IAMP News Bulletin
January 2019

International Association of Mathematical Physics

International Association of Mathematical Physics News Bulletin, January 2019

Contents

Contacts. http://www.iamp.org and e-mail: bulletin@iamp.org

Cover picture: Aarhus University, the site of QMath14

ISSN 2304-7348
News Bulletin (International Association of Mathematical Physics)

Nonlinear Gibbs measure and Bose gases at positive temperature

by PHAN THANH NAM (Munich)

A central problem in mathematical physics is to understand the Bose-Einstein phase transition at the critical temperature from first principles. Most of the existing works focus on Bose gases at zero or very low temperature, where complete condensation is expected. Recently, with Mathieu Lewin and Nicolas Rougerie, we initiated a study of equilibrium Bose gases by approaching the critical temperature from above. It is interesting that we obtain, in a mean-field limit, a superposition state of Bose-Einstein condensates in terms of the nonlinear Gibbs measure.

1 Introduction

Since the first experimental realizations of the Bose-Einstein condensation in 1995 [12, 28], it has been a central problem in mathematical physics to understand the macroscopic behaviors of Bose gases from first principles (i.e. from the many-body Schrödinger Hamiltonian). In particular, while the phase transition at the critical temperature has been understood for the ideal gas from the pioneer work of Bose and Einstein in 1924 [7, 17], extending the analysis to realistic interacting gases remains a major open problem.

From the standard setting in physics, it would be desirable to consider the Bose gases in the thermodynamic limit, but this is very difficult. On the other hand, in the last decades substantial progress has been made in the mean-field limit: this is the situation when the interaction is sufficiently weak such that the interaction energy is comparable to the kinetic energy, leading to well-defined effective theories in the large particle number limit. Among several impressive results, let us mention the rigorous derivations of the non-linear Schrödinger/Gross-Pitaevskii equation in 2000s by Lieb-Seiringer-Yngvason for the ground state [42, 40] and by Erdős-Schlein-Yau [18] for the dynamics. Refinements of the non-linear Schrödinger description, i.e. Bogoliubov's approximation [6], have also been derived recently, see [47, 24, 38, 15, 4] for the low-lying eigenfunctions and [25, 37, 5, 9] for the dynamics. We refer to [41, 46, 22, 45, 32, 2] for reviews.

So far most of the existing works focus on zero or very low temperature states of the Bose gas (well below the Bose-Einstein critical temperature), for which the complete condensation is expected. In 2015, with Lewin and Rougerie [34] we initiated a study of equilibrium Bose gases at above the critical temperature, with the new result being that a convex combination of Bose-Einstein condensates emerges in terms of the Gibbs measure based on the nonlinear Schrödinger energy functional.

The nonlinear Gibbs measure have played a central role in constructive quantum field theory in 1970s, starting from the work of Nelson [44], Glimm-Jaffe [21] and Guerra-Rosen-Simon [26] (see [48, 21, 14] for reviews). Since the works of Lebowitz-Rose-Speer [31], Bourgain [8] and Burq-Thomann-Tzvetkov [10] in 1990s, the Gibbs measure has become a popular tool to study the Cauchy theory for nonlinear Schrödinger equations (or more generally nonlinear dispersive equations) with low-regular data. The Gibbs measure also serves in the background of the recent studies of nonlinear stochastic PDEs, see e.g. Prato-Debussche [13], Lörinczi-Gubinelli [43] and Hairer [27].

What makes the nonlinear Gibbs measure both useful and difficult is that the measure is very singular - it is supported on distributions with low regularity. On the other hand, the linear quantum problem (from first principles) is perfectly regular but it involves noncommutative operators. The derivation of the Gibbs measure from the many-body quantum problem thus requires a rigorous semiclassical analysis in infinite dimensions.

In [34] we could solve the problem only in one dimension. Higher dimensional cases turned out to be extremely difficult, due to the need of subtle renormalization procedures. In 2016, Fröhlich-Knowles-Schlein-Sohinger came up with a groundbreaking work [19] where they derived the renormalized Gibbs measure from a modified quantum model in 2D and 3D. But the question was not solved fully. Very recently, we managed to resolve the full problem in 2D [36]. The 3D case, which is most physically interesting, remains open. The details of the progress will be discussed below.

Mathematical setting. We consider a Bose gas in a domain $\Omega \subset \mathbb{R}^d$ described by the grand-canonical Hamiltonian

$$\mathbb{H}_\lambda = \int_\Omega a_x^*(h-\nu)a_x dx + \frac{\lambda}{2} \iint_{\Omega\times\Omega} a_x^* a_y^* w(x-y) a_x a_y dx dy$$

on the bosonic Fock space

$$\mathfrak{F} = \mathbb{C} \oplus L^2(\Omega) \oplus \ldots \oplus L^2_{\text{sym}}(\Omega^n) \oplus \ldots .$$

Here $h > 0$ is a self-adjoint operator with compact resolvent (the reader might think of the case $h = -\Delta + \text{const}$ in a bounded domain with appropriate boundary conditions, or $h = -\Delta + V(x)$ in $\mathbb{R}^d$), $\nu \in \mathbb{R}$ a chemical potential, and $\lambda \geq 0$ a coupling constant used to adjust the strength of the interaction (the case $\lambda = 0$ corresponds to the idea gas). The field operators a_x^*, a_x are the usual creation/annihilation operators on Fock space which satisfy the canonical commutation relations

$$[a_x, a_y] = 0 = [a_x^*, a_y^*] = 0, \quad [a_x, a_y^*] = \delta(x-y).$$

We assume that the interaction is repulsive, i.e. $w \geq 0$, so that $\mathbb{H}_\lambda$ can be defined as a positive self-adjoint operator on Fock space by Friedrichs method.

The equilibrium state of the Bose gas at a positive temperature $T > 0$ is

$$\Gamma_\lambda = \mathcal{Z}_\lambda^{-1} \exp\left(-\mathbb{H}_\lambda / T\right), \quad \mathcal{Z}_\lambda = \operatorname{Tr}\left[\exp\left(-\mathbb{H}_\lambda / T\right)\right]. \tag{1.1}$$

This so-called *Gibbs state* is the minimizer of the free-energy functional

$$\mathcal{F}_{\lambda,T}[\Gamma] = \operatorname{Tr}\left[\mathbb{H}_\lambda \Gamma\right] + T\operatorname{Tr}[\Gamma \log \Gamma]$$

over all quantum states Γ on the Fock space (i.e. $\Gamma \geq 0$, $\operatorname{Tr}\Gamma = 1$). We are interested in the large temperature/mean-field limit

$$T \to \infty, \quad \lambda = T^{-1},$$

which allows us to rewrite

$$\frac{\mathbb{H}_\lambda}{T} = \int_\Omega b_x^*(-\Delta_x - \nu)b_x\, dx + \frac{1}{2}\iint\limits_{\Omega\times\Omega} b_x^* b_y^* w(x-y) b_x b_y\, dx\, dy$$

with $b_x = a_x/\sqrt{T}$. Since the new field operators commute in the large T limit, the formal semiclassical approximation suggests to replace the quantum fields by classical fields, i.e. to replace operators b_x, b_x^* by functions $u(x), \overline{u(x)}$. This results in a probability measure formally given by

$$d\mu(u) = \text{“}z^{-1}e^{-\mathcal{E}_{\rm H}[u]}\, du\text{”}$$

with the nonlinear Hartree/Schrödinger energy functional

$$\mathcal{E}_{\rm H}[u] = \int_\Omega \overline{u(x)}(h_x - \nu)u(x)\, dx + \frac{1}{2}\iint\limits_{\Omega\times\Omega} |u(x)|^2\, w(x-y)|u(y)|^2\, dx\, dy.$$

In fact, the so-obtained *Gibbs measure* μ is supported outside the energy space of the Schrödinger functional, namely $\mathcal{E}_{\rm H}[u]$ is infinity μ-almost surely. The proper definition of μ is nontrivial, but it has been well understood since 1970s. The main issue in our study is to relate the quantum Gibbs state to the classical Gibbs measure in a rigorous manner.

2 The ideal gas

Let us consider the non-interacting case $\lambda = 0$. Since $h > 0$ has compact resolvent, its has eigenvalues $\lambda_j \to \infty$ with eigenfunctions u_j's. The free Gibbs measure (with $\nu = 0$) is the infinite-dimensional Gaussian measure

$$d\mu_0(u) = \text{“}z_0^{-1}e^{-\langle u, hu\rangle}\, du\text{”} = \bigotimes_{j=1}^{\infty}\left(\frac{\lambda_j}{\pi}e^{-\lambda_j|\alpha_j|^2}\, d\alpha_j\right), \quad \alpha_j = \langle u_j, u\rangle.$$

Here $d\alpha_j$ is the usual Lebesgue measure on $\mathbb{C} \simeq \mathbb{R}^2$. By [50, Lemma 1], μ_0 is well-defined uniquely if its cylindrical projection $\mu_{0,K}$ on $V_K = \mathrm{Span}(u_1, ..., u_K)$ satisfies the tightness condition

$$\lim_{R\to\infty}\sup_K \mu_{0,K}\big(\{u \in V_K \ :\ \|u\| \geq R\}\big) = 0.$$

This condition is verified in the (possibly negative) Sobolev space

$$\mathfrak{H}^{1-p} = \left\{ u = \sum_{j=1}^{\infty}\alpha_j u_j \ :\ \sum_{j=1}^{\infty}\lambda_j^{1-p}|\alpha_j|^2 < \infty \right\}$$

provided that $[h^{-p}] < \infty$ for some $p \geq 1$. By contrast, if $\mathrm{Tr}[h^{-q}] = \infty$, then μ_0 is supported outside $\mathfrak{H}^{1-q}$, i.e. $\mu_0(\mathfrak{H}^{1-q}) = 0$. This zero-one law follows from Fernique's theorem.

For examples, if $h = -\Delta + \mathrm{const}$ on a bounded domain $\Omega \subset \mathbb{R}^d$, then μ_0 is supported on $\mathfrak{H}^s$ for any $s < 1 - d/2$ (in particular, $s < 0$ if $d \geq 2$). If $h = -\Delta + |x|^2$ (harmonic oscillator), then μ_0 is supported on the negative Sobolev spaces $\mathfrak{H}^s$ with $s < 1 - d \leq 0$.

There is a one-to-one correspondence between h and μ_0 via the formula [34, Lemma 3.3]

$$k!\,(h^{-1})^{\otimes k} = \int |u^{\otimes k}\rangle\langle u^{\otimes k}|\,d\mu_0(u), \quad \forall k \geq 0.$$

Here for any u, $|u^{\otimes k}\rangle\langle u^{\otimes k}|$ is only a bounded operator from $(\mathfrak{H}^{1-p})^{\otimes k}$ to $(\mathfrak{H}^{p-1})^{\otimes k}$. However, averaging over the measure μ_0 results in a bounded (indeed compact) operator on the original Hilbert space $\mathfrak{H} = L^2(\Omega)$.

Now we turn to the quantum Gibbs state. The free Gibbs state

$$\Gamma_0 = \mathcal{Z}_0^{-1} \exp\left(-\mathbb{H}_0/T\right)$$

is a quasi-free state and Wick's theorem tells us that

$$-\log \mathcal{Z}_0 = \mathrm{Tr}\log(1 - e^{-h/T})$$

and

$$\Gamma_0^{(k)} = \left(\frac{1}{e^{h/T}-1}\right)^{\otimes k}$$

(projected on the symmetric subspace). Recall that for every quantum state Γ on Fock space and for every $k \geq 1$, $\Gamma^{(k)}$ is a trace class operator on $L^2_{\rm sym}(\Omega^k)$ with kernel

$$\Gamma^{(k)}(x_1,...,x_k;y_1,...,y_k) = (k!)^{-1}\mathrm{Tr}\left[a_{x_1}^*...a_{x_k}^* a_{y_1}...a_{y_k}\Gamma\right] .$$

Thus the free Gibbs state can be related to the free Gibbs measure in the large T limit:

$$\frac{k!}{T^k}\Gamma_0^{(k)} \underset{T\to\infty}{\longrightarrow} k!\,(h^{-1})^{\otimes k} = \int |u^{\otimes k}\rangle\langle u^{\otimes k}|\,d\mu_0(u).$$

The convergence holds strongly in the Schatten space $\mathfrak{S}^p$ by dominated convergence. Our goal is to derive a similar convergence for the interacting gas.

3 Interacting gas in one-dimension

A natural candidate for the interacting Gibbs measure (with $\nu = 0$) is

$$d\mu(u) = \text{“}z^{-1}e^{-\mathcal{E}_{\rm H}[u]}du\text{”} = \text{“}z^{-1}e^{-\mathcal{I}[u]}e^{-\langle u, hu\rangle}du\text{”} = z_r^{-1}e^{-\mathcal{I}[u]}d\mu_0(u)\,, \tag{3.1}$$

where

$$\mathcal{I}[u] = \frac{1}{2}\iint_{\Omega^2} |u(x)|^2 w(x-y)|u(y)|^2\,dxdy \geq 0.$$

In fact, μ is well-defined in this way if $\mathcal{I}[u]$ is finite on a set of μ_0-positive measure.

For example [34], if $h = -\Delta + \mathrm{const}$ on a bounded interval $\Omega \subset \mathbb{R}$ and $w = w_1 + w_2$ with $0 \leq w_1 \in L^\infty$ and w_2 a positive measure with finite mass, then $\mathcal{I}[u]$ is finite μ_0-almost surely since

$$\int \mathcal{I}[u]d\mu_0(u) = \frac{1}{2}\mathrm{Tr}[wh^{-1}\otimes h^{-1}] < \infty.$$

(The trace is taken over symmetric subspace.)

The convergence of the interacting Gibbs state Γ_λ in (1.1) to the so-defined Gibbs measure μ in (3.1) is the content of the following result [34, Theorem 5.3]. Here we omit the chemical potential ν (it is absorbed into h).

Theorem 3.1 (Trace class case)**.** *Let $h > 0$ and $w \geq 0$ satisfy*

$$\mathrm{Tr}[h^{-1}] + \mathrm{Tr}[w\, h^{-1} \otimes h^{-1}] < \infty. \tag{3.2}$$

Then in the limit $T = \lambda^{-1} \to \infty$, we have the convergence of the relative partition function

$$\frac{\mathcal{Z}_\lambda}{\mathcal{Z}_0} \to z_r := \int e^{-\mathcal{I}[u]} d\mu_0(u) \in (0, 1]$$

and the convergence of reduced density matrices

$$\frac{k!}{T^k} \Gamma^{(k)}_{\lambda,T} \to \int_{\mathfrak{H}} |u^{\otimes k}\rangle\langle u^{\otimes k}|\, d\mu(u), \quad \forall k \geq 1$$

strongly in trace-class.

When $\mathfrak{H}$ is a finite-dimensional space, a version of this theorem was proved by Gottlieb [23] (see also [29, 45]). Extending the result to infinite dimensions requires a completely new proof which we explain below.

Variational approach. Our strategy in [34] is based on Gibbs' variational principle. More precisely, if we take the free Gibbs state Γ_0 as a reference state, then Γ_λ is the unique minimizer for the *relative free energy functional*

$$-\log \frac{\mathcal{Z}_\lambda}{\mathcal{Z}_0} = \frac{\mathcal{F}_{\lambda,T}(\Gamma_\lambda) - \mathcal{F}_{0,T}(\Gamma_0)}{T} = \inf_{\substack{\Gamma \geq 0 \\ \mathrm{Tr}_{\mathfrak{F}} \Gamma = 1}} \Big(\mathcal{H}(\Gamma, \Gamma_0) + T^{-2} \mathrm{Tr}[w \Gamma^{(2)}_\lambda] \Big)$$

with the relative entropy

$$\mathcal{H}(\Gamma, \Gamma') = \mathrm{Tr}_{\mathfrak{F}}\big(\Gamma(\log \Gamma - \log \Gamma')\big) \geq 0.$$

Similarly, the Gibbs measure μ is the unique minimizer for the problem

$$-\log z_r = \inf_{\nu \text{ probability measure}} \Big(\mathcal{H}_{\mathrm{cl}}(\nu, \mu_0) + \int \mathcal{I}[u]\, d\nu(u) \Big)$$

with the classical relative entropy

$$\mathcal{H}_{\mathrm{cl}}(\nu, \nu') := \int \frac{d\nu}{d\nu'}(u) \log\left(\frac{d\nu}{d\nu'}(u)\right) d\nu'(u) \geq 0.$$

Our task is to relate the quantum variational problem to the classical one. The upper bound

$$\limsup_{T\to\infty} \left(-\log \frac{\mathcal{Z}_\lambda}{\mathcal{Z}_0} \right) \leq -\log z_r$$

follows from a suitable trial state and well-known techniques for semiclassical analysis in finite-dimensions [39, 49]. The matching lower bound is much more challenging.

De Finetti measure. A variant of the quantum de Finetti theorem in Fock space [34, Theorem 4.2] (whose proof goes back to the analysis of [1, 33]) states that if a sequence of quantum states $\{\Gamma_n\}$ satisfies the a priori bound

$$\limsup_{T_n\to\infty} \mathrm{Tr}\Big|\frac{\Gamma_n^{(k)}}{T_n^k}\Big|^p < \infty, \quad \forall k\ge 1, \tag{3.3}$$

then up to a subsequence of T_n, there exists a Borel probability measure ν on $\mathfrak{H}^{1-p}$ such that

$$\frac{k!}{T_n^k}\Gamma_n^{(k)} \rightharpoonup \int_{\mathfrak{H}^{1-p}} |u^{\otimes k}\rangle\langle u^{\otimes k}|\, d\nu(u)$$

weakly-$*$ in Schatten space $\mathfrak{S}^p$ for every $k\ge 1$. The measure ν is called the de Finetti measure (or Wigner measure) of the quantum states $\{\Gamma_n\}$ with scale T_n^{-1}.

For example, from the direct analysis for the ideal gas, we know that the free Gibbs measure μ_0 is the de Finetti measure of the free Gibbs states Γ_0 with scale T^{-1}. Moreover, we can check that the interacting Gibbs states $\{\Gamma_{\lambda_n}\}$ satisfy the a priori bound (3.3) with $p=1$, thanks to the trace-class condition (3.2). In fact, the reduced density matrices can be connected to the particle number operator on Fock space

$$\mathcal{N} = \int_\Omega a_x^* a_x dx$$

via the formula

$$\mathrm{Tr}[\Gamma_\lambda^{(k)}] = \mathrm{Tr}\left[\binom{\mathcal{N}}{k}\Gamma_\lambda\right] = \frac{1}{k!}\mathrm{Tr}\left[\mathcal{N}(\mathcal{N}-1)...(\mathcal{N}-k+1)\Gamma_\lambda\right].$$

Therefore, the desired estimate (3.3) with $p=1$ is equivalent to a uniform bound on $\mathrm{Tr}[(\mathcal{N}/T)^k\Gamma_\lambda]$, which can be derived from the corresponding bound on the free Gibbs states using the monotonicity $\mathbb{H}_\lambda \ge \mathbb{H}_0$ and the fact that $\mathcal{N}$ commutes with all relevant operators.

Thus up to a subsequence $T_n\to\infty$, there exists a de Finetti measure ν such that

$$\frac{k!}{T_n^k}\Gamma_{\lambda_n}^{(k)} \rightharpoonup \int |u^{\otimes k}\rangle\langle u^{\otimes k}|\, d\nu(u), \quad \forall k\ge 1$$

weakly-* in trace class.

Lower bounds. From the weak convergence of the two-particle density matrices $\Gamma_{\lambda_n}^{(2)}$, it is fairly easy to prove

$$\liminf_{n\to\infty} T_n^{-2}\mathrm{Tr}\left[w\Gamma_{\lambda_n}^{(2)}\right] \ge \frac12\int \langle u^{\otimes 2}, w\, u^{\otimes 2}\rangle\, d\nu(u) = \int \mathcal{I}[u]d\nu(u)$$

by some sort of Fatou's lemma.

Moreover, it turns out that the de Finetti measure is linked nicely with the relative entropy via a Berezin-Lieb-type inequality [34, Theorem 7.1], whose proof goes back to techniques in [3, 39, 49]. In particular, we have

$$\liminf_{n\to\infty} \mathcal{H}(\Gamma_{\lambda_n}, \Gamma_0) \geq \mathcal{H}_{\rm cl}(\nu, \mu_0).$$

Thus from the variational formulas we find that

$$\liminf_{n\to\infty} \left(-\log \frac{\mathcal{Z}_{\lambda_n}}{\mathcal{Z}_0}\right) \geq \mathcal{H}_{\rm cl}(\nu, \mu_0) + \int \mathcal{I}[u]\, d\nu(u) \geq -\log z_r.$$

Conclusion. Combining the upper bound and the matching lower bound, we obtain the convergence of the relative partition function and also conclude that $\nu = \mu$ (the interacting Gibbs measure). We can remove the dependence on the subsequence T_n since the limiting objects are unique. We still have to work to upgrade the weak convergence of density matrices to the strong convergence, but let us skip the details.

From the above proof, we see that the expectation of the particle number $\mathrm{Tr}[\mathcal{N}\Gamma_\lambda]$ is proportional to T, and hence the choice of the coupling constant $\lambda = T^{-1}$ in front of the interaction term really places us in the mean-field limit. When h^{-1} is not trace class (as below), the expectation of the particle number of the Gibbs state grows much faster than T and the situation becomes much more complicated.

4 Interacting gas in higher dimensions

Now we consider the case when $\mathrm{Tr}[h^{-p}] < \infty$ for some $p > 1$ (but not for $p = 1$), which is relevant when $h = -\Delta + V(x)$ in higher dimensions.

