RETURN TO EQUILIBRIUM FOR SMALL QUANTUM SYSTEMS INTERACTING WITH ENVIRONMENT

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Rigorous results about the return to equilibrium for small quantum systems interacting with environment are described.

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1. Introduction

The following statement belongs to conventional wisdom of physics:

In a generic situation, a small system interacting with a large reservoir at temperature T goes to equilibrium at the same temperature.

In the last several years a number of papers appeared that express this idea in terms of mathematical theorems about relatively realistic quantum models [5, 11–13, 15, 17, 18, 20]. In this paper I would like to explain some of the ideas involved in these results.

All these works use techniques which come from a number of distinct domains of physics and mathematics. In fact, the following techniques play a major role in the formulations and proofs of these results.

- Operator algebras:
  - KMS states,
  - Standard forms of W^*-algebras, Liouvilleans.
- Quantum field theory:
  - Quasi-free (Araki-Woods) representations of the CCR.
- Spectral theory:
  - Analysis of embedded point spectrum—Fermi Golden Rule, the Feshbach method.
  - Analysis of absolutely continuous spectrum—the positive commutator (Mourre) method, the analytic deformation method.

Results discussed in this paper belong to algebraic quantum statistical physics. This domain had a period of considerable development in the 70's and 80's of the last century. A large part of research of this period has been summarized in the well-known monographs [6, 7]. It is probably fair to say that the results of [17, 18, 11, 13, 12, 5, 20, 15] belong to the next period of this domain and go considerably beyond what was known before.

Typical results from the previous period could be divided into three categories. In the first, the approach is axiomatic—certain abstract and implicit hypotheses about quantum systems, perhaps intuitively reasonable but difficult to verify in practice, are proven to imply some physically interesting consequences. In the second category, one considers exactly solvable models with quadratic Hamiltonians. The third category involves "mean field models": all N particles interact with one another with the strength proportional to N^{-1} depending only on their "species", but not on their spatial position, then we consider the limit $N \to \infty$; see e.g. [16].

The results that I am going to discuss are different. They concern classes of models that are explicitely defined, not exactly solvable and not of the mean-field kind.

They are supposed to describe a small quantum system described by a finite-dimensional Hilbert space interacting with a free Bose field at a positive temperature (or more generally, at a positive density) through an interaction, usually assumed to be linear in creation/annihilation operators. Similar models are often used in physics (quantum optics or solid state physics). They are also interesting because of their mathematical properties and have been the subject of rigorous research by various authors, e.g. [23, 10, 4]. We will use the name *Pauli–Fierz operators* to denote such models. This name has some historical justification [21] and tradition [3] and has been used consistently by some of the authors [10–12]. Let us however warn the reader that the name Pauli–Fierz Hamiltonian appears in the literature, especially by H. Spohn and his collaborators, also in a slightly different meaning—that of the nonrelativistic Hamiltonian of QED (with matter minimally coupled to photons).

In the results discussed in this paper the perturbation is multiplied by a small but nonzero coupling constant λ. This is related to the fact that perturbation techniques (especially rigorous versions of the Fermi Golden Rule [9]) play an important role in their proofs. Let us stress, however, that the results are not perturbative in the sense that they concern models with $0 < |\lambda| \leq \lambda_0$ for some $\lambda_0 > 0$. Thus they do not involve the weak coupling limit $\lambda \searrow 0$.

The series of results that we would like to discuss originated in the work of Jaksic and Pillet [17, 18]. Jaksic and Pillet used the analytic deformation technique to deal with absolutely continuous spectrum. The papers [11, 13, 12, 5, 20, 15] introduced various refinements, in particular spectral deformation was replaced or supplemented by other techniques (mostly the positive commutator method and in the case of [5] the "renormalization group method"). In my notes I will not discuss these aspects of the results about return to equilibrium. I will describe their operator algebraic background and then I will state an example of a theorem on return to equilibrium, which is a simplified version of the main result of [12].

Methods developed to prove the return to equilibrium can be also applied to the case of a reservoir that does not have a fixed temperature, e.g. consists of several reservoirs at distinct temperatures. They can be used to prove that such systems generically have no invariant normal states. This somewhat negative result has been to our knowledge first noticed in [12] and can be viewed as an expression of another conventional wisdom:

The behaviour of a small system interacting with several reservoirs at distinct temperatures is much more difficult to describe than in the case of a reservoir at a fixed temperature.

In other words, there are good mathematical reasons for the fact that nonequilibrium quantum statistical physics is in a much worse shape than its equilibrium counterpart.

Physically, one expects that a nonequilibrium system converges for large times to a steady state. The result mentioned above shows that in typical situations such a steady state is not described by a normal state on a W^*-algebra—thus the framework that works well for the return to equilbrium is less adequate for nonequilibrium situations.

Note, however, that there are models where nonequilibrium steady states can be described as states on a C^*-algebra. A number of interesting rigorous results about nonequilibrium quantum statistical physics within this formalism have been recently obtained, see [19].

2. Operator algebraic background

In this section we review some concepts that belong to operator algebras and are needed to formulate the results we discuss. There exists a number of excellent references to this subject, among them [6, 7, 24]. We also recommend the paper [13], which contains some of the technical results needed for the return to equilibrium.

2.1. C^*-dynamical systems versus W^*-dynamical systems

In algebraic quantum statistical physics a quantum system can be described by either a C^*- or W^*-dynamical system. Let us recall that:

(1) A C^*-dynamical system is a pair $(\mathfrak{A}, \alpha)$ consisting of a C^*-algebra $\mathfrak{A}$ and a 1-parameter group of $*$automorphisms of $\mathfrak{A}$,

$$\mathbb{R} \ni t \mapsto \alpha_t$$

such that $t \mapsto \alpha_t(A)$ is norm continuous for $A \in \mathfrak{A}$.

