Narodowe
Laboratorium
Technologii
Kwantowych

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Opracowanie utworu pod tytułem:

„Optyka Kwantowa: podstawy technologii kwantowych" w ramach kursu zaawansowanego organizowanego dniach 29.06 – 24.07.09 będącego kontynuacją szkoleń z zakresu zarządzania dużą infrastrukturą badawczą organizowanego przez Narodowe Laboratorium Technologii Kwantowych

INNOWACYJNA GOSPODARKA
NARODOWA STRATEGIA SPÓJNOŚCI

UNIA EUROPEJSKA
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ROZWOJU REGIONALNEGO

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Lecture 1

Why photons?

Classical effects

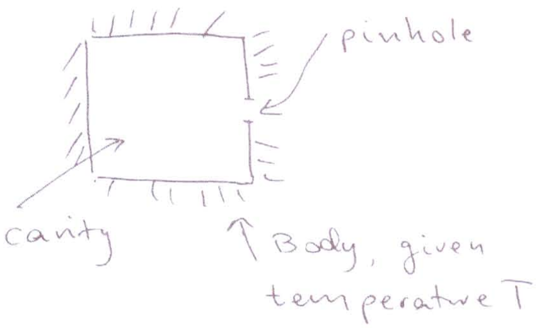
a) Blackbody radiation

Blackbody - absorbs all radiation regardless of frequency

~~Realiz~~

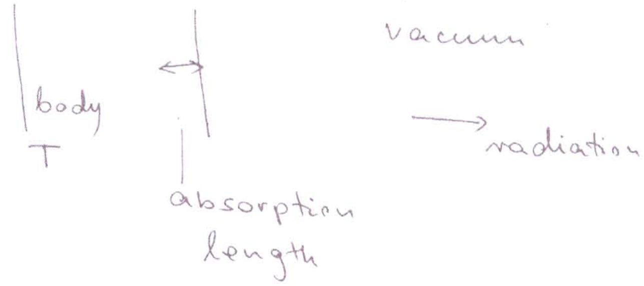
Radiation in thermal equilibrium with black body

Realization



Radiation inside the cavity is in equilibrium with the walls of the cavity, temperature T.

Another possibility



Radiation emitted by the body of temperature T is absorbed and reemitted many times - equilibrates with the body.

Blackbody radiation spectrum (first careful measurements by Lummer ~ 1890)

Note ω_{max} - frequency at which I_ω has maximum

~~Classical approach~~

Real bodies (not ideal black bodies) - similar spectrum

Theory of blackbody radiation based on statistical mechanics and classical electrodynamics

Fundamental law of statistical mechanics

- equipartition of energy – each degree of freedom has energy $k_B T$ $\quad k_B$ – Boltzmann constant

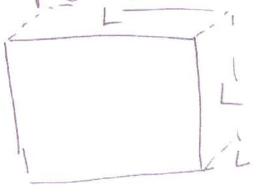
Degrees of freedom for electromagnetic radiation

- plane waves

$$\bar{E}(r,t) = \underset{\substack{\uparrow \\ \text{amplitude}}}{E_0} \; \underset{\substack{\uparrow \\ \text{polarization}}}{\bar{e}} \cos(\bar{k}\bar{r} - \omega t)$$

$$|\vec{k}| = \frac{\omega}{c}$$

Consider plane waves in a box, $L \times L \times L$,



periodic boundary conditions

Allowed wave vectors $\vec{k} = \frac{2\pi}{L}(n_1, n_2, n_3)$

$n_i = \;, \pm 1, \pm 2$ etc.

Number of modes in a given frequency range Δ

$$\sum_{n_1 n_2 n_3} \underset{\substack{\uparrow \\ \text{large } L}}{\simeq} \frac{V}{(2\pi)^3} \int dk_1\, dk_2\, dk_3$$

$$\simeq \frac{V}{(2\pi)^3} \int k^2\, dk \simeq \frac{V}{(2\pi)^3} \int \frac{\omega^2}{c^2} \frac{d\omega}{c}$$

$$\text{density} = \frac{V}{(2\pi)^3 c^3} \omega^2$$

Density of modes $\sim \omega^2$

Hence $I_\omega \sim \omega^2 (k_B T)$ for blackbody radiation.

No maximum, growth to ∞ for ~~...~~ larger

Sharp contradiction with experiment.

The need for Planck's constant.

Observation of ~~...~~ spectrum maxima (Wien's law) ~~...~~ $\omega_{max} \sim T$.

Where does the proportionality constant come from? Possibilities: Boltzmann's constant k_B, speed of light c. Blackbody itself has no structure, cannot introduce new constant.

New fundamental constant is needed.

Modern theory

$$\hbar \omega_{max} = a k_B T$$

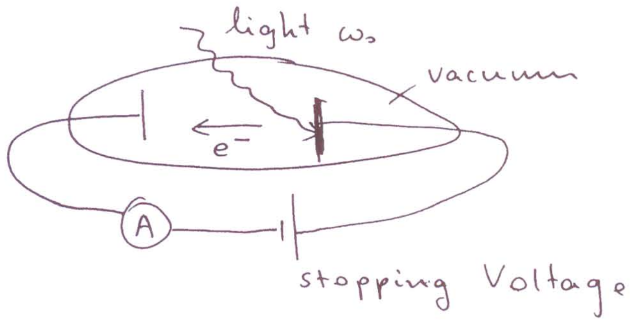
a - numerical factor. $\simeq 2{,}82$

$\hbar$ - Planck's constant $\sim 1.0 \cdot 10^{-34}$ J$\cdot$s

fundamental constant of nature.

b. Photoelectric effect

Metal surface illuminated by light



Observations

- electrons are emitted very shortly after the onset of illumination ($\sim 10^{-9}$ s)
- photocurrent growths linearly with light intensity
- current to cathode decreases with increasing retarding potential, zero current at stopping voltage V_0
- stopping voltage V_0 is linearly proportional to ω_0 and shows a threshold at some frequency ω_{th} (dependent on the metal)

Classical theory of electromagnetism does not explain these observations.