Renormalized Gibbs measure. When h^{-1} is not trace class, the simple definition of the interacting Gibbs measure in (3.1) does not work since $\mathcal{I}[u]$ is infinity μ_0-almost surely for any nontrivial smooth function w (since μ_0 is supported on negative Sobolev spaces). However, it is well-known how to define the interacting Gibbs measure by renormalization.

The basic idea, going back to Nelson [44], is that although the mass $\int |u|^2$ is infinity μ_0-almost surely, this infinity is the same for almost every u. The renormalized interaction energy measure is thus defined formally by

$$\mathcal{D}[u] = \text{“}\frac{1}{2} \iint\limits_{\Omega^2} \left(|u(x)|^2 - \left\langle |u(x)|^2\right\rangle_{\mu_0}\right) w(x-y) \left(|u(y)|^2 - \left\langle |u(y)|^2\right\rangle_{\mu_0}\right) dxdy\text{”}$$

where $\langle\cdot\rangle_{\mu_0}$ denotes the expectation in the free Gibbs measure μ_0. This formula can be made rigorous by replacing

$$|u(x)|^2 - \left\langle |u(x)|^2\right\rangle_{\mu_0} \rightsquigarrow |P_K u(x)|^2 - \left\langle |P_K u(x)|^2\right\rangle_{\mu_0}$$

with $P_K = \mathbb{1}(h \leq K)$ and then taking the limit $K \to \infty$.

The renormalized interacting Gibbs measure is defined by

$$d\mu(u) := z_r^{-1} e^{-\mathcal{D}[u]} d\mu_0(u).$$

When the interaction potential w is of positive type, i.e. its Fourier transform $\widehat{w}$ is nonnegative, the renormalized interaction energy $\mathcal{D}[u]$ is nonnegative and μ is a probability measure on the same space of μ_0.

Renormalized quantum model. By taking into account the quantum analogue of the renormalized interaction, we arrive at the Hamiltonian

$$\begin{aligned}\mathbb{H}_\lambda &= \int_\Omega a_x h_x a_x dx + \frac{\lambda}{2}\iint_{\Omega^2} (a_x^* a_x - \varrho_0(x)) w(x-y)(a_y^* a_y - \varrho_0(y))\, dxdy \\ &= \int_\Omega a_x (h_x - \nu) a_x dx + \lambda \iint_{\Omega^2} a_x^* a_y^* w(x-y) a_x a_y dxdy + E\end{aligned}$$

with $\varrho_0(x) = \mathrm{Tr}[a_x^* a_x \Gamma_0]$ the density of the free Gibbs state and

$$\nu := \lambda\Big[(w * \varrho_0)(x) - \frac{w(0)}{2}\Big]. \tag{4.1}$$

(E is an energy shift which is not relevant to the associated Gibbs state.)

Let us focus on the *homogeneous gas*, when $h = -\Delta + \kappa$ on the torus $\Omega = \mathbb{T}^d$ (with $\kappa > 0$ fixed), the free density

$$\varrho_0 = \sum_{k \in (2\pi\mathbb{Z})^d} \frac{1}{e^{\frac{|k|^2+\kappa}{T}} - 1}$$

is independent of x (by the translation invariance). Thus ν defined in (4.1) is a constant, which plays the role of an adjusted chemical potential to compensate the divergence of the interactions (it is proportional to $\log T$ if $d = 2$, and $\sqrt{T}$ if $d = 3$).

We expect that the Gibbs state $\Gamma_\lambda = \mathcal{Z}_\lambda^{-1} e^{-\mathbb{H}_\lambda/T}$ converges to the so-defined renormalized Gibbs measure μ for $d = 2, 3$. In fact, we have succeeded in treating the two dimensional case in [36].

Theorem 4.1 (Homogeneous gas). *Let $h = -\Delta + \kappa$ on the torus $\mathbb{T}^2$ with a constant $\kappa > 0$. Let $w : \mathbb{T}^2 \to \mathbb{R}$ be an even function satisfying*

$$0 \le \widehat{w}(k), \qquad \sum_{k \in (2\pi\mathbb{Z})^2} \widehat{w}(k)\,(1 + |k|^\alpha) < \infty \quad \textit{for some } \alpha > 0.$$

In the limit $T = \lambda^{-1} \to \infty$ we have $\mathcal{Z}_\lambda/\mathcal{Z}_0 \to z_r \in (0, 1]$ and for all $k \ge 1$,

$$\frac{k!}{T^k}\Gamma_\lambda^{(k)} \to \int |u^{\otimes k}\rangle\langle u^{\otimes k}| d\mu(u)$$

strongly in Schatten space $\mathfrak{S}^p$ for any $p > 1$. Moreover, for $k = 1$ we have

$$\frac{1}{T}\left(\Gamma_\lambda^{(1)} - \Gamma_0^{(1)}\right) \to \int |u\rangle\langle u| \Big(d\mu(u) - d\mu_0(u)\Big)$$

strongly in trace-class.

The proof of Theorem 4.1 is a big jump in difficulty in comparison to Theorem 3.1. Conceptually, now the Gibbs measure is very singular (outside the L^2 space) and the semiclassical limit becomes really complicated. Technically, the chemical potential ν diverges to infinity as $T \to \infty$, ruling out the compactness arguments we used previously for Theorem 3.1.

Thus we have to develop quantitative estimates. It is known that the quantum de Finetti Theorem can be made quantitative: if the one-body Hilbert space is restricted to be finite-dimensional, then the error depends linearly on the dimension [11]. Therefore, we need to project our states to finite dimensional subspaces. Controlling the projection error is the main challenge and it requires new correlation estimates.

Localization of quantum variances. Let us take the spectral projections

$$P = \mathbb{1}(h \le \Lambda), \quad Q = \mathbb{1}(h > \Lambda)$$

for some energy cut-off Λ. We are after an estimate of the form

$$\left\langle \left(\langle \mathcal{N}_Q - \langle \mathcal{N}_Q \rangle_{\Gamma_0} \right)^2 \right\rangle_{\Gamma_\lambda} \le o(T^2), \quad \mathcal{N}_Q = \int_\Omega a_x^* Q_x a_x dx. \tag{4.2}$$

This is a simplified version of what we need to localize the renormalized interaction energy to the low energy modes (indeed we will decompose the interaction potential w in Fourier space and need a version of (4.2) with Q replaced by $Qe^{ik\cdot x}Q$ for any $k \in (2\pi\mathbb{Z})^2$, but let us skip this technical detail). Heuristically, (4.2) means that in the high energy modes, the particles move too fast and the effect of the interaction becomes negligible. Thus the larger Λ, the smaller the localization error. On the other hand, Λ cannot be too large, since we need $\mathrm{Tr} P \ll T$ to control the error of the quantitative quantum de Finetti theorem.

In turns out that in the eligible range of Λ, the expectation of particle number in the high energy modes $\langle \mathcal{N}_Q \rangle_{\Gamma_\lambda}$ grows much faster than T. Therefore, to verify (4.2) it is important to take the cancelation by $\langle \mathcal{N}_Q \rangle_{\Gamma_0}$ into account. The justification of (4.2) needs several new ideas.

Step 1. Our first key estimate is the operator inequality

$$\mathrm{Tr} \left| h^\alpha \frac{\Gamma_\lambda^{(1)} - \Gamma_0^{(1)}}{T} h^\alpha \right| \le C_\alpha \ ,$$

which confirms the physical intuition that the free and interacting Gibbs states do not differ much in the high energy modes. This bound follows from the Feynman-Hellmann principle and a new inequality for the relative entropy. More precisely, by using a perturbation of the variational principle defining the free Gibbs state with h replaced by $h - A$, we obtain

$$\mathrm{Tr}\Big(A\big(\Gamma_\lambda^{(1)} - \Gamma_0^{(1)}\big)\Big) \le \mathcal{H}(\Gamma_\lambda, \Gamma_0) + \mathrm{Tr}\left(A \left(\frac{1}{e^{h-A} - 1} - \frac{1}{e^h - 1} \right)\right)$$

for any one-body self-adjoint operator A satisfying $A \le ch$ with some constant $c < 1$. This implies the desired result by an appropriate choice of A and a Klein-type inequality.

Consequently, we deduce an easier version of (4.2):

$$|\langle \mathcal{N}_Q \rangle_{\Gamma_\lambda} - \langle \mathcal{N}_Q \rangle_{\Gamma_0}| \le o(T).$$

What makes (4.2) much more difficult than the latter estimate is the fact that the Feynman-Hellmann principle does not work with two-body perturbations (two-body perturbations, with a minus sign, may easily destroy the positivity of the Hamiltonian).

Step 2. Our key idea in this step is to reduce two-body estimates to a one-body estimate. More precisely, we rewrite (4.2) as the true variance estimate

$$\left\langle \left(\langle \mathcal{N}_Q - \langle \mathcal{N}_Q \rangle_{\Gamma_\lambda} \right)^2 \right\rangle_{\Gamma_\lambda} \leq o(T^2) \ ,$$

thanks to the previous step, and then approximate the variance by the *linear response* of $\mathcal{N}_Q$:

$$T^{-2} \left\langle \left| \mathcal{N}_Q - \langle \mathcal{N}_Q \rangle_{\Gamma_\lambda} \right|^2 \right\rangle_{\Gamma_\lambda} = T^{-1} \partial_\varepsilon \left(\langle \mathcal{N}_Q \rangle_{\Gamma_{\lambda,\varepsilon}} \right)_{|\varepsilon=0} + o(1) \ , \tag{4.3}$$

where

$$\Gamma_{\lambda,\varepsilon} := \mathcal{Z}_{\lambda,\varepsilon}^{-1} \exp\left(-\frac{1}{T} \left(\mathbb{H}_\lambda - \varepsilon \mathcal{N}_Q \right) \right) .$$

If $\mathcal{N}_Q$ is replaced by an observable $\mathbb{A}$ which commutes with $\mathbb{H}_\lambda$, then we have exactly [30]

$$T^{-2} \langle |\mathbb{A} - \langle \mathbb{A} \rangle_{\Gamma_\lambda}|^2 \rangle_{\Gamma_\lambda} = T^{-1} \partial_\varepsilon \left(\langle \mathbb{A} \rangle_{\Gamma_{\lambda,\varepsilon}} \right)_{|\varepsilon=0} \ .$$

Of course, $\mathcal{N}_Q$ does not commute with $\mathbb{H}_\lambda$ and we have to work to establish (4.3). Nevertheless, the commutator is small in our semiclassical limit, allowing us to control the error in (4.3) as soon as $\mathrm{Tr}[h^{-p}] < \infty$ for some $p < 3/2$. This technical condition holds in 2D but barely fails in 3D.

Step 3. Now the right side of (4.3) is the derivative of a one-body term. In principle, this can be estimated using Taylor's expansion

$$g(\varepsilon) = g(0) + g'(0)\varepsilon + \frac{\varepsilon^2}{2} g''(\theta_\varepsilon)$$

(i.e. we can control $g'(0)$ by $g(\varepsilon) - g(0)$ and g''). What saves the day is that a rough estimate for the second derivative is sufficient for our purpose.

The localization method just discussed allows us to obtain the convergence of the relative partition function. The convergence of reduced density matrices follows from the various estimates developed to prove the energy convergence, plus Pinsker inequalities.

5 Extension to inhomogeneous gas

For the homogeneous gas in Theorem 4.1, the translation invariance simplifies the analysis greatly. Now we turn to the inhomogeneous case. Let us consider the case $h = -\Delta + V$ with a trapping potential V on $\mathbb{R}^d$.

In one dimension and in many physically interesting cases (e.g. the harmonic oscillator $V(x) = |x|^2$), although h^{-1} fails to be trace class and μ_0 is not supported on L^2, it turns out that μ_0 is supported on L^4 and the bare interaction energy $\mathcal{I}[u]$ is still finite μ_0-almost surely. Thus the interacting Gibbs measure is well-defined without renormalization and the problem has been solved in [35].

In higher dimensions, the renormalized Gibbs measure can be defined similarly to the homogeneous case, but now it is more subtle to define the quantum model. When h is not translation invariant, the free density

$$\varrho_0(x) = \Gamma_0^{(1)}(x;x) = \frac{1}{e^{h/T}-1}(x;x)$$

depends on x, and hence the counter term ν defined in (4.1) is not just a chemical potential. This means that Γ_0 is not the right reference state to study Γ_λ.

As discussed by Fröhlich-Knowles-Schlein-Sohinger [19], this issue can be fixed as follows. By changing the reference state to the Gibbs state associated with a general one-body potential V_T, we arrive at the Hamiltonian

$$\mathbb{H}_\lambda = \int_\Omega a_x(-\Delta_x + V_T(x) - \lambda(w * \varrho_0^{V_T})(x))a_x dx + \frac{\lambda}{2}\iint_{\Omega^2} a_x^* a_y^* w(x-y) a_x a_y dx dy\ ,$$

where $\varrho_0^{V_T}(x)$ is the density of the new free Gibbs state

$$\varrho_0^{V_T}(x) = (e^{\frac{-\Delta+V_T}{T}} - 1)^{-1}(x;x).$$

(We ignore the unimportant factor $w(0)/2$ and the energy shift.) The above Hamiltonian coincides with the physical Hamiltonian

$$\mathbb{H}_\lambda = \int_\Omega a_x^*(-\Delta_x + V(x) - \nu)a_x dx + \frac{\lambda}{2}\iint_{\Omega\times\Omega} a_x^* a_y^* w(x-y) a_x a_y dx dy$$

for some chemical potential $\nu \in \mathbb{R}$ if

$$V_T - \lambda w * \varrho_0^{V_T} = V - \nu. \tag{5.1}$$

This nonlinear equation has been solved in [19] by a fixed point argument.

As explained later in [36], the counter-term problem (5.1) also emerges from the equilibrium state of the *nonlinear Hartree free energy functional*

$$\mathcal{F}^{\mathrm{rH}}[\gamma] := \mathrm{Tr}\,[(-\Delta + V - \nu)\gamma] + \frac{\lambda}{2}\iint \gamma(x;x) w(x-y)\gamma(y;y)\,dx\,dy \\ - T\,\mathrm{Tr}\,[(1+\gamma)\log(1+\gamma) - \gamma\log\gamma]\ ,$$

which is deduced from the free energy functional of quasi-free states (the exchange term is unimportant and ignored). This reduced functional has a unique minimizer over all trace class operators $\gamma \geq 0$, which solves the self-consistent equation

$$\gamma^{\mathrm{rH}} = \left\{\exp\left(\frac{-\Delta + V - \nu + \lambda\rho_{\gamma^{\mathrm{rH}}} * w}{T}\right) - 1\right\}^{-1}, \quad \rho_{\Gamma^{\mathrm{rH}}}(x) = \gamma^{\mathrm{rH}}(x;x).$$

Consequently, the function $V_T := \lambda\rho_{\gamma^{\rm rH}} * w + V - \nu$ solves the counter-term problem (5.1).

It turns out that the limit $V_\infty = \lim_{T\to\infty} V_T$ exists [19] and it will be the correct renormalized potential showing up in the limiting Gibbs measure. We have the following simplified version of [36, Theorem 3.4].

Theorem 5.1 (Inhomogeneous gas). *Let $h = -\Delta + V$ on $L^2(\mathbb{R}^2)$ with V growing sufficiently fast at infinity (e.g. $V(x) = |x|^s$ with $s \geq 51$). Let w be an even function satisfying*

$$0 \leq \widehat{w}(k), \qquad \int_{\mathbb{R}^2} \widehat{w}(k)\left(1 + |k|^{1/2}\right) dk < \infty, \qquad \int_{\mathbb{R}^2} |w(x)| V(x)^2\, dx < \infty.$$

Consider the Gibbs state $\Gamma_\lambda = \mathcal{Z}_\lambda^{-1} e^{-\mathbb{H}_\lambda/T}$ with the chemical potential $\nu = \lambda\hat{w}(0)\varrho_0^\kappa - \kappa$ for a large, fixed constant $\kappa > 0$. Then when $T = \lambda^{-1} \to \infty$, for all $k \geq 1$,

$$\frac{k!}{T^k}\Gamma_\lambda^{(k)} \to \int |u^{\otimes k}\rangle\langle u^{\otimes k}| d\mu(u) \qquad (5.2)$$

strongly in the Hilbert-Schmidt topology. Here μ is the renormalized interacting Gibbs measure associated with w and $h_\infty = -\Delta + V_\infty$, with V_∞ be the limit of the counter-term potential in (5.1).

In general V_∞ is different from V, but this is unavoidable (if we want to obtain the potential V in the limiting measure μ, we need to change the potential in the quantum problem). Our growth condition on V is not optimal and we expect that the result holds as soon as h^{-1} is a Hilbert-Schmidt operator.

The proof of Theorem 5.1 is based on the same strategy in the homogeneous case, plus extra techniques to deal with the inhomogeneity and the faster divergence of the chemical potential ν (which is now a polynomial divergence in T instead of the logarithmic one).

6 Concluding remarks

Another approach. In parallel to our study [34, 35, 36], Fröhlich-Knowles-Schlein-Sohinger made an important contribution [19] to the subject via a completely different method. Instead of using the variational approach, they computed directly the reduced density matrices of quantum states using Borel summation method for divergent series. In this way, they recovered the result in one dimension (see also [20] for an investigation of the time-dependent problem). They also proved (5.2) in the 2D and 3D cases, but with the modified quantum state

$$\Gamma_\lambda^\eta = \frac{1}{\mathcal{Z}_\lambda^\eta} \exp\left(-\frac{\eta}{2T}\mathbb{H}_0\right) \exp\left(-\frac{\mathbb{H}_\lambda - 2\eta\mathbb{H}_0}{T}\right) \exp\left(-\frac{\eta}{2T}\mathbb{H}_0\right) \qquad (6.1)$$

for a fixed parameter $0 < \eta < 1$.

If operators $\mathbb{H}_0$ and $\mathbb{H}_\lambda$ would commute, the above state would coincide with the normal Gibbs state. However, this is not the case and the modification brought by η gains a crucial commutativity which is important for the perturbative expansion. Nevertheless, the modification disappears in the limit (since all classical objects commute), and the resulting Gibbs measure is the same as in Theorem 5.1.

In our approach, we treat the normal Gibbs state by gaining some commutativity via new correlation estimates, but we can control the limit only in 2D so far.

Connection to thermodynamic limit. By scaling we can reformulate our result to the thermodynamic limit with a fixed temperature.

Let us consider the ideal Bose gas with $h = -\Delta - \nu_L$ on a torus $[0, L]^d$ with $L \to \infty$. By choosing the chemical potential

$$\nu_L = -\kappa L^{-2}$$

with fixed $\kappa > 0$, the density of the free Gibbs state at a fixed temperature $T_0 > 0$ is

$$\rho_0(L) = \frac{\mathrm{Tr}[\mathcal{N}\Gamma_0]}{L^d} = \frac{1}{L^d} \sum_{k \in 2\pi\mathbb{Z}^d} \frac{1}{e^{\frac{k^2+\kappa}{L^2 T_0}} - 1}.$$

When L becomes large, $\rho_0(L)$ converges to

$$\rho_c(T_0) = \begin{cases} +\infty & \text{in } d = 1, 2 \\ (T_0)^{d/2} \int_{\mathbb{R}^3} \frac{1}{e^{|2\pi k|^2} - 1} dk & \text{in } d = 3. \end{cases}$$

which is exactly the critical density for the Bose-Einstein phase transition with respect to the temperature T_0. Thus in our setting, we are approaching the critical density *from below*. Equivalently, we can fix the density and approach the critical temperature *from above*.

In the 2D interacting case, the chemical potential in the torus $[0, L]^2$ behaves as

$$\widehat{w}(0) \frac{\log(L^2 T_0)}{L^2} - \frac{\kappa}{L^2} ,$$

which converges to 0 slightly slower than that of the ideal gas. Thus our result quantifies the effect of the interaction at the Bose-Einstein phase transition in the coupled limit $L \to \infty$, $\nu_L \to 0$ and gives raise to the renormalized Gibbs measure. We expect that similar results hold in 3D (with $\log(L^2 T_0)$ replaced by $L\sqrt{T_0}$), which is very interesting but still out of reach of our methods.

Finally, let us mention the recent important result of Deuchert-Seiringer-Yngvason [16] on the Bose-Einstein transition in a combined thermodynamic and Gross-Pitaevskii limit in 3D. They approached the critical density faster than that in our setting, and obtained the Bose-Einstein condensation of the interacting gas with the same transition temperature and condensate fraction as the ideal gas.

Hopefully, the recent developments in the mean-field/dilute Bose gas will provide crucial insights to the rigorous understanding of the Bose-Einstein condensation for interacting Bose gases in the true thermodynamic limit.

References

[1] Z. Ammari and F. Nier, *Mean field limit for bosons and infinite dimensional phase-space analysis*, Ann. Henri Poincaré, 9 (2008), pp. 1503–1574.

[2] N. BENEDIKTER, M. PORTA, AND B. SCHLEIN, *Effective Evolution Equations from Quantum Dynamics*, Springer Briefs in Mathematical Physics, Springer, 2016.

[3] F. A. BEREZIN, *Convex functions of operators*, Mat. Sb. (N.S.), 88(130) (1972), pp. 268–276.

[4] C. BOCCATO, C. BRENNECKE, S. CENATIEMPO, AND B. SCHLEIN, *Bogoliubov Theory in the Gross-Pitaevskii limit.* arXiv:1801.01389, 2018.

[5] C. BOCCATO, S. CENATIEMPO, AND B. SCHLEIN, *Quantum many-body fluctuations around nonlinear Schrödinger dynamics*, Ann. Henri Poincaré, 18 (2017), pp. 113–191.

[6] N. N. BOGOLIUBOV, *About the theory of superfluidity*, Izv. Akad. Nauk SSSR, 11 (1947), p. 77.

