Narodowe
Laboratorium
Technologii
Kwantowych

Piotr Deuar

Opracowanie utworu pod tytułem:

"Zimne fermiony" w ramach kursu zaawansowanego, organizowanego w dniach 31.08 – 25.09.09 będącego kontynuacją szkoleń z zakresu eksploatacji i zarządzania dużą infrastrukturą badawczą organizowanego przez Narodowe Laboratorium Technologii Kwantowych

INNOWACYJNA GOSPODARKA
NARODOWA STRATEGIA SPÓJNOŚCI

UNIA EUROPEJSKA
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Kurs letni: inżynieria kwantowa – Zimne fermiony

PIOTR DEUAR

ULTRACOLD FERMI GASES

WARSZAWA

24 SEPTEMBER 2009

PLAN

MORNING: 9⁰⁰
- BASIC PROPERTIES OF FERMI GASES
- IDEAL GAS IN A BOX
- IN A TRAP : THOMAS-FERMI APPROXIMATION

11⁰⁰ TOMASZ KAWALEC

AFTERNOON 15⁰⁰
- BCS SUPERFLUIDITY
- OTHER TOPICS : SPIN-POLARISED GAS ?
  BEC-BCS CROSSOVER ?
  FFLO ?
  DIPOLES ?

USEFUL REFERENCES :
- S. Giorgini et al. Reviews of Modern Physics VOLUME 80 PAGE 1215 (2008)
- KRZYSZTOF SACHA : KONDENSAT BOSEGO-EINSTEINA

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Real Basics

	BOSONS	FERMIONS
SPIN	$0, 1, 2, \ldots$	$\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots$
EXCHANGE	$\Psi(x_1, x_2) = \Psi(x_2, x_1)$	$\Psi(x_1, x_2) = -\Psi(x_2, x_1)$
COMMUTATION	$[\hat{\Psi}(x), \hat{\Psi}^{+}(y)] = \delta(x-y)$	$[\hat{\Psi}(x), \hat{\Psi}^{+}(y)]_{+} = \delta(x-y)$
SUPERFLUID?	IN IDEAL OR INTERACTING GAS	IN INTERACTING GAS ONLY
PARTICLES/ORBITAL	$0, 1, \ldots \infty$	$0, 1$

ANTI-COMMUTATOR

$$[A, B]_{+} = AB + BA$$

Related to properties of relativistic field theory by the <u>Spin-statistics theorem</u> (Markus Fierz 1939) rederived Pauli 1940

<u>QUESTION</u>: What are some fermions that you can make a gas out of?

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Pauli exclusion

- Suppose we have two fermions with positions x & y & one orbital ϕ

- if both particles are in ϕ then wavefunction is:

$$\Psi = \phi(x) \otimes \phi(y)$$

$$= -\phi(y) \otimes \phi(x)$$

$$\hookrightarrow \quad 2\,\phi(x) \otimes \phi(y) = 0 \quad \text{for all } x \,\&\, y$$

$$\hookrightarrow \quad \phi(x) = 0 \quad \text{or} \quad \phi(y) = 0 \quad \text{for all } x \,\&\, y$$

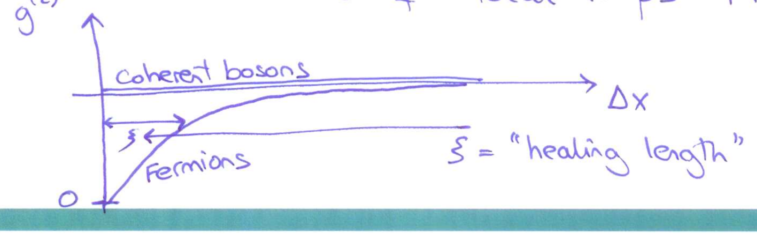
$$\hookrightarrow \quad \phi = 0 \rightarrow \text{NO SUCH STATE}$$

Effect 1: Repulsion between particles

two-body (density-density) correlation function

$$g^{(2)}(\Delta x) = \frac{\langle \hat{\Psi}^{+}(x)\, \hat{\Psi}(x+\Delta x)\, \hat{\Psi}(x+\Delta x)\, \hat{\Psi}(x) \rangle}{n(x) \quad n(x+\Delta x)}$$

$g^{(2)}$ = shape & size of "local lumps" in the gas



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4

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nature Vol 445|25 January 2007|doi:10.1038/nature05513

LETTERS

Comparison of the Hanbury Brown–Twiss effect for bosons and fermions

T. Jeltes[1], J. M. McNamara[1], W. Hogervorst[1], W. Vassen[1], V. Krachmalnicoff[2], M. Schellekens[2], A. Perrin[2], H. Chang[2], D. Boiron[2], A. Aspect[2] & C. I. Westbrook[2]