Einstein's theory (Nobel prize in 1921)

light - independent energy quanta, each quantum carries $\hbar\omega$

Photoelectric effect - one electron interacts with one "quantum". Energy conservation

$$\hbar\omega_0 = \frac{1}{2} m v^2 + \underset{\uparrow \text{work function}}{W}$$

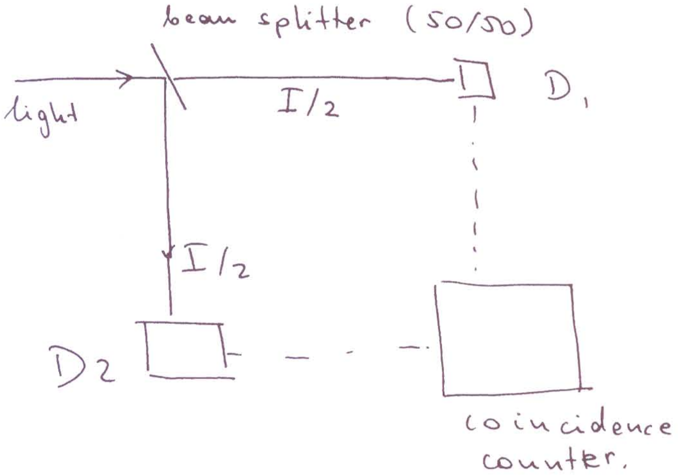
Number of "quanta" — intensity of light beam — proportional to the number of photoelectrons.

Word "photon" invented by Gilbert Lewis in 1926.

c) Hanbury-Brown and Twiss experiment (1956) on photon correlation.

How to design an experiment to detect single photon

- photon - particle in one place
- single photon, two detectors, do they "click" (record a photon) at the same time?



Experiment: measure the number of coincidences relative to the number of individual counts on the detector

Correlation parameter

$$A = \frac{P_c}{P_1 P_2}$$

P_1 - probability of detector 1 responding
P_2 - probability of detector 2 responding

What do we expect?

- If light is a wave and if detectors click randomly and independendly

  then $P_c = P_1 P_2$

  hence $A = \frac{P_c}{P_1 P_2} = 1$

- If light consists of particles

  then ~~[illegible]~~

  $P_c = 0$

  only one detector can detect a particle

  hence $A = 0$ (anticorrelation)

In this way one can distinguish between waves and particles

The result

$$\boxed{A = 2}$$

When one detector ~~[illegible]~~ clicked then it was very probable that the second detector also clicked.

This experiment failed to demonstrate the existence of photons.

How can A be larger than 1 ?

From semiclassical (electrons - quantum, light - classical) theory:

$$P_1 = \alpha_1 I \,\Delta t$$
$$P_2 = \alpha_2 I \,\Delta t$$
$$P_c = \alpha_1 \alpha_2 I^2 (\Delta t)^2$$

If I - constant then

$$A = \frac{P_c}{P_1 P_2} = 1$$

If I - fluctuates in time

$$P_1 = \alpha_1 \langle I \rangle \Delta t$$
$$P_2 = \alpha_2 \langle I \rangle \Delta t$$
$$P_c = \alpha_1 \alpha_2 \langle I^2 \rangle (\Delta t)^2$$

Hence

$$A = \frac{\langle I^2 \rangle}{\langle I \rangle^2} \geqslant 1$$

In case of laser $A \simeq 1$, in case of "classical source" - incadescent bulb etc $A \simeq 2$, large fluctuations.

The problem with Hanbury-Brown and Twiss experime in the light source.

Lecture 2

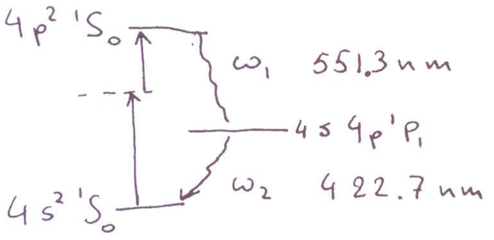
d) Aspect experiments (1986)

Europhysics Lett. 1 (4) 173-179 (1986)

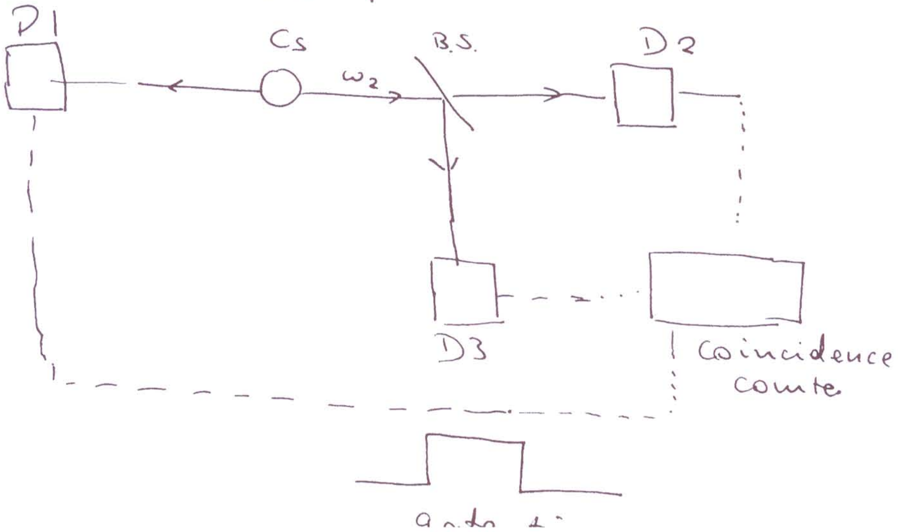
- anticorrelations (similar to Hanbury-Brown, Twiss
- single photon interference.

The problem with the original Hanbury-Brown and Twiss experiment was with the light source.
Aspect build a single photon source.

Ca atoms excited to a s-state



Experiment setup



Photon ω_1 triggers time gate of duration w

Measure coincidences during time w. Reject coincidences for times outside w.

The result

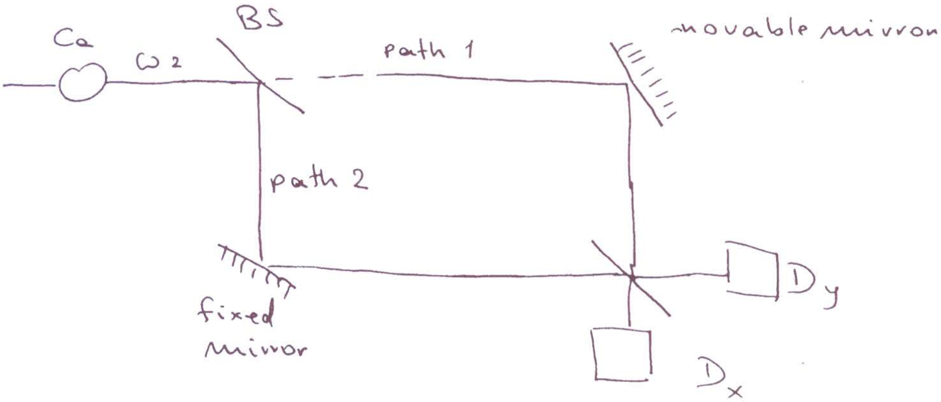
$$A = 0 \quad !$$

Clear evidence for existence of photons.