[7] S. BOSE, *Plancks Gesetz und Lichtquantenhypothese*, Z. Phys., 26 (1924), pp. 178–181.

[8] J. BOURGAIN, *Invariant measures for the 2D-defocusing nonlinear Schrödinger equation*, Comm. Math. Phys., 176 (1996), pp. 421–445.

[9] C. BRENNECKE, P. T. NAM, M. NAPIÓRKOWSKI, AND B. SCHLEIN, *Fluctuations of n-particle quantum dynamics around the nonlinear schrödinger equation*, Ann. Inst. Henri Poincaré Anal. Non Linéaire, in press, arXiv:1710.09743.

[10] N. BURQ, L. THOMANN, AND N. TZVETKOV, *Gibbs measures for the non linear harmonic oscillator*, in Journées EDP Évian 2009, 2009.

[11] M. CHRISTANDL, R. KÖNIG, G. MITCHISON, AND R. RENNER, *One-and-a-half quantum de Finetti theorems*, Comm. Math. Phys., 273 (2007), pp. 473–498.

[12] E. A. CORNELL AND C. E. WIEMAN, *Bose-Einstein condensation in a dilute gas, the first 70 years and some recent experiments*, Rev. Mod. Phys., 74 (2002), pp. 875–893.

[13] G. DA PRATO AND A. DEBUSSCHE, *Strong solutions to the stochastic quantization equations*, Ann. Probab., 32 (2003), pp. 1900–1916.

[14] J. DEREZIŃSKI AND C. GÉRARD, *Mathematics of Quantization and Quantum Fields*, Cambridge University Press, Cambridge, 2013.

[15] J. DEREZIŃSKI AND M. NAPIÓRKOWSKI, *Excitation spectrum of interacting bosons in the mean-field infinite-volume limit*, Annales Henri Poincaré, (2014), pp. 1–31.

[16] A. DEUCHERT, R. SEIRINGER, AND J. YNGVASON, *Bose-Einstein condensation in a dilute, trapped gas at positive temperature.* arXiv:1803.05180.

[17] A. EINSTEIN, *Quantentheorie des einatomigen idealen Gases*, Sitzber. Kgl. Preuss. Akad. Wiss., 1924, pp. 261–267.

[18] L. ERDŐS, B. SCHLEIN, AND H.-T. YAU, *Derivation of the Gross-Pitaevskii equation for the dynamics of Bose-Einstein condensate*, Ann. of Math., 172 (2010), pp. 291–370.

[19] J. FRÖHLICH, A. KNOWLES, B. SCHLEIN, AND V. SOHINGER, *Gibbs measures of nonlinear Schrödinger equations as limits of quantum many-body states in dimensions $d \leq 3$*, Communications in Mathematical Physics, 356 (2017), pp. 883–980.

[20] ——, *A microscopic derivation of time-dependent correlation functions of the 1D cubic nonlinear Schrödinger equation*. arXiv:1703.04465, 2017.

[21] J. GLIMM AND A. JAFFE, *Quantum Physics: A Functional Integral Point of View*, Springer-Verlag, 1987.

[22] F. GOLSE, *On the Dynamics of Large Particle Systems in the Mean Field Limit*, Lecture notes for a course at the NDNS and Applied Dynamical Systems Summer School "Macroscopic and large scale phenomena", Universiteit Twente, Enschede (The Netherlands), arXiv:1301.5494.

[23] A. D. GOTTLIEB, *Examples of bosonic de Finetti states over finite dimensional Hilbert spaces*, J. Stat. Phys., 121 (2005), pp. 497–509.

[24] P. GRECH AND R. SEIRINGER, *The excitation spectrum for weakly interacting bosons in a trap*, Comm. Math. Phys., 322 (2013), pp. 559–591.

[25] M. G. GRILLAKIS, M. MACHEDON, AND D. MARGETIS, *Second-order corrections to mean field evolution of weakly interacting bosons. I*, Commun. Math. Phys., 294 (2010), pp. 273–301.

[26] F. GUERRA, L. ROSEN, AND B. SIMON , *The $P(\Phi)_2$ Euclidean quantum field theory as classical statistical mechanics. I, II.* Ann. of Math. (2) 101 (1975), pp. 111–189 and 191–259.

[27] M. HAIRER, *A theory of regularity structures*, Inventiones Mathematicae, 198 (2014), pp. 269–504.

[28] W. KETTERLE, *When atoms behave as waves: Bose-Einstein condensation and the atom laser*, Rev. Mod. Phys., 74 (2002), pp. 1131–1151.

[29] A. KNOWLES, *Limiting dynamics in large quantum systems*, Doctoral Thesis at ETH Zurich (2009).

[30] R. KUBO, *The fluctuation-dissipation theorem*, Reports on Progress in Physics, 29 (1966), p. 255.

[31] J. L. LEBOWITZ, H. A. ROSE, AND E. R. SPEER, *Statistical mechanics of the nonlinear Schrödinger equation*, J. Statist. Phys., 50 (1988), pp. 657–687.

[32] M. LEWIN, *Mean-Field limit of Bose systems: rigorous results*, in Proceedings of ICMP 2015, arXiv:1510.04407

[33] M. LEWIN, P. NAM, AND N. ROUGERIE, *Derivation of Hartree's theory for generic mean-field Bose systems*, Adv. Math., 254 (2014), pp. 570–621.

[34] ——, *Derivation of nonlinear Gibbs measures from many-body quantum mechanics*, Journal de l'Ecole Polytechnique, 2 (2016), pp. 553–606.

[35] ——, *Gibbs measures based on 1D (an)harmonic oscillators as mean-field limits*, Journal of Mathematical Physics, 59 (2018).

[36] ——, *Classical field theory limit of 2D many-body quantum Gibbs states*, Preprint 2018, arXiv:1810.08370.

[37] M. LEWIN, P. T. NAM, AND B. SCHLEIN, *Fluctuations around Hartree states in the mean-field regime*, Amer. J. Math., 137 (2015), pp. 1613–1650.

[38] M. LEWIN, P. T. NAM, S. SERFATY, AND J. P. SOLOVEJ, *Bogoliubov spectrum of interacting Bose gases*, Comm. Pure Appl. Math., 68 (2015), pp. 413–471.

[39] E. H. LIEB, *The classical limit of quantum spin systems*, Comm. Math. Phys., 31 (1973), pp. 327–340.

[40] E. H. LIEB AND R. SEIRINGER, *Proof of Bose-Einstein Condensation for Dilute Trapped Gases*, Phys. Rev. Lett., 88 (2002), p. 170409.

[41] E. H. LIEB, R. SEIRINGER, J. P. SOLOVEJ, AND J. YNGVASON, *The mathematics of the Bose gas and its condensation*, Oberwolfach Seminars, Birkhäuser, 2005.

[42] E. H. LIEB, R. SEIRINGER, AND J. YNGVASON, *Bosons in a trap: A rigorous derivation of the Gross-Pitaevskii energy functional*, Phys. Rev. A, 61 (2000), p. 043602.

[43] J. LÖRINCZI AND M. GUBINELLI, *Gibbs Measures on Brownian Currents*, Comm. Pure Appl. Math. 62 (2009), pp. 1–56.

[44] E. NELSON, *A quartic interaction in two dimensions*, in Mathematical Theory of Elementary Particles, MIT Press, Cambdridge, Mass., 1966, pp. 69–73.

[45] N. ROUGERIE, *De Finetti theorems, mean-field limits and Bose-Einstein condensation*, Lecture Notes for a course at LMU Munich (2014), arXiv:1506.05263.

[46] B. SCHLEIN, *Derivation of Effective Evolution Equations from Microscopic Quantum Dynamics*, Lecture Notes for a course at the 2008 CMI Summer School "Evolution Equations" at ETH Zurich, arXiv:0807.4307.

[47] R. SEIRINGER, *The excitation spectrum for weakly interacting bosons*, Commun. Math. Phys., 306 (2011), pp. 565–578.

[48] B. SIMON, *The $P(\Phi)_2$ Euclidean (quantum) field theory*, Princeton University Press, Princeton, N.J., 1974. Princeton Series in Physics.

[49] ——, *The classical limit of quantum partition functions*, Comm. Math. Phys., 71 (1980), pp. 247–276.

[50] A. SKOROKHOD, *Integration in Hilbert space*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Springer-Verlag, 1974.

Random growth models: shape and convergence rate[1]

by MICHAEL DAMRON (Atlanta)

Random growth models are fundamental objects in modern probability theory, have given rise to new mathematics, and have numerous applications, including tumor growth and fluid flow in porous media. In this article, we introduce some of the typical models and the basic analytical questions and properties, like existence of asymptotic shapes, fluctuations of infection times, and relations to particle systems. We then specialize to models built on percolation (first-passage percolation and last-passage percolation), giving a self-contained treatment of the shape theorem, and describing conjectured and proven properties of asymptotic shapes. We finish by discussing the rate of convergence to the limit shape, along with definitions of scaling exponents and a sketch of the proof of the KPZ scaling relation.

1 Introduction

1.1 Some typical growth models

The typical setting for a random growth model is as follows. An infection sits at a vertex v of a connected graph $G = (V, E)$ with vertex set V and edge set E. It spreads along the edges according to some random rules, and each vertex of the graph is eventually infected. The infection takes time $T(v, w) < \infty$ to infect a vertex w, and at time t, the infected set of vertices is

$$B(t) = \{x \in V : T(v, x) \leq t\}.$$

Eden model. One of the simplest examples is the Eden model, introduced by Eden [19] in '61, which gives a simplified version of cell reproduction. Eden considered the two-dimensional square lattice $\mathbb{Z}^2$ with its nearest-neighbor edges, but we could consider any graph, say the d-dimensional cubic lattice $\mathbb{Z}^d = \{x = (x_1, \ldots, x_d) : x_i \in \mathbb{Z} \text{ for all } i\}$ with the edges $\mathcal{E}^d = \{\{x, y\} : x, y \in \mathbb{Z}^d, \|x - y\|_1 = 1\}$, where $\|x\|_1 = \sum_i |x_i|$ is the ℓ^1-norm. We begin with a cell occupying the site $0 = (0, \ldots, 0)$, so our occupied set at time 0 is $S_0 = \{0\}$. At time $n \geq 1$, we consider the edge boundary of S_{n-1}

$$\partial_e S_{n-1} = \{\{x, y\} \in \mathcal{E}^d : x \in S_{n-1}, y \notin S_{n-1}\}$$

and select some edge $e_n = \{x_n, y_n\}$ uniformly at random from it. Last, put $S_n = S_{n-1} \cup \{x_n, y_n\}$. (Strictly speaking, what we describe here is a variant of the Eden model, as the original model's rule is to add one *vertex* from the boundary uniformly at random. Our rule effectively adds a vertex with probability proportional to the number of edges connecting it to the current cluster.)

[1]This work was done with support from an NSF CAREER grant. It is an abridgment, with permission from the American Mathematical Society, of Michael Damron, "Random growth models: Shape and convergence rate" to appear in [15].

In the Eden model, our cluster of cells S_{n-1} replicates on the boundary (uniformly at random) and produces a new cell directly outside of S_{n-1} to form our new cluster S_n. It turns out that $\cup_n S_n$ is all of $\mathbb{Z}^d$, and so the infection time from 0 to x (the first value of n such that S_n contains x) is finite. The Eden model can be rephrased in a larger framework, first-passage percolation (FPP), which we will meet soon. As a consequence, one can show a "shape theorem" for S_n: after proper scaling, S_n approaches a limiting shape. Although our rule appears not to bias any direction, the limit shape is not expected to be rotationally invariant (not a Euclidean ball). This is proved for high dimensions (see [27, 7] and Section 2.1 below).

Diffusion-limited aggregation. In the Eden model, we select a boundary vertex y_n by picking a boundary edge uniformly at random. We could change this selection rule, and pick y_n according to some other distribution. In diffusion-limited-aggregation (DLA), introduced in [43], we select y_n according to the "harmonic measure from infinity." Roughly speaking, this corresponds to the hitting distribution of a random walk on the boundary of S_n started from a far away point. In this way we again obtain a sequence of growing sets (S_n). Since the DLA model is notoriously hard to analyze, a simplified version, internal DLA (IDLA) was introduced in [37], where the selection rule is more straightforward. We begin again with $S_0 = \{0\}$. At time n, we run a simple symmetric random walk started at 0, and set y_n (our selected vertex) to be the first vertex adjacent to S_{n-1} that the random walk touches. Precisely, at time n, we let $X_1^{(n)}, X_2^{(n)}, \ldots$ be an i.i.d. sequence of random vectors taking values $\mathbb{P}(X_i^{(n)} = \pm e_j) = 1/(2d)$ for $i \geq 1,\ j = 1, \ldots, d$, where (e_j) are the j standard basis vectors of $\mathbb{R}^d$, set

$$Y_0^{(n)} = 0, \text{ and } Y_i^{(n)} = \sum_{k=1}^{i} X_k^{(n)} \text{ for } i \geq 1,$$

and let y_n be the first element in the sequence $Y_0^{(n)}, Y_1^{(n)}, Y_2^{(n)}, \ldots$ that is not in S_{n-1}. There is no shape theorem proved for DLA, but there is one for IDLA [30] (convergence to a Euclidean ball), and even the rate of convergence to the shape is known [23].

First-passage percolation. FPP can be seen as a generalization of the Eden model, and was introduced [22] by Hammersley and Welsh in '65. In FPP, the infection spreads across edges according to explicit speeds. We let $(t_e)_{e \in \mathcal{E}^d}$ be a collection of i.i.d. nonnegative random variables. The variable t_e is thought of as the passage time of an edge; that is, the amount of time it takes for an infection to cross the edge. A path Γ is a sequence of edges $e_0, \ldots, e_n$ such that each pair e_i and e_{i+1} shares an endpoint, and the passage time of such a Γ is $T(\Gamma) = \sum_{k=0}^{n} t_{e_k}$. The infection takes the path of minimal passage time, so we set the infection time, or passage time, from x to y, vertices in $\mathbb{Z}^d$, to be

$$T(x, y) = \inf_{\Gamma : x \to y} T(\Gamma),$$

where the infimum is over paths Γ starting at x and ending at y. (It is known that under general assumptions, for instance if $\mathbb{P}(t_e = 0) < p_c$, where p_c is the d-dimensional bond percolation threshold, then there is a unique minimizing path — a geodesic — from x to y. However, for some distributions with $\mathbb{P}(t_0 = 0) = p_c$ in dimensions $d \geq 3$, existence of geodesics is unknown.) In FPP, there is a shape theorem, but the limiting shape depends on the distribution of the (t_e)'s. Very little is known about the limit shapes for various distributions, apart from them being convex, compact, and having the symmetries of $\mathbb{Z}^d$. As we will see, it is expected that for most distributions, the limit shape is strictly convex, and certainly not a polygon, but strict convexity is not proved for any distribution, and there are only some two-dimensional examples of limit shapes that are not polygons. For a recent survey on FPP, see [6].

If the weights (t_e) have (rate 1) exponential distribution, the evolution of the ball $B(t)$ as t grows can be shown to be exactly the same as the evolution of the sets (S_n) in the Eden model. More precisely, using the "memoryless property" of the exponential distribution, one can show that the growth of $B(t)$ is the same as in the following algorithm: begin with $B(0) = \{0\}$ and assign i.i.d. exponential random variables to the edges in the set $\partial_e B(0)$. If τ_1 is the minimum of these variables, and it is assigned to edge $e = \{x, y\}$, then set $B(t) = B(0)$ for $t \in [0, \tau_1)$ and $B(\tau_1) = B(0) \cup \{x, y\}$. Next, generate new i.i.d. exponential random variables assigned to the edges in $\partial_e B(\tau_1)$, and set τ_2' to be the minimum of these variables, with $\tau_2 = \tau_1 + \tau_2'$. Once again, set $B(t) = B(\tau_1)$ for $t \in [\tau_1, \tau_2)$ and $B(\tau_2) = B(\tau_1) \cup \{x', y'\}$, where $\{x', y'\}$ is the edge in $\partial_e B(\tau_1)$ with minimal weight. We continue, and at each step, we sample i.i.d. exponentials for the boundary edges of our current set, choose the minimal weight edge (of weight τ), and add its endpoints into our set after we wait for time τ. Since the location of the minimum is uniformly distributed on the boundary, the sequence of sets $B(0), B(\tau_1), B(\tau_2), \ldots$ has the same distribution as the sequence of sets in the Eden model. There is no such representation of the DLA models in terms of FPP.

Last-passage percolation. LPP is a modification of FPP, introduced because of its relationship to the TASEP particle system. The typical setting is $\mathbb{Z}^d$, and one places i.i.d. nonnegative random variables (weights) $(t_v)_{v \in \mathbb{Z}^d}$ on the vertices. A path Γ is a sequence of vertices $v_0, \ldots, v_n$ such that $\|v_i - v_{i+1}\|_1 = 1$ for all i, and one assigns the passage time $T(\Gamma) = \sum_{k=0}^n t_{v_k}$, as in FPP. The difference now is that in LPP we define the passage time between two vertices as the maximal passage time of any path between them. Of course this will generally be infinity unless we restrict ourselves to a finite set of paths, so we consider oriented paths; that is, paths such that all the coordinates of the v_i's are nondecreasing (written $v_i \leq v_{i+1}$). So for any $v \leq w$, we set the infection time $T(v, w)$ to be the maximal passage time of any oriented path from v to w. Once again there is a shape theorem; however, unlike in FPP, the limiting shape is generally compact but not convex. In two dimensions, it is believed that the boundary of the limit shape is the graph of a strictly concave function, but this is again not known. In LPP, however, it is known that the limit shape is not a polygon. For a survey of LPP, see [34].

1.2 Main questions

As we mentioned many times in the last section, a fundamental object of study in random growth models is the limit shape. In a certain sense to be described in the next section, we will have almost surely $B(t)/t \to \mathcal{B}$, where $\mathcal{B}$ is the limit shape and $B(t)$ is the set of infected sites at time t. Directly related to the limit shape are the following two questions.

(1) Set of limit shapes. What does the limit shape look like? Are there explicit descriptions? Is it a Euclidean ball? Is it the ℓ^1 or ℓ^∞ ball? In models (like FPP and LPP) where many limit shapes can arise, what is the collection of all possible limit shapes? What is the dependence in FPP and LPP of the limit shape on the weight distribution?

(2) Convergence to the limit. What is the convergence rate to the limit shape? Precisely, what is the set of functions $f(t)$ such that one has $(t - f(t))\mathcal{B} \subset B(t) \subset (t + f(t))\mathcal{B}$ for t large?

We will focus here on these two questions, but we also mention some of the questions from the other articles in [15].

(3) Distributional limits of passage times. Letting $T(x, y)$ be the infection time of y started at x, are there functions $a(x)$ and $b(x)$ such that

$$\frac{T(0, x) - a(x)}{b(x)} \Rightarrow X$$

as $\|x\|_1 \to \infty$ for some nondegenerate limiting distribution X? In two-dimensional FPP/LPP-type models, the answer is expected (and proved in a couple of cases) to be yes, and X should have the Tracy-Widom distribution from random matrix theory. $b(x)$ should be the order of fluctuations of $T(0, x)$, and embedded in this question is obviously the question: what is the order of fluctuations of $T(0, x)$? In the DLA model, fluctuations are of significantly lower order than in FPP/LPP-type models. Work on these questions has led into concentration of measure, particle systems, integrable probability and exactly solvable systems.

(4) Structure of geodesics. Optimal infection paths are called geodesics. What is the structure of the set of all geodesics? How different are geodesics from straight lines? Infinite geodesics are infinite paths all whose segments are geodesics. How many infinite geodesics are there? Are there doubly-infinite geodesics? These questions are all related to Busemann functions in metric geometry.

The existence (or nonexistence) of doubly-infinite geodesics mentioned above is directly related to the number of ground states of disordered ferromagnetic spin models. We explain the connection to the disordered Ising ferromagnet, which is a variant of the usual Ising model from statistical mechanics. We consider dimension two, and define the dual lattice $(\mathbb{Z}^2_*, \mathcal{E}^2_*) = (\mathbb{Z}^2, \mathcal{E}^2) + (1/2, 1/2)$, which is the usual two-dimensional lattice shifted by $(1/2, 1/2)$. Each "dual edge" e^* in $\mathcal{E}^2_*$ bisects a unique edge e in $\mathcal{E}^2$, so we say that e^* is the edge dual to e. Define a "spin configuration" $\sigma = (\sigma_x)_{x \in \mathbb{Z}^2_*}$ to be an assignment of $+1$ and -1 to every dual vertex; that is, σ is an element of $\{-1, +1\}^{\mathbb{Z}^2_*}$. The interactions between spins are given by "couplings," which are variables assigned to the edges. Accordingly, let $(J_{x,y})_{\{x,y\} \in \mathcal{E}^2_*}$ be a family of independent random variables which are almost surely positive. For any σ and any finite $S \subset \mathbb{Z}^2_*$, we define the random energy of σ relative to the couplings and the set S as

$$\mathcal{H}_S(\sigma) = - \sum_{\{x,y\} \in \mathcal{E}^2_*, x \in S} J_{x,y} \sigma_x \sigma_y.$$

For the standard disordered Ising ferromagnet, the couplings $(J_{x,y})$ are typically chosen to be identically distributed. In this case, it is of great interest to determine the structure and number of ground states for the model. Precisely, a configuration σ is called a ground state for the couplings $(J_{x,y})$ if for each configuration $\tilde{\sigma}$ such that $\tilde{\sigma}_x \neq \sigma_x$ for only finitely many x, one has $\mathcal{H}_S(\sigma) \leq \mathcal{H}_S(\tilde{\sigma})$ for all finite $S \subset \mathbb{Z}^2_*$. One can think of a ground state as a local minimizer of the energy functional. It is not known how many ground states there are for a given realization of couplings, but it is believed that in this two-dimensional model (and sufficiently low-dimensional analogues), there should be only two almost surely. These two are the all-plus and all-minus states.