(2) A W^*-dynamical system is a pair $(\mathfrak{M}, \tau)$ consisting of a W^*-algebra $\mathfrak{M}$ and a 1-parameter group of $*$automorphisms of $\mathfrak{M}$,

$$\mathbb{R} \ni t \mapsto \tau_t$$

such that $t \mapsto \tau_t(A)$ is σ-weakly continuous for $A \in \mathfrak{M}$.

Usually, in the literature on algebraic quantum statistical physics, C^*-dynamical systems are preferred. They allow for an elegant theory explaining the existence of distinct phases of a single quantum system—they are obtained by considering various representations of the same C^*-dynamical system. The C^*-algebraic approach works usually well for fermionic and spin systems. In the case of bosons it is much more problematic, because even the usual dynamics of the local C^*-algebra of the free Bose gas is not norm continuous.

We do not want to be guided by the convenience of mathematical formalism in the choice of physical systems, and we would like to describe bosonic systems. Therefore, we will use the W^*-algebraic approach. Fortunately the language of W^*-algebras is adequate for the description of the return to equilibrium.

2.2. GNS representation

Let $\mathfrak{M}$ be a W^*-algebra.

Suppose that ω is a state on $\mathfrak{M}$. It is well known then there exists a representation $\pi : \mathfrak{M} \to B(\mathcal{H})$ with vector $\Omega \in \mathcal{H}$ cyclic for $\pi(\mathfrak{M})$ such that

$$\omega(A) = (\Omega|\pi(A)\Omega), \qquad A \in \mathfrak{M}.$$

$(\pi, \mathcal{H}, \Omega)$ is called the GNS representation given by ω. If ω is normal, then so is π.

Let τ be a W^*-dynamics on $\mathfrak{M}$. If in addition ω is stationary with respect to τ, then we have a distinguished unitary implementation of τ defined by

$$\begin{aligned} &\pi(\tau_t(A)) = e^{itL}\pi(A)e^{-itL}, \qquad A \in \mathfrak{M}, \\ &L\Omega = 0. \end{aligned} \tag{2.1}$$

2.3. Standard representation

One of the most important concepts of the modern theory of W^*-algebras is the so-called standard representation. We say that a quadruple $(\pi, \mathcal{H}, J, \mathcal{H}_+)$ is a standard representation of a W^*-algebra $\mathfrak{M}$ if $\pi : \mathfrak{M} \to \mathcal{B}(\mathcal{H})$ is a $*$-representation, J is an antiunitary involution on $\mathcal{H}$ and $\mathcal{H}_+$ is a self-dual cone in $\mathcal{H}$ satisfying the following conditions:

(1) $J\pi(\mathfrak{M})J = \pi(\mathfrak{M})'$;
(2) $J\pi(A)J = \pi(A)^*$ for A in the center of $\mathfrak{M}$;
(3) $J\Psi = \Psi$ for $\Psi \in \mathcal{H}_+$;
(4) $\pi(A)J\pi(A)\mathcal{H}_+ \subset \mathcal{H}_+$ for $A \in \mathfrak{M}$.

Every W^*-algebra has a unique (up to unitary equivalence) standard representation.

The existence of a standard representation for countably decomposable W^*-algebras follows from the following theorem.

THEOREM 1. *Let ω be a faithful state, $(\pi, \mathcal{H}, \Omega)$—the corresponding GNS representation, J—the modular conjugation given by the Tomita–Takesaki theory, and*

$\mathcal{H}^+ := \{\pi(A)J\pi(A)\Omega : A \in \mathfrak{M}\}^{\mathrm{cl}}$. *Then* $(\pi, \mathcal{H}, J, \mathcal{H}^+)$ *is a standard representation of* $\mathfrak{M}$.

The standard representation has several important properties.

THEOREM 2. *Every normal state* ω *has a unique vector representative in* $\mathcal{H}_+$, *that means, there is a unique normalized vector* $\Omega \in \mathcal{H}_+$ *such that* $\omega(A) = (\Omega|\pi(A)\Omega)$.

THEOREM 3. *For every* W^**-dynamics* τ *on* $\mathfrak{M}$ *there is a unique self-adjoint operator* L *on* $\mathcal{H}$ *such that*

$$\pi(\tau^t(A)) = e^{itL}\pi(A)e^{-itL}, \qquad e^{itL}\mathcal{H}_+ = \mathcal{H}_+. \tag{2.2}$$

The operator L *is called the Liouvillean of the* W^**-dynamical system* $(\mathfrak{M}, \tau)$. *If the* W^**-dynamics* τ *has a faithful invariant normal state* ω, *the standard form is obtained from the corresponding GNS representation as in Theorem* 1 *and* L *is the operator defined in* (2.1), *then* L *is the Liouvilean of* τ.

2.4. Return to equilibrium

Let $(\mathfrak{M}, \tau)$ be a W^*-dynamical system and L the corresponding Liouvillean. There exists a close relationship between eigenvectors of L and normal invariant states of τ.

THEOREM 4. *Let* ω *be a normal state and* $\Omega \in \mathcal{H}_+$ *its vector representative. Then the following conditions are equivalent:*
(1) ω *is* τ*-invariant,*
(2) $\Omega \in \mathrm{Ker}L$.

