, a) = e^{+ip_\Lambda^\mu a_\mu} \sum_{\sigma'} a_{\sigma'}(\mathbf{p}_\Lambda)\, [D_{\sigma'\sigma}(W(\Lambda, p))]^*\,. \tag{6.111}$$

In the relativistic normalization of the state-vectors, the operators $a^\dagger_\sigma(\mathbf{p})$ and $a_\sigma(\mathbf{p})$ are normalized so that

$$[a_\sigma(\mathbf{p}),\ a^\dagger_{\sigma'}(\mathbf{p}')]_\mp = (2\pi)^3 2E(\mathbf{p})\,\delta_{\sigma\sigma'}\,\delta^{(3)}(\mathbf{p}'-\mathbf{p})\,, \tag{6.112}$$

where $[\ ,\ ]_\mp$ denotes the commutator if these are operators creating/annihilated bosons and anticommutator if fermions. Out of these creation and annihilation operators the generators of the Poincaré group acting in $\mathcal{H}$ can be constructed as their bilinear combinations, provided the Hamiltonian has the form

$$H_0 = \int \frac{d^3\mathbf{p}}{(2\pi)^3 2E(\mathbf{p})}\sum_\sigma E(\mathbf{p})\,a^\dagger_\sigma(\mathbf{p})a_\sigma(\mathbf{p})\,, \tag{6.113}$$

with $E(\mathbf{p}) = \sqrt{\mathbf{p}^2+m^2}$, i.e. if it is the Hamiltonian of massive free relativistic particles. In this case all many-particle state-vectors obtained by the action on $|\mathrm{void}\rangle$ of the creation operators are automatically generalized (nonnormalizable) eigenvectors of the Hamiltonian H_0. The generators P^i and J^i have the same form as in the nonrelativistic theory. The generators K^i acting *in the entire* Hilbert space $\mathcal{H} = \oplus_{N=0}^{\infty}\mathcal{H}^{(N)}$ can also easily be constructed. Similar construction of the Poincaré group generators $\mathbf{K}$ and of the operators $U(\Lambda, a)$ is more complicated in the case of massless, i.e. when in (6.113) $E(\mathbf{p}) = |\mathbf{p}|$, particles but can also be done. In this way a relativistic theory of free particles can be constructed. However the explicit recipe for constructing the Poincaré group generators for a given type of particles is provided only within a different approach based on the canonical quantization of classical free relativistic fields[29] corresponding to these particles; in the formalism presented here the form of the Poincaré group generators can only be postulated and validated by checking their commutation rules. This alternative approach will be discussed in Chapter 11. It will be shown there that the Poincaré group generators are obtained as integrals of the Noether currents associated with space-time symmetries. Moreover, it is precisely the transformation properties of eigenvectors of Hamiltonians of quantized field that allow to give them the particle interpretation.

If the theory is to be a theory of interacting particles, its Hamiltonian must have a more complicated form than the sum of several terms (6.113) corresponding to different types of particles. The algebra of the Poincaré group generators corresponding to the free part of the Hamiltonian (i.e. of the generators satisfying the rules (6.21) with H_0) is then still naturally realized in the Hilbert space built as (appropriately symmetrized/antisymmetrized) tensor products of one-particle states which are generalized eigenvectors of a free Hamiltonian H_0, but if the theory is to be a relativistic quantum theory of interacting particles in the same Hilbert space must also act generators P^i, J^i and K^i satisfying the algebra (6.21) with the full Hamiltonian[30] $H = H_0 + V_{\mathrm{int}}$. If such theory

[29]Lagrangian densities of free relativistic fields are bilinear in field variables. In this case the correspondence of a field and a particle type is unique.

[30]As will be seen in Section 7.5, the generators K^i cannot then be bilinear in the creation and annihilation operators. Again the explicit recipe for constructing the Poincaré group generators is obtained only within the approach based on quantization of classical relativistic fields.

obtained afer adding to the sum of terms like (6.113) an interaction operator V_{int} and constructing the appropriate Poincaré group generators is still a theory of (interacting) particles, one-particle states, which are generalized eigenvectors of the full Hamiltonian H with the same Poincaré transformation properties as discussed in Sections 6.2 and 6.3 exist in the Hilbert space and are complicated superpositions of one- and many-particle eigenvectors of H_0. The sense in which in a theory with interactions included two-particle states like (6.101) and, more generally, multi-particle states can be defined and ascribed the Poincaré transformation properties is discussed in the next section.

