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Bogoliubov Hamiltonians and one-parameter groups of Bogoliubov transformations

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On the bosonic Fock space, a family of Bogoliubov automorphisms corresponding to a strongly continuous one-parameter group of symplectic maps $(\mathrm{R}(t))_{t\in\mathbb{R}}$ is considered. We give conditions that guarantee it to be implemented by a strongly continuous one-parameter group $\mathrm{U}(t)$ of unitary operators. The generator of such $\mathrm{U}(t)$ will be called a Bogoliubov Hamiltonian. Given $(\mathrm{R}(t))_{t\in\mathbb{R}}$, a Bogoliubov Hamiltonian is defined up to an additive constant. We introduce two kinds of Bogoliubov Hamiltonians: type I, characterized by vanishing of the expectation value at the vacuum, and type II, characterized by the fact that its infimum equals zero. We give conditions so that they are well defined. We show that there exist cases when only H_{I} is well defined, even though the classical Hamiltonian is positive (which may be interpreted as a kind of an infrared catastrophe), and when only H_{II} is well defined (which means that one needs to introduce an infinite counterterm in the formula for the Hamiltonian). © *2007 American Institute of Physics.*
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I. INTRODUCTION

Our paper is devoted to a study of *bosonic Bogoliubov Hamiltonians*, that is, of self-adjoint operators on a bosonic Fock space formally given by an expression of the form

$$H=\int h(\xi)a^*(\xi)a(\xi)\mathrm{d}\xi+\frac{1}{2}\int v(\xi,\xi')a^*(\xi)a^*(\xi')\mathrm{d}\xi\mathrm{d}\xi'+\frac{1}{2}\int \bar{v}(\xi,\xi')a(\xi)a(\xi')\mathrm{d}\xi\mathrm{d}\xi'+c. \tag{1.1}$$

Above, $a^*(\xi)/a(\xi)$ define creation/annihilation operators on the bosonic Fock space $\Gamma_s(\mathfrak{h})$, where, just for the sake of notation, we assumed that $\mathfrak{h}=L^2(\Xi,\mathrm{d}\xi)$ for some measure space $(\Xi,\mathrm{d}\xi)$. $h(\xi)$ is a real function on Ξ, $v(\xi,\xi')$ is a complex symmetric function on $\Xi\times\Xi$, and c is a real constant (which may be infinite).

In the whole paper we will consider only the bosonic case; therefore, in what follows we will usually drop the word *bosonic*.

Actually, in what follows we will almost always use a more compact, but perhaps less transparent notation:

$$H=\mathrm{d}\Gamma(h)+\frac{1}{2}a^*(v)+\frac{1}{2}a(v)+c. \tag{1.2}$$

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In the case of a finite dimensional $\mathfrak{h}$, Eq. (1.2) will be actually a correct (not just formal) definition of a Bogoliubov Hamiltonian. In this case, h is interpreted as a self-adjoint operator on $\mathfrak{h}$, and v can be interpreted in two ways: either as an antilinear operator on $\mathfrak{h}$ satisfying $\bar{v}=v^*$ or as a vector in $\otimes_s^2\mathfrak{h}$—a symmetric two-particle vector.

There exist explicit formulas for various quantities related to Bogoliubov Hamiltonians. Therefore, they are often used in physics literature as useful exactly solvable models. There also exists an extensive rigorous literature devoted to Bogoliubov Hamiltonians, starting with the work of Friedrichs.[1] Later many authors, often independent of one another, studied this problem, among them one should mention Berezin,[2] Ruijsenaars,[3,4] Araki[5] and Araki and Yamagami,[6] and more recently, Matsui and Shimada.[7] For parallel results about fermionic Bogoliubov Hamiltonians, we refer the reader, e.g., to Refs. 8–11.

Bogoliubov Hamiltonians are very well understood in the case of a finite number of degrees of freedom. Their theory becomes more difficult when the number of degrees of freedom is infinite (in other words, when the one-particle Hilbert space $\mathfrak{h}$ is infinite dimensional). Note, however, that even in the case of a finite number of degrees of freedom properties of Bogoliubov Hamiltonians are interesting. Actually, one of our results concerning this case—the formula for the infimum of a Bogoliubov Hamiltonian given in Theorem 3.2—seems to appear in the literature for the first time.

The main topic of our paper is the study of Bogoliubov Hamiltonians in the case of an infinite number of degrees of freedom. We will not use formula (1.2) to define Bogoliubov Hamiltonians. This is due to several problems, e.g.,

- v may actually be not an element of $\otimes_s^2\mathfrak{h}$, but only an unbounded linear functional on this space, which means that $a^*(v)$ is not an operator but a form;
- c can be infinite, which means that the definition of H involves an infinite renormalization; and
- it is sometimes tricky to interpret sums of unbounded operators or forms.

In order to make our definition of a Bogoliubov Hamiltonian as general as possible, we start from the "classical system" associated with the bosonic Fock space $\Gamma_s(\mathfrak{h})$. To this end, it is convenient to introduce the space

$$\mathcal{Y} := \{(f,\bar{f}) : f \in \mathfrak{h}\}.$$

$\mathcal{Y}$ is equipped with a symplectic form

$$y_1\sigma y_2 = 2\,\mathrm{Im}(f_1|f_2),$$

where $y_1=(f_1,\bar{f}_1)$, $y_2=(f_2,\bar{f}_2)$. $\mathcal{Y}$ has the interpretation of the "space dual to the classical phase space."

Elements of $\mathcal{Y}$ naturally parametrize the so-called Weyl operators, for $y=(f,\bar{f})$ defined as

$$W(y) := \mathrm{e}^{ia^*(f)+ia(f)}.$$

Let R be a symplectic linear map R on $\mathcal{Y}$. We say that R is implementable iff there exists a unitary operator U on $\Gamma_s(\mathfrak{h})$ such that

$$W(Ry) = UW(y)U^*, \quad y \in \mathcal{Y}. \tag{1.3}$$

In the case of a finite number of degrees of freedom all symplectic transformations are implementable.

A necessary and sufficient criterion that guarantees the implementability of R was given by Shale in Ref. 12. To formulate it note that any linear operator on $\mathcal{Y}$ can be uniquely extended to a complex linear operator on $\mathfrak{h}\oplus\mathfrak{h}$, and thus can be written as a 2×2 matrix. In particular, we can write

$$R = \begin{pmatrix} P & Q \\ \bar{Q} & \bar{P} \end{pmatrix}. \tag{1.4}$$

Shale proved that R is implementable iff Q is Hilbert-Schmidt.

We say that a strongly continuous one-parameter group of symplectic transformations $\mathbb{R} \ni t \mapsto \mathrm{R}(t)$ is implementable iff there exists a strongly continuous one-parameter unitary group $\mathrm{U}(t)$ on $\Gamma_s(\mathfrak{h})$ which implements the group of Bogoliubov automorphisms associated with $\mathrm{R}(t)$:

$$\mathrm{U}(t)W(y)\mathrm{U}(t)^* = W(\mathrm{R}(t)y), \quad y \in \mathcal{Y}. \tag{1.5}$$

Again, in the case of a finite number of degrees of freedom all such $\mathrm{R}(t)$ are implementable. Their generators coincide with the self-adjoint operators of form (1.2).

One of our results—Theorem 4.2—describes a necessary and sufficient criterion for the implementability of a given symplectic group $t \mapsto \mathrm{R}(t)$. It says that $t \mapsto \mathrm{R}(t)$ is implementable iff $Q(t)$ [defined as in Eq. (1.4)] is Hilbert-Shmidt for all t and $\lim_{t\to 0}\|Q(t)\|_2 = 0$.

Another sufficient criterion is described in Theorem 4.3. Its advantage over Theorem 4.2 is that it uses directly the information about the generator of the group $\mathrm{R}(t)$, that is, the operator A such that $\mathrm{R}(t) = \mathrm{e}^{tA}$. Theorem 4.3 can be traced back to Ref. 2.

Now we are able to formulate our definition of a Bogoliubov Hamiltonian. We say that a self-adjoint operator H is a Bogoliubov Hamiltonian iff there exists a strongly continuous symplectic group $t \mapsto \mathrm{R}(t)$ such that

$$W(\mathrm{R}(t)y) = \mathrm{e}^{itH} W(y) \mathrm{e}^{-itH}.$$

Clearly, given an implementable one-parameter group of symplectic transformations, a Bogoliubov Hamiltonian is defined only up to an additive constant. Note that our definition bypasses problems that arise when one tries to define Bogoliubov Hamiltonians starting from expressions of form (1.2). In the case of a finite number of degrees of freedom, every symplectic group is of the form $\mathrm{R}(t) = \mathrm{e}^{tA}$, where $A = i\begin{pmatrix} h & -v \\ \bar{v} & -\bar{h} \end{pmatrix}$, h is self-adjoint and $v^* = \bar{v}$, and then H is given by Eq. (1.2), where c is an arbitrary constant.

One can ask whether there exists a natural choice that allows to fix a Bogoliubov Hamiltonian given a symplectic group $t \mapsto \mathrm{R}(t)$. One of the possibilities is to use the so-called Weyl (or symmetric) quantization of the classical Hamiltonian. The classical Hamiltonian of the group $t \mapsto \mathrm{R}(t)$ can be defined as

$$(\bar{z}, z) \mapsto \chi(\bar{z}, z) := \langle z | hz \rangle + \frac{1}{2}\langle z | v\bar{z} \rangle + \frac{1}{2}\langle \bar{z} | \bar{v} z \rangle. \tag{1.6}$$

The Weyl quantization of Eq. (1.6) equals

$$\begin{aligned} \mathrm{Op}(\chi) = {} & \frac{1}{2}\int h(\xi)a^*(\xi)a(\xi)\mathrm{d}\xi + \frac{1}{2}\int h(\xi)a(\xi)a^*(\xi)\mathrm{d}\xi + \frac{1}{2}\int v(\xi,\xi')a^*(\xi)a^*(\xi')\mathrm{d}\xi\mathrm{d}\xi' \\ & + \frac{1}{2}\int \bar{v}(\xi,\xi')a(\xi)a(\xi')\mathrm{d}\xi\mathrm{d}\xi' \end{aligned} \tag{1.7}$$

and corresponds to $c = \frac{1}{2}\mathrm{Tr}h$ in Eq. (1.2). This choice works well in the case of a finite number of degrees of freedom. In the general case it imposes a severe restriction on h. Therefore in general it is not practical, and usually impossible, to use Weyl-quantized Bogoliubov Hamiltonians and other choices are preferable. In our paper we discuss two of them.

The first choice is what we call the type I Bogoliubov Hamiltonian. In the case of a finite number of degrees of freedom it corresponds to putting $c = 0$. In the general case its definition is less transparent, we present it in Sec. V B directly on the level of $t \mapsto \mathrm{R}(t)$ and the Hamiltonian is defined through the unitary group it generates.

Another possible choice is what we call the type II Bogoliubov Hamiltonian. It is the Bogoliubov Hamiltonian whose spectrum has the infimum at zero. This is possible only if the corresponding classical Hamiltonian is positive.

We describe various criteria that guarantee a symplectic group to possess the type I or type II Bogoliubov Hamiltonian. For instance, in Corollary 5.1 we give conditions when a Bogoliubov Hamiltonian can be defined by the relative boundedness technique—in this case both the type I and II Bogoliubov Hamiltonians can be defined. We also present Theorem 5.3, which describes the essential self-adjointness of type I Bogoliubov Hamiltonians if v is Hilbert-Schmidt. This theorem and its proof are adapted from Ref. 2; note, however, that the proof in Ref. 2 is not fully rigorous. An even more general criterion for the existence of a type I Bogoliubov Hamiltonian, given in Theorem 5.2, can also be traced back to Ref. 2.

There exists a natural class of symplectic one-parameter groups for which one can fully analyze the corresponding Bogoliubov Hamiltonians, describing in simple terms which groups are implementable and for which there exists the type I or type II Bogoliubov Hamiltonian. This is the class that can be written as a direct sum of independent symplectic groups of a single degree of freedom. It turns out, in particular, that in this class there exist situations, when type I Bogoliubov Hamiltonians do not exist; thus the constant c in Eq. (1.2) undergoes an infinite renormalization. There exist also situations, when the classical Hamiltonian is positive, but the quantum Bogoliubov Hamiltonian is unbounded from below; thus we have a version of the infrared catastrophe.

Let us mention that our paper is in some sense parallel to another paper of one of the authors,[13] where a classification of operators of the form

$$H = \int h(\xi)a^*(\xi)a(\xi)\mathrm{d}\xi + \int z(\xi)a^*(\xi)\mathrm{d}\xi + \int \bar{z}(\xi)a(\xi)\mathrm{d}\xi + c,$$

called *van Hove Hamiltonians*, was given. (Here again c can be infinite.) It turns out that there exist two natural choices of van Hove Hamiltonians, which in Ref. 13 were called type I and type II van Hove Hamitonians. Reference 13 presented simple necessary and sufficient criteria that allow us to determine whether for a given h, z a type I or type II van Hove Hamiltonian exists.

In the case of Bogoliubov Hamiltonians it is natural to ask for analogous criteria involving h and v. Our paper gives partial answers to this question.

II. FOCK SPACES AND CLASSICAL HAMILTONIAN

A. Generalities on the Fock space

Let $\mathfrak{h}$ be a Hilbert space with the scalar product $\langle\cdot|\cdot\rangle$ antilinear on the first argument and linear on the second. We denote by $\Gamma_s(\mathfrak{h})$ the bosonic Fock space over the one-particle space $\mathfrak{h}$,

$$\Gamma_s(\mathfrak{h}) \coloneqq \bigoplus_{n=0}^{\infty} \Gamma_s^n(\mathfrak{h}),$$

where $\Gamma_s^n(\mathfrak{h}) \coloneqq \otimes_s^n \mathfrak{h}$ denotes the symmetric n-fold tensor product of $\mathfrak{h}$, $\Omega \coloneqq (1,0,\ldots)$ will denote the vacuum vector, and

$$\Gamma_s^{\mathrm{fin}}(\mathfrak{h}) \coloneqq \{\Psi = (\Psi^{(0)},\ldots,\Psi^{(n)},\ldots) \in \Gamma_s(\mathfrak{h}) | \Psi^{(n)} = 0 \quad \text{for all but a finite number of } n\},$$

the finite particle space.