COROLLARY 1. *The following conditions are equivalent:*
(1) $\dim \mathrm{Ker}L = 0$,
(2) *the* W^**-dynamics* τ_t *has no normal invariant states.*

THEOREM 5. *Suppose that* ω *is faithful. Let* Ω *be its standard vector representative. Then the following statements are equivalent:*
(1) ω *is a unique invariant normal state,*
(2) *the only eigenvalue of* L *is* 0, *it is nondegenerate and* $L\Omega = 0$,
(3) *for any normal state* ϕ *and* $A \in \mathfrak{M}$,

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t \phi(\tau_s(A))ds = \omega(A).$$

If the conditions of the above theorem are true, then we say that the system satisfies the *return to equilibrium in mean.*

THEOREM 6. *Suppose that* ω *is faithful and* L *has no singular spectrum except for a simple eigenvalue at* 0. *Then for any normal state* ϕ *and* $A \in \mathfrak{M}$,

$$\lim_{t\to\infty} \phi(\tau_t(A)) = \omega(A). \tag{2.3}$$

(2.3) is called the property of *return to equilibrium.*

2.5. KMS states

Let τ be a W^*-dynamics and L the corresponding Liouvillean.

A normal state ω is called a β-KMS state iff

$$\omega(AB) = \omega(B\tau_{i\beta}(A)), \qquad A, B \in \mathfrak{M}, \qquad A \text{ analytic for } \tau.$$

Note that β-KMS states are stationary.

A vector Ω is called a β-KMS vector iff $\Omega \in \mathcal{H}^+$ and

$$e^{-\beta L/2} A\Omega = JA^*\Omega, \qquad A \in \mathfrak{M}.$$

Note that β-KMS vectors belong to Ker$L$. They are standard vector representatives of β-KMS states.

2.6. Perturbation theory of W^*-dynamics, Liouvilleans and KMS states

The following classic result of Araki [1, 7] is crucial for our understanding of stability of KMS states.

THEOREM 7. *Let $(\mathfrak{M}, \tau_{\rm fr})$ be a W^*-dynamical system with the Liouvillean $L_{\rm fr}$. (The subscript* fr *stands for "free"). Let $\Omega_{\rm fr}$ be a β-KMS vector for $\tau_{\rm fr}$. Let V be a self-adjoint operator belonging to $\mathfrak{M}$. Then:*

(1) *there exists a perturbed dynamics τ such that*

$$\frac{d}{dt}\tau_t(A) = \frac{d}{dt}\tau_{{\rm fr},t}(A) + i[V, \tau_{{\rm fr},t}(A)],$$

(2) *the Liouvillean of τ equals*

$$L = L_{\rm fr} + \pi(V) - J\pi(V)J,$$

(3) *$e^{-\beta\pi(V)/2}\Omega_{\rm fr}$ is a β-KMS vector for τ.*

Recall that a possibly unbounded self-adjoint operator V is *affiliated* to $\mathfrak{M}$ iff all its bounded Borel functions belong to $\mathfrak{M}$. As proven in [13], Theorem 7 extends to unbounded V affiliated to $\mathfrak{M}$ satisfying some mild technical assumptions.

2.7. Basic example—a type I factor

Finally, let us list the concepts discussed in this section in the case of the most elementary example of a von Neumann algebra—a type I factor. To this end we fix a Hilbert space $\mathcal{K}$ and we present a table, which on the left lists the concepts we discussed and on the right gives their description in the case of a type I factor.

W^*-algebra:	$B(\mathcal{K})$;
Standard Hilbert space:	$\mathcal{K}\otimes\overline{\mathcal{K}} = B^2(\mathcal{K})$;
Standard representation:	$\pi(A) = A \otimes 1_{\mathcal{K}} \simeq A\,\cdot$,
	(multiplication from the left by A);
Standard positive cone:	$B^2_+(\mathcal{K})$;
State:	$\omega(A) = \mathrm{Tr}\,\rho A$, $\rho \in B^1_+(\mathcal{K})$, $\mathrm{Tr}\,\rho = 1$;

Its vector representative:	$\rho^{1/2} \in B^2_+(\mathcal{K})$;
W^*-dynamics:	$\tau_t(A) = e^{itK} A e^{-itK}$, K self-adjoint;
Its Liouvillean:	$L = K \otimes 1 - 1 \otimes K \simeq [K, \cdot]$;
β-KMS state:	$\omega_\beta(A) = (\mathrm{Tr}\, e^{-\beta K})^{-1} \mathrm{Tr}\, e^{\beta K} A$;
β-KMS vector:	$(\mathrm{Tr}\, e^{-\beta K})^{-1/2} e^{\beta K/2}$.

Note that we denoted by $B^2(\mathcal{K})$ the space of (positive) Hilbert–Schmidt operators and by $B^1(\mathcal{K})$ the space of trace class operators. Likewise, we denoted by $B^2_+(\mathcal{K})$ the space of positive Hilbert–Schmidt operators and by $B^1_+(\mathcal{K})$ the space of positive trace class operators.

3. Return to equilibrium for Pauli–Fierz systems

This section gives a self-contained description of a simplified version of the main result of [11–13], which is a typical representative of results on return to equilibrium. The main reasons why I follow [11–13] are the following:

(1) The conditions on the effectiveness of the interaction have the most optimal and, in my opinion, the most elegant form in [12].
(2) The operator algebraic aspects of the problem are in my opinion most adequatly described in [13, 12].