If there exists a nonconstant ground state σ for couplings $(J_{x,y})$, then we can compare it to the all-plus ground state σ_+. We claim that σ and σ_+ cannot have any finite regions of disagreement: there is no finite S such that $\sigma_x = -1$ for all $x \in S$ and $\sigma_y = +1$ for all $y \in \partial S$ (here ∂S refers to the set of vertices in S^c with a neighbor in S). Indeed, if there were such an S, we could apply the energy minimization property to S to find that $\mathcal{H}_S(\sigma) = 0$. However, as long as the couplings are continuously distributed, one can argue that almost surely, there are no finite S with $\mathcal{H}_S(\sigma) = 0$ for some σ. This justifies the claim; from it we see that any nonconstant ground state must have a two-sided and circuitless original lattice path of edges whose dual edges $\{x, y\}$ satisfy $\sigma_x \neq \sigma_y$. That is, any nonconstant ground state can be associated to at least one such doubly-infinite path.

Conversely, one can construct nonconstant ground states if one assumes existence of doubly-infinite geodesics in a related FPP model. Namely, given couplings $(J_{x,y})$ as above, associated to the edges of the dual lattice, one defines a passage-time configuration (t_e) by setting $t_e = J_{x,y}$, where $\{x, y\}$ is the unique edge dual to e. Supposing that there is a doubly-infinite geodesic Γ (this is a doubly-infinite path each of whose subpaths is an optimizing path for T) in this configuration, one can set $\sigma_x = +1$ for all x on one side of Γ and -1 for all x on the other side of Γ, and such σ will be a nonconstant ground state for $(J_{x,y})$.

The relation between ground states and geodesics allows one to carry results between the two models. For example, if one can rule out existence of doubly-infinite geodesics in FPP models, one can deduce nonexistence of nonconstant ground states for the associated spin models.

2 Limit shapes

For the rest of the article, we focus on the first two main growth model questions (limit shapes and convergence rate), and only in the context of FPP and LPP. First we will describe results from FPP including shape theorems and properties of limit shapes. Afterward, we switch to LPP to show which results are similar (shape theorems) and which are different (non-polygonal shapes and exactly solvable cases).

2.1 FPP

We begin with the shape theorem in FPP. First let's recall the model. We are given the d-dimensional cubic lattice $\mathbb{Z}^d$ with the set of its nearest-neighbor edges $\mathcal{E}^d$. We place i.i.d. nonnegative random variables (edge-weights) $(t_e)_{e \in \mathcal{E}^d}$ with common distribution function F on the edges. A path is a sequence $(v_0, e_0, v_1, e_1, \ldots, e_{n-1}, v_n)$ of vertices and edges such that e_i has endpoints v_i and v_{i+1}. The passage time of a path Γ is $T(\Gamma) = \sum_{k=0}^{n-1} t_{e_k}$ and the passage time between vertices $x, y \in \mathbb{Z}^d$ is

$$T(x, y) = \inf_{\Gamma : x \to y} T(\Gamma),$$

where the infimum is over all paths from x to y.

Shape theorem in FPP If $\mathbb{P}(t_e = 0) = 0$ then T is almost surely a metric on $\mathbb{Z}^d$, as it is nonnegative, one has $T(x, y) = 0$ only when $x = y$, and T satisfies the triangle inequality $T(x, y) \leq T(x, z) + T(z, y)$. (When edge-weights can be zero, T is a pseudometric.) For convenience, we also extend T to real points; that is, for $x \in \mathbb{R}^d$, we set $[x]$ to be the unique point in $\mathbb{Z}^d$ with $x \in [x] + [0, 1)^d$. Then the shape theorem is a type law of large numbers for the set

$$B(t) = \{x \in \mathbb{R}^d : T(0, x) \leq t\},$$

saying that $B(t)/t$ approaches a limiting set $\mathcal{B}$.

In the statement of the shape theorem below, $p_c = p_c(d)$ is the threshold for Bernoulli bond percolation on $\mathbb{Z}^d$ and can be defined as follows. Let $\mathbb{P}_p$ be the probability measure under which edge-weights are Bernoulli with parameter $1 - p$: $\mathbb{P}_p(t_e = 0) = p \quad \text{and} \quad \mathbb{P}_p(t_e = 1) = 1 - p$. Putting

$$\theta(p) = \mathbb{P}_p(0 \text{ is in an infinite self-avoiding path of edges } e \text{ with } t_e = 0),$$

one can then use a coupling argument to show that θ is nondecreasing in p. Therefore the number

$$p_c = \sup\{p \in [0, 1] : \theta(p) = 0\}$$

has the properties $\theta(p) > 0$ for $p > p_c$ and $\theta(p) = 0$ for $p < p_c$. It is an important result that $p_c \in (0, 1)$ for all dimensions $d \geq 2$, and we can quickly argue at least that $p_c > 0$. (The argument for $p_c < 1$ can be found in [20], for example.). Write A_n for the event that there is an edge self-avoiding path starting from 0 with n edges e, all satisfying $t_e = 0$. Then for any n,

$$\theta(p) \leq \mathbb{P}_p(A_n) \leq \sum_{\gamma:\#\gamma=n} \mathbb{P}_p(t_e = 0 \text{ for all } e \in \gamma),$$

where the sum is over all self-avoiding paths γ starting from 0 with n edges. Each such γ satisfies $\mathbb{P}_p(t_e = 0 \text{ for all } e \in \gamma) = p^{\#\gamma}$, and since there are at most $2d(2d - 1)^n$ many such paths with n edges (we first choose an edge touching 0, and then at every subsequent step choose an adjacent edge we have not chosen before), we obtain $\theta(p) \leq 2d(2d - 1)^n p^n$. This converges to 0 if $p < 1/(2d - 1)$, and so $p_c \geq 1/(2d - 1)$.

Returning to the shape theorem, the condition $\mathbb{P}(t_e = 0) < p_c$ below is there to ensure that there are not so many zero-weight edges that $T(0, x)$ will grow sublinearly in x. (This occurs for $\mathbb{P}(t_e = 0) \geq p_c$.)

Theorem 2.1. *(Richardson [39], Cox-Durrett [12], Kesten [27]).*
Assume that $\mathbb{E} \min\{t_1, \ldots, t_{2d}\}^d < \infty$, *where* t_i *are i.i.d. copies of* t_e *and* $\mathbb{P}(t_e = 0) < p_c$. *There exists a deterministic, convex, compact set in* $\mathbb{R}^d$, *symmetric about the axes and with nonempty interior, such that for any* $\epsilon > 0$,

$$\mathbb{P}\left((1 - \epsilon)\mathcal{B} \subset B(t)/t \subset (1 + \epsilon)\mathcal{B} \text{ for all large } t\right) = 1.$$

There is also a version for translation ergodic distributions by Boivin [8], first done in less generality by Derriennic, as reported in [27, (9.25)]. One can show that the above is equivalent to: there is a norm g on $\mathbb{R}^d$ such that $\mathcal{B}$ is the unit ball of g, and

$$\limsup_{x \to \infty} \frac{|T(0, x) - g(x)|}{\|x\|_1} = 0 \text{ almost surely}.$$

This can be thought of as $T(0,x) = g(x) + o(\|x\|_1)$ as $x \in \mathbb{R}^d \to \infty$. We will take this approach in the proof: build the norm g and then show the limsup statement above.

Proof of shape theorem. The idea of the proof is to first show "radial" convergence; that is, for a fixed $x \in \mathbb{Z}^d$, to show that

$$g(x) := \lim_n \frac{T(0,nx)}{n} \text{ exists.}$$

To do this, we appeal to the subadditive ergodic theorem. Then we "patch" together convergence in many different directions x to get a uniform convergence. We can see quickly that this limit should at least exist when we take expectations, since by invariance under lattice translations, $\mathbb{E}T(x,y) = \mathbb{E}T(x+z,y+z)$ for all $x,y,z \in \mathbb{Z}^d$, and so by the triangle inequality, for $0 \leq m \leq n$,

$$\mathbb{E}T(0,nx) \leq \mathbb{E}T(0,mx) + \mathbb{E}T(mx,nx) = \mathbb{E}T(0,mx) + \mathbb{E}T(0,(n-m)x).$$

Thus the sequence (a_n) given by $a_n = \mathbb{E}T(0,nx)$ is subadditive and a_n/n must have a limit from the following standard argument (usually referred to as Fekete's lemma): for fixed k and all $n \geq k$,

$$a_n \leq a_k + a_{n-k} \leq \cdots \leq \lfloor n/k \rfloor a_k + a_{n-\lfloor n/k \rfloor k}.$$

Since $n - \lfloor n/k \rfloor k$ is bounded as $n \to \infty$, we divide by n to obtain

$$\limsup_n a_n/n \leq \limsup_n \frac{\lfloor n/k \rfloor}{n} a_k = a_k/k.$$

This is true for all k, so

$$\limsup_n a_n/n \leq \inf_k a_k/k \leq \liminf_k a_k/k,$$

meaning $\lim_n a_n/n$ exists and equals $\inf_k a_k/k$.

To show the limit without expectation is much harder, and we will have to appeal to some machinery that was invented specifically for this problem. Liggett's version [32, Theorem 1.10] of Kingman's subadditive ergodic theorem states:

Theorem 2.2. *Let $\{X_{m,n} : 0 \leq m < n\}$ is an array of random variables satisfying the following assumptions:*

1. *for each n, $\mathbb{E}|X_{0,n}| < \infty$ and $\mathbb{E}X_{0,n} \geq -cn$ for some constant $c > 0$,*
2. *$X_{0,n} \leq X_{0,m} + X_{m,n}$ for $0 < m < n$,*
3. *for each $m \geq 0$, the sequence $\{X_{m+1,m+k+1} : k \geq 1\}$ is equal in distribution to the sequence $\{X_{m,m+k} : k \geq 1\}$, and*
4. *for each $k \geq 1$, $\{X_{nk,(n+1)k} : n \geq 1\}$ is a stationary ergodic process.*

Then $g := \lim_n \frac{1}{n}\mathbb{E}X_{0,n} = \inf_n \frac{1}{n}\mathbb{E}X_{0,n}$ *exists, and*

$$\lim_n \frac{1}{n} X_{0,n} = g \text{ a.s. and in } L^1.$$

We apply this theorem for a fixed $x \in \mathbb{Z}^d$ to the sequence $X_{m,n} = T(mx, nx)$. Item 2 holds by the triangle inequality, whereas 3 and 4 hold by invariance of the environment under integer translations. In item 1 we can take any $c > 0$, since $T \geq 0$ a.s. The only thing to check is that $\mathbb{E}T(0, nx) < \infty$ for each n. By subadditivity, and symmetry, it suffices to check that $\mathbb{E}T(0, e_1) < \infty$. To do this, we construct any $2d$ edge-disjoint deterministic paths $\gamma_1, \ldots, \gamma_{2d}$ from 0 to e_1 and note $\mathbb{E}T(0, e_1) \leq \mathbb{E}\min\{T(\gamma_1), \ldots, T(\gamma_{2d})\}$. Now we can check the following result from [12]:

Lemma 2.3. *If* $\mathbb{E}\min\{t_1, \ldots, t_{2d}\} < \infty$ *for i.i.d. edge-weights* t_i*, then one has* $\mathbb{E}\min\{T(\gamma_1), \ldots, T(\gamma_{2d})\} < \infty$.

Proof. Let k be the maximal number of edges of any γ_i and note that for each i, $\mathbb{P}\left(T(\gamma_i) \geq \lambda\right) \leq k\mathbb{P}(t_e \geq \lambda/k)$, so that if we put $M = \min\{T(\gamma_1), \ldots, T(\gamma_{2d})\}$,

$$\mathbb{P}(M \geq \lambda) = \prod_{i=1}^{2d} \mathbb{P}(T(\gamma_i) \geq \lambda) \leq k^{2d}\left(\mathbb{P}(t_e \geq \lambda/k)\right)^{2d} = k^{2d}\mathbb{P}\left(\min\{t_1, \ldots, t_{2d}\} \geq \lambda/k\right).$$

Therefore

$$\mathbb{E}M = \int_0^\infty \mathbb{P}(M \geq \lambda)\, \mathrm{d}\lambda \leq k^{2d} \int_0^\infty \mathbb{P}\left(\min\{t_1, \ldots, t_{2d}\} \geq \lambda/k\right)\, \mathrm{d}\lambda < \infty. \qquad \square$$

By the subadditive ergodic theorem, then, we define

$$g(x) = \lim_n \frac{T(0, nx)}{n},$$

which is a.s. constant. We next extend g to $\mathbb{Q}^d$ by taking such a rational x and letting $m \in \mathbb{N}$ be such that mx is an integer point. Then set

$$g(x) = \lim_n \frac{T(0, mnx)}{mn} = \frac{1}{m} \lim_n \frac{T(0, n(mx))}{n} = \frac{1}{m} g(mx).$$

g thus defined on $\mathbb{Q}^d$ satisfies the following properties: for $x, y \in \mathbb{Q}^d$,

1. $g(x + y) \leq g(x) + g(y)$,
2. g is uniformly continuous on bounded sets,
3. for $q \in \mathbb{Q}$, $g(qx) = |q|g(x)$,
4. g is symmetric about the axes.

Item 1 follows from the triangle inequality for T, and 4 follows from symmetries of the edge weights. Item 3 is an easy exercise, and 2 follows from 1: for $h = (h_1, \ldots, h_d)$,

$$\begin{aligned}|g(z) - g(z+h)| &\leq \max\{g(h), g(-h)\} = g(h) = g(h_1 e_1 + \cdots + h_d e_d) \\ &\leq g(e_1)(|h_1| + \cdots + |h_d|) \leq \|h\|_1 \mathbb{E}T(0, e_1).\end{aligned}$$

Then g has a continuous extension to $\mathbb{R}^d$. The above properties extend to real arguments, so g is a seminorm. (A norm, except it could have $g(x) = 0$ for some $x \neq 0$.) It is a result of Kesten [27, Theorem 6.1] that g is a norm exactly when $\mathbb{P}(t_e = 0) < p_c$.

Now that we have "radial" convergence to a norm g, we need to patch together convergence in every direction to a type of uniform convergence. Here we do this under the unnecessary but simplifying assumption:

$$\mathbb{P}(t_e \in [a, b]) = 1, \text{ where } 0 < a < b < \infty.$$

This implies $T(0, x), g(x) \in [a\|x\|_1, b\|x\|_1]$ for $x \in \mathbb{Z}^d$. So define the event

$$\Omega' = \{t_e \in [a, b] \text{ for all } e\} \cap \left\{ \lim_n \frac{T(0, nx)}{n} = g(x) \text{ for all } x \in \mathbb{Z}^d \right\}.$$

Note that on the above event we actually have

$$\lim_{\alpha \to \infty} \frac{T(0, \alpha x)}{\alpha} = g(x) \text{ for all } x \in \mathbb{Z}^d. \tag{2.1}$$

(Here α is real instead of just being an integer.) The reason is that we can bound

$$\begin{aligned}&\left| \frac{T(0, \alpha x)}{\alpha} - g(x) \right| \\ \leq\ & \left| \frac{T(0, \lfloor \alpha \rfloor x)}{\lfloor \alpha \rfloor} - \frac{T(0, \alpha x)}{\lfloor \alpha \rfloor} \right| + \left| \frac{T(0, \alpha x)}{\lfloor \alpha \rfloor} - \frac{T(0, \alpha x)}{\alpha} \right| + \left| \frac{T(0, \lfloor \alpha \rfloor x)}{\lfloor \alpha \rfloor} - g(x) \right| \\ \leq\ & \frac{T(\lfloor \alpha \rfloor x, \alpha x)}{\lfloor \alpha \rfloor} + T(0, \alpha x) \left| \frac{1}{\lfloor \alpha \rfloor} - \frac{1}{\alpha} \right| + \left| \frac{T(0, \lfloor \alpha \rfloor x)}{\lfloor \alpha \rfloor} - g(x) \right|.\end{aligned}$$

By the assumed inequality $t_e \leq b$, the first two terms go to zero as $\alpha \to \infty$. The last term approaches zero because of condition of the second intersected event in the definition of Ω'.

Fix $\omega \in \Omega'$, a set which has probability one, for the rest of the argument. We will show the equivalent statement

$$\limsup_{x \to \infty} \frac{|T(0, x) - g(x)|}{\|x\|_1} = 0,$$

so suppose it fails: for some $\epsilon > 0$, there is a sequence (x_n) going to infinity such that

$$|T(0, x_n) - g(x_n)| \geq \epsilon \|x_n\|_1 \text{ for all } n.$$

We may assume by compactness that $x_n / \|x_n\|_1 \to z$ for some z with $\|z\|_1 = 1$. We will show that for some x in a nearby direction to z, one cannot have $T(0, nx)/n \to g(x)$. (See Figure 2.1 for an illustration.)

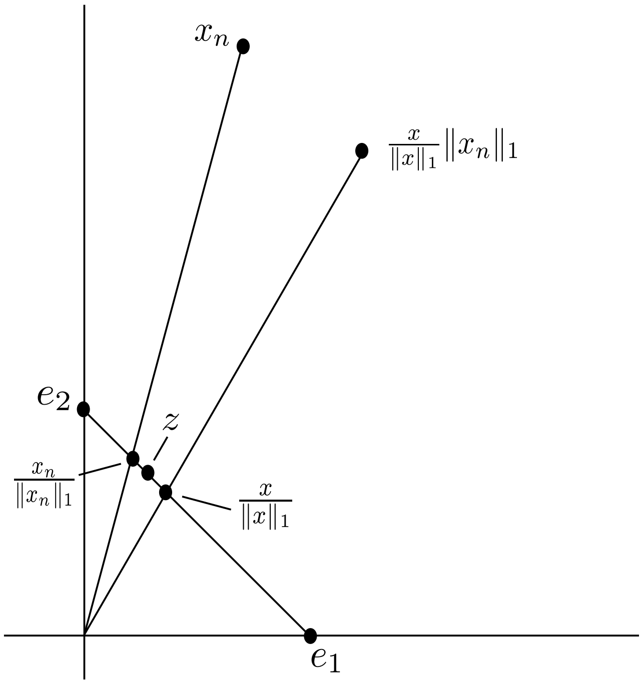

Figure 2.1: Illustration of the argument given in the proof of the shape theorem. The points $\frac{x_n}{\|x_n\|_1}$, z, and $\frac{x}{\|x\|_1}$ are all of ℓ^1-norm 1 and within ℓ^1-distance 2δ of each other. The point x is chosen in $\mathbb{Z}^d$ so that we have radial convergence of $T(0, nx)/n$ to $g(x)$ in this direction. Because x_n is close in direction to x, and our weights are bounded between a and b, the passage time $T(0, x_n)$ cannot be too different (on a linear scale) than the passage time from 0 to the "nearby point" $\frac{x}{\|x\|_1}\|x_n\|_1$. The latter passage time is controlled, since it is from 0 to a multiple of x, so this prohibits the sequence $T(0, x_n)$ from fluctuating too much.

Fix $\delta \in \left(0, \frac{\epsilon}{4b+1}\right)$ and choose $x \in \mathbb{Z}^d$ such that $\left\|\frac{x}{\|x\|_1} - z\right\|_1 < \delta$, so that for n large,

$$\left\|\frac{x_n}{\|x_n\|_1} - \frac{x}{\|x\|_1}\right\|_1 < 2\delta.$$

We will compare the passage time from 0 to x_n to the passage time to the "nearby" point $\|x_n\|_1 x/\|x\|_1$. Then

$$\begin{aligned}
&|T(0, x_n) - g(x_n)| \\
\leq\ & \left|T(0, x_n) - T\left(0, \|x_n\|_1 \frac{x}{\|x\|_1}\right)\right| + \left|T\left(0, \|x_n\|_1 \frac{x}{\|x\|_1}\right) - g\left(\|x_n\|_1 \frac{x}{\|x\|_1}\right)\right| \\
+\ & \left|g\left(\|x_n\|_1 \frac{x}{\|x\|_1}\right) - g(x_n)\right|.
\end{aligned}$$

The first and last terms are bounded by

$$b\left\|\|x_n\|_1 \frac{x}{\|x\|_1} - x_n\right\|_1 = b\|x_n\|_1 \left\|\frac{x}{\|x\|_1} - \frac{x_n}{\|x_n\|_1}\right\|_1 < 2b\delta\|x_n\|_1.$$

However since we have radial convergence in direction x, (2.1) gives

$$T\left(0, \|x_n\|_1 \frac{x}{\|x\|_1}\right) = g\left(\|x_n\|_1 \frac{x}{\|x\|_1}\right) + o(\|x_n\|_1),$$

so for n large, the second term is bounded by $\delta\|x_n\|_1$. In total, $\epsilon\|x_n\|_1 \leq |T(0,x_n) - g(x_n)| \leq (4b+1)\delta\|x_n\|_1 < \epsilon\|x_n\|_1$, a contradiction. □

High dimensions In FPP, the shape theorem gives that $\mathcal{B}$ is convex, compact, with nonempty interior, and with all the symmetries of $\mathbb{Z}^d$. Many different sets have this property, in particular all the ℓ^p balls for $p \in [1,\infty]$. So the shape theorem allows in principle a cube (ℓ^∞ ball) and a diamond (ℓ^1 ball), and therefore says nothing about strict convexity, flat edges, corners, or whether the shape is a polygon.