Summarizing, a second-quantized version of the relativistic mechanics of noninteracting particles can be constructed in the momentum space representation without any reference to wave equations. The momentum space is singled out because the assumed realization of the Poincaré group in the Hilbert space provides us with the set of observables which naturally pertain to the momentum variables. The position space representation, commonly used in nonrelativistic quantum mechanics, in the relativistic case can be at best of only a limited validity.

Appendix D Establishing conventions for the Poincaré transformations

Here we establish how the parameters $\omega_{\mu\nu}$ and a_μ are related to the familiar parameters of boosts, rotations and translations. We take here the *passive* view, i.e. we consider *the same* system viewed from two different frames. To this end the infinitesimal coordinate transformations written in the usual way

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu - a^\mu \approx \left(\delta^\mu{}_\nu - \omega^\mu{}_\nu\right) x^\nu - a^\mu \,, \tag{D.1}$$

should be represented the form

$$x'^\mu \approx x^\mu + \frac{i}{2}\,\omega_{\lambda\nu}\left(\mathcal{J}^{\lambda\nu}_{\rm vec}\right)^\mu{}_\kappa\, x^\kappa - a^\mu \,, \tag{D.2}$$

in which $\mathcal{J}^{\lambda\nu}_{\rm vec}$ are the matrix Lorentz group generators in the vectorial representation

$$\left(\mathcal{J}^{\lambda\nu}_{\rm vec}\right)^\mu{}_\kappa = i\left(g^{\lambda\mu} g^\nu{}_\kappa - g^{\nu\mu} g^\lambda{}_\kappa\right). \tag{D.3}$$

Obviously, the matrices $\mathcal{J}^{\lambda\nu}_{\rm vec}$ satisfy the basic commutation rule (6.19)

$$\left[\mathcal{J}^{\kappa\lambda}_{\rm vec},\ \mathcal{J}^{\mu\nu}_{\rm vec}\right] = i\left(\mathcal{J}^{\kappa\nu}_{\rm vec}\, g^{\lambda\mu} - \mathcal{J}^{\kappa\mu}_{\rm vec}\, g^{\lambda\nu} - \mathcal{J}^{\lambda\nu}_{\rm vec}\, g^{\kappa\mu} + \mathcal{J}^{\lambda\mu}_{\rm vec}\, g^{\kappa\nu}\right).$$

Translations

It readily follows from (D.2) that if x^μ are the coordinates of an event in a frame $\mathcal{O}$, then $x^{\mu\prime}$ are the coordinates of the same event in the frame $\mathcal{O}'$ which is shifted with respect to $\mathcal{O}$ by a vector $\mathbf{a}$ (which in the frame $\mathcal{O}$ has components a^i), and whose clock is late by $\tau = a^0$ with respect to the clock of $\mathcal{O}$.

Rotations

If the frame $\mathcal{O}'$ is rotated with respect to $\mathcal{O}$ by an angle ϕ counterclockwise around the z-axis common for both frames, $\mathcal{O}$ and $\mathcal{O}'$, we have

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \approx \begin{pmatrix} x \\ y \end{pmatrix} + i\phi\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \dots \tag{D.4}$$

The generator $\mathcal{J}^{12}_{\rm vec}$ given by (D.3) is the matrix

$$\left(\mathcal{J}^{12}_{\rm vec}\right)^\mu{}_\kappa = \begin{pmatrix} 0 & & & \\ & 0 & -i & \\ & i & 0 & \\ & & & 0 \end{pmatrix}. \tag{D.5}$$

and comparing (D.2) with (D.4) one finds that

$$\omega_{12} = -\omega_{21} = -\omega^1{}_2 = \omega^2{}_1 = \phi\,. \tag{D.6}$$