Other works on this subject [17, 18, 5, 20, 15] use different techniques to study embedded spectrum. This may lead in some cases to different classes of interactions covered by these results. This aspect of the problem is a very interesting, and we believe, still unfinished chapter of quantum statistical physics, which involves subtle technical questions about spectral analysis of self-adjoint operators. I will not, however, discuss this point in more detail.

3.1. Bose gas at zero temperature

First I will recall basic notation used in second quantization.

Let $\mathcal{Z}$ be a Hilbert space. The bosonic Fock space over the 1-particle space $\mathcal{Z}$ is defined as

$$\Gamma_s(\mathcal{Z}) := \mathop{\oplus}_{n=0}^{\infty} \otimes_s^n \mathcal{Z}.$$

$\Omega := 1 \in \otimes_s^0 \mathcal{Z} = \mathbb{C}$ will denote the vacuum vector. If U is an operator on $\mathcal{Z}$, then $\Gamma(U)$ will denote the usual second quantization of the operator U.

For definiteness, let us assume that $\mathcal{Z} = L^2(\mathbb{R}^d)$. ξ will be used to denote the generic variable in $\mathbb{R}^d$. Let the 1-particle energy be given by $|\xi|$. Thus we consider massless bosons of zero spin. (If we prefer, by minor modifications we can consider e.g. spin 1 bosons in $d = 3$ dimension—the case of photons, etc.). We introduce the usual creation/annihilation operators $a^*(\xi)/a(\xi)$ satisfying the commutation relations

$$[a(\xi), a^*(\xi')] = \delta(\xi - \xi'), \qquad [a^*(\xi), a^*(\xi')] = [a(\xi), a(\xi')] = 0, \qquad (3.4)$$

where δ is the Dirac delta on $\mathbb{R}^d$. (Actually, $a^*(\xi)/a(\xi)$ are not operators but symbols that acquire the meaning of (unbounded) operators after smearing with appropriate functions, such as in (3.5)). We will assume that bosons are described by the Hamiltonian

$$\int d\xi |\xi| a^*(\xi) a(\xi). \tag{3.5}$$

We will refer to (3.5) as the Hamiltonian of the free Bose gas in infinite volume.

It is also useful and physically justified to consider bosons in a finite volume. It is usually believed that for large volume the shape of the "container" and the boundary conditions do not matter much when studying most physical features of such a system. Therefore we assume that they are as simple as possible for mathematical analysis of the problem: the gas is confined to a cube of side length L with periodic boundary conditions. After the Fourier transformation the momenta are contained in the lattice $(\mathbb{Z}/L)^d$. Thus the system is described by the Fock space $\Gamma_s\big((\mathbb{Z}/L)^d\big)$. The creation/annihilation operators are this time true operators and they satisfy the commutation relations (3.4), where δ is now the Kronecker delta. The Hamiltonian is given by the expression (3.5) where we replace $\int d\xi$ with $\sum_{\xi\in(\mathbb{Z}/L)^d}$.

3.2. Bose gas at density ρ—Araki–Woods algebras

Assume that $\mathbb{R}^d \ni \xi \mapsto \rho(\xi)$ is a nonnegative real measurable function describing the density of bosons with the momentum $\xi \in \mathbb{R}^d$. To describe the Bose gas at density ρ one uses a special von Neumann algebra first described by Araki and Woods in [2], see also [8]. It can be defined by its representation in the Hilbert space

$$\mathcal{H}^{\mathrm{AW}} := \Gamma_s(L^2(\mathbb{R}^d) \oplus L^2(\mathbb{R}^d)).$$

We will write $a_{\mathrm{l}}(\xi)$, $a_{\mathrm{l}}^*(\xi)$, $a_{\mathrm{r}}(\xi)$, $a_{\mathrm{r}}^*(\xi)$ for the creation and annihilation operators corresponding to the left and right $L^2(\mathbb{R}^d)$ resp. We define the left/right Araki–Woods creation and annihilation operators

$$\begin{aligned}
a_{\rho,\mathrm{l}}^*(\xi) &:= \sqrt{1+\rho(\xi)}\,a_{\mathrm{l}}^*(\xi) + \sqrt{\rho(\xi)}\,a_{\mathrm{r}}(\xi),\\
a_{\rho,\mathrm{l}}(\xi) &:= \sqrt{1+\rho(\xi)}\,a_{\mathrm{l}}(\xi) + \sqrt{\rho(\xi)}\,a_{\mathrm{r}}^*(\xi),\\
a_{\rho,\mathrm{r}}^*(\xi) &:= \sqrt{\rho(\xi)}\,a_{\mathrm{l}}(\xi) + \sqrt{1+\rho(\xi)}\,a_{\mathrm{r}}^*(\xi),\\
a_{\rho,\mathrm{r}}(\xi) &:= \sqrt{\rho(\xi)}\,a_{\mathrm{l}}^*(\xi) + \sqrt{1+\rho(\xi)}\,a_{\mathrm{r}}(\xi).
\end{aligned}$$

The left Araki–Woods algebra is denoted by $\mathfrak{M}_{\rho,\mathrm{l}}^{\mathrm{AW}}$ and defined as the W^*-algebra generated by the operators

$$W(f) := \exp\Big(i \int (f(\xi) a_{\rho,\mathrm{l}}^*(\xi) + \overline{f}(\xi) a_{\rho,\mathrm{l}}(\xi)) d\xi\Big), \tag{3.6}$$

where $f \in L^2(\mathbb{R}^d)$ satisfies $\int |f(\xi)|^2 \rho(\xi) d\xi < \infty$.