Photons are reflected or transmitted by beam splitter, but not both.

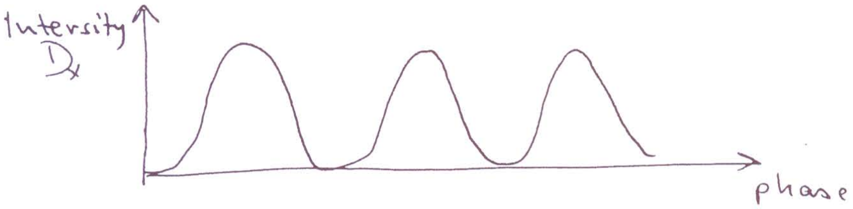
Single photon interference experiment

Mach - Zehnder Interferometer



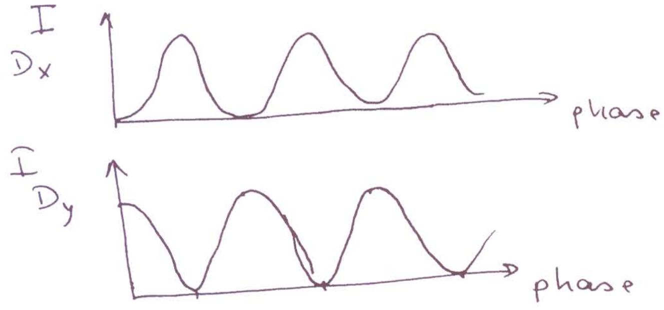
Moving the mirror changes "optical length", phase difference between the paths.

If light is a wave - interference fringes



If light is a particle - photon should follow path 1 or path 2, but not both. No interference.

Result

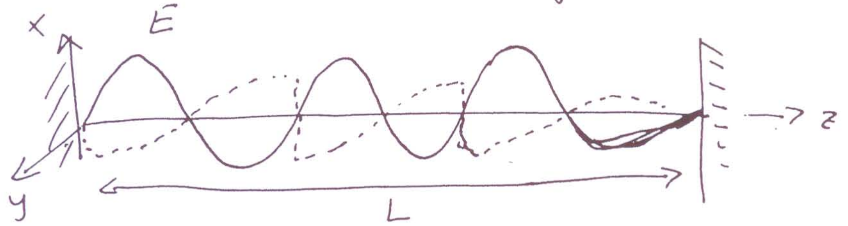


Does the photon go both ways?
Does the photon interfere with itself?

When does photon behave like a particle and when as a wave?

Field quantization

- Consider e.m. field in a cavity with perfectly conducting walls
- Field has a form of standing waves
- consider one standing wave



$$\vec{E} = \left(E_x(z,t),\ 0\ \ 0\right)$$
$$\vec{B} = \left(0,\ B_y(z,t)\ 0\right)$$

$$E_x(z t) = \left(\frac{2\omega^2}{V\varepsilon_0}\right)^{1/2} q(t) \sin kz$$

$$B_y(z.t) = \frac{\mu_0\varepsilon_0}{k}\left(\frac{2\omega^2}{V\varepsilon_0}\right)^{1/2} \dot{q}(t) \cos kz$$

$$k = \frac{\pi n}{L} \qquad n = 1, 2\ldots \qquad \omega = ck$$

V – cavity volume

$q(t)$ – time dependent amplitude

This is one of the modes in the cavity

Energy of the field

is expressed in terms of q and $\dot{q}$

$$H = \frac{1}{2}\left(\dot{q}^2 + \omega^2 q^2\right)$$

Analogy with a harmonic oscillator

$$H_0 = \frac{1}{2}\left(\dot{x}^2 + \omega^2 x^2\right) = \frac{1}{2}\left(p^2 + \omega^2 x^2\right)$$

Quantization $x \to \hat{x}$ operator
$p \to \hat{p}$ operator

with $[\hat{x}, \hat{p}] = i\hbar$.

Quantization:

states: vectors in Hilbert space

physical quantities: operators in Hilbert space

Physical interpretation

Spectral decomposition of operator representi[ng] physical quantity Q:

$$\hat{Q} = \sum \lambda_i P_i \qquad P_i \text{ – projection operators.}$$

Quantity

$(\psi | P_i \psi)$ – probability of getting value λ_i in the measurement of Q. in state ψ

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In case of harmonic oscillator

physical quantities $\hat{x}$, $\hat{p}$ and functions of $\hat{x}$ and $\hat{p}$.

In case of one mode e. m. field

physical quantities:

q and $p \equiv \dot{q}$

Quantization of the field

$$q \longrightarrow \hat{q} \qquad p \longrightarrow \hat{p}$$

with $[\hat{q}, \hat{p}] = i\hbar$

Therefore ~~the~~ electric field $\vec{E}(z)$ and ~~the~~ magnetic ~~become~~ field $\vec{B}(z)$ become noncommuting operators acting in a Hilbert state.

Introduce nonhermitian operators

$\hat{a}^+$ creation

$\hat{a}$ annihilation

$$\hat{a} = \frac{1}{\sqrt{2\hbar\omega}}(\omega\hat{q} + i\hat{p}) \qquad \hat{a}^+ = \frac{1}{\sqrt{2\hbar\omega}}(\omega q - ip)$$

$$[a, a^+] = 1.$$

Fields:

$$E_x = \mathcal{E}_0 (a + a^+) \sin kz \qquad \mathcal{E}_0 = \sqrt{\frac{\hbar\omega}{\varepsilon_0 V}}$$

$$B_y = \mathcal{B}_0 \frac{1}{i}(a - a^+) \cos kz \qquad \mathcal{B}_0 = \frac{\mu_0}{k}\sqrt{\frac{\varepsilon_0 \hbar \omega^3}{V}}$$

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Energy as a funktion of a a^+.

$$H = \hbar\omega\left(a^+ a + \tfrac{1}{2}\right)$$

Time dependence of a, a^+:

Maxwells eq:

$$-\frac{\partial E}{\partial t}\varepsilon_0 + \frac{1}{\mu_0}\operatorname{rot} B = 0$$

$$\frac{\partial B}{\partial t} + \operatorname{rot} E = 0$$

lead to

$$\frac{d\hat{a}}{dt} = -i\omega\hat{a} \qquad \frac{d}{dt}\hat{a}^+ = i\omega\hat{a}^+$$

Observe that these equations are the same as Heisenberg equations:

$$\frac{d\hat{a}}{dt} = \frac{i}{\hbar}\left[H, \hat{a}\right]$$

Number operator $\hat{n}$

definition $\hat{n} = a^+ a$

Eigenstates of $\hat{n}$: $|n\rangle$

eigenstates of energy $H|n\rangle = \hbar\omega(n + \tfrac{1}{2})|n$

Operator $a^+ a$ is positive, hence $n = 0, 1, \ldots$

If $|n\rangle$ eigenstate of $a^+ a$ then

$a^+|n\rangle$ is an eigenstate of $a^+ a$ with

Also

$a|n\rangle$ – eigenstate of a^+a with eigenvalue $n-1$. $(n>1)$

State $|0\rangle$ (with $a|0\rangle = 0$) is the ground state, with energy $\frac{1}{2}\hbar\omega$.