The matrix which realizes a finite (passive) transformation, obtained by integrating (D.2) with $\omega_{12} \neq 0$ and $\mathcal{J}^{12}_{\text{vec}}$ given by (D.5) is

$$R_z(\phi) \equiv e^{i\omega_{12}\mathcal{J}^{12}_{\text{vec}}} = e^{i\phi\mathcal{J}^{z}_{\text{vec}}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\phi & \sin\phi & 0 \\ 0 & -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{D.7}$$

Similarly, if $\mathcal{O}'$ is rotated with respect to $\mathcal{O}$ by an angle θ counterclockwise around the common y-axis, then

$$\begin{pmatrix} x' \\ z' \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ z \end{pmatrix} \approx \begin{pmatrix} x \\ z \end{pmatrix} + i\theta \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \begin{pmatrix} x \\ z \end{pmatrix} + \dots \tag{D.8}$$

The generator $\mathcal{J}^{31}_{\text{vec}}$ given by (D.3) is the matrix

$$\left(\mathcal{J}^{31}_{\text{vec}}\right)^{\mu}{}_{\kappa} = \begin{pmatrix} 0 & & & \\ & 0 & & i \\ & & 0 & \\ & -i & & 0 \end{pmatrix}. \tag{D.9}$$

Exponentiating (D.9), one finds

$$R_y(\theta) \equiv e^{i\omega_{31}\mathcal{J}^{31}_{\text{vec}}} = e^{i\theta\mathcal{J}^{y}_{\text{vec}}} = \begin{pmatrix} 1 & & & \\ & \cos\theta & & -\sin\theta \\ & & 1 & \\ & \sin\theta & & \cos\theta \end{pmatrix}. \tag{D.10}$$

The matrix $R_x(\alpha)$ can be found similarly by exponentiating $\mathcal{J}^{23}_{\text{vec}}$ given by (D.3).

The matrix representing the *active* rotation $R_{\hat{\mathbf{z}}}(\hat{\mathbf{p}})$ which produces $\hat{\mathbf{p}} = (\sin\theta_{\mathbf{p}}\cos\phi_{\mathbf{p}},\ \sin\theta_{\mathbf{p}}\sin\phi_{\mathbf{p}},\ \cos\theta_{\mathbf{p}})$ out of $\hat{\mathbf{z}} = (0,\,0,\,1)$ and which enters the standard transformations L_p (6.49) and (6.71) reads[1]

$$\begin{aligned} R_{\hat{\mathbf{z}}}(\hat{\mathbf{p}}) &= e^{-i\phi_{\mathbf{p}}\mathcal{J}^{z}_{\text{vec}}} \cdot e^{-i\theta_{\mathbf{p}}\mathcal{J}^{y}_{\text{vec}}} = R^{-1}_z(\phi_{\mathbf{p}}) \cdot R^{-1}_y(\theta_{\mathbf{p}}) \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\phi_{\mathbf{p}} & -\sin\phi_{\mathbf{p}} & 0 \\ 0 & \sin\phi_{\mathbf{p}} & \cos\phi_{\mathbf{p}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta_{\mathbf{p}} & 0 & \sin\theta_{\mathbf{p}} \\ 0 & 0 & 1 & 0 \\ 0 & -\sin\theta_{\mathbf{p}} & 0 & \cos\theta_{\mathbf{p}} \end{pmatrix}. \end{aligned} \tag{D.11}$$

[1]To distinguish passive rotations (whose matrices are matrices of changes of bases) from the active ones (which are linear mappings of the vector space into itself whose matrices are written in fixed - usually the same on both "sides" - bases) it would be more appropriate to denote $R_{\hat{\mathbf{z}}}(\hat{\mathbf{p}})$ by another symbol, e.g. $O_{\hat{\mathbf{z}}}(\hat{\mathbf{p}})$, reserving the symbol R for passive rotations.