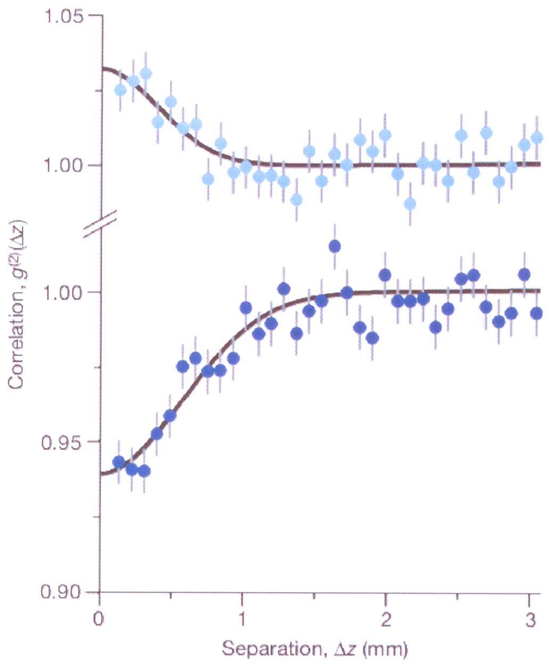

Figure 2 | Normalized correlation functions for ^{4}He* (bosons) in the upper plot, and ^{3}He* (fermions) in the lower plot. Both functions are measured at the same cloud temperature (0.5 μK), and with identical trap parameters. Error bars correspond to the square root of the number of pairs in each bin. The line is a fit to a gaussian function. The bosons show a bunching effect, and the fermions show antibunching. The correlation length for ^{3}He* is expected to be 33% larger than that for ^{4}He* owing to the smaller mass. We find 1/e values for the correlation lengths of 0.75 ± 0.07 mm and 0.56 ± 0.08 mm for fermions and bosons, respectively.

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Effect 2: COLLISIONS SUPRESSED

~~part~~ - in dilute gas the size of the interatomic potential is much less than the healing length ξ

↳ atoms find it hard to get close enough to collide

→ elastic collisions supressed

- three-body collisions are similarly unlikely

↳ 3-body loss strongly reduced in comparison with Bose gas

- few collisions mean slow thermalisation

↳ difficult to cool!

↳ use buffer gas cooling

- attractive interactions are necessary for BCS superfluidity (more on that later)

↳ need two different "species" of fermion or long-range interactions (dipoles)

Effect 3: Fermi sphere (see below)

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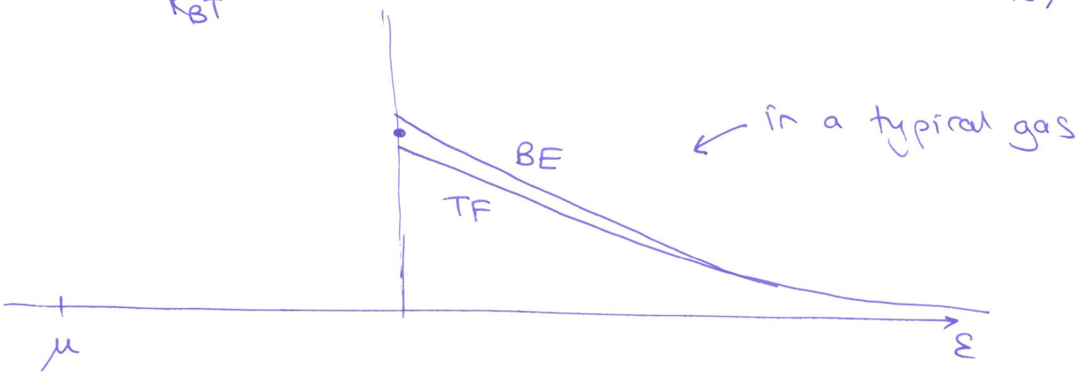
Q: why is fermi / Bose statistics irrelevant for everyday gases?

Bose-Einstein / Fermi-Dirac distribution
$$n(\varepsilon) = \frac{1}{\exp[(\varepsilon-\mu)/k_BT] \pm 1}$$

mean number of particles per orbital with energy ε

if $\frac{\varepsilon-\mu}{k_BT} \gg 1$ have Boltzmann distribution $n(\varepsilon) \sim e^{\frac{\mu-\varepsilon}{k_BT}}$



consider the ideal gas: $\varepsilon = \frac{p^2}{2m} \geq 0$

$$\mu = \left.\frac{\partial E}{\partial N}\right|_{S,V} = k_BT \log\left[\frac{n}{n_Q}\right]$$

$\frac{n}{n_Q}$ = "phase-space density" $\qquad n_Q = \left(\frac{mk_BT}{2\pi\hbar^2}\right)^{3/2}$

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"typical" gas – consider air

$T \sim 300\ K$

$m \sim 30\,u = 5\times10^{-26}\ kg$

$k_B = 1.38\times10^{-23}\ J/K$

$\hbar = 1.054\times10^{-34}\ J.s$

$\rightarrow n_Q \sim 1.62\times10^{32}\ m^{-3}$

density $n \sim \frac{N_A}{22.4\times10^{-3}\ m^3} \sim 2.69\times10^{25}\ m^{-3}$

$\therefore$ phase-space density $= 0.000000166$

$\hookrightarrow \frac{\mu}{k_BT} \approx -15.6$

thus $\frac{\varepsilon-\mu}{k_BT} > 15.6$ which is very far from any differences in statistics

<u>Degenerate Fermi gas</u> : $\frac{n}{n_Q} \gtrapprox 1$

here $\mu > 0$ and a "Fermi sphere" & "Fermi surface" appear at the fermi energy $E_F \cong \mu$ as $T \to 0$