If h is an operator on $\mathfrak{h}$, $\mathrm{d}\Gamma(h)$ will denote the second quantization of h:

$$\mathrm{d}\Gamma(h)\lceil \otimes_s^n \mathfrak{h} \coloneqq \sum_{j=1}^{n} \underbrace{1 \otimes \cdots \otimes 1}_{j-1} \otimes h \otimes \underbrace{1 \otimes \cdots \otimes 1}_{n-j}.$$

The operator $N \coloneqq \mathrm{d}\Gamma(1)$ is called the number operator.

If q is an operator on $\mathfrak{h}$ of norm less than 1, we define $\Gamma(q):\Gamma_s(\mathfrak{h})\rightarrow\Gamma_s(\mathfrak{h})$ by $\Gamma(q)\lceil_{\otimes_s^n\mathfrak{h}}:=q\otimes\cdots\otimes q$.

For any $f\in\mathfrak{h}, a(f)$ and $a^*(f)$ denote the usual annihilation/creation operators on $\Gamma_s(\mathfrak{h})$:

$$a^*(f)\Psi := \sqrt{n+1}f\otimes_s\Psi, \quad \Psi\in\Gamma_s^n(\mathfrak{h}),$$

$$a(f)\Psi := \sqrt{n+1}(\langle f|\otimes 1^{\otimes n})\Psi, \quad \Psi\in\Gamma_s^{n+1}(\mathfrak{h}),$$

where $\langle f|\otimes 1^{\otimes n}:\Gamma_s^{n+1}(\mathfrak{h})\ni f_1\otimes\cdots\otimes f_{n+1}\mapsto\langle f|f_1\rangle f_2\otimes\cdots\otimes f_{n+1}\in\Gamma_s^n(\mathfrak{h})$. These operators are well defined on $\Gamma_s^{\text{fin}}(\mathfrak{h})$ and can be extended to $\mathrm{Dom}(N^{1/2})$.

The following estimates are well known and sometimes called N_τ-estimates.[14–17]

Proposition 2.1: *Let h be a positive self-adjoint operator on $\mathfrak{h}$, and $f\in\mathfrak{h}$. Then, for all $\Psi\in\mathrm{Dom}(\mathrm{d}\Gamma(h)^{1/2})$,*

$$\|a(f)\Psi\|\leq\|h^{-1/2}f\|\,\|\mathrm{d}\Gamma(h)^{1/2}\Psi\|,$$

$$\|a^*(f)\Psi\|\leq\|h^{-1/2}f\|\,\|(1+\mathrm{d}\Gamma(h))^{1/2}\Psi\|.$$

B. Quadratic annihilation and creation operators

Let $v\in\Gamma_s^2(\mathfrak{h})$. We define the annihilation and creation operators associated with v as follows:

$$a^*(v)\Psi := \sqrt{n+2}\sqrt{n+1}v\otimes_s\Psi, \quad \Psi\in\Gamma_s^n(\mathfrak{h}),$$

$$a(v)\Psi := \sqrt{n+2}\sqrt{n+1}(\langle v|\otimes 1^{\otimes n})\Psi, \quad \Psi\in\Gamma_s^{n+2}(\mathfrak{h}),$$

where $\langle v|\otimes 1^{\otimes n}:\Gamma_s^{n+2}(\mathfrak{h})\ni f_1\otimes\cdots\otimes f_{n+2}\mapsto\langle v|f_1\otimes f_2\rangle f_3\otimes\cdots\otimes f_{n+2}\in\Gamma_s^n(\mathfrak{h})$. These operators are well defined on $\Gamma_s^{\text{fin}}(\mathfrak{h})$ and can be extended to $\mathrm{Dom}(N)$.

Proposition 2.2: *Let $\Psi\in\Gamma_s^{\text{fin}}(\mathfrak{h})$, then*

$$\text{(i) } \|a(v)\Psi\|\leq\|v\|\,\|N\Psi\| \text{ and} \tag{2.1}$$

$$\text{(ii) } \|a^*(v)\Psi\|\leq\|v\|\,\|(N+2)^{1/2}(N+1)^{1/2}\Psi\|. \tag{2.2}$$

This result will be a particular case of Propositions 5.1 and 5.2 (Sec. V F).

Note also that if we write $v=\Sigma\lambda_n\phi_n\otimes\psi_n$, where $(\phi_n)_{n\in\mathbb{N}}$, $(\psi_n)_{n\in\mathbb{N}}$ are two orthonormal bases of $\mathfrak{h}$ and $(\lambda_n)_{n\in\mathbb{N}}$ is a sequence of positive numbers [with $\Sigma\lambda_n^2=\|v\|^2_{\Gamma_s(\mathfrak{h})}<+\infty$], then we have

$$a(v)=\sum\lambda_n a(\phi_n)a(\psi_n), \quad a^*(v)=\sum\lambda_n a^*(\phi_n)a^*(\psi_n), \tag{2.3}$$

where on the right hand side (rhs) a and a^* denote the usual annihilation/creation operators.

Before going further, we would like to make the link between elements of the two-particle space and real symmetric Hilbert-Schmidt operators on $\mathfrak{h}$, which will play an important role in the sequel. Let us fix a conjugation $f\mapsto\bar{f}$ on $\mathfrak{h}$. Then, if A is an operator on $\mathfrak{h}$ with domain $\mathcal{D}$, $\bar{A}$ will denote the operator defined by $\bar{A}f:=\overline{A\bar{f}}$ and with domain $\bar{\mathcal{D}}:=\{\bar{f}|f\in\mathcal{D}\}$. We denote by $B^2(\mathfrak{h})$ the set of all Hilbert-Schmidt operators and by $B_s^2(\mathfrak{h})$ the set of real symmetric (i.e., $\bar{v}=v^*$) Hilbert-Schmidt operators. It is well known that $\mathfrak{h}\otimes\mathfrak{h}$ and $B^2(\mathfrak{h})$ are isomorphic [the map $T:\mathfrak{h}\otimes\mathfrak{h}\ni\phi\otimes\psi\mapsto|\phi\rangle\langle\bar{\psi}|\in B^2(\mathfrak{h})$ extends by linearity and defines an isometry]. It is easy to see that $T(\Gamma_s^2(\mathfrak{h}))=B_s^2(\mathfrak{h})$. We will thus often use the same symbol to denote an element of $\Gamma_s^2(\mathfrak{h})$ and a symmetric element of $B^2(\mathfrak{h})$.

One then easily gets the following commutation relations.

Proposition 2.3: *For all $v,v'\in\Gamma_s^2(\mathfrak{h})$, $f\in\mathfrak{h}$,* and *h self-adjoint operator on $\mathfrak{h}$,*

$$[a^*(v),a(f)] = -2a^*(v\bar{f}), \quad [a(v),a^*(f)] = 2a(v\bar{f}), \tag{2.4}$$

$$[a(v),a^*(v')] = 4\mathrm{d}\Gamma(v'v^*) + 2\,\mathrm{Tr}(v^*v'), \tag{2.5}$$

$$[\mathrm{d}\Gamma(h),a^*(v)] = a^*(hv + v\bar{h}), \quad [\mathrm{d}\Gamma(h),a(v)] = -a(hv + v\bar{h}). \tag{2.6}$$

To end this section, we would like to introduce the exponential of the operators $a(v)$ and $a^*(v)$, which will be used to define the unitary operators U_{nat} (Sec. IV A), see, e.g., Ref. 4.

Proposition 2.4: *Let $v \in B_s^2(\mathfrak{h})$.*

(1) *For all $\Psi \in \Gamma_s^{\mathrm{fin}}(\mathfrak{h})$, there exists $s-\lim_{n\to+\infty}\Sigma_{k=0}^n 1/k!(\frac{1}{2}a(v))^k\Psi =: \mathrm{e}^{(1/2)a(v)}\Psi$, and $\mathrm{e}^{(1/2)a(v)}\Psi \in \Gamma_s^{\mathrm{fin}}(\mathfrak{h})$.*

(2) *If $\|v\|_{B(\mathfrak{h})} < 1$, then for all $\Psi \in \Gamma_s^{\mathrm{fin}}(\mathfrak{h})$, there exists $s-\lim_{n\to+\infty}\Sigma_{k=0}^n 1/k!(\frac{1}{2}a^*(v))^k\Psi =: \mathrm{e}^{(1/2)a^*(v)}\Psi$.*

Finally, we have the following.

Proposition 2.5: *Let $(v_l)_{l\in\mathbb{N}}$ be a sequence in $\Gamma_s^2(\mathfrak{h})$ such that $\|v_l\|_{B(\mathfrak{h})} < 1$ for all $l \in \mathbb{N}$ and $\lim_{l\to\infty}\|v_l\| = 0$. Then the operators $\mathrm{e}^{(1/2)a^*(v_l)}$ strongly converge to the identity on $\Gamma_s^{\mathrm{fin}}(\mathfrak{h})$.*

Proof: It suffices to prove it for vectors of the form $\Psi = a^*(f_1)\cdots a^*(f_m)\Omega$. Then for all $n \in \mathbb{N}$, and using Eq. (2.2),

$$\left\|\sum_{k=0}^{n}\frac{1}{k!}\left(\frac{1}{2}a^*(v_l)\right)^k\Psi - \Psi\right\|^2 \le \|f_1\|^2\cdots\|f_m\|^2\sum_{k=1}^{+\infty}\frac{(2k+m+1)!}{(2k)!}\left(\frac{1}{2^k k!}\right)^2\|a^*(v_l)^k\Omega\|^2 \le C_m\|v_l\|^k\|\Psi\|^2,$$

where $C_m < +\infty$. Hence $\|\mathrm{e}^{(1/2)a^*(v_l)}\Psi - \Psi\| \le C_m\|v_l\|^k\|\Psi\|^2$ and the result follows. □

C. Classical system

It is natural to associate with the Fock space $\Gamma_s(\mathfrak{h})$ the corresponding classical system. To this end we introduce two vector spaces:

$$\mathcal{Y} := \{(f,\bar{f}) : f \in \mathfrak{h}\}, \quad \bar{\mathcal{Y}} := \{(\bar{z},z) : z \in \mathfrak{h}\}.$$

The space $\bar{\mathcal{Y}}$ has the meaning of the *classical phase space* of our system. $\mathcal{Y}$ serves as its dual. If $(f,\bar{f}) \in \mathcal{Y}$ and $(\bar{z},z) \in \bar{\mathcal{Y}}$, then the duality is given by

$$2\,\mathrm{Re}\langle z|f\rangle = \langle z|f\rangle + \langle f|z\rangle.$$

Elements of $\mathcal{Y}$ naturally parametrize the so-called field operators and Weyl operators. For $y = (f,\bar{f}) \in \mathcal{Y}$ they are given by

$$\phi(y) := a^*(f) + a(f), \quad W(y) := \mathrm{e}^{ia^*(f)+ia(f)}.$$

If $\beta : \bar{\mathcal{Y}} \to \mathbb{C}$ is a function that belongs to an appropriate class, then its Weyl quantization will be denoted $\mathrm{Op}(\beta)$. Let us list some examples of Weyl quantizations relevant for our paper, where h is an operator on $\mathfrak{h}$ and $v \in \otimes_s^2\mathfrak{h}$:

(1) $\bar{\mathcal{Y}} \ni (\bar{z},z) \mapsto \beta(\bar{z},z) = \langle z|hz\rangle \Rightarrow \mathrm{Op}(\beta) = \mathrm{d}\Gamma(h) + \frac{1}{2}\,\mathrm{Tr}\,h$,

(2) $\bar{\mathcal{Y}} \ni (\bar{z},z) \mapsto \beta(\bar{z},z) = \langle z\otimes z|v\rangle \Rightarrow \mathrm{Op}(\beta) = a^*(v)$, and

(3) $\bar{\mathcal{Y}} \ni (\bar{z},z) \mapsto \beta(\bar{z},z) = \langle v|z\otimes z\rangle \Rightarrow \mathrm{Op}(\beta) = a(v)$.

If $z \in \mathfrak{h}$, then

$$\Omega_z := \mathrm{e}^{-|z|^2/2}\mathrm{e}^{a^*(z)}\Omega = W((-iz, i\bar{z}))\Omega$$

is the coherent vector localized around the point in phase space $(\bar{z},z) \ni \bar{\mathcal{Y}}$. This family of vectors can be used to define the Wick symbol of an operator B acting on $\Gamma_s(\mathfrak{h})$. It is equal to the function on phase space given by

$$s_B(\bar{z},z) := \langle \Omega_z | B\Omega_z \rangle.$$

Let us give examples of Wick symbols relevant for our paper:

(1) $B = \mathrm{d}\Gamma(h) \Rightarrow \bar{\mathcal{Y}} \ni (\bar{z},z) \mapsto s_B(\bar{z},z) = \langle z | hz \rangle$,
(2) $B = a^*(v) \Rightarrow \bar{\mathcal{Y}} \ni (\bar{z},z) \mapsto s_B(\bar{z},z) = \langle z \otimes z | v \rangle$, and
(3) $B = a(v) \Rightarrow \bar{\mathcal{Y}} \ni (\bar{z},z) \mapsto \mathrm{s}_B(\bar{z},z) = \langle v | z \otimes z \rangle$.

