Problems in Quantum Field Theory

Problem 0.1
Prove the following expansion

$$e^{A}B\,e^{-A} = B + [A,\ B] + \frac{1}{2!}\,[A,\ [A,\ B]] + \frac{1}{3!}\,[A,\ [A,\ [A,\ B]]] + \dots$$

Prove also the Baker-Hausdorff operator identity

$$e^{A+B} = e^{A}\,e^{B}\,e^{-\frac{1}{2}[A,\ B]} = e^{-\frac{1}{2}[A,\ B]}\,e^{A}\,e^{B}\,,$$

holding for operators A and B commuting with $[A,\ B]$. Finally, prove the general formula,[1]

$$e^{t(A+B)} = e^{tA}\,\mathrm{T}\exp\left(\int_0^t d\tau\, e^{-\tau A}B\,e^{\tau A}\right),$$

valid for any two operators A and B, in which T denotes the "time" ordered product.
Hints: To prove the expansion solve iteratively the differential equation satisfied by the operator function $C(\lambda) = e^{\lambda A}B\,e^{-\lambda A}$. Similarly, to prove the Baker-Hausdorff formula consider the function $F(\lambda) = e^{-\lambda B}e^{-\lambda A}e^{\lambda(A+B)}$ and simplify the differential equation satisfied by it using the fact that owing to the assumption, in the expansion of $e^{B}A\,e^{-B}$ in powers of the operator B only two first terms are nonvanishing.

Problem 0.2
Let $|\Psi(t)\rangle_S$ be an eigenvector with the eigenvalue $a(t)$ of the Schrödinger picture operator A^S. Show that $|\Psi\rangle_H$ representing the same state in the Heisenberg picture (defined with respect to $t = 0$) is the eigenvector of $A^H(t)$ with the same eigenvalue $a(t)$. Prove also that if $[A^H(t_0),\ B^H(t_0)] = C^H(t_0)$, then the same holds for any t.

Problem 0.3
Find the Heisenberg picture operators $\hat{x}_H(t)$ and $\hat{p}_H(t)$ of a particle of mass M moving in one dimension if
a) it is a free particle ($H = \hat{p}^2/2M$),
b) $H = \hat{p}^2/2M - \hat{x}F(t)$, where $F(t)$ is an external, time dependent force,
c) $H = \hat{p}^2/2M + M\omega^2\hat{x}^2/2$.
In all these cases compute the commutators

$$[\hat{x}_H(t),\ \hat{x}_H(t')]\,,\qquad [\hat{p}_H(t),\ \hat{p}_H(t')]\,,\qquad [\hat{x}_H(t),\ \hat{p}_H(t')]\,.$$

Using the Heisenberg picture operators compute in cases a) and c) the dispersion of the particle's position at the instant t expressing it through matrix elements of some combinations of the position and momentum operators at $t = 0$.

[1] The Baker-Hausdorff formula is its special case with $t = 1$ and $[A,\ [A,\ B]] = [B,\ [A,\ B]] = 0$.

Problem 0.4

Find the Heisenberg picture operators $\hat{x}_H(t)$ and $\hat{p}_H(t)$ of the one-dimensional harmonic oscillator the dynamics of which is set by the time dependent Hamiltonian

$$H(t) = \frac{\hat{p}^2}{2M} + \frac{1}{2}M\omega^2\hat{x}^2 - \hat{x}F(t)\,,$$

using the solution of the corresponding classical equations of motion with the initial conditions $x(0) = x_0$ and $p(0) = p_0$. To this end, recalling that x_0 and p_0 are also canonical variables related to the standard ones, $x(t)$ and $p(t)$, by the canonical transformation (the generating function of which is just the properly understood action I), promote them to operators $\hat{x}_0$ and $\hat{p}_0$ on which the standard commutation rules $[\hat{x}_0,\ \hat{p}_0] = i\hbar$, etc. are imposed and represent them in the standard way in terms of the creation and annihilation operators. Since the classical Hamiltonian written in terms of the canonical variables x_0 and p_0 vanishes (this is precisely what is ensured by solving the Hamilton-Jacobi equation, but one does not need to do it explicitly here), the operators $\hat{x}(t)$ and $\hat{p}(t)$ obtained from the classical solution in which the operators $\hat{x}_0$ and $\hat{p}_0$ are substituted for x_0 and p_0 (expressed, in turn, through the creation and annihilation operators) are just the Heisenberg picture operators. The Heisenberg picture operators $a_H(t)$ and $a_H^\dagger(t)$ can be then read off from the form of $\hat{x}_H(t)$ and $\hat{p}_H(t)$.

A reassuring remark: the description of the problem is long but the steps to do are entirely trivial. After you do it, you will have, perhaps, a better understanding of what "quantization" means.

Problem 0.5

A particle of mass m and electric charge q (in units of $e > 0$) moves in the constant magnetic field $\mathbf{B} = \mathbf{e}_z B$. Find the Heisenberg picture operators $\hat{x}_H(t)$, $\hat{y}_H(t)$ and $\hat{z}_H(t)$ and compute the commutators $[\hat{x}_H(t),\ \hat{x}_H(t')]$, $[\hat{y}_H(t),\ \hat{y}_H(t')]$, $[\hat{x}_H(t),\ \hat{y}_H(t')]$ and $[\hat{x}_H(t),\ \hat{z}_H(t')]$. Do these commutators depend on the choice of the potential $\mathbf{A}$ (the choice of the gauge)? Consider also the operators $\hat{p}^x_H(t)$, $\hat{p}^y_H(t)$, $\hat{p}^z_H(t)$ and their commutators. Do they depend on the gauge?