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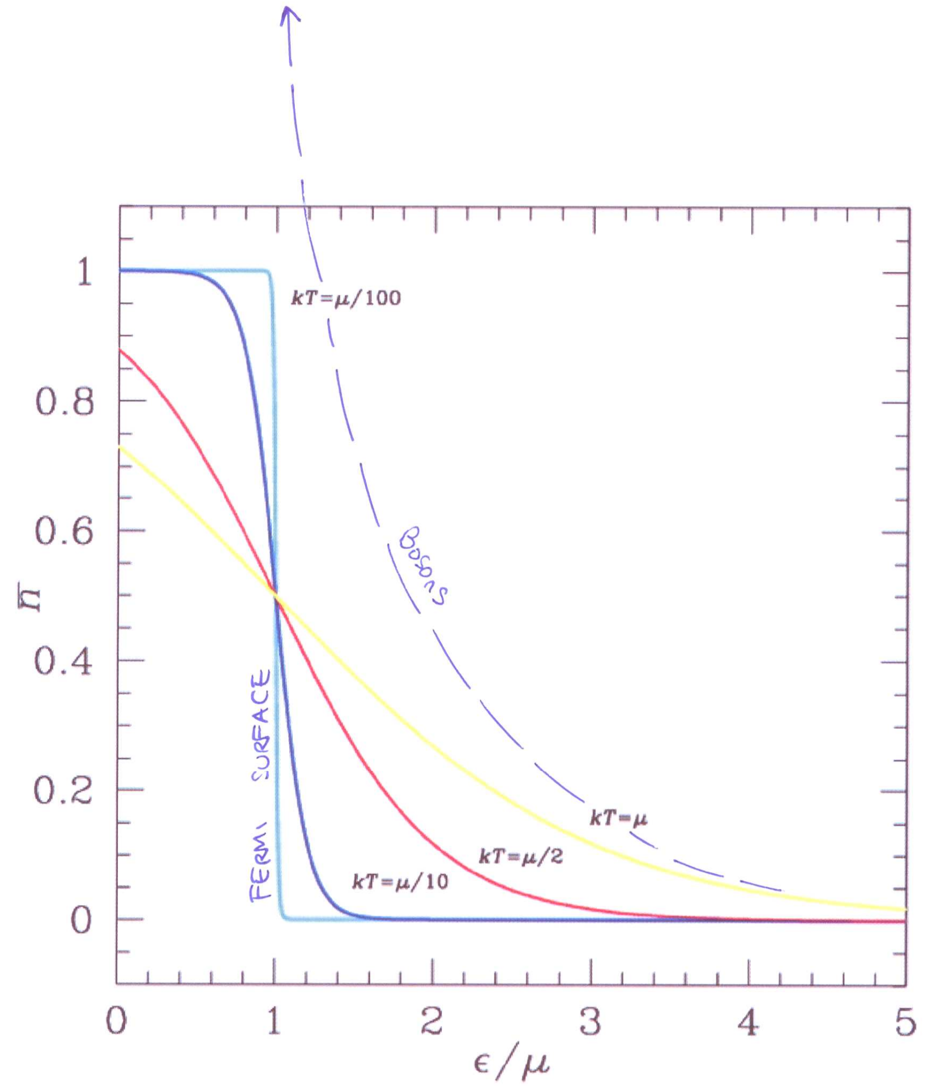

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IDEAL FERMI GAS IN A BOX $T \to 0$

Box length L, 3D, periodic boundary conditions

eigenstates $\phi_{\vec{k}} \sim e^{i\vec{k}\cdot\vec{x}}$

$$\mathcal{E}(\vec{k}) = \frac{p^2}{2m} = \frac{\hbar^2 k^2}{2m}$$

$$\boxed{\begin{array}{l} \text{here: } \vec{k} = [k_x, k_y, k_z] \\ k_j = n_j \Delta k \\ n_j = \ldots, -2, -1, 0, 1, 2, \ldots \\ \Delta k = \dfrac{2\pi}{L} \end{array}}$$

From Fermi-Dirac distribution at $T \to 0$

$$\text{occupation} = \begin{cases} 1 \text{ if } \mathcal{E}(\vec{k}) < \mu \\ 0 \text{ if } \mathcal{E}(\vec{k}) > \mu \end{cases}$$

Want to find <u>Fermi energy</u> $E_F = \mu = k_B T_F$ (Fermi temp.)
as a function of density etc.

To have N fermions: $N = \underset{\text{occupation}}{1} \times \underbrace{\sum_{\substack{\vec{k} \\ \mathcal{E}(k)<\mu}}}_{\text{modes}}$.

let $\vec{k}$ go to polar coordinates k, θ, ϕ

$$N \approx \frac{1}{(\Delta k)^3} \int_{\mathcal{E}(k)<\mu} d^3\vec{k}$$

$$= \frac{V}{(2\pi)^3} \int_0^{k_F} k^2 dk \int_0^{2\pi} d\theta \int_0^{\pi} \sin\phi \, d\phi$$

where $\mu = \dfrac{\hbar^2 k_F^2}{2m}$ defines fermi momentum k_F

$$N = \frac{V k_F^3}{6\pi^2}$$

Kurs letni: inżynieria kwantowa – Zimne fermiony

hence, density $n = \frac{k_F^3}{6\pi^2} \rightarrow k_F^2 = (6\pi^2 n)^{2/3}$

$$\hookrightarrow \boxed{\mu = \frac{\hbar^2 k_F^2}{2m} = \frac{\hbar^2}{2m}(6\pi^2 n)^{2/3}} = k_B T_F$$

compare this to the BEC critical temperature in a box:

$$k_B T_c = \frac{\hbar^2}{2m}\left[\frac{8\pi\sqrt{\pi}}{\zeta(3/2)}\, n\right]^{2/3}$$

→ serious quantum gas effects occur for temperatures scaling as $\propto n^{2/3}$

→ for both $\left\{\begin{array}{ll} \text{high } n \\ \text{low } m \end{array}\right.$ are good for raising critical temperature.