Let $J^{\mathrm{AW}} := \Gamma(\epsilon)$, where ϵ is an antilinear involution on $L^2(\mathbb{R}^d) \oplus L^2(\mathbb{R}^d)$ given by

$$\epsilon(f_1, \overline{f}_2) := (f_2, \overline{f}_1),$$

and let $\mathcal{H}_\rho^{\mathrm{AW},+}$ be the closure of the cone in $\mathcal{H}^{\mathrm{AW}}$ generated by

$$AJA\Omega, \qquad A \in \mathfrak{M}_{\rho,1}^{\mathrm{AW}}.$$

Then $(\mathfrak{M}_{\rho,1}^{\mathrm{AW}}, \mathcal{H}^{\mathrm{AW}}, J^{\mathrm{AW}}, \mathcal{H}_\rho^{\mathrm{AW},+})$ is a von Neumann algebra in a standard form. It describes the Bose gas at density ρ. The state $(\Omega|\cdot\Omega)$ is a quasi-free state for the representation of CCR given by (3.6). The algebra $\mathfrak{M}_{\rho,1}^{\mathrm{AW}}$ is type I for $\rho = 0$, otherwise it is always type III_1.

There exists a unique W^*-dynamics on $\mathfrak{M}_{\rho,1}^{\mathrm{AW}}$ that on the Weyl operators acts as

$$\tau_{\rho,t}^{\mathrm{AW}}(W(f)) := W(e^{it|\xi|}f).$$

Its Liouvillean equals

$$\int d\xi\,|\xi|\,\left(a_1^*(\xi)a_1(\xi) - (a_{\mathrm{r}}^*(\xi)a_{\mathrm{r}}(\xi)\right) = \int d\xi\,|\xi|\,\left(a_{\rho,1}^*(\xi)a_{\rho,1}(\xi) - (a_{\rho,\mathrm{r}}^*(\xi)a_{\rho,\mathrm{r}}(\xi)\right).$$

Thus we obtain a family of W^*-dynamical systems $(\mathfrak{M}_{\rho,1}^{\mathrm{AW}}, \tau_\rho^{\mathrm{AW}})$. These W^*-dynamical systems are often nonisomorphic to one another.

Of course, we can do the same for the cube with periodic boundary conditions. In this case, we start from a function

$$(\mathbb{Z}/L)^d \in \xi \mapsto \rho^L \in [0, \infty[.$$

Thus we obtain a family of algebras in standard representation $(\mathfrak{M}_{\rho,1}^{\mathrm{AW},L}, \mathcal{H}^{\mathrm{AW},L}, J^{\mathrm{AW},L}, \mathcal{H}_\rho^{\mathrm{AW},L,+})$. We also obtain a family of W^*-dynamical systems $(\mathfrak{M}_{\rho,1}^{\mathrm{AW},L}, \tau_\rho^{\mathrm{AW},L})$.

Note that if $\sum_{\xi\in(\mathbb{Z}/L)^d} \rho(\xi) < \infty$, then all of them are unitarily isomorphic to one another. In fact, they are isomorphic to the case of $\rho = 0$, which is just the free Bose gas described in Subsection 3.1. The state $(\Omega|\cdot\Omega)$ is equivalent to the state given by the density matrix

$$\Gamma\left(\gamma(1+\gamma)^{-1}\right) / \operatorname{Tr}\Gamma\left(\gamma(1+\gamma)^{-1}\right).$$

We can view free Bose gas at density ρ in infinite volume as thermodynamic limit of the free Bose gas in finite volume obtained by a special limiting procedure. It is interesting to note that in finite volume the systems we consider can be equivalent and trivial from the point of view of operator algebras (type I), and in the limit they may become nonequivalent and nontrivial (type III).

3.3. Thermal Araki–Woods representations

Let $\beta > 0$ be the inverse temperature. The radiation density given by the Planck law

$$\rho_\beta(\xi) = (e^{\beta|\xi|} - 1)^{-1} \tag{3.7}$$

corresponds to the inverse temperature β. When dealing with (3.7) we will replace the index ρ_β with β in various symbols.

In this case the state $(\Omega|\cdot\Omega)$ satisfies the KMS condition for the dynamics τ_β^{AW}.

In a finite volume, the state $(\Omega|\cdot\Omega)$ on $\mathfrak{M}_{\beta,1}^{\mathrm{AW},L}$ is equivalent to the state given by the density matrix $e^{-\beta H}/\mathrm{Tr}\, e^{-\beta H}$ on the algebra $B(\Gamma_{\mathrm{s}}(L^2(\mathbb{R}^d))$ and the Araki–Woods representation is unitarily equivalent to the GNS representation for this state [8].

3.4. Pauli–Fierz system at zero temperature

Suppose that the Bose gas interacts with a small system described by a finite-dimensional Hilbert space $\mathcal{K}$ and a Hamiltonian K. The Hilbert space of the Pauli–Fierz system at zero temperature is $\mathcal{K}\otimes\Gamma_{\mathrm{s}}(L^2(\mathbb{R}^d))$, where $\Gamma_{\mathrm{s}}(L^2(\mathbb{R}^d))$ denotes the symmetric (bosonic) Fock space over the 1-particle space $L^2(\mathbb{R}^d)$. The free Pauli–Fierz Hamiltonian is

$$H_{\mathrm{fr}} := K\otimes 1 + 1\otimes\int|\xi|a^*(\xi)a(\xi)d\xi,$$

where $a^*(\xi)/a(\xi)$ are the creation/annihilation operators of bosons of momentum $\xi\in\mathbb{R}^d$.