Hint: If it is too difficult to work without specifying explicitly a gauge, set e.g. $\mathbf{A} = \mathbf{e}_y\xi Bx - \mathbf{e}_x(1-\xi)By$ with an arbitrary parameter ξ in order to follow the gauge (in)dependence at least within a restricted class of gauges. To construct the Heisenberg picture operators $\hat{x}_H(t)$, $\hat{y}_H(t)$, $\hat{p}^x_H(t)$, $\hat{p}^y_H(t)$, take the inspiration from Problem 0.4. Remember that the canonical momenta p^x and p^y are not simply given by $m\dot{x}$ and $m\dot{y}$.

Problem 0.6

A particle of mass M and electric charge q (in units of $e > 0$) moves in the electric and magnetic fields represented by the potentials $\varphi(t, \mathbf{r})$ and $\mathbf{A}(t, \mathbf{r})$. Find the equation of motion satisfied by the Heisenberg picture operator $\hat{\mathbf{r}}_H(t)$, that is compute $d^2\hat{\mathbf{r}}_H(t)/dt^2$. Establish how this derivative differs from the classical formula (written here in the Gauss system of units)

$$M\,\frac{d^2\mathbf{r}(t)}{dt^2} = qe\Big[\mathbf{E}(t, \mathbf{r}) + \frac{\mathbf{v}}{c}\times\mathbf{B}(t, \mathbf{r})\Big]\,.$$

Problem 0.7
Express the difference $E_\Omega - E_{\Omega_0}$ of ground state energies of $H = H_0 + \lambda V_{\rm int}$ and of H_0 through the derivative with respect to λ of the operator[2]

$$S_0^\varepsilon \equiv U_I^{-\varepsilon}(+\infty,\,0)\,U_I^\varepsilon(0,\,-\infty) = [U_I^{-\varepsilon}(0,\,+\infty,)]^\dagger U_I^\varepsilon(0,\,-\infty)\,,$$

that is, prove the so-called Sucher formula

$$E_\Omega - E_{\Omega_0} = \frac{1}{2}\,i\hbar\,\varepsilon\,\lambda\,\frac{\partial}{\partial\lambda}\ln\langle\Omega_0|S_0^\varepsilon|\Omega_0\rangle\,,$$

Problem 0.8
By considering the differential equation satisfied by it, find the complete evolution operator $U^\varepsilon(t,0)$, including its phase, corresponding to the Gell-Mann - Low modification $V_{\rm int}\longrightarrow e^{\varepsilon t}V_{\rm int}$ of the Hamiltonian ($\Delta_\omega = \hbar\omega/2$)

$$H = H_0 + V_{\rm int} = \hbar\omega\,a^\dagger a + \Delta_\omega + \lambda\,a^\dagger + \lambda^*\,a\,,$$

of the linearly perturbed harmonic oscillator. Show then by an explicit computatation that the expression (in which $U_I^{\pm\varepsilon}(t,0)$ are the interaction picture evolution operators corresponding to the interaction term adiabatically switched on and off)

$$\lim_{T\to\infty}\left(\lim_{\varepsilon\to0^+}\langle\Omega_0|U_I^{-\varepsilon}(T,0)[U_I^\varepsilon(-T,0)]^\dagger|\Omega_0\rangle\right) \equiv \lim_{T\to\infty}\langle\Omega_0|U_I(T,-T)|\Omega_0\rangle\,,$$

(i.e. the limit $\varepsilon\to0$ is taken first) behaves as

$$\exp\left\{-i\frac{2T}{\hbar}\,(E_\Omega - E_{\Omega_0})\right\}.$$

Hint: In order to ensure the proper transformation to the Heisenberg picture of the basic operators a and $a^\dagger$, the sought evolution operator must have the form

$$U^\varepsilon(t,0) = e^{i\varphi(t)}\,e^{-iH_0t/\hbar}\,e^{h(t)\,a^\dagger - h^*(t)\,a}\,,\qquad h(t) = -\frac{i}{\hbar}\int_0^t d\tau\,\lambda\,e^{(\varepsilon+i\omega)\tau}\,,$$

so only the phase $\varphi(t)$ has to be determined.

Problem 0.9 (TRK sum rule)
The Hamiltonian of a set of N *identical* nonrelativistic spinless particles of mass M has the general form

$$H = \sum_{a=1}^{N}\frac{\mathbf{p}_a^2}{2M} + V(\mathbf{r}_1,\ldots,\mathbf{r}_N)\,.$$

[2] In $U_I^{-\varepsilon}(+\infty,\,0)$ the original time independent interaction is replaced by $\lambda V_{\rm int}\,e^{-\varepsilon t}$ and the limit $\varepsilon\to0^+$ is taken.

Defining the operators

$$O_r(\mathbf{a}) = O_r^\dagger(\mathbf{a}) = \sum_{a=1}^{N} \mathbf{a}\cdot\hat{\mathbf{r}}_a\,, \qquad O_p(\mathbf{a}) = O_p^\dagger(\mathbf{a}) = \sum_{a=1}^{N} \mathbf{a}\cdot\hat{\mathbf{p}}_a\,,$$

in which $\mathbf{a}$ can be any (real for Hermiticity) vector, prove that if $|s\rangle$ is a normalizable (and normalized to unity) eigenvector of H, the following sum rules

$$\sum_l |\langle l\,|O_p(\mathbf{a})|s\rangle|^2 = \frac{M^2}{\hbar^2}\sum_l (E_l - E_s)^2\,|\langle l\,|O_r(\mathbf{a})|s\rangle|^2\ ,$$
$$\sum_l (E_l - E_s)\,|\langle l\,|O_r(\mathbf{n})|s\rangle|^2 = \frac{N\hbar^2}{2M}\,,$$
$$\sum_l (E_l - E_s)\,\left|\langle l\,|e^{iO_r(\mathbf{q})}|s\rangle\right|^2 = N\,\frac{\hbar^2\mathbf{q}^2}{2M}\,,$$