IN A TRAP – THOMAS FERMI APPROXIMATION

Allows one to obtain the 1-particle density and distribution in momentum-space if interactions are small compared to μ

LOCAL DENSITY APPROXIMATION (the original TF approx as opposed to the Boson case)

$$H = \frac{\vec{p}^2}{2m} + U(\vec{r})$$ ← ∴ interaction should be small!

harmonic trap $U(\vec{r}) = \frac{m}{2}\left\{ \omega_x^2 x^2 + \omega_y^2 y^2 + \omega_z^2 z^2 \right\}$

ASSUMPTION 1 : chemical potential is the same regardless of where in the cloud the atom is added.

Note : in the trap $\mu = (6N)^{1/3} \hbar\omega_{ho}$ Different from uniform gas!

$\omega_{ho} = [\omega_x \omega_y \omega_z]^{1/3}$

this is obtained similarly to uniform gas, but the summing over states is more complicated.

ASSUMPTION 2 : locally the gas behaves like a uniform gas → $U(\vec{r})$ must be "smooth enough".

∴ LOCAL FERMI SPHERE with local Fermi Energy

$$\mu = E_F(\vec{r}) + U(\vec{r})$$

since locally the gas behaves like the uniform ideal gas

$$E_F(\vec{r}) = \frac{\hbar^2 k_F(\vec{r})^2}{2m} \quad , \quad n(\vec{r}) = \frac{k_F(\vec{r})^3}{6\pi^2}$$

combining

$$n(\vec{r}) = \frac{1}{6\pi^2}\left[\frac{2m}{\hbar^2} E_F(\vec{r})\right]^{3/2}$$

$$= \frac{1}{6\pi^2}\left[\frac{2m}{\hbar^2}\left(\mu - U(\vec{r})\right)\right]^{3/2}$$

For the harmonic trap

$$n(\vec{r}) = \frac{1}{6\pi^2}\frac{(2m)^{3/2}}{\hbar^3}\left[\mu - \frac{m}{2}\sum_i \omega_i^2 x_i^2\right]^{3/2}$$

TF Radius $R_i = \frac{1}{\omega_i}\sqrt{\frac{2\mu}{m}}$

~~same~~ form as in BEC

- which can be anisotropic.
- is zero for $U(\vec{r}) > \mu$ like BEC T-F profile
- is $\propto (\mu - U(\vec{r}))^{3/2}$ unlike BEC which has $\propto (\mu - U(\vec{r}))^1$

However, the momentum distribution is always isotropic (unlike the BEC)

to see this: As always, occupation of orbitals is given by Fermi-Dirac distribution. However now, there ~~dist~~ is a local distribution at each $\vec{r}$ due to the Local density approximation

$$n(\vec{k},\vec{r}) = \frac{1}{\exp\left\{\left[\frac{\hbar^2 k^2}{2m} + U(\vec{r}) - \mu\right]/k_B T\right\} + 1}$$

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consider space split up into little boxes of volume ΔV
length ℓ

the gas is $\approx$ uniform in each box

here momentum is discretized into values with spacing $\Delta k = \frac{2\pi}{\ell}$

the number of particles with momenta between k and $k+\Delta k$ in the entire system is

$$\tilde{n}(k)(\Delta k)^3 = \sum_{\vec{k} \in [\underbrace{\vec{k}, \vec{k}+\Delta k}_{\text{volume in k-space of sides } \Delta k \text{ in all 3 directions}}]} \times \sum_{\vec{r}} \times n(k,r)$$

sum over boxes

hence

$$\tilde{n}(k) \cong \frac{1}{(\Delta k)^3} \times \sum_{\vec{k} \in [\vec{k}, \vec{k}+\Delta k]} \times \frac{1}{\Delta V} \int d\vec{r} \; n(k,r)$$

there is ≈ 1 such momentum per box

$$= \frac{\ell^3}{(2\pi)^3} \times 1 \times \frac{1}{\ell^3} \int d\vec{r} \, n(k,r)$$

$$= \frac{1}{(2\pi)^3} \int d^3\vec{r} \; n(k,r)$$

now take $T \to 0$ limit

$$\hookrightarrow \quad n(k,r) \approx \Theta\left[\mu - \frac{\hbar^2 k^2}{2m} - U(\vec{r})\right]$$

$$\tilde{n}(k) \approx \frac{1}{(2\pi)^3} \int d^3\vec{r}\, \Theta\left[\frac{\hbar^2}{2m}\left(k_F^2 - k^2\right) - \frac{m}{2} \sum_i \omega_i^2 r_i^2 \right]$$

anisotropic → a pain

let $\omega_i r_i = q_i \quad \rightarrow \quad d^3\vec{r} = \dfrac{d^3\vec{q}}{\omega_x \omega_y \omega_z}$

$$\tilde{n}(k) \approx \frac{1}{(2\pi\omega_{ho})^3} \int d^3\vec{q}\, \Theta\left[\frac{\hbar^2}{2m}\left(k_F^2 - k^2\right) - \frac{m}{2} |\vec{q}|^2 \right]$$

polar coordinates of $\vec{q}$: $|q|$, θ_q ϕ_q

$$= \frac{1}{(2\pi\omega_{ho})^3} \int_0^\infty |q|^2 d|q|\, \Theta\left[\frac{\hbar^2}{2m}\left(k_F^2 - k^2\right) - \frac{m}{2} |q|^2 \right] \int_0^{2\pi} d\theta_q \int_0^{\pi} \sin\phi_q\, d\phi_q$$