D. Symplectic transformations on $\mathcal{Y}$

Let R be a bounded linear map on $\mathcal{Y}$. We can extend it uniquely to a bounded complex linear map on $\mathfrak{h} \oplus \mathfrak{h}$, and thus we can represent it as

$$R = \begin{pmatrix} P & \bar{Q} \\ Q & \bar{P} \end{pmatrix}, \tag{2.7}$$

where P is a linear and Q an antilinear map on $\mathfrak{h}$.

We will often use the symplectic form on $\mathcal{Y}$, which for $y_1 = (f_1, \bar{f}_1)$, $y_2 = (f_2, \bar{f}_2)$ is given by

$$y_1 \sigma y_2 = 2\,\mathrm{Im}(f_1 | f_2).$$

We say that a map R on $\mathcal{Y}$ is symplectic iff it is invertible and preserves the symplectic form σ.

It is easy to see that a map R is symplectic if and only if $RJR^* = R^*JR = J$, where $J = \left(\begin{smallmatrix} i & 0 \\ 0 & -i \end{smallmatrix}\right)$, which is equivalent to

$$P^*P - Q^*Q = 1, \quad PP^* - \overline{QQ^*} = 1,$$

$$\overline{P^*}Q - \overline{Q^*}P = 0, \quad QP^* - \overline{PQ^*} = 0. \tag{2.8}$$

In particular, if R is symplectic, then $P^*P \geqslant 1$ and therefore P is invertible.

One of the central objects of our paper will be strongly continuous groups of symplectic transformations. Let $t \mapsto \mathrm{R}(t)$ be such a transformation and A its generator, that is, $\mathrm{R}(t) = \mathrm{e}^{tA}$. The *classical Hamiltonian of* $t \mapsto \mathrm{R}(t)$ is defined as the function on the phase space given by

$$\bar{\mathcal{Y}} \supset \mathrm{Dom}\, A \ni \bar{y} \mapsto \chi(\bar{y}) := \frac{1}{2} y\sigma A y.$$

[Note that in the case of the classical Hamiltonian we always normalize it so that $\chi(0) = 0$. An analogous normalization will not be always possible in the quantum case.]

Proposition 2.6: *Suppose that A is an operator on $\mathcal{Y}$, whose extension to $\mathfrak{h} \oplus \mathfrak{h}$ can be written as*

$$A = i\begin{pmatrix} h & -v \\ \bar{v} & -\bar{h} \end{pmatrix}, \tag{2.9}$$

where h is a self-adjoint operator with domain $\mathrm{Dom}(h)$, v is a bounded operator such that $v^* = \bar{v}$, and $\mathrm{Dom}(A) = \mathrm{Dom}(h) \oplus \mathrm{Dom}(\bar{h})$. Then A generates a strongly continuous one-parameter group $(\mathrm{R}(t))_{t \in \mathbb{R}}$ of symplectic maps and its classical Hamiltonian equals

$$\bar{\mathcal{Y}} \cap \mathrm{Dom}(\bar{h} \oplus h) \ni (\bar{z}, z) \mapsto \chi(\bar{z}, z) := \langle z | hz \rangle + \frac{1}{2}\langle z \otimes z | v \rangle + \frac{1}{2}\langle v | z \otimes z \rangle. \tag{2.10}$$

III. BOGOLIUBOV HAMILTONIANS IN THE CASE OF A FINITE NUMBER OF DEGREES OF FREEDOM

In this section we assume that $\mathfrak{h}$ has a finite dimension. Let $\mathrm{R}(t) = \mathrm{e}^{tA}$ be a continuous symplectic group on $\mathcal{Y}$. Then A is always of form (2.9), so that the corresponding classical Hamiltonian is given by Eq. (2.10).

In the case of a finite number of degrees of freedom we define Bogoliubov Hamiltonians associated with $t \mapsto R(t)$ by the formula

$$H = \mathrm{d}\Gamma(h) + \frac{1}{2}a^*(v) + \frac{1}{2}a(v) + c, \tag{3.1}$$

where c is an arbitrary real number. Note that when $c=0$ the Wick symbol of H coincides with its classical Hamiltonian: $s_H(\bar{z}, z) = \chi(\bar{z}, z)$.

A. Weyl quantization of the classical Hamiltonian

A special choice of a Bogoliubov Hamiltonian is given by setting $c = \frac{1}{2}\,\mathrm{Tr}\, h$. This leads to the Weyl quantization of the classical Hamiltonian (2.10):

$$\mathrm{Op}(\chi) = \mathrm{d}\Gamma(h) + \frac{1}{2}a^*(v) + \frac{1}{2}a(v) + \tfrac{1}{2}\mathrm{Tr}\, h. \tag{3.2}$$

The following theorem is well known.

Theorem 3.1: *Set*

$$K(t) := \overline{Q(t)P^{-1}(t)}, \quad L(t) := -P^{-1}(t)\bar{Q}(t), \tag{3.3}$$

Then

$$\mathrm{e}^{it\mathrm{Op}(\chi)} = \det(\bar{P}(t))^{-1/2}\mathrm{e}^{-(1/2)a^*(K(t))}\Gamma((P(t)^{-1})^*)\mathrm{e}^{-(1/2)a(L(t))}, \tag{3.4}$$

$$\mathrm{e}^{it\mathrm{Op}(\chi)}W(y)\mathrm{e}^{-it\mathrm{Op}(\chi)} = W(R(t)y), \quad y \in \mathcal{Y}. \tag{3.5}$$

B. Semibounded Bogoliubov Hamiltonians

Theorem 3.2: *In the case of a finite number of degrees of freedom, a Bogoliubov Hamiltonian is bounded from below iff the corresponding classical Hamiltonian is positive. Then we have*

$$\inf \mathrm{sp}\left(\mathrm{d}\Gamma(h) + \frac{1}{2}a^*(v) + \frac{1}{2}a(v)\right) = \frac{1}{4}\mathrm{Tr}\left[\begin{pmatrix} \bar{h}^2 - \bar{v}v & \bar{h}\bar{v} - \bar{v}h \\ hv - v\bar{h} & h^2 - v\bar{v} \end{pmatrix}^{1/2} - \begin{pmatrix} \bar{h} & 0 \\ 0 & h \end{pmatrix}\right]. \tag{3.6}$$

Proof: Let us prove Eq. (3.6). Let d denote the (complex) dimension of $\mathfrak{h}$, hence $\mathcal{Y}$ is a real symplectic space of dimension $2d$. We define, on $\bar{\mathcal{Y}}$, the operator

$$\beta := \frac{1}{2}\begin{pmatrix} v & h \\ \bar{h} & \bar{v} \end{pmatrix}.$$

The operator $\beta(h,v)$ is real symmetric and hence induces a real quadratic form on $\bar{\mathcal{Y}}$,

$$\bar{\mathcal{Y}} \ni (\bar{z},z) \mapsto \langle (z,\bar{z}) | \beta(\bar{z},z) \rangle,$$

which coincides with the classical Hamiltonian of $\mathrm{R}(t)$, denoted by $\chi(\bar{z},z)$ and given by Eq. (2.10). We also denote

$$\sigma = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}.$$

The map $\mathcal{Y}\times\mathcal{Y} \ni (y,y') \mapsto \langle \bar{y} | \sigma y' \rangle = \sigma(y,y')$ is the symplectic form on $\mathcal{Y}$.

Since β is positive real symmetric and σ is real antisymmetric, we can diagonalize them simultaneously, i.e., there is a basis $(y_1,\dots,y_{2d})$ of $\mathcal{Y}$ and positive real numbers $\lambda_1,\dots,\lambda_{2d}$ such that

$$\beta\bar{y}_j = \lambda_j y_j, \tag{3.7}$$

$$\sigma y_{2j-1} = \bar{y}_{2j}, \quad \sigma y_{2j} = -\bar{y}_{2j-1}, \tag{3.8}$$

where, if $y=(f,\bar{f})$, $\bar{y}=(\bar{f},f)$. Let $f_k \in \mathfrak{h}$ be such that $y_k = 1/\sqrt{2}(f_k,\bar{f}_k)$, and let $h_k = |f_k\rangle\langle f_k|$ and $v_k = |f_k\rangle\langle \bar{f}_k|$. Finally, we denote $\beta_k = \beta(h_k,v_k)$ and χ_k the associated quadratic form on $\bar{\mathcal{Y}}$. One then gets, using Eq. (3.7), $\beta = \Sigma_{j=1}^{2d}\lambda_j\beta_j$. Hence

$$\mathrm{Op}(\chi) = \sum_{j=1}^{2d} \lambda_j \, \mathrm{Op}(\chi_j),$$

where

$$\mathrm{Op}(\chi_j) = \mathrm{d}\Gamma(|f_j\rangle\langle f_j|) + \frac{1}{2}(a(f_j\otimes f_j) + a^*(f_j\otimes f_j)) + \frac{1}{2} = \phi(y_j)^2,$$

and where $\phi(y)$ denotes the field operators (Sec. II C), so that

$$\mathrm{Op}(\chi) = \sum_{j=1}^{d} (\lambda_{2j-1}\phi(y_{2j-1})^2 + \lambda_{2j}\phi(y_{2j})^2). \tag{3.9}$$

Now, since $(y_1,\dots,y_{2d})$ diagonalizes σ, we have

$$\sigma(y_{2j},y_{2k}) = \sigma(y_{2j-1},y_{2k-1}) = 0, \quad \text{and } \sigma(y_{2j},y_{2k-1}) = \delta_{jk}.$$

Hence, $[\phi(y_{2j-1}),\phi(y_{2j})] = i$ for all $j\in\{1,\dots,d\}$ while the other field operators commute. Therefore, by Eq. (3.9), and the properties of the harmonic oscillator,

$$\inf \mathrm{Op}(\chi) = \sum_{j=1}^{d} \sqrt{\lambda_{2j-1}\lambda_{2j}}.$$

On the other hand, using Eqs. (3.7) and (3.8), one gets

$$-(\sigma\beta)^2\bar{y}_{2j-1} = \lambda_{2j-1}\lambda_{2j}\bar{y}_{2j-1}, \quad -(\sigma\beta)^2\bar{y}_{2j} = \lambda_{2j-1}\lambda_{2j}\bar{y}_{2j}.$$

Therefore we have

$$\inf \mathrm{Op}(\chi) = \sum_{j=1}^{d} \sqrt{\lambda_{2j-1}\lambda_{2j}} = \frac{1}{2}\mathrm{Tr}\sqrt{-(\sigma\beta)^2}.$$

Finally, a simple computation gives $-(\sigma\beta)^2 = \frac{1}{4}\begin{pmatrix} \bar{h}^2-\bar{v}v & \bar{h}\bar{v}-\bar{v}h \\ hv-v\bar{h} & h^2-v\bar{v} \end{pmatrix}$, from which Eq. (3.6) follows since

$$\mathrm{d}\Gamma(h) + \frac{1}{2}a^*(v) + \frac{1}{2}a(v) = \mathrm{Op}(\chi) - \frac{1}{2}\mathrm{Tr}(h).$$

□

IV. IMPLEMENTABILITY OF ONE-PARAMETER GROUPS OF BOGOLIUBOV AUTOMORPHISMS

From now on, $\mathfrak{h}$ has an arbitary dimension. (Actually, in what follows only the case of infinite dimension is interesting.)

A. Bogoliubov implementer

Let $R=\begin{pmatrix} P & \bar{Q} \\ Q & \bar{P} \end{pmatrix}$ be a symplectic map on $\mathcal{Y}$.

Definition 4.1: *A symplectic map R is called implementable if and only if there exists a unitary operator U on $\Gamma_s(\mathfrak{h})$ such that $UW(y)U^{-1}=W(Ry)$, $\forall y \in \mathcal{Y}$. If it exists, U is called a Bogoliubov implementer of R.*

We define the operators K and L as follows:

$$K \coloneqq \overline{QP^{-1}}, \quad L \coloneqq -P^{-1}\bar{Q}. \tag{4.1}$$

The following result is well known (see Ref. 2, Sec. II.4, and Refs. 3, 4, and 12).

Theorem 4.1: *Let R be a symplectic map. The following are equivalent.*

(1) *R is implementable.*
(2) *$Q \in B^2(\mathfrak{h})$, or equivalently, $[R,J] \in B^2(\mathfrak{h} \oplus \mathfrak{h})$.*

If the above conditions are true, then

(i) *the operators K and L belong to $B_s^2(\mathfrak{h})$ and $\|K\|, \|L\| < 1$;*
(ii) *the operator*

$$U_{\mathrm{nat}} \coloneqq \det(1 - K^*K)^{1/4} \mathrm{e}^{-(1/2)a^*(K)} \Gamma((P^{-1})^*) \mathrm{e}^{-(1/2)a(L)} \tag{4.2}$$

is well defined on $\Gamma_s^{\mathrm{fin}}(\mathfrak{h})$, extends to a unitary operator on $\Gamma_s(\mathfrak{h})$, and implements R;

(iii) *all Bogoliubov implementers of R are proportional to U_{nat}; and*
(iv) *U_{nat} is the only Bogoliubov implementer whose expectation value on the vacuum is positive: $\langle \Omega | U_{\mathrm{nat}} \Omega \rangle = \det(1-K^*K)^{1/4} > 0$.*

We call U_{nat} the *natural Bogoliubov implementer of R*.

B. Bogoliubov dynamics and Bogoliubov Hamiltonians

Suppose $t \mapsto \mathrm{R}(t) = \begin{pmatrix} P(t) & \bar{Q}(t) \\ Q(t) & \bar{P}(t) \end{pmatrix}$ is a strongly continuous one-parameter group of symplectic maps. We denote by $K(t)$ and $L(t)$ the operators defined in Eq. (4.1) associated with $\mathrm{R}(t)$.