hold. The second one, in which it is assumed that $\mathbf{n}^2 = 1$, is called the Thomas-Reiche-Kuhn sum rule. The summations over l, where $|l\rangle$ are eigenvectors of H, mean also integrations over the continuous part of the Hamiltonian spectrum.
Hint: Prove first the identity $O_p(\mathbf{a}) = i(M/\hbar)[H, O_r(\mathbf{a})]$. To prove the TRK rule compute in two ways the $|s\rangle$ state expectation value of the double commutator $[[H, O_r(\mathbf{n})], O_r(\mathbf{n})]$ and to prove the last one work out the operator $e^{-iO_r(\mathbf{q})}He^{iO_r(\mathbf{q})} - H$ using the expansion proved in Problem 0.1 and take the expectation value of both sides in the normalized eigenvector $|s\rangle$ of H.

Problem 0.10
Justify the identity[3]

$$a^\dagger a = \sum_{n=0}^{\infty} |n\rangle\, n\,\langle n|\,,$$

in which $|n\rangle$ are the normalized eigenvectors of the operator $a^\dagger a$, where a and $a^\dagger$ are the standard annihilation and creation operators.

Problem 0.11
The harmonic oscillator of mass M and frequency ω is acted upon by an external force $F(t)$. The oscillator state was at $t = 0$ prepared in the coherent state $|z\rangle$ (the eigenvector of the annihilation operator with the eigenvalue z). Show that the oscillator remains in a coherent state $|z(t)\rangle$ at any instant t and find $z(t)$.

Problem 0.12
The harmonic oscillator of mass M and frequency ω on which acts an external force $F(t)$ vanishing as $t \to \mp\infty$ (i.e. the Hamiltonian as in Problem 0.4; one can consider concrete

[3]This is taken from the BMW, but there the problem is formulated with a misprint...

forces like $F(t) = F_0/(1+t^2/\tau^2)$ or $F(t) = F_0 \exp(-t^2/\tau^2))$ was at $t = -\infty$ in the ground state of H_0. Compute the mean oscillator energy $\overline{E}$ at $t = \infty$ and the energy dispersion squared $\overline{E^2} - \overline{E}^2$ using the formalism of the *in* and *out* operators. Obtain the same result using the S-matrix elements $S_{kl} = \langle k\,\mathrm{out}|l\,\mathrm{in}\rangle$.

Problem 0.13
The work W done on the harmonic oscillator of frequency ω and mass M, the classical motion of which as $t \to -\infty$ is given by $x(t) = A\cos(\omega t + \delta)$, by a force $F(t)$ vanishing as $t \to \mp\infty$ can be computed as the difference of the oscillator energies at $t = \infty$ and and $t = -\infty$. If $F(t) = F_0 \exp(-t^2/\tau^2)$ this work is equal (see Kotkin & Serbo Problem 5.12 or my notes to classical mechanics, Problem 2.13)

$$W = \frac{\pi F_0^2}{2M\omega^2}\,\omega^2\tau^2\, e^{-\frac{1}{2}\omega^2\tau^2} - \sqrt{\pi}\, F_0 A\,\omega\tau\, e^{-\frac{1}{4}\omega^2\tau^2} \sin\delta\,.$$

Find the analogous result in quantum mechanics of the harmonic oscillator, that is compute the mean value $\overline{W}$ of the work done by the force $F(t)$ on the oscillator, the (Schrödinger picture) state-vector $|\psi(t)\rangle$ of which was at $t \to -\infty$ such that

$$\langle\psi(t)|\,\hat{x}\,|\psi(t)\rangle = A\cos(\omega t + \delta)\,.$$

Hint: Express the matrix elemet $\langle\psi(t)|\,\hat{x}\,|\psi(t)\rangle$ in the Heisenberg picture and use the formalism of the *in* and *out* operators.

Problem 0.14
The center of the one-dimensional harmonic oscillator force gets suddenly displaced by the distance d (the time τ in which the displacement takes place is much shorter than $1/\omega$), that is the Hamiltonian undergoes the change

$$H_0 \equiv \frac{\hat{p}^2}{2M} + \frac{1}{2}M\omega^2\hat{x}^2 \longrightarrow H_d \equiv \frac{\hat{p}^2}{2M} + \frac{1}{2}M\omega^2(\hat{x} - d)^2.$$

Compute the probability that after the displacement the oscillator which initially was in the n-th state of H_0 will after the change be found in the m-th state of H_d. Do this using the relation between the corresponding creation and annihilation operators a, $a^\dagger$ a_d, $a_d^\dagger$ (without using explicit wave functions).

Problem 0.15
The frequency $\omega(t)$ of a one-dimensional harmonic oscillator of mass M varies with time according to the formula

$$\omega^2(t) = \omega_0^2 + \Delta\omega_0^2\,\mathrm{arctg}(t/\tau)\,, \qquad \omega_0^2 > \frac{\pi}{2}\Delta\omega_0^2\,,$$

so that $H(-\infty) = H_{(-)}$ and $H(\infty) = H_{(+)}$. Compute in the lowest order of the perturbative expansion the probability of finding the oscillator in the k-th excited state of $H_{(+)}$ if at $t = -\infty$ it was in the n-th eigenstate of $H_{(-)}$.