The interaction is described by a measurable operator-valued function (form-factor) $\mathbb{R}^d\ni\xi\mapsto v(\xi)\in\mathcal{B}(\mathcal{K})$. We assume that the form-factor satisfies

$$\int(1+|\xi|^{-1})\|v(\xi)\|^2 d\xi<\infty. \tag{3.8}$$

The interaction is given by the operator

$$V := \int(v(\xi)\otimes a^*(\xi)+v^*(\xi)\otimes a(\xi))d\xi,$$

and the full Pauli–Fierz Hamiltonian equals

$$H := H_{\mathrm{fr}}+\lambda V,$$

where $\lambda\in\mathbb{R}$. H is self-adjoint on the domain of H_{fr} and bounded from below.

The Pauli–Fierz Hamiltonians arise as an approximation to the standard Hamiltonian of the nonrelativistic QED [11, 4], or as effective Hamiltonians describing interaction of an atom with phonons.

3.5. Pauli–Fierz systems at nonzero density

The Pauli–Fierz algebra at density ρ, $\mathfrak{M}_\rho$, is defined by $\mathfrak{M}_\rho := \mathcal{B}(\mathcal{K})\otimes\mathfrak{M}_{\rho,1}^{\mathrm{AW}}$. To define the dynamics, we need the following assumption.

ASSUMPTION 1. $\int(1+|\xi|^2)(1+\rho(\xi))\|v(\xi)\|^2 d\xi<\infty$.

Set

$$L_{\mathrm{fr}}^{\mathrm{semi}} := K\otimes 1+1\otimes\int\Big(|\xi|a_{\mathrm{l}}^*(\xi)a_{\mathrm{l}}(\xi)-|\xi|a_{\mathrm{r}}^*(\xi)a_{\mathrm{r}}(\xi)\Big)d\xi,$$

$$V_\rho := \int\Big(v(\xi)\otimes a_{\rho,1}^*(\xi)+v^*(\xi)\otimes a_{\rho,1}(\xi)\Big)d\xi,$$

$$L_\rho^{\mathrm{semi}} := L_{\mathrm{fr}}^{\mathrm{semi}}+\lambda V_\rho. \tag{3.9}$$

Note that V_ρ is affiliated to $\mathfrak{M}_\rho$.

PROPOSITION 1. *Assume that Assumption 1 holds. Then the operator L_ρ^{semi} is essentially self-adjoint on* $\mathrm{Dom}(L_{\mathrm{fr}}) \cap \mathrm{Dom}(V_\rho)$ *and*

$$\tau_\rho^t(A) := e^{itL_\rho^{\mathrm{semi}}} A e^{-itL_\rho^{\mathrm{semi}}} \tag{3.10}$$

is a W^*-dynamics on $\mathfrak{M}_\rho$.

We will call the W^*-dynamical system $(\mathfrak{M}_\rho, \tau_\rho)$ the Pauli–Fierz system at density ρ. In the absence of interaction ($\lambda = 0$) we call it the free Pauli–Fierz system.

The identity representation $\mathfrak{M}_\rho \to \mathcal{B}(\mathcal{K} \otimes \Gamma_{\mathrm{s}}(L^2(\mathbb{R}^d) \oplus L^2(\mathbb{R}^d))$ will be called the semistandard representation of the Pauli–Fierz system, to distinguish it from the standard representation described in the next subsection. Similarly, we will call the operator L_ρ^{semi} the Pauli–Fierz semi-Liouvillean at density ρ.

3.6. Pauli–Fierz systems in standard representation

It is easy to describe the standard representation of $\mathfrak{M}_\rho$. Let $\overline{\mathcal{K}}$ be the Hilbert space complex conjugate to $\mathcal{K}$ (see e.g. Section 4.6 in [12]). The standard representation acts on the space

$$\mathcal{K} \otimes \overline{\mathcal{K}} \otimes \Gamma_{\mathrm{s}}(L^2(\mathbb{R}^d) \oplus L^2(\mathbb{R}^d)) \tag{3.11}$$

and is given by $\pi(A \otimes B) := A \otimes 1_{\overline{\mathcal{K}}} \otimes B$ for $A \in \mathcal{B}(\mathcal{K})$, $B \in \mathfrak{M}_{\rho,\mathrm{l}}^{\mathrm{AW}}$. The modular conjugation is given by

$$J\,\Psi_1 \otimes \overline{\Psi}_2 \otimes \Phi := \Psi_2 \otimes \overline{\Psi}_1 \otimes J^{\mathrm{AW}}\Phi.$$

Note that it is useful to consider the two representations of $\mathfrak{M}_\rho$—the semi-standard and the standard representations in a parallel way. The semi-standard representation is simpler whereas the standard representation has special mathematical properties.

We have

$$\pi(V_\rho) = \int v(\xi) \otimes 1 \otimes a_{\rho,\mathrm{l}}^*(\xi)d\xi + \mathrm{hc},$$

$$J\pi(V_\rho)J = \int 1 \otimes \overline{v}(\xi) \otimes a_{\rho,\mathrm{r}}^*(\xi)d\xi + \mathrm{hc}.$$

The Liouvillean of the free Pauli–Fierz system is

$$L_{\mathrm{fr}} = K \otimes 1 \otimes 1 - 1 \otimes \overline{K} \otimes 1 + 1 \otimes 1 \otimes \int \big(|\xi| a_{\mathrm{l}}^*(\xi)a_{\mathrm{l}}(\xi) - |\xi| a_{\mathrm{r}}^*(\xi)a_{\mathrm{r}}(\xi)\big)d\xi.$$

Set

$$L_\rho := L_{\mathrm{fr}} + \lambda\pi(V_\rho) - \lambda J\pi(V_\rho)J.$$

PROPOSITION 2. *Assume that Assumption 1 holds. Then the operator L_ρ is essentially self-adjoint on* $\mathrm{Dom}(L_{\mathrm{fr}}) \cap \mathrm{Dom}(\pi(V_\rho)) \cap \mathrm{Dom}(J\pi(V_\rho)J)$ *and is the Liouvillean of the Pauli–Fierz system* $(\mathfrak{M}_\rho, \tau_\rho)$.