$$= \frac{1}{2\pi^2\omega_{ho}^3} \int_0^{q_{max}} |q|^2 d|q| \qquad \text{with } \frac{m}{2} q_{max}^2 = \frac{\hbar^2}{2m}\left(k_F - k^2\right)$$

$$\hookrightarrow \quad q_{max} = \sqrt{\frac{2\mu}{m}\left(1 - \left(\frac{k}{k_F}\right)^2\right)}$$

$$= \frac{1}{6\pi^2\omega_{ho}^3}\, q_{max}^3$$

$$\tilde{n}(k) = \frac{2\sqrt{2}\,\mu^{3/2}}{6\pi^2\omega_{ho}^3\, m^{3/2}} \times \left[1 - \left(\frac{k}{k_F}\right)^2\right]^{3/2}$$

isotropic!

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PHYSICAL REVIEW
LETTERS

VOLUME 86 | 11 JUNE 2001 | NUMBER 24

Pauli Blocking of Collisions in a Quantum Degenerate Atomic Fermi Gas

B. DeMarco, S. B. Papp, and D. S. Jin*
JILA, National Institute of Standards and Technology and University of Colorado and Physics Department, University of Colorado, Boulder, Colorado 80309-0440
(Received 29 January 2001)

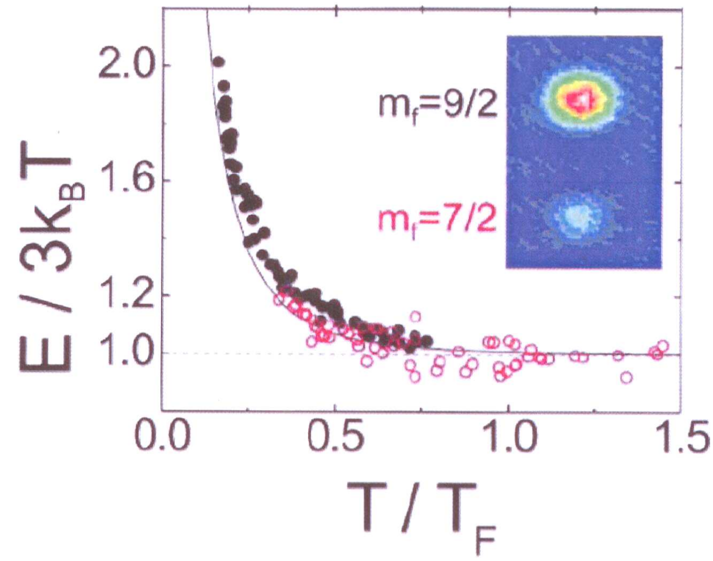

FIG. 3. (Color online) Evidence for quantum degeneracy effects in trapped Fermi gases. The average energy per particle, extracted from absorption images, is shown for two-spin mixtures. In the quantum degenerate regime, the data agree well with the ideal Fermi gas prediction (solid line). The horizontal dashed line corresponds to the result of a classical gas. From De Marco *et al.*, 2001.

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Lecture 2 : BCS THEORY – SUPERFLUIDITY

J. Bardeen L. Cooper J. Schrieffer
Phys. Rev. 108 1175 (1957)
originally for superconductivity = superfluidity of e^-

INTRO :

How is a superfluid (SF) made in the Bose gas?

OFF-DIAGONAL LONG RANGE ORDER (ODLRO) is needed
where $\langle \hat{f} \rangle \neq 0$ for some non-Hermitian $\hat{f}$

– have BEC which provides ODLRO via the
order parameter $\langle \hat{\psi}(x) \rangle = \psi(x)$

condensate wavefunction

– kills a boson
– coherently so $\langle \hat{\psi} \rangle \neq 0$
because there is a well defined phase

In Helium SF in He^4 at 2.18 K
He^3 at ~ 1 mK

How to make a SF with fermions?

suggestion : same way as with bosons.

Q: how to kill a boson when you have fermions?

A: kill 2 fermions.

let $\hat{\Pi}(x)$ be a boson [= fermion pair] anihilator

$$\hat{\Pi}(x) = \hat{\psi}(x)\hat{\psi}(x)$$

if we had ODLRO in $\hat{\Pi}$: $\langle \hat{\Pi}(x) \rangle = \Pi(x) \neq 0$
it should lead to superfluidity just like
(or at least similarly to) with $\varphi(x)$ for bosons.

small problem: due to fermion properties

$$\hat{\psi}(x)\hat{\psi}(x) = -\hat{\psi}(x)\hat{\psi}(x) \quad \text{(exchange } x \leftrightarrow x\text{)}$$

$$\hookrightarrow \quad \hat{\psi}(x)\hat{\psi}(x) = 0 \quad :($$

(theoretically) easy fix : two componets. ↑ and ↓

(cooper) pair : $\hat{\psi}_\uparrow(x)\hat{\psi}_\downarrow(x)$ consists of one ↑ and one ↓

ODLRO : $\langle \hat{\psi}_\uparrow(x)\hat{\psi}_\downarrow(x) \rangle = \Delta(x)$

macroscopic wavefunction of cooper pairs
called "the gap"