Definition 4.2: *A one-parameter symplectic group $\mathrm{R}(t)$ is called implementable if and only if there exists a strongly continuous unitary group $\mathrm{U}(t)$ such that, for all t, $\mathrm{U}(t)$ is a Bogoliubov implementer of $\mathrm{R}(t)$. If $\mathrm{R}(t)$ is implementable, the unitary group $\mathrm{U}(t)$ implementing $\mathrm{R}(t)$ is called a unitary Bogoliubov dynamics and its self-adjoint generator is called a Bogoliubov Hamiltonian [associated with $\mathrm{R}(t)$].*

Since a Bogoliubov implementer of a symplectic map R is unique up to a phase, if $\mathrm{R}(t)$ is implementable, then there exists $c(t) \in \mathbb{C}$, $|c(t)|=1$, such that $\mathrm{U}(t)=c(t)U_{\mathrm{nat}}(t)$, and where $U_{\mathrm{nat}}(t)$ is the natural Bogoliubov implementer of $\mathrm{R}(t)$. $c(t)$ will be called the natural cocycle for $\mathrm{U}(t)$.

One can actually prove that $\mathrm{R}(t)$ is unitarily implementable under very weak assumptions.

Theorem 4.2: *Suppose $\mathrm{R}(t)$ is a strongly continuous one-parameter symplectic group. Then $\mathrm{R}(t)$ is implementable if and only if $\|Q(t)\|_2 < \infty$ for all t and $\lim_{t\to 0}\|Q(t)\|_2 = 0$.*

Proof: Suppose $\mathrm{R}(t)$ is implementable. Using Theorem 4.1, we get that $\|Q(t)\|_2 < \infty$ for all t.

Let us prove that $\|K(t)\|_2 \to 0$ as t goes to zero. Let $\mathrm{U}(t)$ be a strongly continuous unitary group implementing $\mathrm{R}(t)$ and let

$$\alpha_t : \mathcal{B}(\Gamma_s(\mathfrak{h})) \ni B \mapsto \mathrm{U}(t) B \mathrm{U}(t)^* \in \mathcal{B}(\Gamma_s(\mathfrak{h})).$$

Clearly α_t is a weak$*$ continuous one-parameter group of $*$-automorphisms, and $\alpha_t(B) = U_{\mathrm{nat}}(t) B U_{\mathrm{nat}}(t)^*$ since we have $\mathrm{U}(t) = c(t) U_{\mathrm{nat}}(t)$, where $c(t)$ is the natural cocycle for $\mathrm{U}(t)$. Therefore the map

$$\mathbb{R} \ni t \mapsto \mathrm{Tr}(|\Omega\rangle\langle\Omega| \alpha_t(|\Omega\rangle\langle\Omega|)) = \det(1 - K(t)^* K(t))^{1/2}$$

is continuous. Since $\|K(t)\| < 1$, $\det(1 - K(t)^* K(t)) = \mathrm{e}^{\mathrm{Tr}(\log(1 - K(t)^* K(t)))}$. Moreover $K(0) = 0$, so

$$\lim_{t \to 0} \mathrm{Tr}(\log(1 - K(t)^* K(t))) = 0,$$

from which it follows that $\lim_{t\to 0} \|K(t)\|_2^2 = 0$, using $\|K(t)\|_2^2 \leq |\mathrm{Tr}(\log(1 - K(t)^* K(t)))|$.

Now note that

$$\|Q(t)\|_2^2 = \mathrm{Tr}\, K(t)^* K(t)(1 - K(t)^* K(t))^{-1} \leq (1 - \|K(t)\|^2)^{-1} \|K(t)\|_2^2 .$$

But $\|K(t)\|_2 \to 0$ implies $\|K(t)\| \to 0$. Hence $\lim_{t \to 0} \|Q(t)\|_2^2 = 0$. This proves the $\Rightarrow$ part of the theorem.

Let us prove the $\Leftarrow$ part of the theorem. The Shale condition is satisfied for all t; hence, we can construct $U_{\mathrm{nat}}(t)$, the natural implementer associated with $\mathrm{R}(t)$. Let us define the map

$$\mathcal{B}(\Gamma_s(\mathfrak{h})) \ni B \mapsto \alpha_t(B) := U_{\mathrm{nat}}(t) B U_{\mathrm{nat}}(t)^* \in \mathcal{B}(\Gamma_s(\mathfrak{h})).$$

Obviously, for all t, α_t is a weak$*$ continuous $*$-automorphism of $\mathcal{B}(\Gamma_s(\mathfrak{h}))$. Moreover, for all t, $s \in \mathbb{R}$,

$$\alpha_t(\alpha_s(W(y))) = \alpha_{t+s}(W(y)) = W(R(t+s)y), \qquad \forall\, y \in \mathcal{Y}.$$

Since the $*$-algebra generated by the Weyl operators is weak$*$ dense in $\mathcal{B}(\Gamma_s(\mathfrak{h}))$, this proves that α_t forms a one-parameter group of $*$-automorphisms of $\mathcal{B}(\Gamma_s(\mathfrak{h}))$.

In order to prove that it can be implemented by a self-adjoint operator H, it remains to show that this one-parameter group is weak$*$ continuous with respect to t (Ref. 18, Ex 3.2.35). Moreover, using the group property it suffices to prove that it is weak$*$ continuous at $t=0$. For that purpose, we shall prove that $t \mapsto U_{\mathrm{nat}}(t)$ is strongly continuous at $t=0$.

We have

$$K(t)^* K(t) = Q(t) Q(t)^* (1 + Q(t) Q(t)^*)^{-1}.$$

Hence $\|K(t)\|_2 < \infty$ for all t and $\lim_{t\to 0} \|K(t)\|_2 = 0$, i.e., the map $t \mapsto K(t)$ is continuous at $t=0$ in the Hilbert-Schmidt norm [recall that $K(0)=0$]. This together with Proposition 2.5 proves that $t \mapsto U_{\mathrm{nat}}(t)\Omega$ is continuous at $t=0$.

Now, for any $y \in \mathcal{Y}$ one has $U_{\mathrm{nat}}(t) W(y) \Omega = W(\mathrm{R}(t) y) U_{\mathrm{nat}}(t) \Omega$. Hence

$$\begin{aligned} \|U_{\mathrm{nat}}(t) W(y)\Omega - W(y)\Omega\| &\leq \|W(\mathrm{R}(t)y)(U_{\mathrm{nat}}(t)\Omega - \Omega)\| + \|(W(\mathrm{R}(t)y) - W(y))\Omega\| \\ &\leq \|U_{\mathrm{nat}}(t)\Omega - \Omega\| + \|(W(\mathrm{R}(t)y) - W(y))\Omega\|. \end{aligned}$$

The first term of the second line goes to zero as t goes to zero, and the second one as well since $t \mapsto \mathrm{R}(t)$ is strongly continuous and

$$\lim_{n\to+\infty}\|y_n-y\|=0 \Rightarrow s-\lim_{n\to+\infty} W(y_n)=W(y).$$

Thus, we have proven that $U_{\mathrm{nat}}(t)$ is strongly continuous at $t=0$ on $\mathrm{Span}\{W(f)\Omega, f\in\mathfrak{h}\}$. Since this subspace is dense in $\Gamma_s(\mathfrak{h})$ and the $U_{\mathrm{nat}}(t)$ are unitary, this ends the proof. □

C. Generator of unitarily implementable symplectic groups

In this section, we look for conditions on the generator A of $\mathrm{R}(t)$ which make it implementable. The following theorem is essentially due to Berezin[2] (Sec. III.6, Lemma 3).

Theorem 4.3: *Suppose that the assumption of Proposition 2.6 is satisfied. Define*

$$w(t) \coloneqq -i\int_0^t \mathrm{e}^{i\tau h} v \mathrm{e}^{i\tau\bar{h}} \mathrm{d}\tau. \tag{4.3}$$

Suppose also that for all t, $w(t)\in B^2(\mathfrak{h})$, the function $t\mapsto\|w(t)\|_2$ is locally integrable on $\mathbb{R}$ and continuous at $t=0$. Then $\mathrm{R}(t)$ is implementable.

Proof: *Let* $A_0 \coloneqq i\begin{pmatrix} h & 0\\ 0 & -\bar{h}\end{pmatrix}$. Since h is self-adjoint, it generates a one-parameter group of unitary maps $R_0(t)=\mathrm{e}^{tA_0}$. Let us also write $V\coloneqq i\begin{pmatrix} 0 & -v\\ \bar{v} & 0\end{pmatrix}$, so that $A=A_0+V$. We define $V(t)\coloneqq R_0(t)VR_0(-t)$ and $\widetilde{R}(t)\coloneqq \mathrm{R}(t)R_0(-t)$. Since V is bounded, we have

$$\widetilde{R}(t) = 1 + \int_0^t \widetilde{R}(\tau)V(\tau)\mathrm{d}\tau. \tag{4.4}$$

We introduce the following sequence of bounded operators:

$$\widetilde{R}_0(t)=1, \quad \widetilde{R}_{n+1}(t)=\int_0^t \widetilde{R}_n(\tau)V(\tau)\mathrm{d}\tau.$$

In particular we have

$$\widetilde{R}_1(t)=\int_0^t V(\tau)\mathrm{d}\tau=\begin{pmatrix} 0 & w(t)\\ \overline{w(t)} & 0\end{pmatrix}.$$

Hence, $\widetilde{R}_1(t)$ is Hilbert-Schmidt and

$$\|\widetilde{R}_1(t)\|_2=\sqrt{2}\|w(t)\|_2. \tag{4.5}$$

Then we shall prove that, for any $n\geqslant 1$

$$\|\widetilde{R}_{n+1}(t)\|_2 \leqslant (2\|v\|)^n \int_0^t \frac{(t-\tau)^{n-1}}{(n-1)!}\sqrt{2}\|w(\tau)\|_2\mathrm{d}\tau. \tag{4.6}$$

Indeed, using Eq. (4.5), an induction and $\|V(\tau)\|=2\|v\|$ for all τ, we get

$$\begin{aligned}\|\widetilde{R}_{n+2}(t)\|_2 &\leqslant 2\|v\|\int_0^t \|\widetilde{R}_{n+1}(\tau)\|_2\mathrm{d}\tau \leqslant (2\|v\|)^{n+1}\int_0^t \mathrm{d}\tau\int_0^\tau \mathrm{d}s\frac{(\tau-s)^{n-1}}{(n-1)!}\sqrt{2}\|w(s)\|_2\\ &= (2\|v\|)^{n+1}\int_0^t \frac{(t-s)^n}{n!}\sqrt{2}\|w(s)\|_2\mathrm{d}s.\end{aligned}$$

Moreover, we have $\widetilde{R}(t)-1=\Sigma_{n\geqslant 1}\widetilde{R}_n(t)$, hence

$$\|\tilde{R}(t) - 1\|_2 \leq \sqrt{2}\|w(t)\|_2 + 2\sqrt{2}\|v\| \int_0^t \mathrm{e}^{2(t-\tau)\|v\|}\|w(\tau)\|_2 \mathrm{d}\tau < +\infty. \tag{4.7}$$

Since $\mathrm{R}(t)R_0(-t)-1$ is Hilbert-Schmidt, so is $\mathrm{R}(t)-R_0(t)$, which proves that $Q(t)$ is Hilbert-Schmidt. Using Eq. (4.7) and the continuity of $\|w(t)\|_2$ at $t=0$, we get $\lim_{t\to 0}\|\tilde{R}(t)-1\|_2=0$. Thus we have $\lim_{t\to 0}\|\bar{Q}(t)\|_2=0$. □

V. TYPE I AND TYPE II BOGOLIUBOV HAMILTONIANS

A. Renormalized Bogoliubov dynamics

We will denote by $B^1(\mathfrak{h})$ the set of trace class operators on $\mathfrak{h}$ and by $\|\cdot\|_1$ the trace norm.

Theorem 5.1: *Let $t\mapsto \mathrm{R}(t)$ be a symplectic group. We assume that there exists a self-adjoint operator h_{ren} on $\mathfrak{h}$ such that $P(t)\mathrm{e}^{-ith_{\mathrm{ren}}}-1 \in B^1(\mathfrak{h})$ and $\lim_{t\to 0}\|P(t)\mathrm{e}^{-ith_{\mathrm{ren}}}-1\|_1=0$. Then $\mathrm{R}(t)$ is implementable, and the operators*

$$U_{\mathrm{ren}}(t) := \det(\bar{P}(t)\mathrm{e}^{it\bar{h}_{\mathrm{ren}}})^{-1/2}\mathrm{e}^{-(1/2)a^*(K(t))}\Gamma((P(t)^{-1})^*)\mathrm{e}^{-(1/2)a(L(t))} \tag{5.1}$$

form a Bogoliubov unitary dynamics implementing $\mathrm{R}(t)$. Their natural cocycle is given by

$$c_{\mathrm{ren}}(t) = \det(\bar{P}(t)\mathrm{e}^{it\bar{h}_{\mathrm{ren}}})^{-1/2}\det(1-K(t)^*K(t))^{-1/4}. \tag{5.2}$$

Note that the assumption of the theorem is trivially satisfied in finite dimension for any operator h_{ren}.

In the proof of Theorem 5.1, we will make use of the following lemmas.

Lemma 5.1: *Let B be a bounded operator and V a unitary operator such that $BV-1$ is trace class. Then $VB-1$ is trace class and* $\det(BV)=\det(VB)$.