Problem 0.16
An atom, initially, i.e. in the far past, in the state $|i\rangle$, is placed beetween the plates of a capacitor. The electric field $\mathbf{E}(t)$ is switched on and off according to the formula:

$$\mathbf{E}(t) = \mathbf{E}_0 \exp(-t^2/\tau^2)\,.$$

Using the first order of the time-dependent perturbative expansion express the probability of finding the atom in the far future in the state $|f\rangle$ through the matrix element of the electric dipole operator. Using the obtained formula, compute explicitly the probabilities of finding the Hydrogen atom in the far future in the $|2P\rangle$ states, if it was initially in the $|1S\rangle$ state.

Problem 0.17
Compute approximately the probability that an atom, initially in a state $|i\rangle$, will get excited to a state $|f\rangle$ (belonging to the discrete or continuous part of the spectrum of the free atom Hamiltonian) by the variable electric field produced by a heavy charged particle of charge Qe (treated classically and without taking into account its small deflection due to the interaction - making this approximation is possible owing to the neutrality of the atom as a whole) passing near the atom with the impact parameter $b \gg a_{\rm B}$ (b is counted with respect to the atom's nucleus; $a_{\rm B}$ is the Bohr radius) with a constant velocity $\mathbf{v}$. The approximation should consist of truncating the formula suggested below to the lowest l.
Hint: Use the formula

$$\frac{1}{|\mathbf{r}-\mathbf{R}|} = \sum_{l=0}^{\infty} \frac{|\mathbf{r}|^l}{|\mathbf{R}|^{l+1}}\, P_l(\cos\vartheta)\,,$$

where ϑ is the angle between the vectors $\mathbf{r}$ and $\mathbf{R}$ and $P_l(z)$'s are the Legendre polynomials. The formula is written assuming that $|\mathbf{r}| < |\mathbf{R}|$.

Problem 0.18
Combine the result of the preceding Problem with the Thomas-Reiche-Kuhn sum rule (Problem TRK) to estimate the mean energy loss per unit length of the trajectory of a heavy charged (classical) particle which passes through a medium in which there are N identical atoms (each having Z electrons) per unit volume.

Problem 0.19
The Hilbert space of a system is two-dimensional. The Hamiltonian is $H = H_0 + V_{\rm int}$. The two normalized eigenvectors $|1\rangle$ and $|2\rangle$ of H_0 (corresponding to its eigenvalues E_1 and E_2) can be taken for the basis of the Hilbert space. At $t = 0$ the system was prepared in the state $|1\rangle$. Assuming that the matrix elements of $V_{\rm int}$ between the H_0 eigenstates are known, compute the probability of finding the system in the states $|1\rangle$ and $|2\rangle$ at any instant t. Compare the exact result with the one obtained in the lowest order of the time-dependent perturbative expansion.

Problem 0.20
As in the preceding Problem the Hilbert space of a system is two-dimensional. The unperturbed Hamiltonian H_0 has two degenerate (normalized to unity) eigenvectors $|1\rangle$ and $|2\rangle$, both corresponding to the same energy E_0. The perturbation has the form $V_{\rm int} = f(t)O$, where O is some Hermitian operator and $f(t)$ is a c-number function. What is the probability of finding the system at the instant $t > 0$ in the the state $|2\rangle$, if it was prepared at $t = 0$ in the state $|1\rangle$? Establish the conditions in which the transition probability obtained using the first order of the time-dependent perturbative expansion is a good approximation to the one computed exactly. As in the preceding Problem assume that the matrix elements of the operator O between the H_0 eigenstates are known.

Problem 0.21
Consider the evolution of the magnetic moment represented by the operator

$$\hat{\boldsymbol{\mu}} = \mu\,\frac{1}{2}\boldsymbol{\sigma}\,,$$

of a spin $\frac{1}{2}$ particle[4] remaining at rest (this reduces the the system to a two-state one) in a variable magnetic field

$$\mathbf{B}(t) = B_0\,\mathbf{e}_z + B_1\,(\mathbf{e}_x\cos\Omega t + \mathbf{e}_y\sin\Omega t)\,.$$

Find the exact evolution of the state which at $t = 0$ is the lower energy eigenstate of the system's Hamiltonian at that instant. Find the state-vector $\psi(t)$ of the spin (magnetic moment) after the complete rotation of the direction of the magnetic field ($\Omega t = 2\pi$) and taking the adiabatic limit $\Omega \to 0$, $t \to \infty$ with $\Omega t = 2\pi$, identify the Berry's phase.
Hint: Find first the evolution of the state-vector $\psi'(t)$ related to $\psi(t)$ by the time-dependent unitary transformation $S(t) = \exp(\frac{i}{2}\sigma^z\Omega t)$: $\psi'(t) = S(t)\psi(t)$.

Problem 0.22
Applying the Fermi's Golden Rule calculate the probability of the Hydrogen atom ionization by the spatially constant and uniform electric field $\mathbf{E}(t) = 2\mathbf{E}_0\sin\omega t$ (produced e.g. in a capacitor). Use the plane waves as the final states of the electron (to avoid technical complications).

Problem 0.23
Consider the electron bound in the Coulomb potential $-Ze^2/r$ (a Hydrogen-like atom) interacting with the electromagnetic plane wave represented by the vector potential

$$\mathbf{A}(t,\mathbf{r}) = A_0\,\boldsymbol{\epsilon}(\mathbf{k},\lambda)\,e^{-i(\omega t-\mathbf{k}\cdot\mathbf{r})} + c.c.\,,$$

of frequency ω such that $\hbar\omega$ is higher than the atom's ionization energy ($\boldsymbol{\epsilon}(\mathbf{k},\lambda)$ is the unit polarization vector). Approximating the electron asymptotic states by plane waves,

[4] If the particle has a nonzero electric charge Qe ($e > 0$ is the elementary charge) one usually writes $\mu = Q(e\hbar/2mc)g = Qg\mu_{\rm B}$, where $\mu_{\rm B}$ is the Bohr magneton and g some numerical factor ($g = 2$ plus pennies, if the particle is electron); if the particle is neutral (e.g. neutron), one can always write $\mu = g\mu_{\rm B}$ admitting either sign of g.

compute the differential cross section of the atom ionization process as a result of which the electron is found with the momentum in the element $d\Omega_{\mathbf{k}}$ of the solid angle.
Hint: The cross section is given by the transition probability per unit time divided by the flux of incident photons. If the calculation is done within the semiclassical radiation theory, the "flux of photons" should be identified with the energy flux (averaged over the period) carried by the electromagnetic wave divided by the energy $\hbar\omega$ od a single quantum.