Normalization

$$\hat{n}|n\rangle = n|n\rangle$$

$$a|n\rangle = c_n|n-1\rangle \implies c_n = \sqrt{n}$$

Also

$$a^+|n\rangle = \sqrt{n+1}\,|n+1\rangle$$

By iteration $\quad |n\rangle = \frac{(a^+)^n}{\sqrt{n!}}|0\rangle.$

Properties of field in state $|n\rangle$

Energy $\quad \langle n|\cancel{x}H|n\rangle = \hbar\omega(n+\frac{1}{2})$

Electric field $\quad \langle n|E_x|n\rangle = 0 \quad$ because E_x is linear in a, a^+

But electric field squared is different from zero

$$\langle n|E_x^2|n\rangle = 2\mathcal{E}_0^2\sin^2 kz\,(n+\tfrac{1}{2})$$

Dispersion

$$\langle n|E_x^2|n\rangle - \langle E_x\rangle^2 = 2\mathcal{E}_0^2\sin^2 kz\,(n+\tfrac{1}{2})$$

is not zero even in vacuum state.

State $|n\rangle$ – called n photon state.

Lecture 3

Further properties of Fock states

State $|0\rangle$ is the ground state (lowest energy state). Called "vacuum".

States $|n\rangle$ are called n-photon state.

General state of the e.m field (one mode)

$$|\psi\rangle = \sum_{n=0}^{\infty} c_n |n\rangle$$

Remember, that $\langle n|E_x|n\rangle = 0$ (average value of the field E in $|n\rangle$ state is zero.

Commutation between $\hat{E}$ and $\hat{n}$:

$$[\hat{n}, \hat{E}] = \mathcal{E}_0 \sin kz \, (\hat{a}^+ - \hat{a})$$

This leads to uncertainty relation

$$(\Delta n)(\Delta E) \geq \frac{1}{2} \mathcal{E}_0 |\sin kz| \, |\langle a^+ - a\rangle|$$

One cannot know number of photons and the field exactly.

Other states of e.m. field

Classical limit – if the number of photons is large ($n >> 1$) the field shoul become classical.

But this is not so simple, since

$$\langle n | E | n \rangle = 0$$

Coherent states – the "most classical" quantum states

Arguments based on interferece (not reported here) suggested to look for the "classical" states as eigenvectors of a:

$$a|\alpha\rangle = \alpha |\alpha\rangle \qquad \alpha \text{ – complex number}$$

In order to find $|\alpha\rangle$ we expand

$$|\alpha\rangle = \sum_{n=0}^{\infty} c_n |n\rangle$$

$$a|\alpha\rangle = \alpha|\alpha\rangle = \sum_{n=0}^{\infty} c_n \sqrt{n}\, |n-1\rangle$$

therefore

$$c_n \sqrt{n} = \alpha\, c_{n-1}$$

Solution

$$c_n = \frac{\alpha^n}{\sqrt{n!}}\, c_0$$

Coefficient $c_0 \Rightarrow$ from normalization

$$\langle \alpha | \alpha \rangle = 1 = |c_0|^2 \sum_{n=0}^{\infty} \frac{|\alpha|^{2n}}{n!} = |c_0|^2 e^{|\alpha|^2}$$

So

$$c_0 = e^{-1/2 |\alpha|^2}$$

and

$$|\alpha\rangle = \sum_{n=0}^{\infty} e^{-\frac{1}{2}|\alpha|^2} \frac{\alpha^n}{\sqrt{n!}} |n\rangle$$

Physical meaning of α

Average value of the field

$$\langle \alpha | E | \alpha \rangle = \sqrt{\frac{\hbar\omega}{2\varepsilon_0 V}} \sin kz \, (\alpha + \alpha^*)$$

Thus $(\alpha + \alpha^*)$ – amplitude of the field

The phase of α – phase of the field.

Dispersion of the field

$$(\Delta E)^2 = \frac{\hbar\omega}{2\varepsilon_0 V} \quad \text{regardless of } n.$$

Field in this state has a well defined amplitude (up to a small dispersion, small means smaller than average value).

Distribution of photon number in coherent state

$$P_n = |\langle n | \alpha \rangle|^2 = e^{-|\alpha|^2} \frac{|\alpha|^{2n}}{n!}$$

Coherent states are <u>not</u> orthogonal to each other

$$|\langle \beta | \alpha \rangle|^2 = e^{-|\alpha - \beta|^2}$$

Homework: find $\langle \beta | \alpha \rangle$.

Coherent states are overcomplete, i.e. arbitrary state is a linear combination of coherent states:

$$|\psi\rangle = \int \frac{d^2\alpha}{\pi} |\alpha\rangle\langle\alpha|\psi\rangle$$

Homework: prove the completness of coherent states.

The problem of quantum phase

The field in a coherent state has amplitude and phase defined up to small dispersion.

Is it possible to find states with a well defined phase (no dispersion) ~~and~~ amplitude may have large dispersion?

Dirac suggested to decompose $\hat{a}$ and $\hat{a}$ into product of "amplitude" and "phase".

$$\hat{a} = e^{i\hat{\varphi}}\sqrt{\hat{n}} \qquad \hat{a}^+ = \sqrt{\hat{n}}\, e^{-i\hat{\varphi}}$$

But $a|n\rangle = \sqrt{n}\,|n-1\rangle$

hence $(e^{i\hat{\varphi}})|n\rangle = |n-1\rangle$, but $(e^{i\hat{\varphi}})|0\rangle =$

Also $a^+|n\rangle = \sqrt{n+1}\,|n+1\rangle$

hence $(e^{-i\hat{\varphi}})|n\rangle = |n+1\rangle$.