Lemma 5.2: *Let $K,L\in B^2(\mathfrak{h})$ such that $\bar{K}=K^*$, $\bar{L}=L^*$ and $\|K\|<1$, $\|L\|<1$. Then*

$$\langle \mathrm{e}^{-(1/2)a^*(L)\Omega}|\mathrm{e}^{-(1/2)a^*(K)}\Omega\rangle = \det(1-L^*K)^{-1/2}.$$

Proof: Since K is Hilbert-Schmidt and $\bar{K}=K^*$, there exists an orthonormal basis of $\mathfrak{h}$, $(f_n)_n$, and a sequence λ_n such that $K=\Sigma\lambda_n|f_n\rangle\langle\bar{f}_n|$. Similarly, we can write $L=\Sigma\mu_m|g_m\rangle\langle\bar{g}_m|$. Therefore, we have

$$\langle \mathrm{e}^{-(1/2)a^*(L)}\Omega|\mathrm{e}^{-(1/2)a^*(K)}\Omega\rangle = \prod_{m,n}\langle \mathrm{e}^{-(1/2)\mu_m a^*(g_m)^2}\Omega|\mathrm{e}^{-(1/2)\lambda_n a^*(f_n)^2}\Omega\rangle$$
$$= \prod_{m,n}\sum_j\left(-\frac{1}{2}\right)^{2j}\frac{\bar{\mu}_m^j\lambda_n^j(2j)!}{(j!)^2}\langle g_m|f_n\rangle^{2j} = \prod_{m,n}(1-\bar{\mu}_m\lambda_n\langle g_m|f_n\rangle^2)^{-1/2}.$$

Now, it suffices to see that $L^*K=\Sigma_{m,n}\bar{\mu}_m\lambda_n\langle g_m|f_n\rangle|\bar{g}_m\rangle\langle\bar{f}_n|$. □

Proof of Theorem 5.1: $P(t)\mathrm{e}^{-ith_{\mathrm{ren}}}-1\in B^1(\mathfrak{h})$, hence so is $P(t)P(t)^*-1$. Using Eq. (2.8) this proves that $\mathrm{Q}(t)\in B^2(\mathfrak{h})$. Similarly, $\lim_{t\to 0}\|P(t)\mathrm{e}^{-ith_{\mathrm{ren}}}-1\|_1=0$ implies that $\lim_{t\to 0}\|Q(t)\|_2=0$. This proves that $\mathrm{R}(t)$ is implementable (Theorem 4.2).

Since $U_{\mathrm{nat}}(t)$ is unitary and $U_{\mathrm{ren}}(t)=c_{\mathrm{ren}}(t)U_{\mathrm{nat}}(t)$, to prove that $U_{\mathrm{ren}}(t)$ is a Bogoliubov implementer, it suffices to show that $|c_{\mathrm{ren}}(t)|=1$. Using Eq. (2.8), we have $1-K(t)^*K(t)=(\overline{P(t)P(t)^*})^{-1}$. Now

$$|\det(\bar{P}(t)\mathrm{e}^{it\bar{h}_{\mathrm{ren}}})^{-1/2}| = |\det(\bar{P}(t)\mathrm{e}^{it\bar{h}_{\mathrm{ren}}})\overline{\det((\bar{P}(t)\mathrm{e}^{it\bar{h}_{\mathrm{ren}}})^*)}|^{-1/4}$$
$$= |\det(\bar{P}(t)\mathrm{e}^{it\bar{h}_{\mathrm{ren}}})\det(\mathrm{e}^{-it\bar{h}_{\mathrm{ren}}}\overline{P(t)^*})|^{-1/4} = |\det(\overline{P(t)P(t)^*})|^{-1/4}.$$

We now prove that the operators $U_{\mathrm{ren}}(t)$ form a one-parameter group. As for $U_{\mathrm{nat}}(t)$, for all s and t there exists $\alpha(t,s)\in\mathbb{R}$ such that $U_{\mathrm{ren}}(t)U_{\mathrm{ren}}(s)=\mathrm{e}^{i\alpha(t,s)}U_{\mathrm{ren}}(t+s)$. Using Lemmas 5.1 and 5.2 we have

$$\begin{aligned}\langle\Omega|U_{\mathrm{ren}}(t)U_{\mathrm{ren}}(s)\Omega\rangle &= \det(\bar{P}(t)\mathrm{e}^{it\bar{h}_{\mathrm{ren}}})^{-1/2}\det(\bar{P}(s)\mathrm{e}^{is\bar{h}_{\mathrm{ren}}})^{-1/2}\langle \mathrm{e}^{-(1/2)a^*(L(t))}\Omega|\mathrm{e}^{-(1/2)a^*(K(s))}\Omega\rangle\\ &= \det(\bar{P}(t)\mathrm{e}^{it\bar{h}_{\mathrm{ren}}})^{-1/2}\det(\bar{P}(s)\mathrm{e}^{is\bar{h}_{\mathrm{ren}}})^{-1/2}\det(1-L(t)^*K(s))^{-1/2}\\ &= (\det(\mathrm{e}^{it\bar{h}_{\mathrm{ren}}}\bar{P}(t))\det(1+\bar{P}(t)^{-1}Q(t)\bar{Q}(s)\bar{P}(s)^{-1})\det(\bar{P}(s)\mathrm{e}^{is\bar{h}_{\mathrm{ren}}}))^{-1/2}\\ &= \det(\mathrm{e}^{it\bar{h}_{\mathrm{ren}}}(\bar{P}(t)\bar{P}(s)+Q(t)\bar{Q}(s))\mathrm{e}^{is\bar{h}_{\mathrm{ren}}})^{-1/2}\\ &= \det(\bar{P}(t+s)\mathrm{e}^{i(t+s)\bar{h}_{\mathrm{ren}}})^{-1/2} = \langle\Omega|U_{\mathrm{ren}}(t+s)\Omega\rangle.\end{aligned}$$

Therefore $\mathrm{e}^{i\alpha(t,s)}=1$ and $U_{\mathrm{ren}}(t)$ is a one-parameter group.

Finally we have to prove that $U_{\mathrm{ren}}(t)$ is strongly continuous. Using the group property together with the same argument as in Theorem 4.2, it suffices to prove that $t\mapsto U_{\mathrm{ren}}(t)\Omega$ is continuous at $t=0$. Now, $t\mapsto K(t)$ is continuous in the Hilbert-Schmidt norm since $\mathrm{R}(t)$ is implementable (Theorem 4.2), and, by assumption, $t\mapsto P(t)\mathrm{e}^{-ith_{\mathrm{ren}}}$ is continuous in the trace norm at $t=0$, thus so is the map $t\mapsto\det(P(t)\mathrm{e}^{-ith_{\mathrm{ren}}})$, which ends the proof. □

B. Bogoliubov dynamics of type I

Definition 5.1: *Let $t\mapsto\mathrm{R}(t)$ be an implementable symplectic group. We say that it is type* I *if and only if there exists a self-adjoint operator h on $\mathfrak{h}$ such that $P(t)f$ is differentiable at* 0 *for a dense subspace of $f\in\mathfrak{h}$, $(\mathrm{d}/\mathrm{d}t)P(t)f\lceil_{t=0}=ihf$, $P(t)\mathrm{e}^{-ith}-1\in B^1(\mathfrak{h})$, for all $t\in\mathbb{R}$, and* $\lim_{t\to 0}\|P(t)\mathrm{e}^{-ith}-1\|_1=0$. *In this case, h is defined uniquely, and we set*

$$U_{\mathrm{I}}(t) := \det(\bar{P}(t)\mathrm{e}^{it\bar{h}})^{-1/2}\mathrm{e}^{-(1/2)a^*(K(t))}\Gamma((P(t)^{-1})^*)\mathrm{e}^{-(1/2)a(L(t))}, \tag{5.3}$$

$$c_{\mathrm{I}}(t) := \det(\bar{P}(t)\mathrm{e}^{it\bar{h}})^{-1/2}\det(1-K(t)^*K(t))^{-1/4}. \tag{5.4}$$

Definition 5.2: *The operator $H_{\mathrm{I}}=(1/i)(\mathrm{d}/\mathrm{d}t)U_{\mathrm{I}}(t)\lceil_{t=0}$ is called a Bogoliubov Hamiltonian of type* I.

C. Bogoliubov dynamics of type II

Definition 5.3: *An implementable symplectic group is of type* II *if and only if it has a bounded from below Bogoliubov Hamiltonian (and hence all its Bogoliubov Hamiltonians are bounded from below).*

Definition 5.4: *If $\mathrm{R}(t)$ is a symplectic group of type* II, *we define the Bogoliubov Hamiltonian of type* II *to be the unique associated Bogoliubov Hamiltonian whose infimum of spectrum is* 0. *We denote it by H_{II}. The corresponding Bogoliubov unitary dynamics is denoted by $U_{\mathrm{II}}(t)=\mathrm{e}^{itH_{\mathrm{II}}}$.*

Clearly, every Bogoliubov Hamiltonian of type II has a positive Wick symbol.

D. Berezin's criterion for the existence of type I Bogoliubov Hamiltonian

We would like in this section to give some sufficient conditions on the generator A of a symplectic group $\mathrm{R}(t)$ so that it is of type I.

The following theorem can be traced back to Ref. 2 (Sec. III.6, Lemma 4).

Theorem 5.2: *Suppose that A satisfies the assumptions of Proposition 2.6 and $w(t)$ is defined as in* Eq. (4.3). *Suppose also that*

(1) *$w(t)$ is Hilbert-Schmidt and $t\mapsto\|w(t)\|_2$ is locally integrable and continuous at zero.*
(2) *$\bar{v}w(t)$ is trace class and $t\mapsto\|\bar{v}w(t)\|_1$ is locally integrable and continuous at zero.*

Then $\mathrm{R}(t)$ *is of type* I.

Proof: We will use the notation introduced in the proof of Theorem 4.3.

We have

$$\widetilde{R}_2(t) = \int_0^t \widetilde{R}_1(\tau)V(\tau)\mathrm{d}\tau = \begin{pmatrix} \int_0^t w(\tau)\mathrm{e}^{-i\tau\bar{h}}\bar{v}\mathrm{e}^{-i\tau h}\mathrm{d}\tau & 0 \\ 0 & \int_0^t \overline{w(\tau)}\mathrm{e}^{i\tau h}v\mathrm{e}^{i\tau\bar{h}}\mathrm{d}\tau \end{pmatrix}.$$

Now,

$$\begin{aligned}\left\| \int_0^t \overline{w(\tau)}\mathrm{e}^{i\tau h}v\mathrm{e}^{i\tau\bar{h}}\mathrm{d}\tau \right\|_1 &= \left\| \int_0^t \mathrm{d}\tau \int_0^\tau \mathrm{d}s\mathrm{e}^{-is\bar{h}}\bar{v}\mathrm{e}^{-ish}\mathrm{e}^{i\tau h}v\mathrm{e}^{i\tau\bar{h}} \right\|_1 \\ &\leqslant \int_0^t \mathrm{d}s \left\| \int_s^t \mathrm{d}\tau\bar{v}\mathrm{e}^{i(\tau-s)h}v\mathrm{e}^{i(\tau-s)\bar{h}} \right\|_1 \\ &= \int_0^t \mathrm{d}s\|\bar{v}w(t-s)\|_1 =: \rho(t) < +\infty.\end{aligned}$$

Therefore $\widetilde{R}_2(t)$ is trace class and $\|\widetilde{R}_2(t)\|_1 \leqslant 2\rho(t)$. In the same way as in the proof of Theorem 4.3, we have that $\widetilde{R}(t)-1-\widetilde{R}_1(t)$ is trace class. In particular, $P(t)\mathrm{e}^{-ith}-1$ is trace class.

Finally, $\lim_{t\to 0}\|\bar{v}w(t)\|_1=0$ implies $\lim_{t\to 0}\|\widetilde{R}(t)-1-\widetilde{R}_1(t)\|_1=0$, and hence of $\lim_{t\to 0}\|P(t)\mathrm{e}^{-ith}-1\|_1=0$ in a similar way as in Theorem 4.3. □

E. Essential self-adjointness of type I Bogoliubov Hamiltonians

Formally, it is easy to see that the Bogoliubov Hamiltonian of type I is given by Eq. (1.2) with $c=0$. We can make this precise when v is Hilbert-Schmidt.

Theorem 5.3: *Suppose v is Hilbert Schmidt. Then*

(1) The operator

$$\mathrm{d}\Gamma(h) + \frac{1}{2}(a^*(v) + a(v)) \tag{5.5}$$

is essentially self-adjoint on $\mathcal{D} := \Gamma_s^{\mathrm{fin}}(\mathfrak{h}) \cap \mathrm{Dom}(\mathrm{d}\Gamma(h))$.

(2) U_{I} *and* c_{I} *defined as in* Eqs. (5.3) and (5.4) *satisfy*

$$U_{\mathrm{I}}(t) = \mathrm{e}^{(i/2)\mathrm{Tr}(\int_0^t Q(s)v\bar{P}(s)^{-1}\mathrm{d}s)}\mathrm{e}^{-(1/2)a^*(K(t))}\Gamma((P(t)^{-1})^*)\mathrm{e}^{-(1/2)a(L(t))}, \tag{5.6}$$

$$c_{\mathrm{I}}(t) = \mathrm{e}^{(i/2)\mathrm{Re}(\mathrm{Tr}(\int_0^t Q(s)v\bar{P}(s)^{-1}\mathrm{d}s))}. \tag{5.7}$$

(3) Equation (5.5) *is the Bogoliubov Hamiltonian of type I: that is, if we set Eq. (5.5) equal H_{I}, then* $\mathrm{e}^{itH_{\mathrm{I}}}=U_{\mathrm{I}}(t)$.

(4) *Let H be a Bogoliubov Hamiltonian associated with* $\mathrm{R}(t)$. *Then* $\Omega \in \mathrm{Dom}\, H$ *and* $H=H_{\mathrm{I}}$ *iff* $\langle\Omega|H\Omega\rangle=0$.

Note that, since $\Gamma_s^{\mathrm{fin}}(\mathfrak{h}) \subset \mathrm{Dom}(a^*(v)+a(v))$, the operator H_{I} is therefore essentially self-adjoint on $\mathrm{Dom}(\mathrm{d}\Gamma(h)) \cap \mathrm{Dom}(a^*(v)+a(v))$. The strategy of the proof for the essential self-adjointness comes from Ref. 2 (Theorem 6.1) and goes back to Carleman.[19] However, as we mentioned in the Introduction, the proof in Ref. 2 is not completely rigorous. A similar result has also been proven in Ref. 20 when h is bounded.