BCS HAMILTONIAN : TRUE $\hat{H}$

consider $\hat{K} = \hat{H} - \mu \sum_\sigma \hat{N}_\sigma$ the "Kamiltonian"

$$\hat{K} = \sum_\sigma \int d^3r \left\{ \hat{\varphi}^+_\sigma(r)\, H_0 \hat{\varphi}_\sigma(r) + \frac{g}{2}\, \hat{\varphi}^+_\sigma(r)\, \hat{\varphi}^+_{-\sigma}(r)\, \hat{\varphi}_{-\sigma}(r)\, \hat{\varphi}_\sigma(r) \right\}$$

$\sigma = \pm\frac{1}{2}$

contact interactions
attractive : $g < 0$

$$H_0 = \frac{-\hbar^2}{2m}\nabla^2 - \mu$$

[uniform gas] equal μ for both components σ
& same number of ~~spins~~ atoms in each spin

EFFECTIVE $\hat{H}$ ($\hat{K}$)

Want an "effective" mean field theory that describes single fermions in a mean field produced by the rest

↳ want only pairs of operators $\hat{\varphi}$, $\hat{\varphi}^+$

Postulate:

$$\hat{K}_{ef} = \int d^3r \Big\{ \sum_\sigma \Big[\hat{\varphi}^+_\sigma(r)\, H_0 \hat{\varphi}_\sigma(r) + W(r)\, \hat{\varphi}^+_\sigma(r) \hat{\varphi}_\sigma(r)$$

$$+\frac{1}{2}F(r)\, \hat{\varphi}^+_\sigma(r)\, \hat{\varphi}_{-\sigma}(r) + h.c.$$

$$+\frac{1}{2}\Delta(r)\, \hat{\varphi}^+_\sigma(r)\, \hat{\varphi}^+_{-\sigma}(r) + h.c. \Big]\Big\}$$

with $\Delta(r)$, $W(r)$, $F(r)$ to be determined

How to choose Δ, W, F ??

1) consider the free energy based on $\hat{K}_{ef}$

$F_{ef} = \langle \hat{K}_{ef} \rangle_{ef} - TS$ using the basis that diagonalizes $\hat{K}_{ef}$

because of this, F_{ef} jest stacjonarna

$\delta F_{ef} = 0$ with respect to eigenstates.

2) consider the free energy based on $\hat{K}$ (full) : F calculated in the $\hat{K}_{ef}$ basis. $F = \langle \hat{K} \rangle_{ef} - TS$

To have $\delta F = 0$ also we want

$$\delta F = \delta F_{ef}$$

[Then $\hat{K}_{ef}$ is chosen so that its eigenstates make the full free energy minimized as well.]

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what is δF_{ef} ?

$$\delta F_{ef} = \int d^3r \sum_\sigma \Big\{ \delta\langle \hat\varphi^+_\sigma(r) H_0 \hat\varphi_\sigma(r)\rangle + W(r)\,\delta\langle \hat\varphi^+_\sigma(r)\hat\varphi_\sigma(r)\rangle$$

$$+\tfrac{1}{2}F(r)\,\delta\langle \hat\varphi^+_\sigma(r)\hat\varphi^+_{-\sigma}(r)\rangle + \tfrac{1}{2}F(r)^*\,\delta\langle \hat\varphi^+_{-\sigma}(r)\hat\varphi_\sigma(r)\rangle$$

$$+\tfrac{1}{2}\Delta(r)\,\delta\langle \hat\varphi^+_\sigma(r)\hat\varphi^+_{-\sigma}(r)\rangle + \tfrac{1}{2}\Delta(r)^*\,\delta\langle \hat\varphi_{-\sigma}(r)\hat\varphi_\sigma(r)\rangle \Big\}$$

$$- T\delta S$$

what is δF ?

n.b. $\langle F\rangle$ involves $\langle \hat\varphi^+_\sigma(r)\hat\varphi^+_{-\sigma}(r)\hat\varphi_{-\sigma}(r)\hat\varphi_\sigma(r)\rangle$

using the "generalised" Wick theorem
[applies to statistical ensembles of the form $e^{-\hat K_{ef}/k_BT}$ with quadratic $\hat K_{ef}$]

$$= \langle \varphi^+_\sigma(r)\varphi^+_{-\sigma}(r)\rangle\langle \varphi_{-\sigma}(r)\varphi_\sigma(r)\rangle + \langle \varphi^+_\sigma(r)\varphi_\sigma(r)\rangle\langle \varphi^+_{-\sigma}(r)\varphi_{-\sigma}(r)\rangle$$

$$- \langle \varphi^+_\sigma(r)\varphi_{-\sigma}(r)\rangle\langle \varphi^+_{-\sigma}(r)\varphi_\sigma(r)\rangle$$

hence...

$$\delta F = \int d^3r \sum_\sigma \Big\{ \delta\langle \psi_\sigma^+ H_0 \psi_\sigma \rangle + \frac{g}{2}\langle \psi_\sigma^+ \psi_{-\sigma}^+ \rangle \delta\langle \psi_{-\sigma}\psi_\sigma\rangle$$

$$+ \frac{g}{2}\delta\langle \psi_\sigma^+ \psi_{-\sigma}^+ \rangle\langle \psi_{-\sigma}\psi_\sigma\rangle + \frac{g}{2}\langle \psi_\sigma^+ \psi_\sigma\rangle \delta\langle \psi_{-\sigma}^+\psi_{-\sigma}\rangle + \frac{g}{2}\delta\langle \psi_\sigma^+\psi_\sigma\rangle\langle \psi_{-\sigma}^+\psi_{-\sigma}\rangle$$