Combining

$$(e^{i\hat{\varphi}})(e^{-i\hat{\varphi}}) = I \quad \text{unit operator}$$

but $(e^{-i\hat{\varphi}})(e^{i\hat{\varphi}}) = I - |0\rangle\langle 0|$

Operator $(e^{i\hat{\varphi}})$ is <u>not</u> unitary

$\hat{\varphi}$ is <u>not</u> hermitian.

Eigenstates of $(e^{i\hat{\varphi}})$

$$(e^{i\hat{\varphi}})|\phi\rangle = e^{i\phi}|\phi\rangle$$

solution

$$|\phi\rangle = \sum^{\infty} e^{in\phi}\,|n\rangle$$

In spite of all these problems one may define phase distribution in state $|\psi\rangle$

$$P(\phi) = \frac{1}{2\pi} \left| \langle \phi | \psi \rangle \right|^2$$

$$\cancel{= \frac{1}{2\pi}}$$

For $|\psi\rangle = \sum b_n |n\rangle$ we get

$$P(\phi) = \frac{1}{2\pi} \left| \sum_{n=0}^{\infty} e^{-in\phi} b_n \right|^2 .$$

~~For~~

Phase distribution for photon number states:

$$P(\phi) = \frac{1}{2\pi}$$

Photon number states have uniform phase distribution.

Displacement operators

Define displacement operator $D(\beta)$

(β – complex number)

$$D(\beta) = \exp(\beta a^+ - \beta^* a)$$

Note that $\beta a^+ - \beta^* a$ – antihermition

hence $D(\beta)$ – unitary operator.

Can be presented as

$$D(\beta) = e^{-\frac{1}{2}|\beta|^2} e^{\beta a^+} e^{-\beta^* a}$$

(Baker Hausdorf formula)

Homework: prove this formula; make use of commutation relations betwen a and a^+.

Action of $D(\beta)$ on the vacuum:

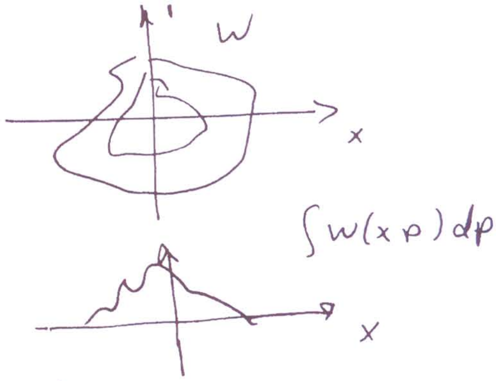
$$D(\beta)|0\rangle = e^{-\frac{1}{2}|\beta|^2} e^{\beta a^+}|0\rangle$$

$$= e^{-\frac{1}{2}|\beta|} \sum_{n=0}^{\infty} \frac{\beta^n}{\sqrt{n!}} |n\rangle$$

coherent state.

Homework: express $D(\beta)D(\alpha)$ by a ~~shift~~ displacement operator $D(\gamma)$ (group

Pictorial representation



Projection of W on the x axis,

$P(x) = \int W(x\,p)\,dp$ is not sufficient to reproduce $W(x\,p)$.

Also projection of W on the p axis

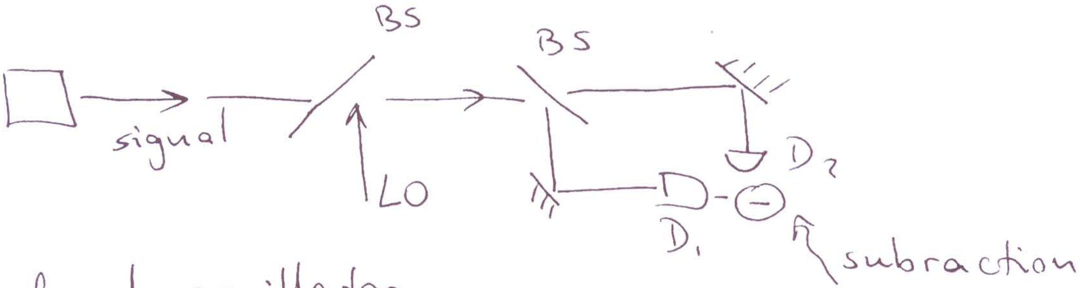
$Q(p) = \int W(x\,p)\,dx$ is not sufficient.

However one can study other projections, on axes

$x\cos\theta + p\sin\theta \quad , \quad 0 < \theta < 2\pi$

As it turnes out all projections (~~all~~ for all θ) allow to find $W(xp)$. This is the idea of quantum state tomography.

Measurements



LO - local oscillator

BS - beam splitter

Detectors D1 and D2 measure the photocurrent
Subtraction eliminates the classical (LO) behavior.
What is left is

$$\hat{X}_\varphi = \hat{N}_\varphi / \sqrt{2 n_{LO}}$$

$\bar{n}_{LO}$ – mean photoelectric number produced by LO pulses

$$x_\varphi = x\cos\varphi + p\sin\varphi \qquad x = \frac{a+a^+}{\sqrt{2}}$$

$$p_\varphi = -x\sin\varphi + p\cos\varphi \qquad p = \frac{a-a^+}{\sqrt{2}\, i}$$

One measures the distribution of $\hat{x}_\varphi$ by countig photons in D1 and D2 for a given phase of LO relative to the signal.

Let $W(x p)$ — Wigner function of the signal.
The distribution function $P(x_0 \varphi)$ of quantity $x_\varphi = x\cos\varphi + p\sin\varphi$ is

$$P(x_0 \varphi) = \int dx\, dp\ \delta(x_\varphi - x_0)\, W(x p)$$

(probability that x_φ has a value x_0).

Change variables to

$$\cancel{x'_\varphi = x\cos\varphi + p\cos\varphi}$$

$$\cancel{p'}$$

$$x' = x\cos\varphi + p\sin\varphi$$
$$p' = -x\sin\varphi + p\cos\varphi$$

The integral becomes

$$P(x_\varphi\,\varphi) = \int dx'\,dp'\;\delta(x' - x_0)\,W(x'\cos\varphi - p'\sin\varphi,\; x'\sin\varphi + p'\cos\varphi)$$

$$= \int dp'\,W(x_0\cos\varphi - p'\sin\varphi,\; x_0\sin\varphi + p'\cos\varphi).$$

Thus the measurements give an integral of W, as given above.

Reconstruction of W from $P(x_\varphi\,\varphi)$.