Proof of Theorem 5.3 (1): We consider the symmetric operator H defined as Eq. (5.5) on the domain $\mathcal{D}$ and we prove that for all $z \in \mathbb{C}, z \notin \mathbb{R}, \mathrm{Ker}(H^*-z)=\{0\}$.

We denote by P_n the orthogonal projection onto $\Gamma_s^n(\mathfrak{h})$. In particular, for any vector $\Psi, P_n\Psi \in \Gamma_s^{\mathrm{fin}}(\mathfrak{h})$. We also define, for all $\epsilon \in \mathbb{R}$,

$$\Psi_\epsilon := (1 - i\epsilon \mathrm{d}\Gamma(h))^{-1}\Psi.$$

For any $\epsilon \neq 0, \Psi_\epsilon \in \mathrm{Dom}(\mathrm{d}\Gamma(h))$ and $\lim_{\epsilon\to 0}\Psi_\epsilon = \Psi$. Moreover, since the operator $\mathrm{d}\Gamma(h)$ leaves the subspace $\Gamma_s^n(\mathfrak{h})$ invariant, we have $P_n\Psi_\epsilon = (P_n\Psi)_\epsilon \in \mathcal{D}$ for all n and $\epsilon \neq 0$.

Let us now fix $z \notin \mathbb{R}$ and let $\Phi \in \mathrm{Ker}(H^* - z)$. For all n we have

$$z\|P_n\Phi\|^2 = z\langle P_n\Phi|\Phi\rangle = \lim_{\epsilon\to 0} z\langle P_n\Phi_\epsilon|\Phi\rangle = \lim_{\epsilon\to 0}\langle P_n\Phi_\epsilon|H^*\Phi\rangle = \lim_{\epsilon\to 0}\langle HP_n\Phi_\epsilon|\Phi\rangle,$$

where in the last equality we have used the fact that $P_n\Phi_\epsilon \in \mathcal{D}$. Similarly, we have $\bar{z}\|P_n\Phi\|^2 = \lim_{\epsilon\to 0}\langle\Phi|HP_n\Phi_{-\epsilon}\rangle$. Therefore,

$$2i\,\mathrm{Im}\,z\|P_n\Phi\|^2 = \lim_{\epsilon\to 0}\Bigg(\langle \mathrm{d}\Gamma(h)P_n\Phi_\epsilon|\Phi\rangle - \langle\Phi|\mathrm{d}\Gamma(h)P_n\Phi_{-\epsilon}\rangle + \frac{1}{2}\langle(a(v)+a^*(v))P_n\Phi_\epsilon|\Phi\rangle - \frac{1}{2}\langle\Phi|(a(v)+a^*(v))P_n\Phi_{-\epsilon}\rangle\Bigg).$$

Since P_n commutes with $\mathrm{d}\Gamma(h)$, the two first terms of the right hand side cancel. Moreover, the operator $(a(v)+a^*(v))P_n$ is bounded. So finally we get, with the convention $P_{-1}=P_{-2}=0$,

$$\begin{aligned}4i\,\mathrm{Im}\,z\|P_n\Phi\|^2 &= \langle(a(v)+a^*(v))P_n\Phi|\Phi\rangle - \langle\Phi|(a(v)+a^*(v))P_n\Phi\rangle \\ &= \langle a(v)P_n\Phi|P_{n-2}\Phi\rangle + \langle a^*(v)P_n\Phi|P_{n+2}\Phi\rangle - \langle a(v)P_{n+2}\Phi|P_n\Phi\rangle - \langle a^*(v)P_{n-2}\Phi|P_n\Phi\rangle.\end{aligned}$$

We now sum the previous identity for $0 \leq n \leq N$, which gives

$$4i\,\mathrm{Im}\,z\sum_{n=0}^{N}\|P_n\Phi\|^2 = \langle a^*(v)P_N\Phi|P_{N+2}\Phi\rangle + \langle a^*(v)P_{N-1}\Phi|P_{N+1}\Phi\rangle - \langle a(v)P_{N+2}\Phi|P_N\Phi\rangle - \langle a(v)P_{N+1}\Phi|P_{N-1}\Phi\rangle.$$

Therefore, for all $N \in \mathbb{N}$, and using Proposition 2.2, we have

$$\begin{aligned}4|\mathrm{Im}\,z|\sum_{n=0}^{N}\|P_n\Phi\|^2 &\leq \|v\|_2(2(N+2)\|P_N\Phi\|\|P_{N+2}\Phi\| + 2(N+1)\|P_{N-1}\Phi\|\|P_{N+1}\Phi\|) \\ &\leq (N+2)\|v\|_2(\|P_{N-1}\Phi\|^2 + \|P_N\Phi\|^2 + \|P_{N+1}\Phi\|^2 + \|P_{N+2}\Phi\|^2).\end{aligned}$$

Suppose $\Phi \neq 0$. Hence there exists N_0 such that $\Sigma_{n=0}^{N_0}\|P_n\Phi\|^2 = C > 0$, and for all $N \geq N_0, \Sigma_{n=0}^{N}\|P_n\Phi\|^2 \geq C$. So we have, for all $N \geq N_0$,

$$\frac{4|\mathrm{Im}\,z|C}{N+2} \leq \|v\|_2\sum_{j=N-1}^{N+2}\|P_j\Phi\|^2.$$

If now we sum over N this inequality, the right hand side converges (and is less than $4\|v\|_2\|\Phi\|^2$), while the left hand side diverges. Hence $\Phi=0$ and H_{I} is essentially self-adjoint on $\mathcal{D}$. □

To prove (2), we will use the following well-known abstract result.

Lemma 5.3: *Let $\mathcal{H}$ be a Hilbert space and $\mathbb{R} \ni t \to \varphi(t) \in \mathcal{H}$ be a weakly continuous function such that, for all $t \in \mathbb{R}$, $\|\varphi(t)\| = \|\varphi(0)\|$. Then $\varphi(t)$ is norm continuous.*

Proof of Theorem 5.3 (2): If v is Hilbert-Schmidt, the assumptions of Theorem 5.2 are satisfied, so that $\mathrm{R}(t)$ is of type I. According to the definition of $\mathrm{U}_{\mathrm{I}}(t)$, we have to prove that, for all t,

$$\det(\bar{P}(t)e^{it\bar{h}}) = e^{-i\int_0^t \mathrm{Tr}(Q(s)v\bar{P}(s)^{-1})\mathrm{d}s}. \tag{5.8}$$

One has $V(t)=i\begin{pmatrix} 0 & -e^{ith}ve^{it\bar{h}} \\ e^{-it\bar{h}}\bar{v}e^{-ith} & 0 \end{pmatrix}$. It is clear that $t\mapsto e^{ith}ve^{it\bar{h}}$ is continuous in the weak operator topology, and therefore in the weak sense in $B^2(\mathfrak{h})$ considered as a Hilbert space [i.e., for all $K\in B^2(\mathfrak{h}), t\mapsto \mathrm{Tr}(Ke^{ith}ve^{it\bar{h}})$ is continuous]. Moreover, since e^{ith} is unitary, we have $\|e^{ith}ve^{it\bar{h}}\|_2 = \|v\|_2$. Hence, using Lemma 5.3, $e^{ith}ve^{it\bar{h}}$ is continuous in the Hilbert-Schmidt norm, and so is $V(t)$.

We have

$$\tilde{R}(t) - 1 = \tilde{R}_1(t) + \int_0^t (\tilde{R}(\tau) - 1)V(\tau)\mathrm{d}\tau,$$

where, recall, $\tilde{R}_1(t)=\int_0^t V(\tau)\mathrm{d}\tau$. Therefore, $t\mapsto \tilde{R}(t)-1$ is continuous as a function in $B^2(\mathfrak{h})$.

Using

$$\tilde{R}(t) - 1 - \tilde{R}_1(t) = \int_0^t (\tilde{R}(\tau) - 1 - \tilde{R}_1(\tau))V(\tau)\mathrm{d}\tau + \int_0^t \mathrm{d}\tau V(\tau)\int_0^\tau \mathrm{d}\tau_1 V(\tau_1),$$

we obtain that $t\mapsto \tilde{R}(t)-1-\tilde{R}_1(t)$ is continuously differentiable in the trace class norm. Consequently, $\bar{P}(t)e^{it\bar{h}}-1$ is differentiable in the trace class topology. Hence, $\det(\bar{P}(t)e^{it\bar{h}})$ is differentiable and

$$\frac{\mathrm{d}}{\mathrm{d}t}\det(\bar{P}(t)e^{it\bar{h}}) = \mathrm{Tr}\left(\frac{\mathrm{d}}{\mathrm{d}t}(\bar{P}(t)e^{it\bar{h}})(\bar{P}(t)e^{it\bar{h}})^{-1}\right)\det(\bar{P}(t)e^{it\bar{h}}) = -\, i\,\mathrm{Tr}(Q(t)v\bar{P}(t)^{-1})\det(\bar{P}(t)e^{it\bar{h}}),$$

which proves Eq. (5.8), and where we used Eq. (4.4) in the second line.

The proof of Eq. (5.7) follows from Eqs. (5.4) and (5.8) and the fact that $\det(1-K(t)^*K(t))$ is positive. □

Recall that $L(t)=-P(t)^{-1}\bar{Q}(t)$. When v is Hilbert-Schmidt, the operator $v\bar{L}(t)$ is trace class and $\mathrm{Tr}(Q(s)v\bar{P}(s)^{-1})=-\mathrm{Tr}(v\bar{L}(s))$. Therefore,

$$c_{\mathrm{I}}(t) = e^{-(i/2)\mathrm{Re}(\int_0^t \mathrm{Tr}(v\bar{L}(s))\mathrm{d}s)}. \tag{5.9}$$

Lemma 5.4: *Suppose v is Hilbert-Schmidt. Then the map $t\mapsto L(t)$ is differentiable in the Hilbert-Schmidt topology.*

Proof: In the same way as above, we can prove that $R_0(-t)R(t)-1$ is differentiable in $\mathcal{V}$. Hence, $e^{-ith}P(t)-1$ is differentiable in the trace class norm, thus $e^{-ith}P(t)$ is norm differentiable and hence so is $P(t)^{-1}e^{ith}$. Moreover $e^{-ith}\bar{Q}(t)$ is differentiable in the Hilbert-Schmidt norm, so that $L(t)=-P(t)^{-1}\bar{Q}(t)=-(e^{-ith}P(t))^{-1}e^{-ith}\bar{Q}(t)$ is differentiable in the Hilbert-Schmidt norm. □

Lemma 5.5: *Suppose v is Hilbert-Schmidt. Then $\langle\Omega|U_{\mathrm{nat}}(t)\Omega\rangle$ is continuously differentiable.*

Proof: Using Eq. (5.4) and (5.7) we have $\langle\Omega|U_{\mathrm{nat}}(t)\Omega\rangle=e^{(1/2)\,\mathrm{Im}(\int_0^t \mathrm{Tr}(v\bar{L}(s))\mathrm{d}s)}$. The differentiability then follows from Lemma 5.4. □

Proof of Theorem 5.3 (3): It remains to prove that $e^{itH_{\mathrm{I}}}=U_{\mathrm{I}}(t)$. For that purpose, we prove that $e^{itH_{\mathrm{I}}}$ is a Bogoliubov dynamics implementing $\mathrm{R}(t)$ so that it equals $U_{\mathrm{I}}(t)$ up to a phase factor. Then we prove that this phase is 1.

Let us write $A=A_0+V$ as in the proof of Theorem 4.3 Both A_0 and V are generators of a one-parameter group of symplectic maps. Computing in terms of convergent power series on $\Gamma_s^{\mathrm{fin}}(\mathfrak{h})$ we obtain, for all $t\in\mathbb{R}$,

$$e^{(i/2)t(a^*(v)+a(v))}W(y)e^{-(i/2)t(a^*(v)+a(v))} = W(e^{tV}y).$$

But, since h is self-adjoint, it is well known (see, e.g., Ref. 21) that

$$\mathrm{e}^{it\mathrm{d}\Gamma(h)}W(y)\mathrm{e}^{-it\mathrm{d}\Gamma(h)} = W(\mathrm{e}^{tA_0}y).$$

Thus, using the Trotter product formula,

$$\mathrm{e}^{itH_\mathrm{I}} = s - \lim_{n\to\infty}(\mathrm{e}^{it\mathrm{d}\Gamma(h)/n}\mathrm{e}^{[it(a^*(v)+a(v))]/2n})^n = s - \lim_{n\to\infty}(\mathrm{e}^{[it(a^*(v)+a(v))]/2n}\mathrm{e}^{it\mathrm{d}\Gamma(h)/n})^n.$$

In the same way, we have $\mathrm{R}(t)=s-\lim_{n\to\infty}(\mathrm{e}^{tA_0/n}\mathrm{e}^{tV/n})^n$. Hence,

$$\mathrm{e}^{itH_\mathrm{I}}W(y)\mathrm{e}^{-itH_\mathrm{I}} = s - \lim_{n\to\infty}(\mathrm{e}^{it\mathrm{d}\Gamma(h)/n}\mathrm{e}^{[it(a^*(v)+a(v))]/2n})^n W(y)(\mathrm{e}^{[-it(a^*(v)+a(v))]/2n}\mathrm{e}^{-it\mathrm{d}\Gamma(h)/n})^n$$

$$= s - \lim_{n\to\infty} W((\mathrm{e}^{tA_0/n}\mathrm{e}^{tV/n})^n y) = W(\mathrm{R}(t)y).$$