3.7. Thermal Pauli–Fierz systems

Let $\beta > 0$ be the inverse temperature. A Pauli–Fierz system whose radiation density is given by the Planck law (3.7) is called a thermal Pauli–Fierz system at inverse temperature β. Due to the specific form of the Planck law Assumption 1 takes a somewhat simpler form and is equivalent to another assumption.

ASSUMPTION 2. $\int(|\xi|^2 + |\xi|^{-1})\|v(\xi)\|^2 d\xi < \infty$.

Again, with a slight abuse of notation instead of the subscript ρ_β we will use β, so L_β and τ_β^t now stand for L_{ρ_β} and $\tau_{\rho_\beta}^t$ etc. The free Pauli–Fierz system has a unique β-KMS state given by the density matrix

$$\frac{e^{-\beta K}}{\mathrm{Tr}\, e^{-\beta K}} \otimes |\Omega)(\Omega), \tag{3.12}$$

where Ω is the vacuum on $\Gamma_s(L^2(\mathbb{R}^d) \oplus L^2(\mathbb{R}^d))$. Let $\gamma_\beta := e^{-\beta K/2}/(\mathrm{Tr}\, e^{-\beta K})^{1/2}$. Then $\gamma_\beta \otimes \Omega$ is the standard vector representative of (3.12). Using the main result of [13] one can easily show the following theorem:

THEOREM 8. *Assume that Assumption 2 holds. Then for all $\lambda \in \mathbb{R}$ and $\beta \in]0, \infty[$ the Pauli–Fierz system $(\mathfrak{M}_\beta, \tau_\beta)$ has a unique β-KMS state. Moreover, $\gamma_\beta \otimes \Omega \in$ $\mathrm{Dom}(e^{-\beta(L_{\mathrm{fr}}+\lambda\pi(V_\beta))/2})$ and*

$$e^{-\beta(L_{\mathrm{fr}}+\lambda\pi(V_\beta))/2}\gamma_\beta \otimes \Omega \Big/ \|e^{-\beta(L_{\mathrm{fr}}+\lambda\pi(V_\beta))/2}\gamma_\beta \otimes \Omega\|$$

is the corresponding β-KMS vector.

3.8. Return to equilibrium

In this subsection we state simplified versions of the main result of [12]. We use the following notation. $\mathrm{sp}(K)$ denotes the spectrum of K. The spectral projection of K onto $k \in \mathbb{R}$ will be denoted by $1_k(K)$ and $v^{k_1,k_2}(\xi) = 1_{k_1}(K)v(\xi)1_{k_2}(K)$. Obviously, $v^{k_1,k_2}(\xi) = 0$ unless $k_1, k_2 \in \mathrm{sp}(K)$. $p \in \mathbb{R}_+$ denotes the radial coordinate. S^{d-1} is the $(d-1)$-dimensional unit sphere, $\omega \in S^{d-1}$ is the angle coordinate and $d\omega$ is the surface measure on S^{d-1}.

Let $\mathcal{F}^+$ be the set of positive differences of eigenvalues of K. (In physical terms, these are the Bohr frequencies of the small system—the energies of photons that can be emitted).

An important role will be played by a certain subset $\mathfrak{N}$ of bounded operators on $\mathcal{K}$ defined as follows: $B \in \mathcal{B}(\mathcal{K})$ belongs to $\mathfrak{N}$ iff for almost all $\omega \in S^{d-1}$ we have

$$\begin{aligned} B \sum_{k\in \mathrm{sp}(K)} v^{k-p,k}(p\omega) &= \sum_{k\in \mathrm{sp}(K)} v^{k-p,k}(p\omega)B, \qquad p \in \mathcal{F}^+, \\ B^* \sum_{k\in \mathrm{sp}(K)} v^{k-p,k}(p\omega) &= \sum_{k\in \mathrm{sp}(K)} v^{k-p,k}(p\omega)B^*, \qquad p \in \mathcal{F}^+, \\ B \sum_{k\in \mathrm{sp}(K)} \lim_{p\downarrow 0} p^{-1/2} v^{k,k}(p\omega) &= \sum_{k\in \mathrm{sp}(K)} \lim_{p\downarrow 0} p^{-1/2} v^{k,k}(p\omega)B. \end{aligned} \tag{3.13}$$

Obviously, $1 \in \mathfrak{N}$. Note also that $\mathfrak{N}$ is a $*$-algebra invariant with respect to $e^{itK}\cdot e^{-itK}$.