Problem 0.24
Solve the preceding problem using the quantum theory of radiation, i.e. compute the cross section of the process in which the atom gets ionized as a result of absorbing one photon (the initial state consists of the atom in the ground state and one photon of momentum $\hbar\mathbf{k}$ and polarization $\lambda = \pm 1$). Average the cross section over polarizations of the initial photon.

Problem 0.25
Consider the quantum three dimensional isotropic harmonic oscillator of mass M and frequency ω carrying the electric charge q (in units of $e > 0$). Compute the width (or the lifetime) of the oscillator $|n_x, n_y, n_z\rangle$ states using the electric dipole approximation. When is this approximation reliable?

Problem 0.26
Estimate the probability per unit time of the spontaneous electric dipole transition (with the emission of one photon) between the Hydrogen atom $2S_{1/2}$ and $2P_{1/2}$ states. The difference of energies of these two energy levels is $\Delta E = 4.4 \times 10^{-6}$ eV or 1057 MHz (the conversion factor is $2\pi\hbar = h$ - why the oldfashioned h ? "Because such is the power of tradition!"). This splitting, called the Lamb shift, is due to higher order corrections to the Hydrogen atom energy spectrum which are calculable only within full Quantum Electrodynamics (details of this calculation are, however, entirely irrelevant for the present Problem).

Problem 0.27
Derive the general formula giving in the dipole approximation the probability per unit time that the Hydrogen atom excited to the atomic level characterized by the quantum numbers (n, l) makes a spontaneous transition (with the emission of one photon) to another level characterized by the numbers (n', l'). Average the transition probability over the m_l and sum it over the m'_l quantum numbers. Give the lifetime of the $2P$ Hydrogen atom states.

Problem 0.28
A neutron at rest is placed in the constant uniform magnetic field $\mathbf{B}$. This results in splitting its spin up and spin down states. The difference of energies of these split states

is $\Delta E = 2|\boldsymbol{\mu}||\mathbf{B}|$, where $|\boldsymbol{\mu}|$ is the neutron magnetic moment equal[5] $(e\hbar/2M_n c)\kappa_n$ with the appropriate dimensionless factor $\kappa_n = -1.91$. Assuming that the neutron is initially in the higher energy state and using

$$V_{\rm int} = -\boldsymbol{\mu}\cdot\hat{\mathbf{B}}(\mathbf{0})\,,$$

as the interaction, compute the probability per unit time w_{fi} of the spontaneous transition to the lower energy level. Give the answer in the form $w_{fi} = (\dots) \times [|\mathbf{B}|/\text{Gauss}]^a$ or $w_{fi} = (\dots) \times [|\mathbf{B}|/\text{Tesla}]^a$ (if you are a legalist) with the appropriate power a.
Remark: Setting to zero the space argument of the $\hat{\mathbf{B}}$ field operator results from approximating the wave function of the neutron at rest by a Gaussian packet strongly peaked at $\mathbf{r} = \mathbf{0}$.

Problem 0.29
Compute the probability per unit time of the spontaneous transition with the emission of one photon between the triplet and singlet Hydrogen atom $1S$ states. The triplet state has energy higher than the singlet one by $\Delta E = 1420$ MHz (that is - notice again the old-fashioned conversion factor - $\Delta E = 1420\,\text{MHz}\,\cdot h \equiv 1420 \cdot 2\pi\hbar \approx 6 \times 10^{-6}$ eV). This so-called hyperfine splitting is due to the interaction between the electron and nucleus (proton) spins.
Remark: Write the wave functions of the singlet atomic state in the form

$$\Psi_{\rm singlet} = \frac{1}{\sqrt{2}}\left[\begin{pmatrix}1\\0\end{pmatrix}_e \otimes \begin{pmatrix}0\\1\end{pmatrix}_p - \begin{pmatrix}0\\1\end{pmatrix}_e \otimes \begin{pmatrix}1\\0\end{pmatrix}_p\right] \frac{2}{a_{\rm B}^{3/2}}\, e^{-r/a_{\rm B}} Y_{00}\,,$$

and analogously the functions of the triplet state and take for the interaction

$$V_{\rm int} = \frac{e}{2M_e c}\, g_e\, \frac{\hbar}{2}\, (\boldsymbol{\sigma}_e \otimes I_p)\cdot\hat{\mathbf{B}}(\hat{\mathbf{r}})\,,$$

(I_p is a unit 2×2 matrix acting on the proton spinors; the proton magnetic moment coupling to the magnetic field operator $\hat{\mathbf{B}}$ can be neglected because the proton magnetic moment is much smaller than that of the electron.