$$- \frac{g}{2}\delta\langle \psi_\sigma^+ \psi_{-\sigma}\rangle\langle \psi_{-\sigma}^+\psi_\sigma\rangle - \frac{g}{2}\langle \psi_\sigma^+\psi_{-\sigma}\rangle \delta\langle \psi_{-\sigma}^+\psi_\sigma\rangle \Big\} - T\delta S$$

equating $\delta F = \delta F_{ef}$ get the "optimised" choice:

$$\Delta(r) = g\langle \hat{\psi}_{-\sigma}(r)\,\hat{\psi}_\sigma(r)\rangle \longleftarrow \text{"Gap equation"}$$

$$W(r) = g\langle \hat{\psi}_\sigma^+(r)\,\hat{\psi}_\sigma(r)\rangle$$

$$F(r) = -g\langle \hat{\psi}_{-\sigma}^+(r)\,\hat{\psi}_\sigma(r)\rangle$$

NOTES:

i) if $F \neq 0$ then spin is not conserved in system (not what we have in mind)

↳ ASSUME $F = 0$

ii) $\Delta(r)$ can be identified with the pairing wavefunction suggested earlier.

iii) $W(r)$ is usually omitted → it is not very important for superfluidity. (see below)

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The full description then involves $\hat{K}_{ef}$ and the Gap equation

they must be solved self-consistently.

UNIFORM GAS

Better in k-space

$$\hat{\psi}_\sigma(\vec{r}) = \frac{1}{(2\pi)^{3/2}} \int d^3\vec{k}\, e^{i\vec{k}\cdot\vec{r}}\, \hat{\psi}_\sigma(k)$$

$$\Delta(r) = \text{const.} = \Delta$$

obtain:

$$\hat{K}_{ef} = \int d^3\vec{k} \left\{ \mathcal{E}(k)\, \hat{\varphi}^{+}_\sigma(k)\, \hat{\varphi}_\sigma(k) + \frac{\Delta}{2} \hat{\varphi}^{+}_\sigma(k)\, \hat{\varphi}^{+}_{-\sigma}(-k) + \frac{\Delta^*}{2} \hat{\varphi}_{-\sigma}(k)\, \hat{\varphi}_\sigma(-k) \right\}$$

with $\mathcal{E}(k) = \frac{\hbar^2 k^2}{2m} - \mu + W$

W is seen as a harmless correction to μ

Note the pairing of k & −k

in creation operators : $\hat{\varphi}_{\sigma}(k) \sim \frac{\hat{a}_{\sigma k}}{\sqrt{(\Delta k)^3}}$

$$\hat{K}_{ef} = \sum_k \left\{ \varepsilon(k) \sum_\sigma a^+_{\sigma k} a_{\sigma k} + \Delta\, a^+_{\uparrow k} a^+_{\downarrow -k} + \Delta^* a_{\downarrow -k} a_{\uparrow k} \right\}$$

$$= \sum_k \left[a^+_{\uparrow k} ,\ a_{\downarrow -k} \right] \Omega \begin{bmatrix} a_{\uparrow k} \\ a^+_{\downarrow, -k} \end{bmatrix} + C$$

$$C = \sum_k \varepsilon(k)$$

$$\Omega = \begin{bmatrix} \varepsilon_k & \Delta \\ \Delta^* & -\varepsilon_k \end{bmatrix}$$

and Gap equation is

$$\Delta = g \sum_k \langle a_{\downarrow -k}\, a_{\uparrow k} \rangle$$

$\hat{K}_{ef}$ can be diagonalised into

$$\hat{K}_{ef} = \sum_k E_k \left[b^+_{\uparrow k} b_{\uparrow k} + b^+_{\downarrow -k} b_{\downarrow -k} \right] + \tilde{C}$$

with $\tilde{C} = \sum_k (\varepsilon(k) - E_k)$

$E_k = \sqrt{\varepsilon(k)^2 + \Delta^2}$ from eigenvalues of Ω

by the Bogoliubov transformation

$$\begin{cases} a_{\uparrow k} = b_{\uparrow k} u_k \mp b^+_{\downarrow -k} v_k \\ a_{\downarrow -k} = b_{\uparrow k} v_k + b^+_{\downarrow -k} u_k \end{cases}$$

with fermi operators $\hat{b}_{\sigma k}$

~~eigenvectors~~ commutation relations on a, a^+

give $\cancel{u_k v_m^x - v_k u_m^x = 0}$

$$u_k u_m^x + v_k v_m^x = \delta_{km}$$

squaring eigenvalue eqn. for u_k obtain

$$(E_k u_k)^2 = (\varepsilon_k u_k + \Delta v_k)^2$$

$$(\varepsilon_k^2 + \Delta^2) u_k^2 = \varepsilon_k^2 u_k^2 + \Delta^2 v_k^2 + 2\varepsilon_k \Delta u_k v_k$$

$\hookrightarrow$ obtain

$$u_k v_k = \frac{\Delta}{2E_k}$$

$$u_k^2 = \frac{1}{2}\left(1 + \frac{\varepsilon_k}{E_k}\right)$$

$$v_k^2 = \frac{1}{2}\left(1 - \frac{\varepsilon_k}{E_k}\right)$$

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this leads to

$$\Delta = -g \sum_k \frac{\Delta}{2\sqrt{\varepsilon_k^2 + \Delta^2}}$$ Gap equation

using the fact that we are in the ground state and $\hat{b}_{\sigma k} |GS\rangle = 0$

this can be solved to give

$$\Delta \cong \left(\frac{2}{e}\right)^{7/3} E_F \exp\left[- \frac{\pi^{1/3}}{2|a| (6n)^{1/3}} \right]$$

where $a =$ scattering length

$$g = \frac{4\pi\hbar^2 a}{m}$$

also $T_c \approx 0.28\, E_F \exp[\ldots]$

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GROUND STATE $|GS\rangle$

HAS NO BOGOLIUBOV EXCITATIONS.