For (t_e) that are i.i.d. and, say, continuous, the following properties are expected for $\mathcal{B}$.
(1) $\mathcal{B}$ is strictly convex. That is, for any distinct $x, y \in \mathcal{B}$ and $\lambda \in (0,1)$ the point $\lambda x + (1-\lambda)y$ is in the interior of $\mathcal{B}$. Thus $\mathcal{B}$ should have no flat facets (like a polygon has).
(2) $\mathcal{B}$ has no "corners." That is, the boundary of $\mathcal{B}$ should be differentiable. One way to say this is in terms of supporting hyperplanes. A hyperplane in $\mathbb{R}^d$ is a set of the form $\{x = (x_1,\ldots,x_d) : x \cdot y = a\}$ for some $y \in \mathbb{R}^d$ and $a \in \mathbb{R}$ (where '$\cdot$' is the standard dot product $x \cdot y = \sum_{i=1}^d x_i y_i$). A hyperplane H is supporting for $\mathcal{B}$ at $z \in \partial\mathcal{B}$ if H contains z but $\mathcal{B}$ intersects at most one component of H^c. By the Hahn-Banach Theorem, since $\mathcal{B}$ is convex and bounded, each $z \in \partial\mathcal{B}$ has a supporting hyperplane. We say that $\partial\mathcal{B}$ is differentiable if each $z \in \partial\mathcal{B}$ has exactly one supporting hyperplane.
(3) Combining the above two cases, but weaker, the set $\mathcal{B}$ should not be a polygon. To state this precisely, we say that $\mathcal{B}$ should have infinitely many extreme points. An extreme point $x \in \mathcal{B}$ is not on the interior of any line segment with endpoints in $\mathcal{B}$. Precisely, whenever we write $x = \lambda z + (1-\lambda)y$ with $z, y \in \mathcal{B}$ and $\lambda \in (0,1)$, we have $z = y$. So $\mathcal{B}$ has infinitely many extreme points.
(4) The boundary $\partial\mathcal{B}$ should have uniformly positive curvature. In other words, near every boundary point, the boundary should locally look like the boundary of a Euclidean ball with bounded radius.

Here we state the main high-dimensional asymptotic result: for distributions with no atom at 0 but with 0 in the support (like exponential), for high dimensions, the limit shape is not an ℓ^p ball for $p = 1, 2, \infty$. Since we saw that the Eden model is equivalent to FPP with exponential weights, this shows the same result for the Eden model in high dimensions. For the statement, we define the balls $\mathbf{D}, \mathbf{B}, \mathbf{C}$ to be the ℓ^1, ℓ^2, and ℓ^∞-balls of radius $g(e_1)^{-1}$. We will also make the following assumptions from [7]: for some $a \geq 0$,

$$\mathbb{E}t_e < \infty,\ F(0) = 0,\ \text{and}\ \left|\frac{\mathbb{P}(t_e \leq x)}{x} - a\right| = O(|\log x|^{-1}) \text{ as } x \downarrow 0. \tag{2.2}$$

The following theorem is from Auffinger-Tang [7], which weakens various assumptions (widens the class of distributions in particular) of the work of previous authors. Some earlier work was done by Kesten [27], Dhar [16], Couronné-Enriquez-Gerin [11], and Martinsson [36].

Theorem 2.4. *Assume* (2.2)*. For all large d, in FPP on $\mathbb{Z}^d$ with fixed weights (t_e), one has* $\mathbf{D} \subset \mathcal{B} \subset \mathbf{C}$ *and* $\mathcal{B} \neq \mathbf{B}, \mathbf{C},$ *or* $\mathbf{D}$.

The proof proceeds by showing the asymptotic

$$\lim_{d\to\infty} \frac{g(e_1)d}{\log d} = \frac{1}{2a}.$$

For example, if $\mathcal{B}$ were the ℓ^1-ball, one would have for $e = (1, \ldots, 1)$, $g(e) = dg(1/d, \ldots, 1/d) = dg(e_1) \sim \frac{\log d}{2a}$. However one can show that $g(e) \leq C$ for some constant C. In fact, combining the work of Martinsson [36] and Couronné-Enriquez-Gerin [11], one can show that $\lim_{d\to\infty} g(e)$ exists for the exponential distribution and is related to the nonzero solution of $\coth \alpha = \alpha$.

It is not hard to give the bound $g(e) \leq 1$ for exponential weights. Construct a path γ with vertices $x_0, x_1, \ldots$ as follows: set $x_0 = 0$ and for $n \geq 0$, starting from x_n, take the minimal-weight edge of the d different edges leading in directions $e_1, \ldots, e_d$ (the positive coordinate directions) and call its endpoint x_{n+1}. If $X_1, X_2, \ldots$ are the weights of the first edge, second edge, and so on, then X_i is the minimum of d i.i.d. exponential random variables, so is an exponential with mean $1/d$. Thus putting $H_n = \{x = (x_1, \ldots, x_d) : \sum_i x_i = n\}$ one obtains $\mathbb{E}T(0, H_n) \leq \sum_{i=1}^n \mathbb{E}X_i = n/d$. But one can show using the shape theorem that

$$g(e) = \lim_n \frac{\mathbb{E}T(0, ne)}{n} = \lim_n \frac{\mathbb{E}T(0, H_{dn})}{n} \leq \lim_n \frac{\sum_{i=1}^{nd} \mathbb{E}X_i}{n} = 1\ .$$

Flat edges. A main question in FPP is: which compact convex sets are realizable as limit shapes? The question above is completely open in the i.i.d. setting. Interestingly, though, this is solved by Häggström and Meester '95 [21] in the case of stationary (not necessarily i.i.d.) passage times. The following result shows that there are ergodic models of FPP whose limit shapes are polygons or Euclidean balls. Therefore conjectured properties from the i.i.d. setting like strict convexity are not true in such generality.

Theorem 2.5 (Häggström-Meester)**.** *Any non-empty compact, convex set C that has the symmetries of $\mathbb{Z}^d$ is a limit shape for some FPP model with weights distributed according to a stationary (under translations of $\mathbb{Z}^d$) and ergodic measure.*

The previous section focused on high dimensions. What can we say about low dimensions? For one special class of distributions, Durrett and Liggett [18] were able to say much more: that the limit shape is not strictly convex. Of course, we believe strict convexity in the continuous weight case, so these distributions have atoms. Their "flat edge" result holds for higher dimensions as well, but we can give a precise description of it in two dimensions, and further work has been done by Marchand [33], Zhang [45, 46], and Auffinger-Damron [2].

Recalling that F is the distribution of our weights, let $\mathcal{M}_p$ be the set of distributions F that satisfy the following:

- $F(x) = 0$ for all $x < 1$ and $F(1) = p \geq \vec{p}_c$, where $\vec{p}_c$ is the two-dimensional oriented bond percolation threshold (approximately .70548), and $\int x \, \mathrm{d}F(x) = \mathbb{E}t_e < \infty$.

In [18], it was shown that if $F \in \mathcal{M}_p$ then the limit shape $\mathcal{B}$ has some flat edges. The precise location of these edges was found in [33]. To describe this, write $\mathbf{B}_1$ for the closed ℓ^1 unit ball, $\mathbf{B}_1 = \{(x,y) \in \mathbb{R}^2 \ : \ |x| + |y| \leq 1\}$, and write $int\ \mathbf{B}_1$ for its interior. For $p > \vec{p}_c$ let α_p be the asymptotic speed of oriented percolation (see [17]), define the points

$$M_p = \left(\frac{1}{2} - \frac{\alpha_p}{\sqrt{2}}, \frac{1}{2} + \frac{\alpha_p}{\sqrt{2}}\right) \text{ and } N_p = \left(\frac{1}{2} + \frac{\alpha_p}{\sqrt{2}}, \frac{1}{2} - \frac{\alpha_p}{\sqrt{2}}\right) \tag{2.3}$$

and let $[M_p, N_p]$ be the line segment in $\mathbb{R}^2$ with endpoints M_p and N_p. For symmetry reasons, the following theorem is stated only for the first quadrant.

Theorem 2.6 (Durrett-Liggett [18], Marchand [33]). *Let $F \in \mathcal{M}_p$ in two dimensions. Then $\mathcal{B} \subset \mathbf{B}_1$. Also,*

1. *If $p < \vec{p}_c$ then $\mathcal{B} \subset int\ \mathbf{B}_1$.*
2. *If $p > \vec{p}_c$ then $\mathcal{B} \cap [0,\infty)^2 \cap \partial \mathbf{B}_1 = [M_p, N_p]$.*
3. *If $p = \vec{p}_c$ then $\mathcal{B} \cap [0,\infty)^2 \cap \partial \mathbf{B}_1 = (1/2, 1/2)$.*

The angles corresponding to points in the line segment $[M_p, N_p]$ are said to be in the *percolation cone*. The existence of a flat edge for the limit shape can be explained heuristically. The definition of the oriented percolation threshold $\vec{p}_c$ is as follows, taking $\mathbb{P}_p$ to be the distribution of i.i.d. edge-weights (η_e) on $\mathcal{E}^2$ with probability p to be 1 and $1-p$ to be 2:

$$\vec{p}_c = \sup\{p : \mathbb{P}_p(\exists \text{ infinite oriented path } \Gamma \text{ with } \eta_e = 1 \text{ for } e \in \Gamma) = 0\}.$$

(Here, oriented means as usual that the vertices of the path, in order, have non-decreasing coordinates.) By monotonicity of the probability in p, one has that for $p > \vec{p}_c$, if $F \in \mathcal{M}_p$, then there is positive probability of existence of an infinite oriented path of all 1-weights starting from 0. In fact one can even show the stronger statement that for any angle θ in the percolation cone, there is positive probability for an infinite oriented path of 1-weights starting from 0 and going in direction θ (the angles of the vertices on the path converge to θ). Since no edge-weights have value below 1, any finite segment of such an infinite path must be a minimizing path (geodesic). But distance along these geodesics correspond to the ℓ^1-distance on $\mathbb{Z}^2$, so in such directions, the limit shape must correspond to the ℓ^1-ball. It is important to point out that it is unknown (but expected to be false) whether there are other distributions whose limit shapes have flat edges. Even in the case of $\mathcal{M}_p$, the flat part of the percolation cone ends at M_p and N_p; however, that does not exclude the limit shape from having further flat spots. This is not expected though.

Let $\beta_p := 1/2 + \alpha_p/\sqrt{2}$, that is, define β_p as the x coordinate of N_p. Convexity and symmetry of the limit shape imply that $1/g(e_1) \geq \beta_p$. A non-trivial statement about the edge of the percolation cone came in 2002 when Marchand [33, Theorem 1.4] proved that this inequality is in fact strict: $1/g(e_1) > 1/2 + \alpha_p/\sqrt{2}$. In other words, Marchand's result says that the line that goes through N_p and is orthogonal to the e_1-axis is not a tangent line of $\partial\mathcal{B}$. The following theorem builds on Marchand's result and technique and says that at the edge of the percolation cone, one cannot have a corner.

Theorem 2.7 (Auffinger-Damron [2]). *Let $F \in \mathcal{M}_p$ for $p \in [\vec{p}_c, 1)$. The boundary $\partial\mathcal{B}$ is differentiable at M_p and N_p.*

The above theorem implies that any distribution in $\mathcal{M}_p$ has a non-polygonal limit shape. The first example of a non-polygonal limit shape was discovered by Damron-Hochman [13], where it was also shown that for any $\epsilon > 0$, there exist i.i.d. distributions of (t_e) for which the extreme points of the corresponding limit shape are ϵ-dense in the boundary.

2.2 LPP

LPP is defined similarly to FPP, but taking the maximum over oriented paths, instead of the infimum over all paths. On $\mathbb{Z}^d$, we assign i.i.d. site-weights $(t_v)_{v\in\mathbb{Z}^d}$ with common distribution F. These weights no longer need to be nonnegative, but for simplicity we will take them to be so. An oriented path Γ with vertices $x_0, x_1, \ldots, x_n$ has the property that $x_i \leq x_{i+1}$ coordinatewise. By convention, we identify the path Γ with its vertices, but we exclude the initial point. The passage time of such a path is $T(\Gamma) = \sum_{k=0}^{n} t_{v_k}$, and the last-passage time $T(x, y)$ between vertices $x \leq y$ is

$$T(x, y) = \max_{\Gamma: x \to y} T(\Gamma),$$

where the maximum is over oriented paths from x to y. Note that there are only finitely many paths under consideration, so we can take a maximum. It is important to note that when x, y do not satisfy $x \leq y$ or $y \leq x$, then $T(x, y)$ is not defined. Also, by convention, $T(x, x) = 0$ for all x.

Due to directedness of the model and the fact that we are taking a maximum, T has somewhat different properties than those in FPP. One still has for $x \leq y$, $T(x, y) \geq 0$ and if $t_v > 0$ for all v, then $T(x, y) > 0$ when $x \neq y$. Due to excluding the initial point from all our paths, we have a super-additivity property of T that corresponds to the triangle inequality in FPP:

$$\text{for } x \leq y \leq z,\ T(x, z) \geq T(x, y) + T(y, z).$$

Due to this super-additivity, the limiting shape in LPP is not convex, since the corresponding "shape function" g will be super-additive. For its definition, we again appeal to the subadditive ergodic theorem, noting that $-T$ is subadditive. The only difficulty is to come up with conditions under which the limit is finite. The following version of "radial convergence" for T to a limit comes from Martin [34].

Theorem 2.8. *Suppose that*

$$\int_0^\infty (1 - F(x))^{1/d}\, dx < \infty. \tag{2.4}$$

Then for each $x \geq 0$ in $\mathbb{R}^d$, the following (deterministic) limit exists a.s. and in L^1:

$$g(x) = \lim_n \frac{T(0, [nx])}{n} < \infty,$$

where $[nx]$ is the point of $\mathbb{Z}^d$ with $nx \in [nx] + [0,1)^d$. The function g is continuous on $\{x : x \geq 0\}$ (including at the boundaries), satisfies $g(x+y) \geq g(x) + g(y)$, is invariant under permutations of the coordinates, and $g(ax) = ag(x)$ for $a \geq 0$.

Although $\mathbb{E}t_v < \infty$ is sufficient to guarantee that $\mathbb{E}T(0,x) < \infty$ for all $x \geq 0$ (just bound $T(0,x)$ above by the sum of all weights t_y with $0 \leq y \leq x$), the subadditive ergodic theorem this time gives

$$g(x) = \sup_{n \geq 1} \frac{\mathbb{E}T(0,nx)}{n},$$

and to deduce that $g(x) < \infty$, we need to know that the means $\mathbb{E}T(0,nx)$ do not grow faster than linearly. The corresponding issue in FPP is that the means $\mathbb{E}T(0,nx)$ do not grow sublinearly, which is guaranteed by $F(0) < p_c$. Here the corresponding condition is not on $F(0)$ but, as we are in LPP, it is on the tail of F, and is the integrability condition (2.4).

As in FPP, one can upgrade radial convergence to a sort of shape theorem [34]. As before, put $\mathcal{B} = \{x \geq 0 : g(x) \leq 1\}$.

Theorem 2.9 (LPP shape theorem)**.** *Assume* (2.4) *and put $B(t) = \{x \geq 0 : T(0,x) \leq t\}$. For any $\epsilon > 0$, one has*

$$\mathbb{P}\left((1-\epsilon)\mathcal{B} \subset B(t)/t \subset (1+\epsilon)\mathcal{B} \text{ for all large } t\right) = 1.$$

Near the boundary Just as in FPP, not much is known about the limiting shape $\mathcal{B}$. It is expected as before to have differentiable boundary (at least when the weights are continuous) with positive curvature, and certainly not to be a polygon. The flat edge result from FPP carries over to LPP: the analogous condition on our distribution F is $F(1) = 1$, $F(1^-) = 1 - p$, where $p > \vec{p}_c$. Such distributions F of weights t_v have $\mathbb{P}(t_v > 1) = 0$ but $\mathbb{P}(t_v = 1) = p > \vec{p}_c$. The value c is the highest weight of an edge, and any oriented path of all c-weights will be an optimal infection path. Thus again in the percolation cone, the boundary of the limit shape will agree with that of the ℓ^1-ball.

Somewhat surprisingly, in two dimensions, the limit shape is shown not to be a polygon for most distributions. This is in contrast to the situation in FPP, where this is only known for distributions in $\mathcal{M}_p$. The LPP result is a corollary of a "universality" of the shape function g near the boundary. The following result, proved by Martin [35], shows that the asymptotics of g near the boundary of the quarter plane $\{x : x \geq 0\}$ are explicit, and only depend on the mean and variance of F. From these asymptotics we can extract non-polygonality of the limit shape.

Theorem 2.10. *Consider $d = 2$. Write μ for the mean of F and σ^2 for the variance of F. If F satisfies* (2.4)*, then $g(1,a) = \mu + 2\sigma\sqrt{a} + o(\sqrt{a})$ as $a \downarrow 0$.*

Here, $g(1,a)$ is the function g evaluated at the point $(1,a)$, and as $a \downarrow 0$, this point approaches the boundary of the quarter plane. Note that $g(1,0) = \mu$, since the passage time from 0 to ne_1 must be achieved along the one oriented path connecting these points, and its passage time is the sum of $n+1$ i.i.d. random variables with distribution F. Thus the law of large numbers gives the value of g at $(1,0)$. Therefore the above result says $g(e_1 + ae_2) - g(e_1) = 2\sigma\sqrt{a} + o(\sqrt{a})$ as $a \downarrow 0$.

To see that the above result implies that the limit shape is not a polygon, suppose for a contradiction that the limit shape is a polygon. Then it must have finitely many extreme points, and the boundary of the shape between the extreme points consists of straight line segments. If the limit shape is a triangle, then by symmetry, g must be a multiple of the ℓ^1-norm, but then $g(e_1 + ae_2) - g(e_1) = a$ for $a > 0$ and the above asymptotics are violated. Otherwise, there is a closest extreme point w to the point $(1, a)$, and the limit shape boundary must be a line segment between w and $(1, a)$. In this case, $g(e_1 + ae_2) - g(e_1) = ca$ for some real c and a small enough. But this again violates the above asymptotics.

Exactly solvable cases in two dimensions The most famous case of LPP is when the distribution F of the site-weights is exponential in two dimensions. Here, there is a direct mapping from the growth of $B(t)$ to a particle system called the Totally Asymmetric Simple Exclusion Process (TASEP). TASEP is defined loosely as follows. We imagine that at each site z of $\mathbb{Z}$ with $z \leq 0$, there sits a particle at time 0. Associated to each particle is a Poisson process, and when the process increments, the particle attempts to move to the site directly to the right. If there is already a particle there, the move is suppressed, and the particle stays in its current location. The particle that is initially at site 0 is allowed to move unrestricted (since there are never any particles to the right of it), but the other particles may sometimes be blocked by particles to their rights. Our convention is that the particle at 0 immediately moves to the right at time zero. That is, at time zero, there is a particle at site 1, and particles at sites $-k$ for $k \geq 1$.

What is the relation between TASEP and LPP with exponential weights? To begin, the procession of the first particle in TASEP is the same as the infection in LPP along the positive e_1-axis from 0. Indeed, the infection appears at site 0 at time 0, just as the first particle in TASEP moves to the right. It then infects e_1 after an independent exponential time, just as the same particle in TASEP moves again to the right. Generally, the infection time from 0 to ne_1 is achieved through the path that proceeds directly down the positive e_1-axis, and occurs when the first particle in TASEP reaches site $n + 1$.

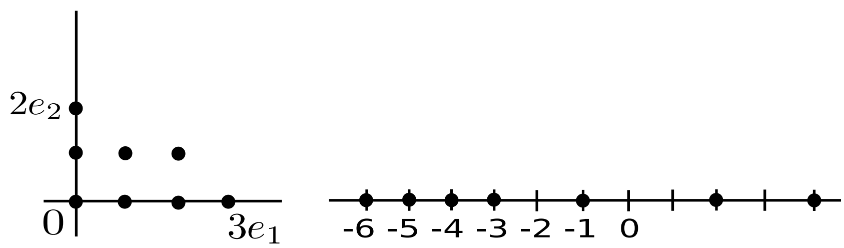

Figure 2.2: Illustration of the correspondence between LPP and TASEP. In the left (LPP), the infection has moved 4 steps to the right on the e_1-axis, and so on the right (TASEP) first particle has moved four steps to the right, from 0 to 4. Similarly, the infection has taken three steps at the second level, and the second particle has moved three steps to the right from -1 to 2. Last, the third TASEP particle has taken one step from -2 to -1.

At the second level, the infection of site e_2 occurs an independent exponential time after the infection appears at 0. This corresponds to the second particle in TASEP moving into the space left open after the first particle moves. Generally, the n-th step of the k-th particle in TASEP corresponds to the site $ke_2 + (n-1)e_1$ being infected from 0. To see this, we can derive the following relation in LPP: for $x_1, x_2 > 0$, one has

$$T(0,(x_1,x_2)) = t_{(x_1,x_2)} + \max\{T(0,(x_1-1,x_2)), T(0,x_1,x_2-1)\}.$$

This is because the infection from 0 reaches (x_1, x_2) through either (x_1-1, x_2) or (x_1, x_2-1) (whichever is infected last), and after the one of these sites with maximal passage time from 0 is infected, (x_1, x_2) must wait $t_{(x_1,x_2)}$ additional time. Similarly, in TASEP, for the k-th particle to make its n-th step, it must wait an independent exponential time after both of the following events occur: (a) the $k-1$-st particle makes its n-th step and (b) the k-th particle makes its $n-1$-st step. See Figure 2.2 for an example.

Because of this coupling, we can represent the passage times in LPP in terms of "currents" in TASEP. For example, $T(0,(n,n)) =$ time for the n-th particle to make its $n+1$-st step. But this is exactly the time needed for n particles to pass through the origin. Thus if we define c_t as the "current through zero at time n"; that is, the number of particles having passed through 0 by time t, one has $T(0,(n,n)) \leq t \Leftrightarrow c_t \geq n$. In '81, in a pioneering work, Rost [40] showed exact asymptotics for variables like this current, and this translates directly to an exact formula for the limit shape in exponential LPP.