[5] The magnetic moment of a spin 1/2 particle of mass M_p is usually written as

$$\boldsymbol{\mu} = \frac{e}{2M_p c}\, 2(Q_p + \kappa_p)\, \frac{\hbar}{2}\, \boldsymbol{\sigma} \equiv \frac{e\hbar}{2M_p c}\, (Q_p + \kappa_p)\, \boldsymbol{\sigma}\,.$$

The quantity $e\hbar/2M_p c$ where M_p is the proton (or neutron) mass is called the *nuclear magneton* (in analogy to the Bohr magneton $e\hbar/2M_e c$ of the electron) whereas $g_p = 2(Q_p + \kappa_p)$ is the gyromagnetic factor. Electron g_e equals $-(2+$ pennies); proton $g_p = 5.58$ (i.e $\kappa_p = 1.79$). "Magnetons" include the factors of 2 (which are commonly, but not quite correctly, believed to come out from the Dirac equation), that is, the value of the electron magnetic moment is (almost) $|\boldsymbol{\mu}| = e\hbar/2M_e c$. Similarly if proton were a pointlike particle its magnetic moment would be $|\boldsymbol{\mu}| = e\hbar/2M_p c$; large departure of the proton factor $Q_p + \kappa_p$ from unity shows that proton is not elementary. It is the factor κ_n (i.e. the nonelementary nature of neutron) which is entirely responsible for the neutron magnetic moment ($Q_n = 0$).

Problem 0.30
Taking the electron spin into account (i.e. using the Pauli equation instead of the Schrödinger one) compute the probability per unit time of the (magnetic dipole) transition between the $2S_{1/2}$ and $1S_{1/2}$ states in the Hydrogen atom with the emission of one photon.
Remarks: Approximate the operator $e^{-i\mathbf{k}\cdot\hat{\mathbf{r}}}$ by $-\frac{1}{2}(\mathbf{k}\cdot\hat{\mathbf{r}})^2$ - in the relevant matrix element the first two terms of the expansion give zeros. Sum the transition matrix element squared over the final state photon polarizations, over the final state electron spin directions and average it over the initial state electron spin projections.

Problem 0.31
Find the contribution of the interaction term[6]

$$V^{(2)}_{\rm int} = \frac{e^2}{2Mc^2}\,\hat{\mathbf{A}}^2(\hat{\mathbf{r}})\,,$$

to the probability per unit time of the spontaneous transition between the $2S$ and $1S$ levels of the Hydrogen atom with the emission of two photons.
Remark: Approximate the matrix element of the operator $e^{-i(\mathbf{k}_1+\mathbf{k}_1)\cdot\hat{\mathbf{r}}}$ between the $2S$ and $1S$ atomic states by expanding the exponent up to the second order (the first two terms of this expansion give vanishing contributions). If the complete calculation is too complicated, try at least to make the estimate by finding the power of $\alpha_{\rm EM} = 1/137$ and of other dimensional factors (like $\hbar$, c, M_e) to which this rate is proportional.

Problem 0.32
Compute the ratio of the intensities of the two first lines of the Balmer series of spectral lines emitted by the Hydrogen atom. The first, called H_α, line of this series is due to the transitions $|3S\rangle \to |2P\rangle$, $|3P\rangle \to |2S\rangle$, and $|3D\rangle \to |2P\rangle$, while the second, H_β, line is due to the transitions $|4S\rangle \to |2P\rangle$, $|4P\rangle \to |2S\rangle$, and $|4D\rangle \to |2P\rangle$.

Problem 0.33
Taking the electron spin into account (i.e. using the Pauli equation instead of the Schrödinger one) compute (in order e^4) the differential cross section of the low frequency ($\hbar\omega_{\mathbf{k}} \ll M_ec^2$) photon Compton scattering on free electron at rest. Average the cross section over the two possible spin projections of the initial electron and sum over the spin projections of the final electron. Compare the result with the Klein-Nishijina formula.

Problem 0.34
Compute (in order e^4) the differential and total cross sections (averaged over the initial photon polarization and summed over the polarizations of the final photon) of the elastic low frequency ($\hbar\omega_{\mathbf{k}} \ll Mc^2$) photon scattering on a spinless charged particle of mass M bound in the spherical harmonic oscillator potential $V(r) = \frac{1}{2}M\omega_0^2r^2$ (elastic i.e. without

[6] The complete calculation of the rate of the two-photon $2S \to 1S$ transition should take into account through the second order perturbative expansion the contribution of the term $V^{(1)}_{\rm int} = (e/Mc)\hat{\mathbf{A}}(\hat{\mathbf{r}})\cdot\hat{\mathbf{p}})$ which is also formally of the same order.

exciting the oscillator which is assumed to be initially in the ground state). Work in the dipole approximation, i.e. set equal to unity the $\hat{\mathbf{r}}$ dependent exponential factors in the photon field operator.

Problem 0.35

Compute the differential cross sections of the low frequency ($\hbar\omega_{\mathbf{k}} \ll M_n c^2$) photon scattering on a free neutron in a definite spin state and no neutron spin flip (the final neutron has the same spin direction as the initial one) and with the spin flip. Average these cross sections over the polarizations of the incoming photon and sum over the polarizations of the final one. Compute also the total cross sections (i.e. integrate over the directions of the final photon). The recoil of the neutron can be neglected - use only the spin part of the neutron wave function and take the photon field operator at $\mathbf{r} = \mathbf{0}$.