Take the Fourier transform of $P(x_0\,\varphi)$ over x_0

$$\int P(x_0\,\varphi)\,e^{ikx_0}\,dx_0 =$$

$$= \int dp'\,dx_0\;W(x_0\cos\varphi - p'\sin\varphi,\; x_0\sin\varphi + p'\cos\varphi)\,e^{ikx_0}$$

$$k > 0$$

Change variables

$$y = x_0\cos\varphi - p'\sin\varphi$$
$$q = x_0\sin\varphi + p'\cos\varphi$$

we get

$$\int dx_0 dp' \, e^{ik(y\cos\varphi + q\sin\varphi)} \, W(y q)$$

Take $k_1 = k\cos\varphi$ $k_2 = \sin\varphi$

Obsreve that this is nothing else but Fourier transform of $W(yq)$. Inverting we get

$$W(xp) = \int dx_0 \, e^{i|k|x_0} \, ?(x_0 \varphi) \, e^{-ik\cos\varphi\, x - ik\sin\varphi\, p} \frac{k\,dk_1\,d\varphi}{(2\pi)^2}$$

This allows to find $W(xp)$

Phys Rev Lett 70 1244 (1993) for more details.

EPR paradox (Einstein, Podolsky, Rosen) 1932

Eistein never liked QM, he belived it was an incomplete theory. He posed a "gedanken" experiment to illustrate

arguments

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- Now Alice aligns her SG along x axis. Same for Bob's particle

  Two variables, S_x and S_z, have definite values, in spite of $[S_x, S_z] \neq 0$.

According to Einstein, a better theory, with local hidden variables is needed.

Bell's inequalities (1964)

Logical argument in the form of an inequality as a method to test the conclusions from the EPR argument.

Many versions of Bell's inequalities

"Original"

$$|C_{HV}(\hat{a}\hat{b}) - C_{HV}(\hat{a}\hat{c})| \leq 1 + C_{HV}(\hat{b},\hat{c})$$

Narodowe
Laboratorium
Technologii
Kwantowych

Last lecture

Proof of Bell's inequality

Remember. Alice measures her spin, results

$A = +1$ if spin is "up", relative to $\bar{a}$
$A = -1$ if spin is "down", relative to $\bar{a}$

Bob measures his spin

$B = 1$ if spin is "up", relative to $\bar{b}$
$B = -1$ if spin is "down", relative to $\bar{b}$

Locality, means the A does not depend on $\bar{b}$,
B does not depend on $\bar{a}$

Hidden variables, denoted λ, have distribution $\rho(\lambda)$

Correlation of A and B within hidden variables theory

$$C_{HV}(a,b) = \int d\lambda \, \rho(\lambda) \, A(a\lambda) . B(a\lambda)$$

In the state we consider

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle_A |\downarrow\rangle_B - |\downarrow\rangle_A |\uparrow\rangle_B\right)$$

$A(\bar{a}\lambda) = -B(\bar{a},\lambda)$ (if Alice measures 1 then Bob measures -1 and vice versa).

Derivation

$$C_{HV}(a,b) - C_{HV}(a,c) =$$

$$= \int d\lambda\, \rho(\lambda) \left[A(a,\lambda) B(b,\lambda) - A(a,\lambda) B(c,\lambda) \right]$$

in the case of the state considered

$$= - \int d\lambda\, \rho(\lambda) \left[A(a,\lambda) A(b\lambda) - A(a\lambda) A(c\lambda) \right]$$

$$= - \int d\lambda\, \rho(\lambda) \left[A(a,\lambda) A(b,\lambda) \left(1 - A(b,\lambda) A(c\lambda) \right) \right.$$

because $A(b\lambda)^2 = 1$

Now, since $A(a\lambda) A(b\lambda) = +1$ or -1

we get

$$\left| C_{HV}(a,b) - C_{HV}(ac) \right| \leq \left| \int d\lambda \left[1 - A(b,\lambda) A(c\lambda) \right] \rho(\lambda) \right.$$

$$= 1 + C_{HV}(bc).$$

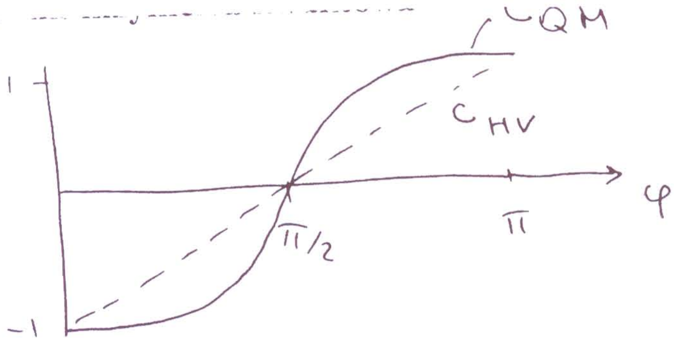
because $B(b\lambda) = -A(b\lambda)$

Conclusion

Any local hidden variable theory is not consistent with quantum physics for a EPR or similar experiment

$$C_{HV}(a,b) \neq C_{QM}(a,b)$$

for all a, b and all states.



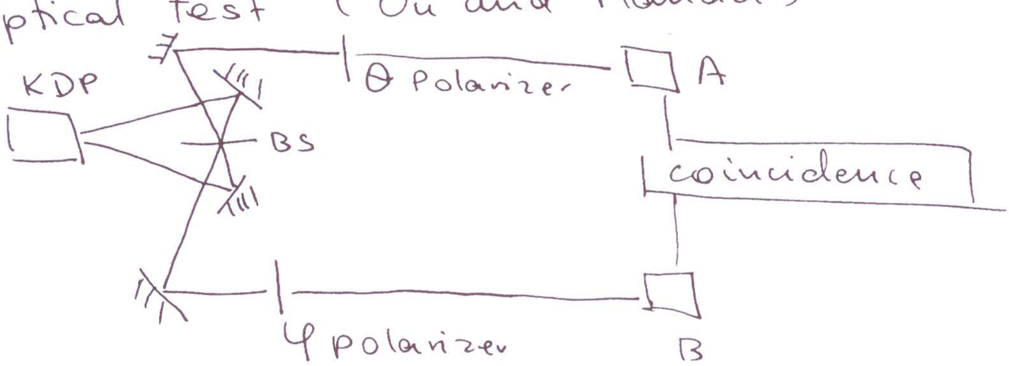
Aspect: The clear violation of the Bell inequalities leads to the conclusive rejection of theories that are simultaneously realistic and local.

Other Bell's inequality

$$-2 \le C_{HV}(\theta\varphi) + C_{HV}(\theta'\varphi) + C(\theta\varphi') - C(\theta'\varphi') \le 2$$

$C(\theta\varphi)$ – correlation function, two state systems ~~for~~ "polarizations" along θ and φ in Alice and Bob respectivelly

Optical test (Ou and Mandel)

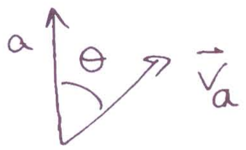


Example of a local hidden variable theory

Treat electrons as rotating particles with spin $\vec{V}_a$.