This proves that $\mathrm{e}^{itH_\mathrm{I}}$ is a Bogoliubov dynamics implementing $\mathrm{R}(t)$. Hence $\mathrm{e}^{itH_\mathrm{I}}$ and $U_\mathrm{I}(t)$ are equal up to a phase factor. In order to prove that this phase is one, we will show that they have the same natural cocycle, as was noticed in, e.g., Ref. 20. By Eq. (5.9), we know that

$$U_\mathrm{I}(t) = \mathrm{e}^{-(i/2)\mathrm{Re}(\int_0^t \mathrm{Tr}(v\bar{L}(s))\mathrm{d}s)}U_\mathrm{nat}(t).$$

Let now $\rho(t)\in\mathbb{R}$ be such that $U_\mathrm{nat}(t)=\mathrm{e}^{i\rho(t)}\mathrm{e}^{itH_\mathrm{I}}$. Note that $\Omega\in\mathcal{D}$, hence,

$$\mathrm{e}^{i\rho(t)} = \frac{\langle\Omega|U_\mathrm{nat}(t)\Omega\rangle}{\langle\Omega|\mathrm{e}^{itH_\mathrm{I}}\Omega\rangle}$$

is continuously differentiable by Lemma 5.5. Moreover, for all $t,\langle\Omega|U_\mathrm{nat}(t)\Omega\rangle\in\mathbb{R}$, thus

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\Omega|U_\mathrm{nat}(t)\Omega\rangle = i\rho'(t)\langle\Omega|U_\mathrm{nat}(t)\Omega\rangle + \langle U_\mathrm{nat}(t)^*\Omega|iH_\mathrm{I}\Omega\rangle \qquad \in\mathbb{R},$$

and hence

$$\rho'(t)\langle\Omega|U_\mathrm{nat}(t)\Omega\rangle = -\mathrm{Im}\langle U_\mathrm{nat}(t)^*\Omega|iH_\mathrm{I}\Omega\rangle = -\frac{1}{2}\mathrm{Im}\langle U_\mathrm{nat}(t)^*\Omega|ia^*(v)\Omega\rangle. \tag{5.10}$$

Therefore

$$\rho'(t) = -\mathrm{Im}\frac{\langle U_\mathrm{nat}(t)^*\Omega|ia^*(v)\Omega\rangle}{2\langle\Omega|U_\mathrm{nat}(t)\Omega\rangle} = -\frac{1}{2}\mathrm{Im}\langle e^{-(1/2)a^*(L(t))}\Omega|ia^*(v)\Omega\rangle = \frac{1}{4}\mathrm{Im}\langle a^*(L(t))\Omega|ia^*(v)\Omega\rangle.$$

Now, using Eq. (2.5), we have

$$\langle\Omega|[a(L(t)),a^*(v)]\Omega\rangle = 2\,\mathrm{Tr}(L(t)^*v).$$

But $L(t)^*=\bar{L}(t)$, therefore

$$\rho(t) = \frac{1}{2}\int_0^t \mathrm{Re}\,\mathrm{Tr}(v\bar{L}(s))\mathrm{d}s, \tag{5.11}$$

and

$$\mathrm{e}^{itH_\mathrm{I}} = \mathrm{e}^{-(i/2)\mathrm{Re}(\int_0^t\mathrm{Tr}(v\bar{L}(s))\mathrm{d}s)}U_\mathrm{nat}(t).$$

□

F. Relative boundedness of quadratic annihilation and creation operators

In this section, we consider $\frac{1}{2}(a^*(v)+a(v))$ as a perturbation of $\mathrm{d}\Gamma(h)$ and derive a condition so that it is relatively bounded with respect to it.

Theorem 5.4: *Let h be a positive self-adjoint operator on $\mathfrak{h}$ and $(h^{-1/2}\otimes h^{-1/2})v \in \mathfrak{h}\otimes_s\mathfrak{h}$ and $h^{-1/2}v \in B(\mathfrak{h})$ (note that we use different meanings of v in these two conditions!). Then $a(v)+a^*(v)$ is $\mathrm{d}\Gamma(h)$ bounded with relative bound less than $2\|(h^{-1/2}\otimes h^{-1/2})v\|$.*

Using the Kato-Rellich Theorem (Ref. 22, Theorem X.39), one then immediately gets the following.

Corollary 5.1: *Under the same assumption, if moreover $\|(h^{-1/2}\otimes h^{-1/2})v\|<1$ then the operator*

$$H \coloneqq \mathrm{d}\Gamma(h) + \frac{1}{2}(a(v) + a^*(v)) \tag{5.12}$$

is self-adjoint on $\mathrm{Dom}(\mathrm{d}\Gamma(h))$ and bounded from below. In particular, the symplectic group $\mathrm{R}(t)$ generated by A given by Eq. (2.9) is both of types I and II, and Eq. (5.12) coincides with the type I Bogoliubov Hamiltonian H_{I}.

Let us mention that the above condition resembles the condition one can find for the Van Hove and the Pauli-Fierz Hamiltonians (see, e.g., Refs. 13 and 16), where a perturbation linear in the annihilation and creation operators, instead of quadratic, is involved.

Lemma 5.6: *Suppose that $v \in \mathrm{Dom}(h^{-1/2}\otimes h^{-1/2})$. Then, there exist orthonormal bases of $\mathfrak{h}$ $(\xi_n)_n,(\chi_n)_n$ and positive numbers μ_n such that $(h^{-1/2}\otimes h^{-1/2})v=\Sigma_n\mu_n\xi_n\otimes\chi_n$. Moreover, for all $\Psi \in \Gamma_s^{\mathrm{fin}}(\mathfrak{h})$,*

$$a(v)\Psi = \sum_n \mu_n a(h^{1/2}\xi_n)a(h^{1/2}\chi_n)\Psi.$$

Proof: It follows from Eq. (2.3) and the fact that $v=\Sigma\mu_n h^{1/2}\xi_n\otimes h^{1/2}\chi_n$. □

We now prove bounds on $a(v)$ and $a^*(v)$ which generalize the ones obtained in Proposition 2.2 and which are in the spirit of the N_τ-estimate of Proposition 2.1.

Proposition 5.1: *Suppose $v \in \mathrm{Dom}(h^{-1/2}\otimes h^{-1/2})$. Then for all $\Psi \in \mathrm{Dom}(\mathrm{d}\Gamma(h))$,*

$$\|a(v)\Psi\| \leq \|(h^{-1/2}\otimes h^{-1/2})v\|\,\|\mathrm{d}\Gamma(h)\Psi\|.$$

Proof: Using Lemma 5.6, we have

$$\|a(v)\Psi\|^2 = \left\|\sum_n \mu_n a(h^{1/2}\xi_n)a(h^{1/2}\chi_n)\Psi\right\|^2 \leq \sum_n \mu_n^2 \sum_n \|a(h^{1/2}\xi_n)a(h^{1/2}\chi_n)\Psi\|^2.$$

Hence, using $\Sigma_n\mu_n^2=\|(h^{-1/2}\otimes h^{-1/2})v\|^2$ and Proposition 2.1, we get

$$\begin{aligned}\|a(v)\Psi\|^2 &\leq \|(h^{-1/2}\otimes h^{-1/2})v\|^2 \sum_n \langle a(h^{1/2}\chi_n)\Psi|\mathrm{d}\Gamma(h)a(h^{1/2}\chi_n)\Psi\rangle\\ &= \|(h^{-1/2}\otimes h^{-1/2})v\|^2 \sum_n (\langle a(h^{1/2}\chi_n)\Psi|a(h^{1/2}\chi_n)\mathrm{d}\Gamma(h)\Psi\rangle - \langle a(h^{1/2}\chi_n)\Psi|a(h^{3/2}\chi_n)\Psi\rangle)\\ &= \|(h^{-1/2}\otimes h^{-1/2})v\|^2(\|\mathrm{d}\Gamma(h)\Psi\|^2 - \langle\Psi|\mathrm{d}\Gamma(h^2)\Psi\rangle),\end{aligned}$$

where in the last line we used the following identities:

$$\sum a^*(h^{1/2}\chi_n)a(h^{1/2}\chi_n) = \mathrm{d}\Gamma(h) \quad \text{and} \quad \sum a^*(h^{1/2}\chi_n)a(h^{3/2}\chi_n) = \mathrm{d}\Gamma(h^2).$$

□

Proposition 5.2: *Suppose $v \in \mathrm{Dom}(h^{-1/2}\otimes h^{-1/2})$ and $h^{-1/2}v \in B(\mathfrak{h})$. For any $\epsilon>0$, for all $\Psi \in \mathrm{Dom}(\mathrm{d}\Gamma(h))$,*

$$\|a^*(v)\Psi\|^2 \leq (\|(h^{-1/2}\otimes h^{-1/2})v\|^2 + 2\epsilon\|h^{-1/2}v\|^2)\|\mathrm{d}\Gamma(h)\Psi\|^2 + \left(2\|v\|^2 + \frac{2}{\epsilon}\|h^{-1/2}v\|^2\right)\|\Psi\|^2.$$

Proof of Proposition 5.2: The following identity follows directly from Eq. (2.5).

$$\|a^*(v)\Psi\|^2 = \|a(v)\Psi\|^2 + 4\langle\Psi|\mathrm{d}\Gamma(vv^*)\Psi\rangle + 2\|v\|^2\|\Psi\|^2.$$

One can write $vv^* = h^{1/2}(h^{-1/2}v)(h^{-1/2}v)^* h^{1/2}$. Now, $h^{-1/2}v$ is bounded. Thus $vv^* \leqslant \|h^{-1/2}v\|^2 h$, and so

$$\langle\Psi|\mathrm{d}\Gamma(vv^*)\Psi\rangle \leqslant \|h^{-1/2}v\|^2\|\mathrm{d}\Gamma(h)^{1/2}\Psi\|^2 \leqslant \frac{\epsilon}{2}\|h^{-1/2}v\|^2\|\mathrm{d}\Gamma(h)\Psi\|^2 + \frac{1}{2\epsilon}\|h^{-1/2}v\|^2\|\Psi\|^2,$$

which ends the proof. □

Proof of Theorem 5.4: It follows directly from Propositions 5.1 and 5.2. □

VI. A CONCRETE EXAMPLE: THE DIAGONAL CASE

In this section we consider the simplest "infinite dimensional" case. Namely, $\mathfrak{h} \coloneqq L^2(\mathbb{N})$ with its canonical basis $(e_n)_{n\in\mathbb{N}}$ and h and v are both diagonal, i.e.,

$$h \coloneqq \sum_n h_n|e_n\rangle\langle e_n|, \quad v \coloneqq \sum_n v_n|e_n\rangle\langle \bar{e}_n|, \tag{6.1}$$

and where the h_n are real numbers so that h is self-adjoint. We can identify $\mathfrak{h}\oplus\mathfrak{h}$ with $\oplus_n\mathbb{C}^2$ and the operator A with $\oplus_n i\left[\begin{smallmatrix} h_n & -v_n \\ \bar{v}_n & -h_n \end{smallmatrix}\right]$.

Note that Eq. (6.1) is equivalent to assuming that $hv = v\bar{h}$ and that there exists a basis of eigenvectors of h.

Our goal is to describe, in this simple situation, what are the one-parameter symplectic groups $t\mapsto \mathrm{R}(t)$ which are implementable, which are those of type I, and those of type II. More precisely, we will prove the following.

Theorem 6.1: *Consider on $L^2(\mathbb{N})$ the operators h and v defined by Eq. (6.1).*

(i) *$t\mapsto\mathrm{R}(t)$ defines a strongly continuous one-parameter group of symplectic maps if and only if v is h-bounded with relative bound strictly less than 1, i.e., there exists $a\in[0,1[$ and $b\geqslant 0$ such that for all $n\in\mathbb{N}, |v_n|\leqslant a|h_n|+b$.*

(ii) *$t\mapsto\mathrm{R}(t)$ is implementable if and only if $\Sigma|v_n|^2/(1+h_n^2) < +\infty$.*

(iii) *$t\mapsto\mathrm{R}(t)$ is of type I if and only if $\Sigma|v_n|^2/(1+|h_n|) < +\infty$.*

(iv) *$t\mapsto\mathrm{R}(t)$ is of type II if and only if $h_n\geqslant|v_n|$ for all n and $\Sigma|v_n|^2/(h_n+h_n^2) < +\infty$.*

Remark 6.1: *(iv) shows that there are examples where the quantum Bogoliubov Hamiltonian is unbounded below although the classical Hamiltonian is positive: when $\Sigma|v_n|^2/(h_n+h_n^2)$ diverges. Moreover, (ii) shows that this is due to the small eigenvalues of h. It may thus be interpreted as a kind of infrared catastroph as we mentioned in the Introduction. On the other hand, (ii) and (iii) show that the need for an infinite renormalization is due to ultraviolet divergencies.*

A. Bogoliubov transformations of a single degree of freedom

Let us start with a description of the case of a single degree of freedom. Let $\mathfrak{h}=\mathbb{C}$ and $A = i\left(\begin{smallmatrix} h & -v \\ \bar{v} & -\bar{h} \end{smallmatrix}\right)$, where $h\in\mathbb{R}$ and $v\in\mathbb{C}$. One can compute explicitly the operators $P(t)$ and $Q(t)$.