Theorem 2.11. *Let (t_v) be exponentially distributed with mean 1. Then $g(x) = g(x_1, x_2) = (\sqrt{x_1} + \sqrt{x_2})^2$. The limit shape boundary is*

$$\left\{(x,y) \in \mathbb{R}^2 : x, y \geq 0 \text{ and } \sqrt{x} + \sqrt{y} = 1\right\}.$$

So we can see directly that the limit shape is not a polygon, contains no flat segments, and has no "corners." In the case of geometrically distributed weights with parameter p [10, 24, 41], there is an exact formula as well, showing

$$g(x_1,x_2) = \frac{1}{p}(x_1 + x_2 + 2\sqrt{x_1 x_2(1-p)}).$$

In both of these cases, finer asymptotics are available [25] by relating the law of the passage time to the largest eigenvalue of a random matrix ensemble. Specifically for $x = (x_1, x_2)$ with $x_1 > 0$ and $x_2 > 0$,

$$\frac{T(0,nx) - ng(x)}{n^{1/3}} \Rightarrow Z \text{ as } n \to \infty$$

for a nondegenerate variable Z. This shows that the fluctuations of T about g are of order $n^{1/3}$, and this is expected for general distributions in both LPP and FPP in two dimensions. Unfortunately, despite decades of work, the best available bounds in the general case for $|T(0,nx) - ng(x)|$ are of order $\sqrt{n}$.

3 Rate of convergence and scaling exponents

In this section, we restrict to the case of FPP, although similar theorems are provable in LPP.

3.1 Decomposition of error and strengthened shape theorems in FPP

The question we address here is: is it possible to improve the shape theorem to one with ϵ that depends on t? In other words, can we find ϵ_t such that $\epsilon_t \downarrow 0$ quickly and so that $\mathbb{P}\left((1-\epsilon_t)\mathcal{B} \subset B(t)/t \subset (1+\epsilon_t)\mathcal{B} \text{ for all large } t\right) = 1$? This is a question about the rate of convergence in the shape theorem. Restating it in terms of the norm g, we can ask the question in the following form: writing $T(0,x) = g(x) + o(\|x\|)$, how large is the term $o(\|x\|)$? The standard way to study this is to decompose

$$o(\|x\|) = [T(0,x) - \mathbb{E}T(0,x)] \;+\; [\mathbb{E}T(0,x) - g(x)]$$

into a random fluctuation term and a nonrandom fluctuation term.

The random fluctuation term is typically analyzed using techniques from concentration of measure — many techniques (transportation inequalities, entropy methods, influence inequalities, isoperimetry, etc.) have been developed to study deviations of a function $f(X_1, X_2, \ldots)$ of independent variables (X_n) away from its mean or its median. Despite all these methods, however, current bounds on the first term are far from the predictions.

The nonrandom term is purely deterministic, and can be written as

$$\|x\| \left[\frac{\mathbb{E}T(0,x)}{\|x\|} - g\left(\frac{x}{\|x\|}\right)\right] \geq 0.$$

The term in the parenthesis is of the form $\mathbb{E}T(0,nx)/n - g(x)$ for x on the unit circle. We have seen that for $x \in \mathbb{Z}^d$, the sequence $(\mathbb{E}T(0,nx))$ is subadditive and so, when divided by n, converges. So quantifying this nonrandom error is really a problem of estimating the rate of convergence of a_n/n to its limit for a subadditive sequence (a_n). Unfortunately there are no general methods for this, but in the context of lattice models (like FPP), techniques have been developed to bound these nonrandom fluctuations in terms of the random ones. Specifically, if one has a concentration inequality of the form

$$\mathbb{P}\left(T(0,x) - \mathbb{E}T(0,x) < -\lambda\|x\|^\alpha\right) \leq \exp\left(-C\lambda^\beta\right),\ \lambda \geq 0$$

for constants $C, \alpha, \beta > 0$, then one can show an upper bound of the type

$$\mathbb{E}T(0,x) - g(x) \leq C\|x\|^\alpha(\log\|x\|)^\delta,$$

implying that nonrandom fluctuations should be no larger than random fluctuations. Indeed, using Gaussian concentration inequalities, one has the following version of Alexander's [1] result from Damron-Kubota [14].

Theorem 3.1. *Assume that* $\mathbb{E}\min\{t_1, \ldots, t_d\}^2 < \infty$ *and* $\mathbb{P}(t_e = 0) < p_c$. *Then for some* $C > 0$, *one has*

$$g(x) \leq \mathbb{E}T(0,x) \leq g(x) + C\sqrt{\|x\|\log\|x\|} \text{ for all } \|x\| > 1.$$

On the other hand, evidence from [5] implies that random fluctuations should be no larger than nonrandom fluctuations. Therefore if we posit the existence of exponents such that

$$T(0,x) - \mathbb{E}T(0,x) \sim \|x\|^\chi \text{ and } \mathbb{E}T(0,x) - g(x) \sim \|x\|^\gamma$$

then one should have $\chi = \gamma$. (Note that Theorem 3.1 is a version of $\gamma \le 1/2$. Also, since the first term is random, it may be measured as $\mathrm{Var}\, T(0,x) \sim \|x\|^{2\chi}$ or in terms of a concentration inequality.) Unfortunately this is far from the state of art, as under various general assumptions (exponential moments for the passage times, for instance), the best existing bounds are

$$0 \le \chi \le 1/2 \text{ and } -1/2 \le \gamma \le 1/2.$$

(Under strong assumptions on existence of a fluctuation exponent $\chi < 1/2$, one can show $\chi = \gamma$ [5].) As is proved for exactly solvable models of LPP, we believe that in two dimensions, $\chi = \gamma = 1/3$. It is reasonable to expect that these (equal) numbers decrease strictly with dimension, and approach 0 as dimension tends to infinity.

In summary, we have a strengthened shape theorem of the following type. Some improvements to the assumptions have been made by Tessera [42] and Damron-Kubota [14] more recently, in particular generally replacing the log with $\sqrt{\log}$. The term $\sqrt{t}$ comes from the bounds $\chi, \gamma \le 1/2$.

Theorem 3.2 (Rate of convergence bound in the shape theorem). *Assume that $\mathbb{E}e^{\alpha t_e} < \infty$ for some $\alpha > 0$. There is $C > 0$ such that*

$$\mathbb{P}\left((t - C\sqrt{t \log t})\mathcal{B} \subset B(t) \subset (t + C\sqrt{t\log t}) \textit{ for all large } t \right) = 1.$$

It is reasonable to believe, as we saw above, that a stronger shape theorem may hold, with $\sqrt{t}$ replaced by $t^\chi = t^\gamma$ for the fluctuation exponents explained above.

3.2 Scaling exponents and the KPZ relation

There are no accepted definitions of the exponents χ and γ from the last section. One can define, as in [5], directional p-fluctuation exponents ($p \ge 1$) in direction $x \in \mathbb{Z}^d$ as

$$\underline{\chi}_p(x) = \liminf_{n\to\infty} \frac{\log \|T(0,nx) - \mathbb{E}T(0,nx)\|_p}{\log n}$$

and

$$\overline{\chi}_p(x) = \limsup_{n\to\infty} \frac{\log \|T(0,nx) - \mathbb{E}T(0,nx)\|_p}{\log n},$$

where $\|X\|_p = (\mathbb{E}|X|^p)^{1/p}$, and nonrandom fluctuation exponents

$$\underline{\gamma}(x) = \liminf_{n\to\infty} \frac{\log(\mathbb{E}T(0,nx) - g(nx))}{\log n} \text{ and } \overline{\gamma}(x) = \limsup_{n\to\infty} \frac{\log(\mathbb{E}T(0,nx) - g(nx))}{\log n}.$$

At the moment, it is not known if $\underline{\chi}_p(x) = \overline{\chi}_p(x)$ for any p or x generally, or if $\underline{\gamma}(x) = \overline{\gamma}(x)$. We can precisely state the bounds from the last section in terms of these exponents: it is known, combining work of Alexander [1], Kesten [28], and Auffinger-Damron-Hanson [5] that under an exponential moment assumption (this can be weakened in various cases),

$$0 \le \underline{\chi}_p(x) \le \overline{\chi}_p(x) \le 1/2 \text{ for all } p \ge 1 \text{ and } x \ne 0$$

and

$$-1 \leq \underline{\gamma}(x) \text{ and } -1/2 \leq \overline{\gamma}(x) \leq 1/2 \text{ for all } x \neq 0.$$

Our lack of information on the model leads to other (stronger) direction-independent definitions. We will present these below while discussing the so-called KPZ scaling relation.

The last exponent we need is the "wandering exponent," which measures the maximal displacement of the (random) geodesics from Euclidean straight lines. Following Chatterjee [9], for any nonzero x, let $D(0, x)$ be the maximal Euclidean distance between any point on a geodesic (minimizing path for T) from 0 to x and the line segment connecting 0 and x. It is reasonable to believe that

$$\mathbb{E}D(0, x) \sim \|x\|^{\xi}$$

for some (dimension-dependent) $\xi = \xi(d)$. The predictions are that in two dimensions, $\xi = 2/3$, that ξ decreases with d to 1/2, but might always be $> 1/2$. In fact, these statements can be read directly off of the similar predictions for χ, along with the conjectured KPZ scaling relation

$$\chi = 2\xi - 1.$$

This relation is expected to be "universal." That is, it does not depend on the distribution of the t_e's, and does not even depend on the dimension d, as long as the distribution of the t_e's is reasonable, say with no atoms, and has enough moments. There are heuristic arguments from physics for this relation in [29], but to date, there is no proof that is valid under only edge-weight assumptions. One major difficulty is that there is no accepted definition of exponents. For instance, the exponents $\overline{\chi}_p$ and $\underline{\chi}_p$ defined above always exist, but unless one can show they are equal, they are not so helpful. Furthermore, the current bounds on the exponent ξ depend even on which definition is taken! In Newman-Piza [38] and Licea-Newman-Piza [31], the only work giving general bounds on ξ, one has versions of

$$\xi \geq 1/(d+1),\ \xi \geq 1/2,\ \xi \geq 3/5, \text{ and } \xi \leq 3/4,$$

depending on the definition of ξ. For example, the first bound is valid for a quite general definition of ξ, the second for a point-to-line geodesic wandering exponent, the third for a more restricted point-to-line exponent, and the fourth only in directions of "positive curvature" of the limiting shape. So, for instance, it is not even known at this point if $\mathbb{E}D(0, ne_1) = o(n)$.

Newman-Piza provided the first rigorous argument for the inequality $\chi \geq 2\xi - 1$, and it is essentially from this inequality and the known bound $\chi \leq 1/2$ that they derive $\xi \leq 3/4$. It took another 16 years for a rigorous argument, due to Chatterjee, for the other inequality, $\chi \leq 2\xi - 1$, even under strong assumptions of existence of exponents. A couple of months later, a simplified proof due to Auffinger-Damron [4] appeared, which removed a technical assumption on the valid class of distributions. We stress though that these inequalities are still conditional: they assume existence of exponents χ and ξ.

Below we will give a proof sketch for the KPZ relation. We begin with Chatterjee's exponents.

Definition 3.3. χ_u is the smallest number such that for any $\chi' > \chi_u$, there exists $\alpha > 0$ such that

$$\sup_{x \neq 0} \mathbb{E} \exp\left(\alpha \frac{|T(0, x) - \mathbb{E}T(0, x)|}{\|x\|^{\chi'}}\right) < \infty,$$

and χ_ℓ is the largest number such that for any $\chi'' < \chi_\ell$, one has

$$\inf_{x \neq 0} \frac{\mathrm{Var}\, T(0,x)}{\|x\|^{2\chi''}} > 0.$$

ξ_u is the smallest number such that for any $\xi' > \xi_u$, there exists $\beta > 0$ such that

$$\sup_{x \neq 0} \mathbb{E} \exp\left(\beta \frac{D(0,x)}{\|x\|^{\xi'}} \right) < \infty$$

and ξ_ℓ is the largest number such that for any $\xi'' < \xi_\ell$,

$$\inf_{x \neq 0} \frac{\mathbb{E} D(0,x)}{\|x\|^{\xi''}} > 0.$$

There are two important things to notice about these definitions. First, they are not directional, as they all take supremum or infimum over all directions x. So, for instance, if we assume that $\chi_u = \chi_\ell$ (as we will in the theorem below), then we are assuming that random fluctuations are the same in all directions, which rules out the case of the class $\mathcal{M}_p$ from Section 2.1 (see [46] for more details). For continuous distributions, though, this should be true. Next, the upper exponents are somewhat stronger than we might want, as they incorporate information about exponential concentration of the variables $T(0,x)$ and $D(0,x)$. In other words, these definitions imply (by an application of Markov's inequality for the exponential): for $\chi' > \chi_u$, there exists $C_1, c_2 > 0$ such that for all x

$$\mathbb{P}\left(|T(0,x) - \mathbb{E}T(0,x)| > \|x\|^{\chi'} \right) \leq C_1 \exp\left(-\|x\|^{c_2}\right),$$

and for $\xi' > \xi_u$, there are C_3, c_4 such that for all x,

$$\mathbb{P}\left(D(0,x) \geq \|x\|^{\xi'} \right) \leq C_3 \exp\left(-\|x\|^{c_4}\right). \tag{3.1}$$

In fact, by using Alexander's technique (as shown in [9]), the first inequality can be upgraded to

$$\mathbb{P}\left(|T(0,x) - g(x)| > \|x\|^{\chi'} \right) \leq C_1 \exp\left(-\|x\|^{c_2}\right). \tag{3.2}$$

Now we state the KPZ relation that has been rigorously proved to date, combining the results of Chatterjee [9] and Auffinger-Damron [4].

Theorem 3.4 (Scaling relation)**.** *Assume that $\mathbb{P}(t_e = 0) < p_c$. Then if $\chi := \chi_\ell = \chi_u$ and $\xi := \xi_\ell = \xi_u$, one has $\chi = 2\xi - 1$.*

The above relation has been extended to a positive temperature model, directed polymers in a random environment [3]. Next, curvature exponents κ for the boundary of the limit shape have been defined in [4], and it is believed that $\kappa = 2$. This corresponds to a boundary which is locally like Euclidean ball. For other curvature exponents, arguments from [4] suggest that a different relation holds: $\chi = \kappa\xi - (\kappa - 1)$. Last, versions of this relation have been proved in two dimensions in the exactly solvable version of LPP and other continuum models, where it is known that $\chi = 1/3$ and $\xi = 2/3$. In fact, the argument of [4] for the upper bound on χ is similar to the one by Wüthrich [44] and later by Johansson [26].

Sketch of proof of Theorem 3.4. We will give the proofs in the e_1 direction for simplicity. For the lower bound, suppose for a contradiction that $\xi > \frac{1+\chi}{2}$ and choose $\xi' \in \left(\frac{1+\chi}{2}, \xi\right)$. We will show that

$$\mathbb{P}(D(0, ne_1) > n^{\xi'}) \leq e^{-n^c} \tag{3.3}$$

for some $c > 0$. This itself suggests a contradiction, since $\xi' < \xi$. A more careful argument would actually give $\mathbb{E}D(0, ne_1) = O(n^{\xi'})$, which contradicts $\xi' < \xi = \xi_\ell$.

To show (3.3), put L to be the line segment connecting 0 and ne_1, and consider the set $S = \{x \in \mathbb{Z}^d : \|x - u\| < n^{\xi'} \text{ for some } u \in L\}$, with $\partial S = \{y \in \mathbb{Z}^d \setminus S : \|y - x\|_1 = 1 \text{ for some } x \in S\}$. If $D(0, ne_1) > n^{\xi'}$, then there is $z \in \partial S$ such that z is on a geodesic from 0 to ne_1, and so $A_z = \{T(0, ne_1) = T(0, z) + T(z, ne_1)\}$ occurs. Therefore $\mathbb{P}(D(0, ne_1) > n^{\xi'}) \leq \sum_{z \in \partial S} \mathbb{P}(A_z)$. One can now show that there is $c > 0$ depending on χ' such that

$$\mathbb{P}(A_z) \leq e^{-n^c} \text{ for all } z \in \partial S. \tag{3.4}$$

Summing over all z, we then obtain (3.3).

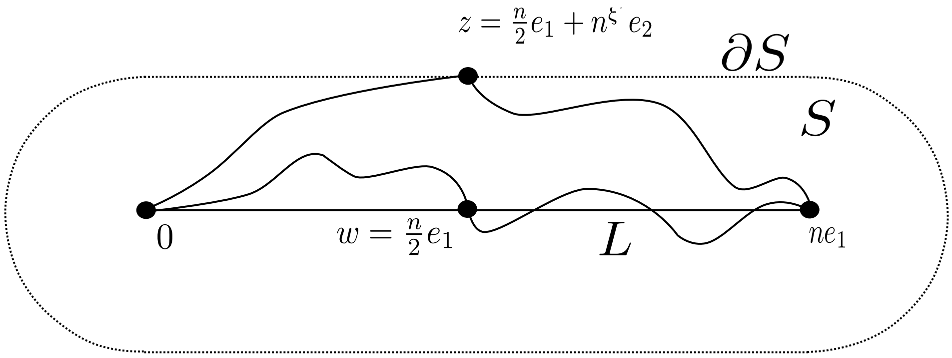

Figure 3.1: Illustration of the argument for the inequality $\chi \geq 2\xi - 1$. S is the set of vertices within distance $n^{\xi'}$ of L, the line segment connecting 0 and ne_1, and its boundary ∂S contains z. The idea is to show that with high probability, the passage time from 0 to ne_1 through z is larger than the passage time through $(n/2)e_1$. When this occurs, z cannot be on a geodesic from 0 to ne_1.

We will not show (3.4) for all z, but only for $z = (n/2)e_1 + n^{\xi'}e_2$, which may not be an integer point, but we will pretend it is, to avoid notational complexity. In this case, if A_z occurs, we write

$$\begin{aligned} 0 &\geq T(0, ne_1) - T(0, (n/2)e_1) - T((n/2)e_1, ne_1) \\ &= [T(0, z) - T(0, (n/2)e_1)] + [T(z, ne_1) - T((n/2)e_1, ne_1)]\,. \end{aligned} \tag{3.5}$$

(See Figure 3.1.) These bracketed terms have the same distribution, so we will only bound one of them. Write $w = (n/2)e_1$ and estimate

$$T(0, z) - T(0, w) = g(z) - g(w) + [T(0, z) - g(z)] + [T(0, w) - g(w)]\,. \tag{3.6}$$

To estimate $g(z) - g(w)$, we write it as

$$g\left((n/2)e_1 + n^{\xi'}e_2\right) - g((n/2)e_1) = \frac{n}{2}\left[g\left(e_1 + \frac{2n^{\xi'}}{n}e_2\right) - g(e_1)\right].$$

Here we must use information about the limit shape, namely that in certain directions it is known to have "positive curvature." It is known (see [9]) that there exists a point on the limit shape boundary near which the boundary is locally positively curved, and instead of doing the argument in that direction, we assume this condition in direction e_1. It amounts to the statement that there exist $\delta, c_1 > 0$ such that if v is a vector with $\|v\| < \delta$ and $v \perp e_1$, then $g(e_1 + v) - g(e_1) \geq c_1\|v\|^2$. We apply this in the above equation with $v = \frac{2n^{\xi'}}{n}e_2$. This is less than δ in norm for n large because we may assume that $\xi' < 1$. (If not, then $\xi \geq 1$, and in this case, the KPZ inequality reads $\chi \geq 1$, which is false, as known exponential concentration bounds imply that $\chi \leq 1/2$.) So we obtain

$$g(z) - g(w) \geq c_1\frac{n}{2}\left(\frac{2n^{\xi'}}{n}\right)^2 = c_2 n^{2\xi'-1}.$$

Returning to (3.6), we obtain $T(0,z) - T(0,w) \geq c_2 n^{2\xi'-1} + [T(0,z) - g(z)] + [T(0,w) - g(w)]$.

The same development works for the other term of (3.5), and we find that if A_z occurs, then

$$\begin{aligned} 0 \geq [T(0,z) - g(z)] + [T(0,w) - g(w)] + [T(z,ne_1) - g(ne_1 - z)] \\ + [T(w,ne_1) - g(ne_1 - w)] + 2c_2 n^{2\xi'-1}. \end{aligned}$$

This means that at least one of the four bracketed terms is at least $(c_2/2)n^{2\xi'-1}$ in absolute value. Thus using symmetry,

$$\mathbb{P}(A_z) \leq 2\mathbb{P}(|T(0,z) - g(z)| \geq (c_2/2)n^{2\xi'-1}) + 2\mathbb{P}(|T(0,w) - g(w)| \geq (c_2/2)n^{2\xi'-1}).$$

By our choice of ξ', we have $2\xi' - 1 > \chi$. We have assumed exponential concentration above scale χ' (see (3.2)), so these probabilities are (stretched) exponentially small in n. In other words, each one is smaller than e^{-n^c} for some $c > 0$. This shows (3.4) and completes the sketch of the bound $\chi \geq 2\xi - 1$.