Problem 0.36

Check by direct calculation that if the wave function $\psi(t, \mathbf{r})$ of a particle of mass M satisfies the Schrödinger equation with the potential $V(\mathbf{r})$, then the wave function

$$\begin{aligned}\psi'(t, \mathbf{r}) &= \exp\left(-\frac{i}{\hbar}\mathbf{V}\cdot(-M\mathbf{r} + t\mathbf{P})\right)\psi(t, \mathbf{r}) \\ &= \exp\left(-i\frac{M\mathbf{V}^2}{2\hbar}t\right)\exp\left(\frac{i}{\hbar}M\mathbf{V}\cdot\mathbf{r}\right)\psi(t, \mathbf{r} - \mathbf{V}t)\,,\end{aligned}$$

of the (actively) boosted system satisfies the Schrödinger equation with $V'(\mathbf{r} - \mathbf{V}t)$. How are related the probability density and the probability currents constructed out of $\psi'(t, \mathbf{r})$ and $\psi(t, \mathbf{r})$? Repeat the check in the abstract language of state-vectors, that is show that if $i\hbar d|\psi(t)\rangle/dt = [\hat{\mathbf{p}}^2/2M + V(\mathbf{r})]|\psi(t)\rangle$, then

$$|\psi'(t)\rangle = e^{-i\frac{M\mathbf{V}^2}{2\hbar}t}\, e^{\frac{i}{\hbar}M\mathbf{V}\cdot\hat{\mathbf{r}}}\, e^{-\frac{i}{\hbar}t\mathbf{V}\cdot\hat{\mathbf{p}}}|\psi(t)\rangle\,,$$

satisfies $i\hbar d|\psi'(t)\rangle/dt = H'|\psi'(t)\rangle$ with $H' = \hat{\mathbf{p}}^2/2M + V(\hat{\mathbf{r}} - t\mathbf{V})$.

Problem 0.37

The nucleus of the Hydrogen atom (in the $1S$ state) gets a sudden kick and starts moving with the velocity $\mathbf{v}$. Assuming that the time τ of the action of the kicking force is very short compared to all relevant characteristic times (including $a_{\rm B}/|\mathbf{v}|$, where $a_{\rm B}$ is the Bohr radius) derive a general formula valid for an N-electron atom of finding the atom in a concrete stationary state. Applying it to the Hydrogen-like one-electron atom compute explictly the probability that it will not remain in the initial $1S$ state.

Hint: Transform the initial wave function of the electron to the frame in which the nucleus is at rest after having received the kick.

Problem 0.38

Using the commutation rules of the rotation group generators $[J^x,\ J^y] = iJ^z$ etc., show that

$$e^{-i\phi J^z}e^{-i\theta J^y}e^{+i\phi J^z} = e^{-i\theta(J^y\cos\phi - J^x\sin\phi)}\,.$$

Write down also other similar relations with the generators J^x, J^y and J^z.

Problem 0.39
Show that if the (active) rotation represented by the 3×3 orthogonal matrix O is generated through the formula[7]

$$M\cdot(\sigma_i\, r^i)\cdot M^\dagger = \sigma_j(O^j_{\ i}\, r^i)\,,$$

by the 2×2 matrix M belonging to the $SU(2)$ group and if the rotation matrix $\tilde{O}$ is related in the same way to another $SU(2)$ matrix $\tilde{M}$, then the$SU(2)$ matrix $\tilde{M}\cdot M$ generates in this way the rotation matrix $\tilde{O}\cdot O$.

Problem 0.40
A vector $\mathbf{V}$ rotated by the angle ϕ around the axis $\mathbf{n}$ (where $|\mathbf{n}|=1$) can be written as

$$\begin{aligned}\mathbf{V}' &= \mathbf{V}\cos\phi + \mathbf{n}\,(\mathbf{n}\cdot\mathbf{V})(1-\cos\phi) + \mathbf{n}\times\mathbf{V}\,\sin\phi\\ &\approx \mathbf{V} + \boldsymbol{\phi}\times\mathbf{V} - \frac{1}{2}\boldsymbol{\phi}^2\,\mathbf{V} + \frac{1}{2}\,\boldsymbol{\phi}\,(\boldsymbol{\phi}\cdot\mathbf{V}) + \dots\end{aligned}$$

where $\boldsymbol{\phi}\equiv\phi\mathbf{n}$. Justify this formula. Find the vector $\boldsymbol{\phi}$ corresponding to the composition of two successive infinitesimal rotations characterized by $\boldsymbol{\phi}_1$ and $\boldsymbol{\phi}_2$ of a vector $\mathbf{V}$. Using the result find the structure constants of the rotation group. Show also that the matrix[8]

$$[O_{\rm vec}(\phi,\mathbf{n})]^i_{\ j} = \delta^{ij}\cos\phi + (1-\cos\phi)\,n^i n^j + \epsilon^{ikj} n^k\sin\phi\,,$$

such that $V'^i = [O_{\rm vec}(\phi,\mathbf{n})]^i_{\ j}V^j$, is just the matrix $\exp\left(-i\phi^k\mathcal{J}^k_{\rm vec}\right)$, where $(\mathcal{J}^k_{\rm vec})^i_{\ j} = i\epsilon^{ikj}$ are the rotation group generators in the defining (vector) representation.

Problem 0.41
Let $O(\theta,\mathbf{n})$ with $\mathbf{n}^2=1$ be the (active) rotation around the direction $\mathbf{n}$ by the angle θ. Show that ($\mathbf{k}^2=1$)

$$O(\theta,\mathbf{n})\cdot O(\psi,\mathbf{k})\cdot O^{-1}(\theta,\mathbf{n}) = O(\psi, O_{\rm vec}(\theta,\mathbf{n})\cdot\mathbf{k})\,,$$

where $O_{\rm vec}$ means the rotation realized on vectors (in this formula $O(\theta,\mathbf{n})$ stand for an abstract rotation which can be realized in any vector space, in particular in a Hilbert space, by an appropriate symetry operator).

[7]In view of notation used in the applications of the $SL(2,C)$ group representations to spinors, it is convenient to operate with two sets of the Pauli matrices σ^i and $\bar{\sigma}^i$ which can be written in the "co-" and "contravariant forms: $\sigma^i=-\sigma_i=-\bar{\sigma}^i=\bar{\sigma}_i$, where σ^i are the three "standard" Pauli matrices.