- If $|h|<|v|$

$$P(t) = \cosh(t\sqrt{|v|^2-h^2}) + ih\frac{\sinh(t\sqrt{|v|^2-h^2})}{\sqrt{|v|^2-h^2}} \quad \text{and } Q(t) = i\bar{v}\frac{\sinh(t\sqrt{|v|^2-h^2})}{\sqrt{|v|^2-h^2}}. \tag{6.2}$$

- If $|h|=|v|$

$$P(t) = 1 + ith \quad \text{and } Q(t) = it\bar{v}. \tag{6.3}$$

- If $|h|>|v|$

$$P(t)=\cos(t\sqrt{h^2-|v|^2})+ih\frac{\sin(t\sqrt{h^2-|v|^2})}{\sqrt{h^2-|v|^2}} \quad \text{and} \quad Q(t)=i\bar{v}\frac{\sin(t\sqrt{h^2-|v|^2})}{\sqrt{h^2-|v|^2}}. \tag{6.4}$$

From Theorem 3.2 we know that $\mathrm{R}(t)$ is always of type I with $H_{\mathrm{I}}=\mathrm{d}\Gamma(h)+\frac{1}{2}(a^*(v)+a(v))$. Moreover, it is of type II if and only if its classical symbol is positive, i.e., $\forall z\in\mathbb{C},\mathrm{Re}(h|z|^2+vz^2)\geqslant 0$, which is equivalent to $h\geqslant|v|$. The Bogoliubov Hamiltonian of type II then writes, according to Eq. (3.6),

$$H_{\mathrm{II}}=H_{\mathrm{I}}-\frac{1}{2}(\sqrt{h^2-|v|^2}-h). \tag{6.5}$$

B. Proof of Theorem 6.1

Lemma 6.1: *We consider the operator on $\mathbb{C}^2$ given by the matrix $A:=\left[\begin{smallmatrix} a & b \\ \bar{b} & \bar{a}\end{smallmatrix}\right]$. Then*

$$\|A\|=|a|+|b|, \quad \|A^{-1}\|=||a|-|b||^{-1}.$$

Proof:

$$A^*A=\begin{bmatrix} |a|^2+|b|^2 & 2\bar{a}b \\ 2a\bar{b} & |a|^2+|b|^2\end{bmatrix}=(|a|+|b|)^2\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}-2|a||b|\begin{bmatrix}1 & \alpha \\ \bar{\alpha} & 1\end{bmatrix}, \tag{6.6}$$

where $\bar{a}b=\alpha|a||b|$. Clearly, Eq. (6.6) has the spectrum $\{(|a|+|b|)^2,(|a|-|b|)^2\}$. ■

The following lemma proves Theorem 6.1 (i).

Lemma 6.2: *The following are equivalent:*

(1) *There exists ω, M such that for all $\lambda>\omega$,*

$$\frac{1}{\sqrt{\lambda^2+h_n^2}-|v_n|}\leqslant\frac{M}{\lambda-\omega}.$$

(2) *There exists $0\leqslant a<1$ and $b\geqslant 0$ such that*

$$|v_n|\leqslant a|h_n|+b.$$

Proof: (1)⇒(2). $1=\lim_{\lambda\to\infty}(\lambda-\omega)/(\sqrt{\lambda^2+h_n^2}-|v_n|)\leqslant M$. Hence $M\geqslant 1$. Now Eq. (1) implies

$$M|v_n|-\omega\leqslant M\sqrt{\lambda^2+h_n^2}-\lambda.$$

Elementary computations yield

$$\inf_{\lambda\geqslant\omega} M\sqrt{\lambda^2+h_n^2}-\lambda=\begin{cases} |h_n|\sqrt{M^2-1}, & \dfrac{|h_n|}{\sqrt{M^2-1}}\geqslant\omega \\ M\sqrt{\omega^2+h_n^2}-\omega, & \dfrac{|h_n|}{\sqrt{M^2-1}}\leqslant\omega. \end{cases}$$

The first case gives

$$|v_n|\leqslant\frac{\sqrt{M^2-1}}{M}|h_n|+\frac{\omega}{M}.$$

The second case means that $|h_n|\leqslant\sqrt{M^2-1}\,\omega$. Therefore, $|v_n|\leqslant M|\omega|$.

(2)⇒(1). For any $\lambda > 0$ we have

$$\sqrt{\lambda^2 + h_n^2} - |v_n| \geq \sqrt{\lambda^2 + h_n^2} - a|h_n| - b = a(\sqrt{\lambda^2 + h_n^2} - |h_n|) + (1-a)\sqrt{\lambda^2 + h_n^2} - b \geq (1-a)\lambda - b.$$

For $\lambda > b/(1-a)$, the rhs is positive. Thus with $\omega := b/(1-a)$ and $M := (1-a)^{-1}$,

$$\frac{1}{\sqrt{\lambda^2 + h_n^2} - |v_n|} \leq \frac{M}{\lambda - \omega}.$$

□

Lemma 6.3: *If the operator A generates a one-parameter group on $\oplus_n \mathbb{C}^2$, then the conditions of Lemma 6.2 hold.*

Proof: By the Hille-Philips Theorem, if A generates a group, then there exist M and ω such that for $|\lambda| > \omega$,

$$\|(\pm\lambda - A)^{-1}\| \leq M(|\lambda| - \omega)^{-1}.$$

But $\pm\lambda - A = \left[\begin{smallmatrix} \pm\lambda - ih & iv \\ -iv & \pm\lambda + ih \end{smallmatrix}\right]$ and therefore

$$\|(\pm\lambda - A)^{-1}\| = \sup_n \left|\sqrt{\lambda^2 + h_n^2} - |v_n|\right|^{-1}.$$

□

Lemma 6.4: *Let $|h_n| \leq c$, $|v_n| \leq c$. Then A generates* a *group. This group is implementable iff $\Sigma |v_n|^2 < \infty$. Moreover, it is in this case type* I.

Proof: Let $A_n = i\left(\begin{smallmatrix} h_n & -v_n \\ \bar{v}_n & -\bar{h}_n \end{smallmatrix}\right)$. We easily check that for any t_0, there exists M such that

$$\sup_n \|e^{tA_n}\| \leq M, \quad |t| \leq t_0.$$

But $\|e^{tA}\| = \sup_n \|e^{tA_n}\|$. Therefore, A generates a group.

By Theorem, 5.3, if $\Sigma |v_n|^2 < \infty$, then $\mathrm{R}(t)$ is type I, hence implementable.

Suppose now $\mathrm{R}(t)$ is implementable. This implies $\Sigma_n |Q_n(t)|^2 < \infty$. The conditions $|h_n| \leq c$ and $|v_n| \leq c$ imply that $\sqrt{|h_n^2 - |v_n|^2|}$ is bounded. Hence, for small enough t,

$$|Q_n(t)|^2 \geq c_0 t |v_n|^2,$$

for some $c_0 > 0$. Therefore, $\Sigma_n |v_n|^2 < \infty$. □

Lemma 6.5: *Let $|h_n| \geq c > 0$ and $|v_n| \leq a_1 |h_n|$, $a_1 < 1$. Then A generates a group. This group is implementable iff $\Sigma |v_n|^2 / h_n^2 < \infty$. Moreover, if $h_n \geq 0$ then it is type* II.

Proof: Suppose first that $\mathrm{R}(t)$ is implementable. Hence the map $t \mapsto \mathrm{Tr}(\log(1 - K(t)^* K(t)))$ is continuous (see the proof of Theorem 4.2) and therefore is locally integrable. We have

$$\int_0^T \mathrm{Tr}(\log(1 - K(t)^* K(t)))\mathrm{d}t = -\int_0^T \sum_n \log\left(1 + \frac{|v_n|^2}{h_n^2 - |v_n|^2} \sin^2(t\sqrt{h_n^2 - |v_n|^2})\right)\mathrm{d}t.$$

But we have $0 \leq |v_n|^2/(h_n^2 - |v_n|^2) \leq c_0$. Thus, for some $c_1 > 0$,

$$\begin{aligned} \infty &> \int_0^T \sum_n \log\left(1 + \frac{|v_n|^2}{h_n^2 - |v_n|^2} \sin^2(t\sqrt{h_n^2 - |v_n|^2})\right)\mathrm{d}t \\ &\geq c_1 \int_0^T \sum_n \frac{|v_n|^2}{h_n^2 - |v_n|^2} \sin^2(t\sqrt{h_n^2 - |v_n|^2}) \\ &= c_1 \sum \frac{|v_n|^2}{h_n^2 - |v_n|^2}\left(\frac{T}{2} - \frac{\sin(2T\sqrt{h_n^2 - |v_n|^2})}{4\sqrt{h_n^2 - |v_n|^2}}\right) \quad \text{for all } T. \end{aligned} \tag{6.7}$$

Noting that $\sqrt{h_n^2-|v_n|^2}\geq c_2>0$ we see that the second term in the large parentheses is bounded. Hence, for large T, for some $c_3>0$, Eq. (6.7) is greater than or equal to

$$c_3\sum\frac{|v_n|^2}{h_n^2-|v_n|^2}\geq c_4\sum\frac{|v_n|^2}{h_n^2},$$

with $c_4>0$. Hence $\Sigma|v_n|^2/h_n^2<+\infty$.

Assume now $|h_n|\geq c$, $|v_n|\leq a_1|h_n|$, $a_1<1$ and $\Sigma|v_n|^2/h_n^2<+\infty$. We can write $P(t)=\oplus_n P_n(t)$, where

$$\begin{aligned}P_n(t)\mathrm{e}^{-it\sqrt{h_n^2-|v_n|^2}}-1&=\left(\cos t\sqrt{h_n^2-|v_n|^2}+ih_n\frac{\sin t\sqrt{h_n^2-|v_n|^2}}{\sqrt{h_n^2-|v_n|^2}}\right)\mathrm{e}^{-it\sqrt{h_n^2-|v_n|^2}}-1\\&=\left(\frac{h_n}{\sqrt{h_n^2-|v_n|^2}}-1\right)(\sin^2 t\sqrt{h_n^2-|v_n|^2}+i\sin t\sqrt{h_n^2-|v_n|^2}\cos t\sqrt{h_n^2-|v_n|^2}).\end{aligned}$$

Hence

$$|P_n(t)\mathrm{e}^{-it\sqrt{h_n^2-|v_n|^2}}-1|\leq c_1\left|\frac{h_n}{\sqrt{h_n^2-|v_n|^2}}-1\right|=c_1\frac{|v_n|^2}{\sqrt{h_n^2-|v_n|^2}(h_n+\sqrt{h_n^2-|v_n|^2})}\leq c_2\frac{|v_n|^2}{h_n^2}. \tag{6.8}$$

Thus $P(t)\mathrm{e}^{-it\sqrt{h^2-v^*v}}-1$ is trace class. Hence, det $P(t)\mathrm{e}^{-it\sqrt{h^2-v^*v}}$ is well defined.

It is easy to see that the left hand side of Eq. (6.8) goes to zero as $t\to 0$. Therefore, applying the Lebesgue Theorem we see that $\lim_{t\to 0}\|P(t)\mathrm{e}^{-it\sqrt{h^2-v^*v}}-1\|_1=0$. Thus, $\mathrm{R}(t)$ is implementable by Theorem 5.1 and the operators

$$U_{\mathrm{ren}}(t) := \det(\bar{P}(t)\mathrm{e}^{it\overline{\sqrt{h^2-v^*v}}})^{-1/2}\mathrm{e}^{-(1/2)a^*(K(t))}\Gamma((P(t)^{-1})^*)\mathrm{e}^{-(1/2)a(L(t))}$$

form a Bogoliubov dynamics.

Suppose moreover that the h_n are positive. We shall prove that the generator H_{ren} of $U_{\mathrm{ren}}(t)$ is bounded below. Formally H_{ren} writes

$$H_{\mathrm{ren}}=\sum_{n\in\mathbb{N}}\left[\mathrm{d}\Gamma(h_n|e_n\rangle\langle e_n|)+\frac{1}{2}a^*(v_n|e_n\rangle\langle e_n|)+\frac{1}{2}a(v_n|e_n\rangle\langle e_n|)-\frac{1}{2}(\sqrt{h_n^2-|v_n|^2}-h_n)\right].$$

Note that each term in the sum is actually the Bogoliubov Hamiltonian of type II of the single degree of freedom of index n and is therefore positive. Now, one can easily prove, using Proposition 2.5, that the sequence of positive selfadjoint operators $(H_N)_N$ defined by

$$H_N := \sum_{n\leq N}\left[\mathrm{d}\Gamma(h_n|e_n\rangle\langle e_n|)+\frac{1}{2}a^*(v_n|e_n\rangle\langle e_n|)+\frac{1}{2}a(v_n|e_n\rangle\langle e_n|)-\frac{1}{2}(\sqrt{h_n^2-|v_n|^2}-h_n)\right]$$

converges to H_{ren} in the strong resolvent sense [one shows that the corresponding unitary groups converge strongly to $U_{\mathrm{ren}}(t)$]. Hence (Ref. 23, Theorem VIII.24) $\inf H_{\mathrm{ren}}\geq\lim_{N\to\infty}\inf H_N\geq 0$. ■

Lemma 6.6: *Let* $|v_n|\leq h_n$. *Then*

$$\sum(h_n-\sqrt{h_n-|v_n|^2})<\infty\quad\text{iff}\sum\frac{|v_n|^2}{h_n}<\infty.$$

Proof: We have

$$h_n - \sqrt{h_n - |v_n|^2} = \frac{|v_n|^2}{h_n + \sqrt{h_n - |v_n|^2}}.$$

Clearly,

$$\frac{|v_n|^2}{2h_n} \leqslant \frac{|v_n|^2}{h_n + \sqrt{h_n - |v_n|^2}} \leqslant \frac{|v_n|^2}{h_n}.$$

□

Proof of Theorem 6.1: (1) ⇒ was proven in Lemmas 6.2 and 6.3.

The property $|v_n| \leqslant a|h_n| + b$, $0 \leqslant a < 1$, implies that there exists c_1, c_2 and $0 \leqslant a_1 < 1$ such that either $|h_n| \leqslant c_1$, $|v_n| \leqslant c_1$ or $|h_n| \geqslant c_2$ and $|v_n| \leqslant a_1 |h_n|$. Thus, we can split the one-particle space into the direct sum of two terms, on the first the assumptions of Lemma 6.4 are satisfied and on the second those of Lemma 6.5. Now (i) ⇐ and (ii) follows from Lemmas 6.4 and 6.5.

Noting that H_{I} and H_{II} are simultaneously defined iff $|v_n| \leqslant h_n$ and $\Sigma(h_n - \sqrt{h_n - |v_n|^2}) < \infty$, we see that (iii) and (iv) follow if in addition we take into account Lemma 6.6. □

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