We turn to the other inequality, $\chi \leq 2\xi - 1$. For technical reasons, we assume that $\chi > 0$; the other case can be proved using a different argument [9]. Suppose it is false and chose χ', χ'', ξ' such that

$$2\xi - 1 < 2\xi' - 1 < \chi' < \chi < \chi''. \tag{3.7}$$

We define the variable δT, which first appeared in Licea-Newman-Piza [31]: $\delta T = T - T'$, where $T = T(0, ne_1)$ and $T' = T(n^{\xi'}e_2, ne_1 + n^{\xi'}e_2)$. Because $\xi' > \xi$, these passage times are nearly independent, as they are with high probability equal to two passage times restricted to disjoint sets of edges ("tubes" of width $n^{\xi'}/2$ centered on the straight lines connecting their endpoints). Using the exponential concentration assumption on $D(0, ne_1)$ (from (3.1)), we then obtain if T'' is an independent copy of T,

$$\text{Var}\, T = \frac{1}{2}\mathbb{E}(T - T'')^2 \leq C_3\mathbb{E}(T - T')^2.$$

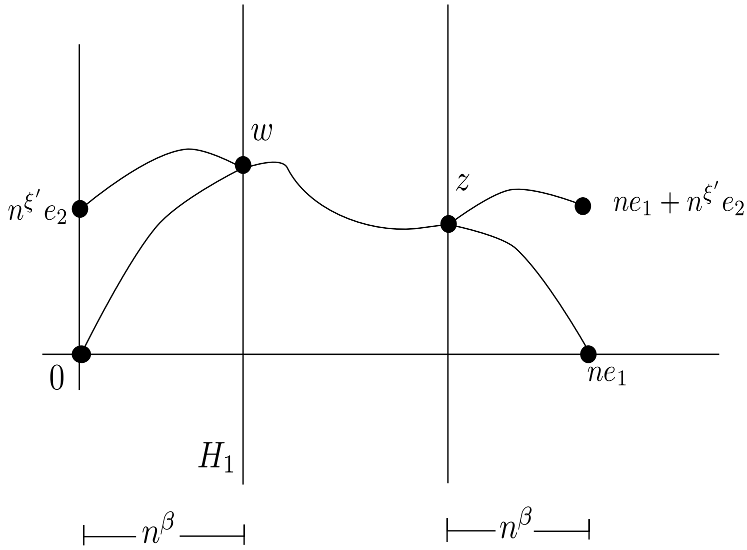

Figure 3.2: Illustration of the argument for the inequality $\chi \le 2\xi - 1$. The path that connects $n^{\xi'}e_2$ to $ne_1 + n^{\xi'}e_2$ through w (its first intersection with H_1) and z is a geodesic. The path that connects 0 to ne_1 through w and z is a possibly suboptimal path, and this produces inequality (3.9).

Since $\chi' < \chi$, we find

$$n^{2\chi'} \le C_4 \mathbb{E}(T - T')^2. \tag{3.8}$$

The next step is to upper bound $(T - T')^2$. Let w be the first intersection of a geodesic from $n^{\xi'}e_2$ to $n^{\xi'}e_2 + ne_1$ with the hyperplane $H_1 = \{y : y \cdot e_1 = n^\beta\}$ (for some $\beta < 1$ but very close to 1), and let z be the first intersection of this geodesic with the hyperplane $\{y : y \cdot e_1 = n - n^\beta\}$. Note that

$$\begin{aligned} T = T(0, ne_1) &\le T(0, w) + T(w, z) + T(z, ne_1) \\ &= T(0, w) - T(n^{\xi'}e_2, w) + T(n^{\xi'}e_2, ne_1 + n^{\xi'}e_2) \\ &+ T(z, ne_1) - T(z, ne_1 + n^{\xi'}e_2), \end{aligned}$$

so that

$$T - T' \le \left[T(0, w) - T(n^{\xi'}e_2, w)\right] + \left[T(z, ne_1) - T(z, ne_1 + n^{\xi'}e_2)\right]. \tag{3.9}$$

With (exponentially) high probability, w and z are not further than distance $C_5 n^{\xi'}$ from the e_1-axis, since $\xi' > \xi$ (see (3.1)). So using symmetry, we obtain with high probability

$$T - T' \le A + B,$$

where A and B have the same distribution as

$$D = \max\left\{|T(0, w) - T(n^{\xi'}e_2, w)| : w \in H_1, |w \cdot e_2| \le C_5 n^{\xi'}\right\}.$$

Reversing the roles of T and T', we obtain the same inequality for $T' - T$, and so

$$\mathbb{E}(T - T')^2 \leq C_6 \mathbb{E} D^2.$$

We last have to bound $\mathbb{E}D^2$. This is a maximum over many different passage times (to all $w \in H_1$ with $\|w\| \leq C_5 n^{\xi'}$). However, since we have assumed that the fluctuation exponent χ exists in a strong sense (there is exponential concentration — see (3.2)), it is possible to replace this maximum with simply one passage time, and $\mathbb{E}D^2$ will increase by only a logarithmic factor. Thus we can write

$$\mathbb{E}(T - T')^2 \leq C_7 (\log n) \mathbb{E}\left[T(0,w) - T(n^{\xi'} e_2, w)\right]^2, \tag{3.10}$$

where $w = n^\beta e_1 + n^{\xi'} e_2$. As in the proof of the other inequality, we decompose this difference as

$$\begin{aligned} T(0,w) - T(n^{\xi'} e_2, w) &= [T(0,w) - \mathbb{E}T(0,w)] - [T(n^{\xi'} e_2, w) - \mathbb{E}T(n^{\xi'} e_2, w)] \\ &\quad + [\mathbb{E}T(0,w) - g(w)] - [\mathbb{E}T(n^{\xi'} e_2, w) - g(w - n^{\xi'} e_2)] \\ &\quad + g(w) - g(w - n^{\xi'} e_2). \end{aligned}$$

And once again, our exponential concentration assumption (3.2) allows us to upper bound all the terms in the first two lines (with high probability) by $\|w\|^{\chi''}$. If $\xi \geq 1$, then our main inequality $\chi \leq 2\xi - 1$ is simply $\chi \leq 2\xi - 1$, where $2\xi - 1$ is ≥ 1, and we already know this to be true (as $\chi \leq 1/2$), so we can assume that $\xi < 1$. In this case, we can also enforce

$$\xi < \xi' < \beta < 1, \tag{3.11}$$

and we obtain that $\|w\| = \|n^\beta e_1 + n^{\xi'} e_2\| \leq C_8 n^\beta$. Therefore from (3.8) and (3.10),

$$n^{2\chi'} \leq C_9 (\log n) \left(C_{10} n^{\beta\chi''} + g(w) - g\left(w - n^{\xi'} e_2\right)\right)^2. \tag{3.12}$$

Again, we analyze the difference in g by mandating a curvature assumption. We calculate

$$g(w) + g(w - n^{\xi'} e_2) = g\left(n^\beta e_1 + n^{\xi'} e_2\right) - g(n^\beta e_1) = n^\beta \left(g\left(e_1 + n^{\xi' - \beta} e_2\right) - g(e_1)\right).$$

Our curvature condition here is the opposite of that in the previous inequality $\chi \leq 2\xi - 1$. That is, we assume that there are $C_{11}, \delta > 0$ such that if u satisfies $\|u\| < \delta$ and $u \perp e_1$, then $g(e_1 + u) - g(e_1) \leq C_{11} \|u\|^2$. Fortunately since the limit shape is convex, one can show that this inequality holds in almost every direction, so we will assume it in the e_1 direction, as it is written. Since $\xi' < \beta$, one has $n^{\xi' - \beta} < \delta$ for large n, and we obtain

$$g(w) - g(w - n^{\xi'} e_2) \leq C_{11} n^\beta n^{2\xi' - 2\beta'} = C_{11} n^{2\xi' - \beta}.$$

Last, we plug this back into (3.12) for $n^{2\chi'} \leq C_9 (\log n)(C_{10} n^{2\beta\chi''} + C_{11} n^{2(2\xi' - \beta)})$. This is true for all n large, so

$$2\chi' \leq \max\left\{2\beta\chi'', 2(2\xi' - \beta)\right\}.$$

This holds for all $\beta, \chi', \chi'', \xi'$ satisfying (3.7) and (3.11).

So take $\chi'' \downarrow \chi$ and $\chi' \uparrow \chi$ for fixed β, ξ' for

$$2\chi \leq \max\{2\beta\chi, 2(2\xi' - \beta)\}.$$

As $\beta < 1$, we find $\chi \leq 2\xi' - \beta$. Now take $\beta \uparrow 1$ and $\xi' \downarrow \xi$ to obtain $\chi \leq 2\xi - 1$. □

References

[1] K. Alexander. (1997). Approximation of subadditive functions and convergence rates in limiting-shape results. *Ann. Probab.*, 1, 30–55.

[2] A. Auffinger. and M. Damron. (2013). Differentiability at the edge of the limit shape and related results in first passage percolation. *Probab. Theory Relat. Fields*, 156, 193–227.

[3] A. Auffinger and M. Damron. (2013). The scaling relation $\chi = 2\xi - 1$ for directed polymers in a random environment. *ALEA Lat. Am. J. Probab. Math. Stat.*, 10, 857–880.

[4] A. Auffinger and M. Damron. (2014). A simplified proof of the relation between scaling exponents in first-passage percolation. *Ann. Probab.*, 42, 1197–1211.

[5] A. Auffinger, M. Damron, and J. Hanson. (2015). Rate of convergence of the mean for sub-additive ergodic sequences. *Adv. Math.*, 285, 138–181.

[6] A. Auffinger, M. Damron, and J. Hanson. (2017). 50 years of first-passage percolation. University Lecture Series, 68. *American Mathematical Society, Providence, RI.*

[7] A. Auffinger and S. Tang. (2015). On the time constant in high dimensional first passage percolation. *Electron. J. Probab.*, 24, 1–23.

[8] D. Boivin. (1990). First passage percolation: the stationary case. *Probab. Theory Relat. Fields*, 86, 491–499.

[9] S. Chatterjee. (2013). The universal relation between scaling exponents in first-passage percolation. *Ann. Math.*, 177, 663–697.

[10] H. Cohn, N. Elkies, and J. Propp. (1996). Local statistics for random domino tilings of the Aztec diamond. *Duke. Math. J.*, 85, 117–166.

[11] O. Couronné., N. Enriquez, and L. Gerin. (2011). Construction of a short path in high-dimensional first passage percolation. *Electron. Commun. Probab.*, 16, 22–28.

[12] J. T. Cox and R. Durrett. (1981). Some limit theorems for percolation with necessary and sufficient conditions. *Ann. Probab.*, 9, 583–603.

[13] M. Damron and M. Hochman. (2013). Examples of non-polygonal limit shapes in i.i.d. first-passage percolation and infinite coexistence in spatial growth models. *Ann. Appl. Probab.*, 23, 1074–1085.

[14] M. Damron and N. Kubota. (2014). Rate of converge in first-passage percolation under low moments. *Stoch. Proc. Appl.*, 126, 3065–3076.

[15] Random growth models. Papers from the AMS Short Course on Random Growth Models held in Atlanta, GA, January 2-3, 2017. Edited by M. Damron, F. Rassoul-Agha, and T. Seppäläinen. Proceedings of Symposia in Applied Mathematics, 72. *American Mathematical Society, Providence, RI,* 2018. ix+256 pp.

[16] D. Dhar. (1986). Asymptotic shape of Eden clusters, *On growth and form. Ed. H. E. Stanley and N. Ostrowsky, Martinus Nijhoff.*, 288–292.

[17] R. Durrett. (1984). Oriented percolation in two dimensions. *Ann. Probab.*, 12, 999–1040.

[18] R. Durrett and T. Liggett. (1981). The shape of the limit set in Richardson's growth model. *Ann. Probab.*, 9, 186–193.

[19] M. Eden. A two-dimensional growth process. 1961 *Proc. 4th Berkeley Sympos. Math. Statist. and Prob., Vol IV,* 223-239, *Univ. California Press, Berkeley, Calif.*

[20] G. Grimmett. (1999). *Percolation*. 2nd edition. *Springer, Berlin.*

[21] O. Häggström and R. Meester. (1995). Asymptotic shapes for stationary first passage percolation. *Ann. Probab.*, 23, 1511–1522.

[22] J. Hammersley and D. Welsh. First-passage percolation, subadditive processes, stochastic networks, and generalized renewal theory. 1965 *Proc. Internat. Res. Semin., Statist. Lab., Univ. California, Berkeley, Calif.,* 61–110, *Springer-Verlag, New York.*

[23] D. Jerison, L. Levine, and S. Sheffield. (2012). Logarithmic fluctuations for internal DLA. *J. Amer. Math. Soc.*, 25, 271–301.

[24] W. Jockusch, J. Propp, and P. Shor. (1998). Random domino tilings and the arctic circle theorem. preprint available at *arXiv:9801068*.

[25] K. Johansson. (2000). Shape fluctuations and random matrices. *Commun. Math. Phys.*, 209, 437–476.

[26] K. Johansson. (2000). Transversal fluctuations for increasing subsequences on the plane. *Probab. Theory Relat. Fields*, 116, 445–456.

[27] H. Kesten. (1986). Aspects of first passage percolation. *École d'Été de Probabilités de Saint Flour XIV,* Lecture Notes in Mathematics, 1180, 125–264.

[28] H. Kesten. (1993). On the speed of convergence in first-passage percolation. *Ann. Appl. Probab.*, 3, 296–338.

[29] J. Krug. (1987). Scaling relation for a growing surface. *Phys. Rev. A*, 36, 5465–5466.

[30] G. F. Lawler, M. Bramson, and D. Griffeath. (1992). Internal diffusion limited aggregation. *Ann. Probab.*, 20, 2117–2140.

[31] C. Licea, C. Newman, and M. Piza. (1996). Superdiffusivity in first-passage percolation. *Probab. Theory Relat. Fields* 106, 559–591.

[32] T. Liggett. (1985). An Improved Subadditive Ergodic Theorem. *Ann. Probab.*, 13, 1279–1285.

[33] R. Marchand. (2002). Strict inequalities for the time constant in first passage percolation. *Ann. Appl. Probab.*, 12, 1001–1038.

[34] J. B. Martin. (2006). Last-passage percolation with general weight distribution. *Markov Process. Related Fields*, 12, 273–299.

[35] J. B. Martin. (2004). Limiting shape for directed percolation models. *Ann. Probab.*, 32, 2908–2937.

[36] A. Martinsson. (2018). First-passage percolation on cartesian power graphs. *Ann. Probab*, 46, 1004–1041.

[37] P. Meakin and J. M. Deutch. (1986). The formation of surfaces by diffusion-limited annihilation. *J. Chem. Phys.*, 85, 2320.

[38] C. Newman and M. Piza. (1995). Divergence of shape fluctuations in two dimensions. *Ann. Probab.*, 23, 977–1005.

[39] D. Richardson. (1973). Random growth in a tessellation. *Proc. Cambridge Philos. Soc.*, 74, 515–528.

[40] H. Rost. (1981). Nonequilibrium behaviour of a many particle process: density profile and local equilibria. *Z. Wahrsch. Verw. Gebiete*, 58, 41–53.

[41] T. Seppäläinen. (1998). Hydrodynamic scaling, convex duality and asymptotic shapes of growth models. *Markov Process. Related Fields*, 4, 1–26.

[42] R. Tessera. (2014). Speed of convergence in first passage percolation and geodesicity of the average distance. *Ann. Inst. Henri Poincaré Probab. Stat.*, 54, 569–586.

[43] T. A. Witten Jr and L. M. Sander. (1981). Diffusion-Limited Aggreation, a kinetic critical phenomenon. *Phys. Rev. Lett.*, 47, 1400.

[44] M. V. Wüthrich. (1998). Scaling identity for crossing Brownian motion in a Possonian potential. *Probab. Theory Relat. Fields,* 112, 299–319.

[45] Y. Zhang. (1995). Supercritical behaviors in first-passage percolation. *Stoch. Proc. Appl.*, 59, 251–266.

[46] Y. Zhang. (2008). Shape fluctuations are different in different directions. *Ann. Probab.*, 36, 331–362.

Review of Ludwig Faddeev Memorial Volume. A Life in Mathematical Physics. Editors: M-L. Ge, A. J. Niemi, K. K. Phua and L. A. Takhtajan.

by JAN DEREZINSKI (Warsaw)

Ludwig Faddeev (1934-2017) was one of the greatest and most versatile mathematical physicists of the 20th century. World Scientific published recently a volume commemorating his life and scientific work.

The volume starts with a few short personal memories about Ludwig written by researchers that knew him and were influenced by him. Mathematical physics of the late 20th century has a very curious history. Ludwig was one of the central players in this history. Therefore, I found these texts very interesting. For instance, I learned a few facts and anecdotes from the contribution by Daniel Sternheimer. I also enjoyed a short text by Frank Wilczek, who described the reception of Faddeev-Popov ghosts in the high energy physics community in the early 70's.

Most of the volume consists of research articles devoted to various areas related to Ludwig's scientific work. They give a broad panorama of modern mathematical physics. For obvious reasons I will not comment on all 21 contributions contained in the volume. Let me describe briefly the content of a few of them.

D. R. Yafaev: "Analytic Scattering Theory for Jacobi Operators and Bernstein-Szegö Asymptotics of Orthogonal Operators".

Tridiagonal one-sidedly infinite matrices define linear operators on l^2 called *Jacobi operators*. They can be viewed as discrete versions of *Sturm-Liouville operators* on a half-line, (also called *Schrödinger operators* on a half-line).

In this elegant and readable article, Dima Yafaev investigates scattering theory for Jacobi operators. In many respects it is similar to the (well-known) scattering theory for one-dimensional Schrödinger operators. As Dima wrote, "The theory of [1-dimensional Schrödinger operators with short-range potentials] is to a large extent due to L. Faddeev".

Many properties familiar from the context of Schrödinger operators have close counterparts in the context of Jacobi operators. For instance, the *short-range condition* for perturbations is essentially the same and the existence of *wave operators* follows by the same argument in both cases.

Each Jacobi operator determines the so-called *3-term recurrence relation* for a certain family of orthogonal polynomials. Therefore, results about Jacobi operators can be translated into results on orthogonal polynomials.

C.-T. Chan, A. Mironov, A. Morozov, and A. Sleptsov: "Orthogonal Polynomials in Mathematical Physics".

I was surprised to learn that orthogonal polynomials "were originally introduced in the 19th century in a place, which may look strange from today's perspectives: in the theory of continued fraction, which is now far away from mainstreams in mathematical physics". Of course, orthogonal polynomials have many other applications.

In this article the authors sketch the theory of orthogonal polynomials on several levels. First they discuss their general theory. They stress the importance of the three-term relation satisfied by orthogonal polynomials (a concept, which I have already mentioned when discussing Yafaev's contribution).

Then they discuss some special classes of polynomials. The best known one is the class of the so-called *classical polynomials*. This class consists of *Jacobi polynomials*, which constitute the generic case, and *Hermite* and *Laguerre polynomials*, which are limits of Jacobi polynomials. Following Tom Koornwinder, the authors propose to call them the *very classical polynomials*, since there exists a much wider class that deserves the name *classical polynomials*. All of them can be expressed in terms of generalized hypergeometric series or basic hypergeometric series. In this class the generic case consists of the so-called *Askey-Wilson polynomials*. Their classification goes under the name of the *Askey scheme*. They satisfy either differential or difference equations.

Orthogonal polynomials have various applications in mathematical physics. Apart from the obvious ones, known to every student of physics, there are interesting applications to conformal field theory and matrix models, which the authors briefly discuss.

A. R. Its and A. Prokhorov: "On some Hamiltonian Properties of the Isomonodromic Tau Functions".

One of the most intriguing families of special functions is the family of *Painlevé functions*. They are solutions of one of 6 types of 2nd order nonlinear equations called *Painlevé equations*. The article of Its and Prokhorov systematically describes the theory that underlies these equations. Each of 6 Painlevé equations is related to a certain system of linear equations for two unknowns with up to 4 singularities on the Riemann sphere. Isomonodromy deformation equation for each of these systems can be written as a commuting system of Hamiltonian equations in appropriate Darboux coordinates. A central role in this analysis is played by the so-called *Jimbo-Miwa-Ueno* τ *function*, which is an analytic function on the universal covering of the parameter space. The authors show that if we adopt the Hamiltonian framework, then the τ function is closely related to the classical action for Painlevé equations.

A. S. Cattaneo, P. Mnev and N. Reshetikin: "Poisson Sigma Model and Semiclassical Quantization of Integrable Systems".

The starting point of this article is a symplectic manifold with a *prequantization line bundle* and a *real polarization*, so that one can use the formalism of *geometric quantization*. They consider two *quantum integrable systems*, transversal to one another. The semiclassical asymptotics of the scalar product of eigenfunctions of these two systems can be elegantly expressed in terms of the classical data. This scalar product depends on an arbitrary phase factor. This arbitrariness disappears when one considers the *transition probability* (the square of the absolute value of the scalar product), or more generally, the so-called *cyclic amplitudes*. The authors propose to rewrite cyclic amplitudes in terms of a path integral of a certain natural *Poisson sigma model*. The model has gauge invariance, and therefore to compute its path integrals one needs to use the *Batalin-Vilkoviski formalism*.

The authors consider also the famous *Kontsevich's star product* which yields a deformation quantization on an arbitrary Poisson manifold. They show how to express Kontsevich's star product in terms of path integrals of essentially the same Poisson sigma model.

J. Fröhlich: "Chiral Anomaly, Topological Field Theory and Novel States of Matter"

Most of the article is devoted to 2-dimensional systems in an external magnetic field. In low temperatures such systems exhibit a striking property: their conductance has many plateaux that correspond to fractional charges. This is the famous *fractional Hall effect*.

Jürg Fröhlich sketches an explanation of this phenomenon involving an effective *Chern-Simons theory* supported at the boundary of the sample. He argues that vertex operators should correspond to "quantum numbers" belonging to an A-, D- or E-root lattice. This leads to concrete predictions of the *Hall conductivity*. The author argues that these predictions closely match the experimental data.

The author also discusses 3-dimensional systems. Various concepts, coming mostly from high energy physics, lead to interesting implications for solid state physics. Let me mention some of these concepts: the *topological term* $\vec{E} \cdot \vec{B}$, an *axion field* à la Pecei-Quinn, *axion domain walls* and a *"Mott transition"* from an insulator to a state with non-vanishing bulk conductivity.