Hidden variable $\Rightarrow$ orientation of $\vec{V}$.

Consider Alice detector



θ angle between the spin and $\bar{a}$ (direction selected by the detector)

For each electron in the quantum ensemble $\Rightarrow \theta$.

(random variable with constant distribution,

$$\rho(\theta) = \frac{1}{\pi}, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

Description of the state $\vec{V}_b = -\vec{V}_a$ (spin correlatio

Measurements:

$$A = \text{sign}(\cos\theta)$$

$$B = \text{sign}(-\cos(\theta - \varphi))$$

therefore $AB = \text{sign}(-\cos\theta\cos(\theta-\varphi))$

Therefore

$$C_{HV}(ab) = \frac{1}{\cancel{2}\pi}\int d\theta \, \text{sign}(-\cos\theta\cos(\theta-\varphi)) =$$

$$= \frac{2}{\pi}\varphi - 1$$

Set $\theta = 22.5^\circ$ $\quad \theta' = 67.5^\circ$

$\varphi = 45^\circ$ $\quad \varphi' = 0$

Measured value of

$$S = (C(\theta\varphi) - C(\theta\varphi') + C(\theta'\varphi') + C(\theta'\varphi) - C(\theta' -) - C(- \varphi) \leq 0$$

Exp. result

$$S = 11 \pm 2$$

Violation of hidden variables theories.

Test, 11 września 2009

1. Wymień co najmniej jeden efekt świadczący o istnieniu fotonów

2. Zaznacz prawdziwe zdanie

Stany Focka fotonów $|n\rangle$ są ortogonalne $\langle n|n'\rangle = \delta_{nn'}$

Stany Focka tworzą zupełny układ stanów

Średnia wartość pole elektrycznego w stanie Focka $|n\rangle$ jest proporcjonalna do $\sqrt{n}$

Stany koherentne fotonów $|\alpha\rangle$ są ortogonalne
$$\langle \alpha' | \alpha \rangle = \delta_2(\alpha - \alpha')$$

Średnia wartość pola elektrycznego w stanie koherentnym $|\alpha\rangle$ jest proporcjonalna do α
(możesz założyć, że α jest rzeczywiste)

3. Wybierz i zaznacz stan splątany układu dwóch fotonów ($|n\rangle$ oznacza stan Focka)

$|1\rangle_A |2\rangle_B$, $\frac{1}{\sqrt{2}}\left(|1\rangle_A |2\rangle_B + e^{i\theta}|2\rangle_A |1\rangle_B\right)$

$\rho = |1\rangle_A |2\rangle_B \, {}_B\langle 2| \, {}_A\langle 1| + |2\rangle_A |1\rangle_B \, {}_B\langle 1| \, {}_A\langle 2|$

4. Wyznacz średnią liczbę fotonów o częstości ω w jednomodowym stanie cieplnym o temperaturze T.

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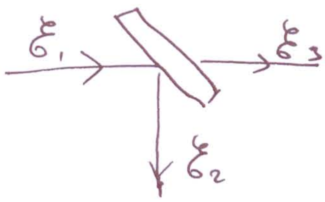
Kurs letni: inżynieria kwantowa

Lecture
Quantum theory of beam splitter

Classically



Solve classical equations, $\xi_2 = r\,\xi_1$, $\xi_3 = t\,\xi_1$

For simplicity $|r|^2 = \frac{1}{2} = |t|^2$ (50/50 beam splitter)

From energy conservation $|\xi_1|^2 = |\xi_2|^2 + |\xi_3|^2$

or $|r|^2 + |t|^2 = 1$

Quantum mechanics

$$a_2 = r\,a_1 \qquad a_3 = t\,a_1$$

find commutators

$$[a_2, a_2^\dagger] = |r|^2 [a_1, a_1^\dagger] = |r|^2$$

$$[a_3, a_3^\dagger] = |t|^2$$

$$[a_2, a_3^\dagger] = r t^* \neq 0.$$

Wrong! We forgot about the fourth "port",

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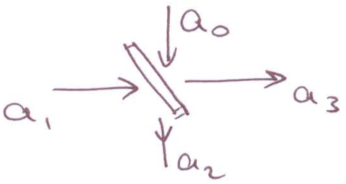
Kurs letni: inżynieria kwantowa

Correct picture



$$a_2 = r a_1 + t a_0$$
$$a_3 = t a_1 + r a_0$$

classical relations $|r|^2 + |t|^2 = 1$

$$r^* t + r t^* = 0$$

Example (50/50) $a_2 = \frac{1}{\sqrt{2}}(a_0 + i a_1)$, $a_3 = \frac{1}{\sqrt{2}}(i a_0 + a_1)$

Inverse relations $a_1 = \frac{1}{\sqrt{2}}(i a_2 + a_3)$ $a_0 = \frac{1}{\sqrt{2}}(a_2 + i a_3)$

This concludes the quantum theory. Examples follow.

Given input, what is the output?

a) vacuum $a_1 |0\rangle_0 |0\rangle_1 = 0$ $a_0 |0\rangle_0 |0\rangle_1 = 0$

so $a_2 |0\rangle_0 |0\rangle_1 = 0$ $a_3 |0\rangle_0 |0\rangle_1 = 0$

it remains a vacuum.

b) one photon state $|0\rangle_0 |1\rangle_1 = a_1^+ |0\rangle_0 |0\rangle_1$

$$= \frac{1}{\sqrt{2}}(-i a_2^+ + a_3^+) |0\rangle_2 |0\rangle_3 = -\frac{i}{\sqrt{2}} |1\rangle_2 |0\rangle_3 + \frac{1}{\sqrt{2}} |0\rangle_2 |1\rangle_3$$

Single photon and vacuum $\Rightarrow$ transmitted or reflected with equal probabilities.

This is entangled state: cannot be written as a a product of individual modes 2 and 3.