$\therefore$ want $\hat{b}_{\sigma k} |GS\rangle = 0 \quad \forall \sigma, k.$

the following construction works:

$$|GS\rangle \sim \prod_k \hat{b}_{\sigma k} \hat{b}_{-\sigma -k} |0\rangle$$

vacuum

$$= \alpha \prod_k (-v_k) \left[u_k + v_k a^+_{-\sigma -k} a^+_{\sigma k} \right] |0\rangle$$

irrelevant factor

$$|GS\rangle = \alpha \prod_k \left[u_k + v_k a^+_{-\sigma -k} a_{\sigma k} \right] |0\rangle$$

vacuum

cooper pair

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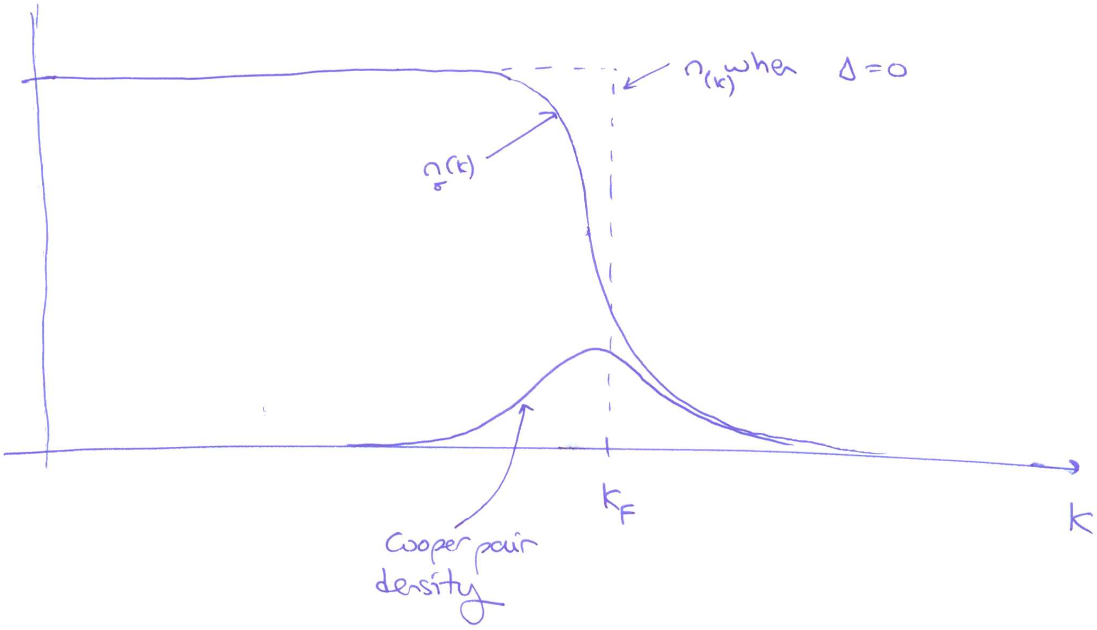
Fermion density $n_\sigma(k) = \langle a^+_{\sigma k} a_{\sigma k} \rangle = v_k^2$

Cooper pair density $|\langle a_{\sigma k} a_{\bar{\sigma} -k} \rangle|^2 = u_k^2 v_k^2$



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ARTICLES

Vortices and superfluidity in a strongly interacting Fermi gas

M. W. Zwierlein[1], J. R. Abo-Shaeer[1]†, A. Schirotzek[1], C. H. Schunck[1] & W. Ketterle[1]

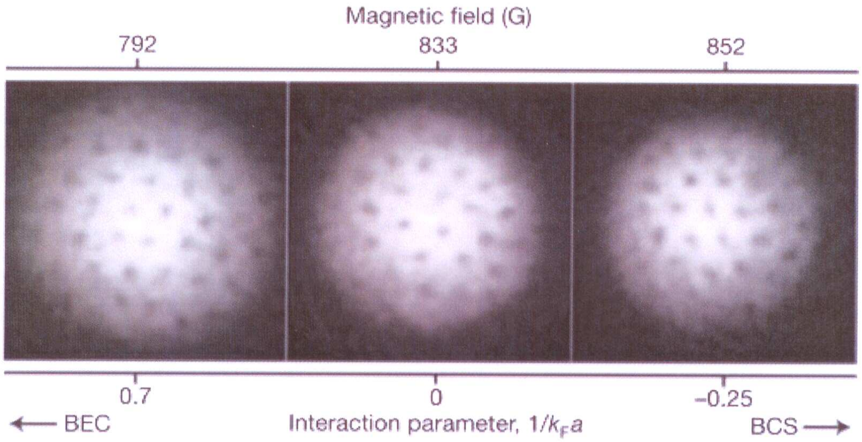

Figure 3 | Optimized vortex lattices in the BEC–BCS crossover. After a vortex lattice was created at 812 G, the field was ramped in 100 ms to 792 G (BEC-side), 833 G (resonance) and 853 G (BCS-side), where the cloud was held for 50 ms. After 2 ms of ballistic expansion, the magnetic field was ramped to 735 G for imaging (see text for details). The field of view of each image is 880 μm × 880 μm.