* * *

The editors wrote in the Preface: "Ludwig Faddeev (...) believed that mathematical, and in particular geometrical beauty must be considered an essential feature of any theoretical description of Nature that is correct". The articles contained in this volume (including those that I did not describe in my review) show that mathematical beauty is indeed an important ingredient of mathematical physics.

Let me quote the editors once again: "Ludwig Faddeev was a scientific giant whose impact significantly extends from pure mathematics to theoretical physics". In fact, Ludwig made significant contributions to many areas, including a few about which there are no articles in the memorial volume. The topics covered in the volume are mostly related to the later period of Ludwig's research. I would like to use this occasion to describe three topics of his (early) research that are not reflected in the memorial volume, and which I find important.

1. Faddeev equations and the quantum 3-body problem.

The *Faddeev equations* [8] express the resolvent of the 3-body Schrödinger operator in terms of resolvents of its 2-body subsystems. They are especially useful when we want to study the boundary value of the resolvent at the real line. These boundary values play an important role in scattering theory, e.g. they are the main ingredient of the well-known expressions for scattering amplitudes.

N-body systems possess an interesting and physically relevant scattering theory. The existence of *wave operators* for various channels is quite easy to prove and was known already in the 60's. Their *asymptotic completeness* was an open problem for a long time. [8] contained the first proof of asymptotic completeness for 3 body systems. The proof was based on the so-called *stationary approach*, whose main tool is the study of the boundary values of the resolvent. Unfortunately, Faddeev's proof needed rather strong assumptions, including some implicit assumptions on the subsystems. Later on improved proofs of asymptotic completeness following Faddeev's strategy appeared, e.g. Hagedorn's proof for 4-body systems [9]. To my knowledge, all of them required implicit assumptions about subsystems, except for the proof contained in the unpublished PhD thesis [12] of M. Loss under direction of I. M. Sigal about 3-body systems with rather restrictive conditions on the potentials.

Later on mathematical physicists working on N-body scattering theory turned away from from the stationary method, and hence also from Faddeev equations and their generalizations. The *time-dependent approach*, which does need complicated resolvent equations, lead finally to a proof of asymptotic completeness for any number of particles, also in the presence of long-range potentials. Among the main contributors, let me mention Enss, Mourre, Sigal, Soffer, Graf and myself, see [4] and references therein.

The Faddeev equations, even if they turned out not to be as useful for the proof of asymptotic completeness as originally expected, are important. They are in fact often used by practitioners, especially in nuclear physics. Unfortunately, in recent years they rarely appear in the rigorous literature. I expect they will come back to mathematical physics, since they are a natural tool to study e.g. scattering amplitudes of three-body systems.

2. Quantum 3-body problem with point interactions.

Physicists often prefer to use point interactions instead of potentials—they involve only one parameter, which is more convenient to fit to experimental data.

For 2-body systems point interactions are well understood. In dimension 1 they can be viewed as the delta function multiplied by a coupling constant. In dimension 2 and 3 the delta function needs to be renormalized, and the interaction depends on one parameter called the *scattering length*. In dimension 4 and more there are no point interactions at all.

Ter-Martirosyan and Skornyakov proposed a Hamiltonian for 3 particles with point interactions in 3 dimensions. Unfortunately, this Hamiltonian turns out to have non-zero defficiency indices. Physically, this absence of self-adjointness is a consequence of triple collisions. Faddeev and Minlos [14, 15] showed that every self-adjoint extension of this Hamiltonian has a sequence of negative eigenvalues going down to $-\infty$, and thus is not bounded from below.

N-body point interactions have been a subject of interesting research also in recent years. The main direction of investigations seems to be finding additional conditions on the particles that guarantee the boundedness from below. For instance, one can consider two identical fermions of mass 1 together with a third particle of a different species of mass m. This Hamiltonian has several distinct qualitative behaviors ("phases") depending on the value of m and of the scattering length. In most, but not all, of these phases, this Hamiltonian is either self-adjoint and bounded from below or has bounded from below self-adjoint extensions. Let me list a few recent references on N-body point interactions: [13, 3, 17, 18].

In my opinion N-body Hamiltonians with point interactions are interesting and worth studying also when they are not bounded from below. Boundedness from below is not a necessary condition for physical relevance: after all, quantum systems in a lab are often unstable, e.g. they live for a few miliseconds. Such a short lifetime is often enough to study their properties. The Faddeev equations and their generalizations are probably a natural tool in the analysis of such Hamiltonians.

3. The infrared problem in QED.

The infrared problem in quantum physics has two aspects. One aspect is the long range of the Coulomb potential. As a result, the usual definition of the scattering operator fails. It is well understood how to fix this problem in the setting of the Schrödinger equation, even for

N-body systems: One needs to modify the free evolution by adding an appropriate logarithmic modification. This construction goes back to Dollard [6] and leads to a well-defined scattering operator

The other problem is the appearance of non-Fock representations of canonical commutation relations in asymptotic states. Figuratively speaking, the electron is accompanied with a "cloud of soft photons". This is also well understood in simple examples, where the photon field is quantized and the electric current is classical. This goes back to an old work of Bloch and Nordsiek [1]

Faddeev and Kulish [10] proposed an asymptotic condition for QED that combines a Dollard-type modified dynamics and non-Fock representations of the photon field. This is an interesting proposal, even though to my understanding it is not fully rigorous.

There exist some attempts to understand the infrared problem in toy models of quantum field theory, which are not Poincaré covariant. These models include the so-called *nonrelativistic QED*, the massless *Pauli-Fierz and Nelson model* [2, 5, 7, 16].

Fully relativistic interacting models can be defined in the perturbative setting by the *Epstein-Glaser method*. This method involves two steps: the first solves the ultraviolet problem by constructing the S-matrix as a formal power series in distributions smeared out with a coupling function vanishing at infinity. Taking the limit of a constant coupling function is called the *adiabatic limit*. The adiabatic limit is relatively easy for massive particles. For massless theories only the so-called *weak adiabatic limit* is known, see a recent paper by Duch [11] and references therein. A more satisfactory understanding of asymptotic states, scattering amplitudes and cross-sections would involve the so-called *strong adiabatic limit*. Unfortunately, the strong adiabatic limit seems much more difficult—one cannot even formulate it without imposing some kind of asymptotic conditions in the spirit of Faddeev-Kulish.

Acknowledgement. The financial support of the National Science Center, Poland, under the grant UMO-2014/15/B/ST1/00126, is gratefully acknowledged.

References

[1] F. Bloch and A. Nordsieck. Phys. Rev., 52: 54 (1937); F. Bloch, A. Nordsieck, Phys. Rev. 52, 59, (1937).

[2] T. Chen, J. Fröhlich, A. Pizzo: Infraparticle Scattering States in Non-Relativistic QED: I. The Bloch-Nordsieck Paradigm, Commun. Math. Phys. 294, (2010) 761-825.

[3] M. Correggi, G. Dell'Antonio, D. Finco, A. Michelangeli, A. Teta: Stability for a system of N fermions plus a different particle with zero-range interactions. Rev. Math. Phys. 24, 1250017 (2012).

[4] J. Dereziński and C. Gerard: Scattering theory of classical and quantum N-particle systems Texts and Monographs in Physics, Springer, 1997.

[5] J. Dereziński and C. Gerard: Scattering theory of infrared divergent Pauli-Fierz Hamiltonians, Ann. H. Poincaré 5 (2004) 523-577.

[6] Dollard, J.: Asymptotic convergence and Coulomb interaction, J. Math. Phys. 5 (1964) 729-738.

[7] W. Dybalski: “From Faddeev-Kulish to LSZ. Towards a non-perturbative description of colliding electrons” arXiv:1706.09057.

[8] L.Faddeev: “Mathematical Aspects of the Three Body Problem in Quantum Scattering Theory”, Steklov Institute 1963 (Russian); English translation: Israel Program for Scientific Translation, 1965.

[9] G. Hagedorn: Asymptotic completeness for classes of two, three and four particle Schrödinger operators, Trans. AMS 258 (1980) 1-75.

[10] P.P.Kulish and L.D.Faddeev, “Asymptotic condition and infrared divergence in quantum electrodynamics”, Theor. Math, Phys. 4 (1970) 745.

[11] P. Duch: “Weak Adiabatic Limit in Quantum Field Theories with Massless Particles”, Annales Henri Poincare 19 (2018), 875–935.

[12] M. Loss and I.M. Sigal: The three-body problem with threshold singularities, ETH preprint.

[13] R.A. Minlos “On point-like interaction of three particles: two fermions and another particle II”, Moscow Math. J. 14 (2014) 617-637.

[14] R.A. Minlos, L.D. Faddeev: On the point interaction for a three-particle system in quantum mechanics, Soviet Phys. Dokl. 6 (1962), 1072–1074.

[15] R.A. Minlos, L.D. Faddeev: Comment on the problem of three particles with point interactions Soviet Phys. Jept. 14 (1962), 1315–1316.

[16] G. Morchio, F. Strocchi: “The infrared problem in QED: A lesson from a model with Coulomb interaction and realistic photon emission”, arXiv:1410.7289.

[17] T. Moser and R. Seiringer, Stability of a Fermionic N + 1 Particle System with Point Interactions, Comm. Math. Phys. 356, 329–355 (2017).

[18] T. Moser and R. Seiringer, Stability of the 2 + 2 Fermionic System with Point Interactions, Math Phys Anal Geom (2018) 21: 19.

QMath14: Mathematical Results in Quantum Physics

12–16 August 2019, Aarhus University

The international conference *QMath14: Mathematical Results in Quantum Physics* will be held at the Department of Mathematics, Aarhus University, during August 12-16, 2019. The international advisory board consists of László Erdős (IST, Austria), Pavel Exner (Academy of Sciences, Czech Republic), Søren Fournais (Aarhus University, Denmark), Claude-Alain Pillet (University of Toulon, France), and Jan Philip Solovej (University of Copenhagen, Denmark). Local organizers are Søren Fournais and Jacob Schach Møller from Aarhus University, and Horia Cornean from Aalborg University.

This will be the fourteenth edition of a series of conferences on *Mathematical Results in Quantum Theory* (or QMath) initiated in 1987 by Pavel Exner and Petr Šeba.

Throughout the years, the main goal of these meetings has been to not only gather people with a strong interest in mathematical quantum mechanics, but also to create a forum where one can discuss and explore new quantum phenomena and develop new tools when the standard ones fail or become outdated.

Together with the *Spectral Days* and the Congress of IAMP, the QMath series has been one of the three major events regularly organized under the auspices of the International Association of Mathematical Physics.

There will be five main topical fields covered by the conference:

1. Spectral and scattering theory of one-particle deterministic Schrödinger operators;
2. Quantum information including entanglement issues and quantum computing;
3. Many-particle systems including self-interactions and interaction with external radiation fields;
4. Random ergodic large systems including spectral and dynamical localization;
5. Condensed matter theory including topological effects and quantum transport.

Each topic will have a dedicated Special Session consisting of several invited and contributed talks. The eleven plenary speakers who have already accepted the invitation are top specialists in their fields and work in recognized universities spread on three continents. Their names and affiliations can be consulted on the event's homepage, see the link above.

We expect between 150 and 200 participants, with the hope that many of them would be junior researchers. In this respect we plan to organize an introductory mini-course (tutorial) in Quantum Information Theory, a field which has not been central to the previous QMath conferences but which has experienced an explosive development during the last decade.

The mini-course will take place on Monday morning and will be given by David Perez-Garcia (UCM, Madrid). The actual program of the conference will start on Monday after lunch and end on Friday at noon. We will have two plenary talks every day, followed by invited speakers and contributed talks in the various special sessions. There will also be a poster session.

There will be a conference fee of 1250 DKK (Early Registration Fee, approx EUR 170), which will cover the cost of conference materials, coffee breaks, and lunches.

Most of the conference related practical information is available on the conference homepage. More tourism and travel info about Aarhus can be found at *Visit Aarhus*.

We hope to see many of you in Aarhus in August 2019.

On behalf of the organizing committee

Søren Fournais

Announcement: IUPAP conference funding

The deadline for applications for IUPAP (International Union of Pure and Applied Physics) support for conferences to be held in 2020 is June 1, 2019.

Please consult the IUPAP policies and guidelines for conference support when preparing your application (see http://iupap.org/sponsored-conferences/conference-policies).

News from the IAMP Executive Committee

Recent conference announcements

Stochastic and Analytic Methods in Mathematical Physics

Sept. 2-7, 2019. Yerevan, Armenia.

This conference is partially supported by IAMP.

http://math.sci.am/conference/sammp2019

Quantum Random Walks, Quantum Graphs and their Spectra in Mathematics, Computer Science and Physics.

This conference is partially supported by IAMP.

Aug. 4-9, 2019. Lake Como School of Advanced Studies, Italy.

Mathematics of interacting QFT models

July 1-5, 2019. University of York, UK.

This conference is partially supported by IAMP.

Integrable Probability Summer School

May 27- June 8, 2019. University of Virginia, Charlottesville Virginia, USA.

Deadline for applications: March 1, 2019.

http://vipss.int-prob.org/

Mathematical Physics at the Crossings

Celebrating the 65th Brithday of George Hagedorn.

This conference is partially supported by IAMP.

May 20-24, 2019. Virginia Tech, USA.

http://www.math.vt.edu/HagedornFest/

From Quantum to Classical

April 22-26, 2019. CIRM, Luminy, France.

This conference is partially supported by IAMP.

https://conferences.cirm-math.fr/1982.html

Algebraic and Geometric Aspects in Quantum Field Theory

April 16-18, 2019. Freiburg im Breisgau, Germany

http://home.mathematik.uni-freiburg.de/murro/AGAQFT/index.html

Open positions

Full Professorship in Mathematical Physics at Heidelberg University

The Department of Mathematics and Computer Science and the Department of Physics and Astronomy of Heidelberg University invite applications for a tenured **Professorship (W3) in Mathematical Physics**. This professorship has been established in the context of the new cluster of excellence STRUCTURES, within the framework of the German Excellence Strategy, and it is to play a major role in connecting research in the two departments and establishing the interdisciplinary academic environment of the cluster of excellence.

Applicants should have made important, internationally recognized contributions to mathematical physics. Special attention will be given to candidates who can demonstrate their ability to enhance and complement the existing ties between mathematics and physics at Heidelberg University, in particular between analysis/stochastics and complex classical or quantum systems.

The new professor will be a member of the Department of Mathematics and Computer Science or the Department of Physics and Astronomy, depending on his/her profile and preferences, and will be co-opted to the respective other department. The successful candidate is expected to teach on all levels in his/her department and to contribute specialized courses aimed at students of both departments.

Prerequisites for application are a university degree and (in accordance with Article 47 of the Higher Education Law of the State of Baden-Württemberg) a habilitation, a successfully evaluated junior professorship or equivalent qualifications.

The faculty intends to increase the number of women in teaching and research; women are therefore expressly invited to apply. Disabled persons with the same qualifications will be given preference.

Applications including curriculum vitae, description of scientific interests, list of publications (no reprints), and a record of teaching activities should be submitted electronically only (preferably as a *single* pdf file) until March 28, 2019, to Dekan der Fakultät für Physik und Astronomie, Universität Heidelberg, Im Neuenheimer Feld 226, D-69120 Heidelberg, Germany, dekanat@physik.uni-heidelberg.de. Applications should include an exposé, which describes concepts for interdisciplinary cooperation and a research plan.

For further information, please contact Manfred Salmhofer, salmhofer@uni-heidelberg.de.

Several PhD positions at Bielefeld University

The German-Korean International Research Training Group (IRTG) 2235 “Searching for the regular in the irregular: Analysis of singular and random systems” funded by the Deutsche Forschungsgemeinschaft (DFG), offers several PhD positions at Bielefeld University, starting April 1, 2019.

The IRTG is a joint research program established by the Faculty of Mathematics at Bielefeld University, Germany, and the Department of Mathematical Sciences at Seoul National University, South Korea. In a truly international and competitive environment, doctoral students will study singular and random systems.

The IRTG concentrates on advanced techniques from the mathematical field of Analysis together with latest developments in neighboring fields such as Mathematical Physics, Geometry, and Probability Theory. The focus will be on the mathematical analysis of problems which generically exhibit singular features or randomness. The topics include nonlinear wave equations, integro-differential equations, oscillator models, random matrices, generalized Dirichlet and magnetic energy forms, analysis on manifolds and fractal metric spaces. The IRTG offers a structured course program in English and a six-months research stay at Seoul National University. Deadline for applications is Jan. 31, 2019.

More details on the position, the required qualification and the application procedure, please see

https://irtg.math.uni-bielefeld.de/doctoral_positions/open

One postdoctoral position in Mathematical Quantum Field Theory at Aarhus University and Aalborg University

The Department of Mathematics at Aarhus University and the Department of Mathematical Sciences at Aalborg University invite applications for one postdoctoral position in Mathematical Physics funded by the Independent Research Fund Denmark via the project grant Mathematical Aspects of Ultraviolet Renormalization.

The position is for 31 months and is available from the first of April 2019. In 2020, the successful candidate will work at Aalborg University while he or she will spend the remaining part of the employment period at Aarhus University.

Applicants should have a PhD degree in Mathematics or Theoretical Physics obtained at most 4 years before the actual starting date of the position. The research project is located in the intersection of mathematical quantum field theory, spectral theory, and stochastic analysis, and applicants are expected to show corresponding research promise.

Applications including a cover letter, a curriculum vitae, and a list of publications should be send as one pdf file to Jacob Moller (jacob@math.au.dk) or Oliver Matte (oliver@math.aau.dk). 2-3 letters of recommendation should be send separately. The deadline for applications is 15 February 2019.

Postdoctoral Fellowship in Combinatorial and Algebraic Structures in Prague

The Center of Advanced Applied Sciences of the Czech Technical University in Prague announces a postdoctoral position in mathematics in the research team Combinatorial and algebraic structures. The position is for one year, starting on April 1st, 2019 (the precise date is negotiable), with the possibility of extension to a second year upon mutual agreement. The gross salary is approximately 45 000 CZK monthly before tax. There are no teaching duties associated with the position.

Applicants should have a PhD in mathematics, mathematical physics or theoretical computer science (or equivalent) obtained preferably after January 1, 2014. They must show strong research potential in at least one of the following fields: algebraic number theory and its application in nonstandard numeration systems, combinatorics on words and symbolic dynamical systems, structure theory of Lie algebras, their representations and applications to integrable systems.

An experience in the project topic area is an advantage but not necessary. The applications including 1. curriculum vitae (including list of publications), 2. brief research statement (past, current and future interests), 3. two letters of recommendation; should be sent by e-mail to Edita Pelantová (edita.pelantova@fjfi.cvut.cz) and Libor Snobl (Libor.Snobl@fjfi.cvut.cz). All documents should be submitted as pdf files. The letters of recommendation should be sent directly by the persons providing the reference. Complete application packages should be delivered before February 15, 2019; applications submitted later will be taken into consideration only if no suitable candidate is found among the candidates applying in due time. For any further information about the position please contact Edita Pelantová and Libor Snobl at the e-mail addresses above.

PhD position available in Grenoble

Principal Investigator: Nicolas Rougerie
Project: ERC Starting grant CORFRONMAT
Duration: 3 years, starting in the fall of 2019

Description: Applications are invited for a CNRS PhD position in mathematical physics. The position is based in Grenoble, at the "Laboratoire de Physique et Modélisation des Milieux Condensés" and is funded by the ERC strating grant "Correlated frontiers of many-body quantum mathematics and condensed matter physics".

We search for candidates with a strong research potential, preferably with a background in one or several of the following fields: Functional analysis, Spectral theory, Partial differential equations, Many-body quantum mechanics, Condensed matter physics.

Suggested research topics include but are not limited to: The many-body problem for quantum particles with fractional statistics (anyons), rigorous studies of fractional quantum Hall states, mean-field limits of bosonic equilibrium states.

Financial conditions: The salary is approximately 1300E after tax per month. The doctoral researcher will have access to funding for travel. The successful candidate will not have any teaching obligations, but can (and will be encouraged to) apply at the Université Grenoble-Alpes for a part-time teaching position.

Applications: Deadline for the first call of the application is April 2019. We reserve the right to leave the position open, to extend the application period and to consider candidates who have not submitted applications during the application period. Applications should be submitted by email to Nicolas Rougerie (nicolas.rougerie@lpmmc.cnrs.fr). Please provide: CV, a short letter explaining your motivations. An internship within the ERC project (e.g. in the context of a master thesis) is also a possibility during the spring of 2019.

For more information on these positions and for an updated list of academic job announcements in mathematical physics and related fields visit

http://www.iamp.org/page.php?page=page_positions

Benjamin Schlein (IAMP Secretary)

PHAN THANH NAM
Mathematics Institute
Ludwig Maximilian University
Theresienstrasse 39
80333 Munich, Germany
nam@math.lmu.de

MICHAEL DAMRON
School of Mathematics
Georgia Institute of Technology
Atlanta, GA 30332-0160, USA
mdamron6@math.gatech.edu

JAN DEREZINSKI
Dep. Math. Meth. in Phys.
Faculty of Physics
University of Warsaw
Pasteura 5
02-093 Warszawa, Poland
jan.derezinski@fuw.edu.pl

SØREN FOURNAIS
Department of Mathematics
Ny Munkegade 118
building 1530, 415
8000 Aarhus C, Denmark
fournais@math.au.dk

BENJAMIN SCHLEIN
Institut für Mathematik
Universität Zürich
Winterthurerstrasse 190
8057 Zürich, Switzerland
secretary@iamp.org

EVANS HARRELL
School of Mathematics
Georgia Institute of Technology
Atlanta, GA 30332-0160, USA
bulletin@iamp.org