Using the formula (4.84) the rotation $R_{\hat{\mathbf{z}}}(\hat{\mathbf{p}})$ can be also represented in the form

$$R_{\hat{\mathbf{z}}}(\hat{\mathbf{p}}) = R(\theta_{\mathbf{p}}, \mathbf{n}) \cdot R(\phi_{\mathbf{p}}, \hat{\mathbf{z}})\,, \tag{D.12}$$

where the second rotation (by $\theta_{\mathbf{p}}$) is performed around the axis $\mathbf{n} = -\hat{\mathbf{x}}\sin\phi_{\mathbf{p}} + \hat{\mathbf{y}}\cos\phi_{\mathbf{p}}$. The first rotation (by $\phi_{\mathbf{p}}$ around the $\hat{\mathbf{z}}$ axis) is then ineffective in acting on the momentum along the z-axis (and it is for this reason that in many textbooks as $R_{\hat{\mathbf{z}}}(\hat{\mathbf{p}})$ one takes not (D.11) but (D.11) multiplied from the right by $\exp(+i\phi_{\mathbf{p}}\mathcal{J}^z_{\rm vec})$, so that this ineffective rotation is absent), while the second rotation makes $\hat{\mathbf{p}}$ out of $\hat{\mathbf{z}}$ directly, in one move.

Boosts

Let the frame $\mathcal{O}'$ be boosted with respect to $\mathcal{O}$ in the x direction with velocity v (as measured in the frame $\mathcal{O}$). Then

$$\begin{aligned} t' &= \gamma(t - vx) \approx t - vx + \ldots \\ x' &= \gamma(x - vt) \approx x - vt + \ldots \end{aligned} \tag{D.13}$$

with $\gamma = 1/\sqrt{1-v^2}$. The generator $\mathcal{J}^{01}_{\rm vec}$ is the matrix

$$\left(\mathcal{J}^{01}_{\rm vec}\right)^{\mu}{}_{\kappa} = \begin{pmatrix} 0 & i & & \\ i & 0 & & \\ & & 0 & 0 \\ & & 0 & 0 \end{pmatrix}. \tag{D.14}$$

Inserting (D.14) into (D.2) and comparing with (D.13) one finds that

$$\omega_{01} = -\omega^{01} \approx v\,. \tag{D.15}$$

Finite (passive) boosts along the x-axis are realized by the matrix

$$\Lambda = e^{i\omega_{01}\mathcal{J}^{01}_{\rm vec}} = e^{i\omega_{01}\mathcal{K}^{x}_{\rm vec}} = \begin{pmatrix} \mathrm{ch}\,\omega_{01} & -\mathrm{sh}\,\omega_{01} & & \\ -\mathrm{sh}\,\omega_{01} & \mathrm{ch}\,\omega_{01} & & \\ & & 1 & 0 \\ & & 0 & 1 \end{pmatrix}, \tag{D.16}$$

so that the finite boost takes the form

$$\begin{aligned} t' &= \mathrm{ch}\,\omega_{01}\,(t - \mathrm{th}\,\omega_{01}\,x)\,, \\ x' &= \mathrm{ch}\,\omega_{01}\,(x - \mathrm{th}\,\omega_{01}\,t)\,, \end{aligned} \tag{D.17}$$

from which it follows that

$$\mathrm{th}\,\omega_{01} = v\,, \qquad \mathrm{ch}\,\omega_{01} = \frac{1}{\sqrt{1-v^2}}\,. \tag{D.18}$$

Similarly, $(\mathcal{J}^{03}_{\rm vec})^{\mu}{}_{\kappa}$ is the matrix

$$(\mathcal{J}^{03}_{\rm vec})^{\mu}{}_{\kappa} = \begin{pmatrix} & & 0 & i \\ & & 0 & 0 \\ 0 & 0 & & \\ i & 0 & & \end{pmatrix}, \tag{D.19}$$

The active boosts B_z appearing in the standard transformations L_p (6.49) acting on the standard four-momentum of a particle of mass M has therefore the explicit form

$$B_z(\omega_{03}) = e^{-i\omega_{03}\mathcal{J}^{03}_{\rm vec}} = e^{-i\omega_{03}\mathcal{K}^{z}_{\rm vec}} = \text{ch}\,\omega_{03}\begin{pmatrix} 1 & 0 & 0 & \text{th}\,\omega_{03} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \text{th}\,\omega_{03} & 0 & 0 & 1 \end{pmatrix}, \tag{D.20}$$

in which $\text{sh}\,\omega_{03} = |\mathbf{p}|/M$, $\text{ch}\,\omega_{03} = E(\mathbf{p})/M$. In turn B_z entering L_p (6.71) appropriate for a massless particle has a similar form with $\omega_{03} = \ln u = \ln(|\mathbf{p}|/\kappa)$ (so that $\text{ch}\,\omega_{03} = (u^2+1)/2u$, $\text{sh}\,\omega_{03} = (u^2-1)/2u$).