[8]Since it is desirable to denote differently active rotations (which are linear mappings of the vector space into itself and, hence, their matrices are written in a fixed basis) and passive rotations (matrices of which are matrices of changes of bases i.e. matrices of the identity mapping but written in two different bases - see my famous Algebra notes), we choose to denote the active ones by O (from polish - let proud Poland getting up from knees contribute also to physics - "obrót").

Problem 0.42
Using the result of Problem 0.41 show that the (active) rotation $O(\alpha,\beta,\gamma)$ parametrized by three Euler angles and composed of three successive rotations: first by the angle α around the axis $\mathbf{n}_1\equiv\mathbf{e}_z$, then by the angle β around the axis $\mathbf{n}_2\equiv-\mathbf{e}_x\sin\alpha+\mathbf{e}_y\cos\alpha$ and finally by γ around the axis $\mathbf{n}_3\equiv\mathbf{e}_x\cos\alpha\sin\beta+\mathbf{e}_y\sin\alpha\sin\beta+\mathbf{e}_z\cos\beta$ (these are the famous three moves of the paw - who attended my Classical Mechanics course, knows what I mean) is equivalent to the composition of three other successive rotations: first by γ around $\mathbf{e}_z$, then by β around $\mathbf{e}_y$ and finally by α again around $\mathbf{e}_z$:

$$O(\gamma,\mathbf{n}_3)\cdot O(\beta,\mathbf{n}_2)\cdot O(\alpha,\mathbf{n}_1)=O(\alpha,\mathbf{e}_z)\cdot O(\beta,\mathbf{e}_y)\cdot O(\gamma,\mathbf{e}_z)\,.$$

Show also formally, that is treating the matrices O (of the active rotations) as matrices of the linear mappings of the vector space into itself which are given in the fixed basis $\mathbf{e}_i$ and matrices of the passive rotations as matrices of the changes of the bases (that is matrices of the identity mapings but written in different bases), that (what should be obvious) in the reference frame rotated by the angles α, β and γ the components of the rotated vector are the same as the compotents of the original vector in the original reference frame.

Problem 0.43
A left-invariant measure $d\mu(g)$ on a group G has the property

$$\int d\mu(g)\,f(g)=\int d\mu(g)\,f(g'g)$$

(g denotes an element of G and $f(g)$ is a function defined on the group G). In a concrete parametrization $g=g(\boldsymbol{\theta})$ of the group elements by some parameters θ_a with $a=1,\dots,n$, where n is the dimension of the Lie algebra of G the measure is given by $d\mu(g)=d^n\boldsymbol{\theta}\,\rho(\boldsymbol{\theta})$. Using the general formula

$$\rho(\boldsymbol{\theta})=\rho(\mathbf{0})\,\det{}^{-1}\left(\frac{\partial h_a(\boldsymbol{\theta},\tilde{\boldsymbol{\theta}})}{\partial\tilde{\theta}_b}\right)_{\tilde{\theta}_b=0}\,,$$

in which $h(\boldsymbol{\theta},\tilde{\boldsymbol{\theta}})$ is the group composition function appropriate for the chosen parametrization $g=g(\boldsymbol{\theta})$ of the group elements, find the left-invariant measure (i.e. the density ρ) on the rotation group $SO(3)$ in the parametrization given by the components of the vector $\boldsymbol{\phi}=(\phi^x,\phi^y,\phi^z)$ defined in Problem 0.40. Compute the rotation group volume adopting the usual convention according to which $\rho(\mathbf{0})=1$.

Problem 0.44
Let the two-parameter group G of transformations of the real axis $\mathbb{R}$ be defined by the formula

$$x'=(1+\xi_1)\,x+\xi_2\,,$$

Using the general formula quoted in Problem 0.43 and its counterpart appropriate for right-invariant measures, find both these measures on the group G. Are they identical?

Problem 0.45
Prove that if the dimension n of the group G is odd, the formula $d\mu(g) \equiv d^n\boldsymbol{\xi}\,\rho(\boldsymbol{\xi})$ with

$$\rho(\xi^1,\ldots,\xi^n) \propto \epsilon^{i_1 i_2 \ldots i_n}\,\mathrm{tr}\left(O^{-1}\cdot\frac{\partial O}{\partial \xi^{i_1}}\cdot O^{-1}\cdot\frac{\partial O}{\partial \xi^{i_2}}\cdot\ \ldots\cdot O^{-1}\cdot\frac{\partial O}{\partial \xi^{i_n}}\right),$$

where $O(\boldsymbol{\xi})$ is a matrix representation of the group element parametrized by the parameters ξ^i, defines on G a left-invariant measure (if n is even, the measure defined in this way vanishes as a result of the antisymmetry of $\epsilon^{i_1 i_2 \ldots i_n}$ and the cyclicity of the trace). Use this result to find explicitly the density $\rho(\alpha,\beta,\gamma)$ of the left-invariant measure on the $SO(3)$ and $SU(2)$ groups parametrized by the three Euler angles α, β and γ.

Problem 0.46
Show that the left-invariant measure on a compact group G parametrized by the parameters θ_a, $a = 1,\ldots,o$

$$g(\boldsymbol{\theta}) = \exp\left(-i\theta_a Q^a\right),$$

where Q^a, $a = 1,\ldots,o$, are the group generators in some representation, takes, infinitesimally close to the identity transformation, the simple form $d^o\theta$.

Problem 0.47
Show that the operator of the electric quadrupole moment $\mathbf{Q}^{ij} = -e(3\hat{\mathbf{r}}^i\hat{\mathbf{r}}^j - \hat{\mathbf{r}}^2\delta^{ij})$ is a tensor operator corresponding to $j = 2$. Between which states of the Hydrogen atom are electric quadrupole transitions possible?