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Kurs letni: inżynieria kwantowa

c) coherent state

Initial state $\quad |0\rangle_0 |\alpha\rangle_1 = D_1(\alpha)\, |0\rangle_0 |0\rangle_1$

with $\quad D_1(\alpha) = \exp(\alpha a_1^\dagger - \alpha^* a_1)$

Beam splitter

$$|0\rangle_0 |\alpha\rangle_1 = \exp\left[\frac{\alpha}{\sqrt{2}}(-i a_2^+ + a_3^+) - \frac{\alpha^*}{\sqrt{2}}(i a_2 + a_3)\right] |0\rangle_2 |0\rangle_3$$

$$= \exp\left[-i\frac{\alpha_2^*}{\sqrt{2}} a_2 + i\frac{\alpha_2}{\sqrt{2}} a_2^+\right] \exp\left[\frac{\alpha}{\sqrt{2}} a_3^+ - \frac{\alpha^*}{\sqrt{2}} a_3\right] |0\rangle_2 |0\rangle_3$$

$$= \left|\frac{-i\alpha}{\sqrt{2}}\right\rangle_2 \left|\frac{\alpha}{\sqrt{2}}\right\rangle_3$$

coherent state in both modes, phase shift.

d) two photons, one at each state

$$|1\rangle_1 |1\rangle_0 = a_1^+ a_0^+ |0\rangle_2 |0\rangle_3 =$$

$$= \frac{1}{\sqrt{2}}(a_2^+ - i a_3^+)(-i a_2^+ + a_3^+)\frac{1}{\sqrt{2}} |0\rangle_2 |0\rangle_3 =$$

$$= \frac{1}{2}(-i)(a_2^+ a_2^+ + a_3^+ a_3^+) |0\rangle_2 |0\rangle_3$$

$$= -i\frac{1}{\sqrt{2}}\left(|2\rangle_2 |0\rangle_3 + |0\rangle_2 |2\rangle_3\right).$$

Discussion. Case b). $\quad |0\rangle_0 |1\rangle_1 \longrightarrow \frac{1}{\sqrt{2}}\left(|0\rangle_2 |1\rangle_3 - i |1\rangle_2 |0\rangle_3\right)$

Look for probability of finding photon in ~~state~~ mode 2.

Construct density matrix

$$\rho = \frac{1}{2}\left(|0\rangle_2 |1\rangle_3 - i |1\rangle_2 |0\rangle_3\right)\left({}_2\langle 0|\, {}_3\langle 1| + i\, {}_2\langle 1|\, {}_3\langle 0|\right)$$

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and take trace over states of mode 3 (reduced density matrix):

$$\rho_2 = \mathrm{Tr}_3\, \rho = \frac{1}{2}\left(|0\rangle_2 {}_2\langle 0| + |1\rangle_2 {}_2\langle 1|\right)$$

No off diagonal terms, no entanglement.

$\langle 0|\rho|0\rangle = \frac{1}{2}$ $\qquad$ $\langle 1|\rho|1\rangle = \frac{1}{2}$

Probability of finding vacuum $= \frac{1}{2}$,
probability of finding one photon $= \frac{1}{2}$.

Probability of finding one photon in state (mode) 3

$= 1/2$

Joint probability to find photon at ~~mo~~ port 2 and 3

$${}_2\langle 1|\,{}_3\langle 1|\left(\frac{1}{2}\right)\left(-i\,|1\rangle_2|0\rangle_3 + |0\rangle_2|1\rangle_3\right)\left(i\,{}_2\langle 1|\,{}_3\langle 0| + {}_2\langle 0|\,{}_3\langle 1|\right)|1\rangle_2|1\rangle_3$$

$= 0$

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Kurs letni: inżynieria kwantowa

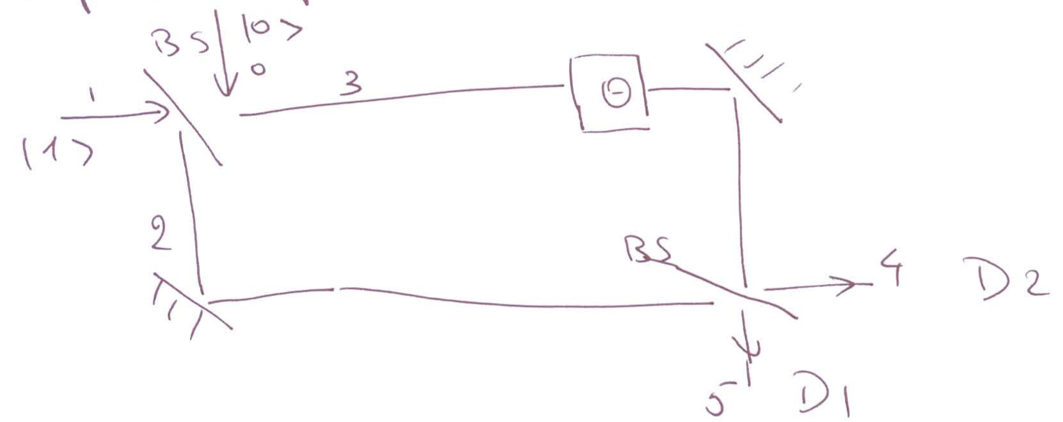
case ~~4~~.d Probability of finding photons in arm 2 and 3

Two photons in mode 2 $\frac{1}{2}$

Two photons in mode 3 $\frac{1}{2}$

One photon in mode 2 and one photon in mode 3; 0

Aspect experiment with one photon interference



start with $|0\rangle_0 |1\rangle_1$

First beam splitter

$$|0\rangle_0|1\rangle_1 \rightarrow \frac{1}{\sqrt{2}}\left(-i\,|1\rangle_2|0\rangle_3 + |0\rangle_2|1\rangle_3\right)$$

Phase shift

$$\frac{1}{\sqrt{2}}\left(-i\,|1\rangle_2|0\rangle_3 + e^{i\theta}|0\rangle_2|1\rangle_3\right)$$

only in this term phase shift is effective.

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Second beam splitter

$$|0\rangle_2 |1\rangle_3 \longrightarrow \frac{1}{\sqrt{2}}\left(|0\rangle_4 |1\rangle_5 - i|1\rangle_4 |0\rangle_5\right)$$

$$|1\rangle_2 |0\rangle_3 \longrightarrow \frac{1}{\sqrt{2}}\left(|1\rangle_4 |0\rangle_5 - i|0\rangle_4 |1\rangle_5\right)$$

All together

$$\frac{1}{2}\left[(e^{i\theta}-1)|0\rangle_4|1\rangle_5 - i(e^{i\theta}+1)|1\rangle_4|0\rangle_5\right] = |\psi\rangle$$

Probability to detect a photon at D1 (mode 5)

$$|{}_5\langle 1|{}_4\langle 0|\psi\rangle|^2 = \left|\frac{1}{2}(e^{i\theta}-1)\right|^2 = \frac{1}{2}(1-\cos\theta)$$

Probability to detect a photon at D2 (mode 4)

$$|\langle 0|\langle 1|\psi\rangle|^2 = \frac{1}{2}(1+\cos\theta)$$

Probability to detect photon at D1 and D2 $= 0$.