The meaning of the parameters $\omega_{\mu\nu}$ and a_μ is always the same, irrespectively of the specific representation of the Poincaré group generators. One can check the formulae given above by considering e.g. the ordinary quantum mechanical scalar wave function $\psi(t,\mathbf{x})$. If the frame $\mathcal{O}'$ is shifted with respect to $\mathcal{O}$ by a vector $\mathbf{a}$ then for a scalar function one should have

$$\psi'(x') = \psi(x) \equiv \psi(x'+a)\,, \tag{D.21}$$

i.e. the shape $\psi'(\cdot)$ of the function as determined in $\mathcal{O}'$ should be the same as the shape of $\psi(\cdot)$ but its maxima, minima etc. should occur for shifted values of its argument. So, for infinitesimal a^i, and renaming $x' \to x$, one has

$$\psi'(x) = \psi(x) + a^i\,\frac{\partial\psi(x)}{\partial x^i} + \dots \tag{D.22}$$

which can be written in the form

$$\psi'(x) = \psi(x) + ia^i\hat{P}^i\psi(x) + \dots \tag{D.23}$$

with $\hat{P}^i$ the momentum operator

$$\hat{P}^i = -i\frac{\partial}{\partial x^i} \equiv +i\frac{\partial}{\partial x_i}\,. \tag{D.24}$$

Hence, the wave function ψ' in the frame $\mathcal{O}'$ is obtained from the wave function ψ in $\mathcal{O}$ by

$$\psi'(x) = e^{+ia^i\hat{P}^i}\psi(x) = e^{-ia_i\hat{P}^i}\psi(x)\,. \tag{D.25}$$

Similarly, if the clock of $\mathcal{O}'$ is late with respect to the clock of $\mathcal{O}$ by τ, that is, if $t' = t - \tau$, the time shape of a function ψ' must be the same, i.e.

$$\psi'(t') = \psi(t) = \psi(t' + \tau)\,. \tag{D.26}$$

This implies (dropping the prime) that

$$\psi'(t) = \psi(t+\tau) = \psi(t) - i\tau\, i\frac{\partial}{\partial t}\psi(t) + \ldots = \psi(t) - i\tau\, H\psi(t) + \ldots \tag{D.27}$$

Thus, with $a^\mu = (\tau, a^i)$ we have

$$U(1,\ a) = e^{-ia_\mu P^\mu}. \tag{D.28}$$

If the axes of $\mathcal{O}'$ are rotated counterclockwise with respect to the axes of $\mathcal{O}$ by an angle θ then for a scalar function ψ, using (D.4) and the same arguments as above, one must have

$$\psi'(x', y') = \psi(x, y) = \psi(x' - \theta y',\, y' + \theta x')\,, \tag{D.29}$$

or, dropping the primes,

$$\psi'(x, y) = \psi(x, y) + \theta\left(x\,\frac{\partial}{\partial y} - y\,\frac{\partial}{\partial x}\right)\psi(x, y) + \ldots \tag{D.30}$$

(where x, y are the contravariant coordinates). This can be written in the form

$$\begin{aligned}\psi'(x, y) &= \psi(x, y) + i\theta\left(x\hat{P}^y - y\hat{P}^x\right)\psi(x, y) + \ldots \\ &= \psi(x, y) + i\theta J^z\psi(x, y) + \ldots\end{aligned} \tag{D.31}$$

Hence,

$$U(R_z(\theta),\ 0) = e^{i\theta J^z} = e^{\frac{i}{2}\omega_{\mu\nu}J^{\mu\nu}} = e^{i\omega_{12}J^z}. \tag{D.32}$$