THEOREM 9. *Suppose that Assumption 2 holds and the following conditions are satisfied:*
(1)

$$\int \|\partial_p^3 p^{-1+d/2}\langle p\rangle^{1/2} v(p\omega)\|^2 dp d\omega < \infty,$$

$$\partial_p^j p^{-1+d/2} v(p\omega)\Big|_{p=0} = (-1)^j \partial_p^j p^{-1+d/2} v^*(p\omega)\Big|_{p=0}, \quad j = 0, 1, 2, \quad \omega \in S^{d-1}.$$

(2) $\mathfrak{N} = \mathbb{C}1$.
Then for any $0 < \beta < \infty$ *there exists* $\lambda_0(\beta) > 0$ *such that for* $0 < |\lambda| < \lambda_0(\beta)$ *the Pauli–Fierz Liouvillean* L_β *has no singular spectrum except for a simple eigenvalue at zero. Consequently, under the above conditions the system* $(\mathfrak{M}_\beta, \tau_\beta)$ *has the property of return to equilibrium.*

Condition (1) is the regularity assumption. Note that it allows for quite singular infrared behaviour of the form-factor. For example, assume that $v(\xi)$ is smooth outside of zero and of compact support. Then (1) holds if around zero

$$v(\xi) = v_0 |\xi|^{1-d/2}, \tag{3.14}$$

where $v_0 \in \mathcal{B}(\mathcal{K})$ is self-adjoint.

Condition (2) is the effective coupling assumption.

Let us mention that the above formalism can be applied to nonthermal radiation densities. For instance, if the small system interacts with several reservoirs at distinct temperatures, each satisfying the conditions of Theorem 9, then the Liouvillean has no singular spectrum. By Corollary 1 this implies that under these assumptions the Pauli–Fierz system $(\mathfrak{M}_\rho, \tau_\rho)$ has no normal states, see [12, 9].

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REFERENCES

[1] H. Araki: Relative Hamiltonian for faithful normal states of a von Neumann algebra, *Pub. RIMS Kyoto Univ.* **9** (1973), 165.

[2] H. Araki and E. J. Woods: Representations of the canonical commutation relations describing a nonrelativistic infinite free Bose gas, *J. Math. Phys.* **4** (1963), 637.

[3] P. Blanchard: Discussion mathematique du modèle de Pauli et Fierz relatif à la catastrophe infrarouge, *Commun. Math. Phys.* **15** (1969), 156.

[4] V. Bach, J. Fröhlich and I. Sigal: Quantum electrodynamics of confined non-relativistic particles, *Adv. Math.* **137** (1998), 299.

[5] V. Bach, J. Fröhlich and I. Sigal: Return to equilibrium, *Journ. Math. Phys.* **41** (2000), 3985.

[6] O. Brattelli and D. W. Robinson: *Operator Algebras and Quantum Statistical Mechanics, Volume 1,* Springer, Berlin, second edition 1987.

[7] O. Brattelli and D. W. Robinson: *Operator Algebras and Quantum Statistical Mechanics, Volume 2,* Springer, Berlin, second edition 1996.

[8] J. Dereziński: Introduction to Representations of Canonical Commutation and Anticommutation Relations, in *Large Coulomb Systems – QED*, eds J. Derezinski and H. Siedentop, Lecture Notes in Physics 695, Springer 2006.
[9] J. Dereziński and R. Früboes: Fermi Golden Rule and Open Quantum Systems, in *Open Quantum Systems III. Recent Developments*, eds S. Attal, A. Joye, C.-A. Pillet, Lecture Notes in Mathematics 1882, Springer 2006.
[10] J. Dereziński and C. Gérard: Asymptotic completeness in quantum field theory. Massive Pauli-Fierz Hamiltonians, *Rev. Math. Phys.* **11** (1999), 383.
[11] J. Dereziński and V. Jakšić: Spectral theory of Pauli-Fierz operators, *J. Func. Anal.* **180** (2001), 241.
[12] J. Dereziński and V. Jakšić: Return to equilibrium for Pauli-Fierz systems, *Ann. H. Poincaré* **4** (2003), 739–793.
[13] J. Dereziński, V. Jakšić and C. A. Pillet: Perturbation theory of W^*-dynamics, Liouvilleans and KMS-states, *Rev. Math. Phys.* **15** (2003), 447–489.
[14] M. Fannes, B. Nachtergale and A. Verbeure: The equilibrium states of the spin-boson model, *Commun. Math. Phys.* **114** (1988), 537.
[15] J. Fröhlich and M. Merkli: Another Return of 'Return to Equilibrium' *Commun. Math. Phys.* **251** (2004), 235–262.
[16] Hepp, K., Lieb, E.: Phase transitions in reservoir-driven open systems with applications to lasers and superconductors, *Helv. Phys. Acta* **46** (1973), 573–602.
[17] V. Jakšić and C. A. Pillet: On a model for quantum friction II: Fermi's golden rule and dynamics at positive temperature, *Commun. Math. Phys.* **176** (1996), 619.
[18] V. Jakšić and C. A. Pillet: On a model for quantum friction III: Ergodic properties of the spin-boson system, *Commun. Math. Phys.* **178** (1996), 627.
[19] V. Jakšić, Y. Ogata and C.A. Pillet: The Green-Kubo formula and the Onsager reciprocity relations in quantum statistical mechanics, *Commun. Math. Phys.* **265** (2006), 721.
[20] M. Merkli: Positive commutators in non-equilibrium quantum statistical mechanics, *Commun. Math. Phys.* **223** (2001), 327.
[21] W. Pauli and M. Fierz: *Nuovo Cimento* **15** (1938), 167.
[22] D. Ruelle: Natural nonequilibrium states in quantum statistical mechanics, *J. Stat. Phys.* **98** (2000), 57 .
[23] H. Spohn: Ground states of the spin-boson Hamiltonian, *Commun. Math. Phys.* **123** (1989), 277.
[24] S. Stratila: *Modular Theory in Operator Algebras*, Abacus Press, Turnbridge Wells 1981.