JOURNAL OF MATHEMATICAL PHYSICS **46**, 063511 (2005)

Stationary van Hove limit

Jan Dereziński[a] and Rafał Früboes[b]
Department of Mathematical Methods in Physics, Warsaw University, Hoża 74, 00-682, Warsaw, Poland

(Received 22 December 2004; accepted 14 March 2005; published online 7 June 2005)

The weak coupling (van Hove) limit of one parameter groups of contractions is studied by the stationary approach. We show that the resolvent of the properly renormalized and rescaled generator of a contractive semigroup has a limit as the coupling constant goes to zero. This limit is the resolvent of the generator of a certain contractive semigroup. Our results can be viewed as a stationary counterpart to the well known results about the weak coupling limit obtained by the time-dependent approach, due to Davies. We compare both approaches. © *2005 American Institute of Physics.* [DOI: 10.1063/1.1904509]

I. INTRODUCTION

Let $\mathcal{X}$ be a Banach space with a distinguished bounded projection P. Suppose that U_t^λ is a one parameter strongly continuous group of isometries on $\mathcal{X}$ generated by $L_\lambda := L_0 + \lambda Q$. Assume that P commutes with the free dynamics $U_t^0 = \mathrm{e}^{tL_0}$ or equivalently P commutes with L_0. Our main object of interest is the reduced dynamics

$$\mathbb{R} \ni t \mapsto PU_t^\lambda P$$

as an operator on Ran P.

The reduced dynamics was studied in a series of papers[4,5] and in the book[6] by Davies. First he showed that the reduced dynamics after appropriately rescaling, i.e.,

$$t \mapsto PU_{t/\lambda^2}^\lambda P,$$

can be approximated, as the coupling constant λ goes to zero, by a certain one parameter semigroup on Ran P depending on λ. The generator of this semigroup is a quadratic polynomial in λ^{-1}. By the convergence we mean that for each fixed time t the norm of the difference between the resulting semigroup and the reduced rescaled dynamics tends to zero as $|\lambda|$ becomes smaller. We will call this result, for the reasons that soon become clear, the pointwise (in time) van Hove limit with the first order term.

The second result obtained by Davies describes the case with $PQP=0$. He proved that the reduced dynamics with a rescaled time renormalized by the free evolution, i.e.,

$$t \mapsto U_{-t/\lambda^2}^0 PU_{t/\lambda^2}^\lambda P, \tag{1.1}$$

has a limit, for each fixed time t, as the coupling constant λ goes to zero. The limit is a one parameter semigroup, independent of λ. The generator of the resulting semigroup is often called the Davies generator. We will call this limit the pointwise (in time) van Hove limit without the first order term.

One can distinguish two approaches to semigroups, the time-dependent approach and the stationary approach. The former concentrates on the study of semigroups themselves. The latter

[a]Electronic mail: jan.derezinski@fuw.edu.pl
[b]Electronic mail: rafal.fruboes@fuw.edu.pl

focuses at the resolvent of their generators. Davies used the time-dependent approach, both in the choice of the assumptions for his results and in their statements. In our paper we use mostly the stationary approach.

Our main results are contained in three theorems. First in Theorem 3.1 we study, the stationary van Hove limit for the reduced dynamics as the coupling constant λ goes to zero. More precisely, we describe the asymptotics of the rescaled resolvent of L_λ reduced by P by the resolvent of a certain operator $A_{\lambda,0}$. This is the stationary counterpart to the first result of Davies.

Theorem 3.2 describes the case without the first order term $(PQP=0)$. We additionally assume that the spectrum of PL_0P consist of isolated points. We obtain a simple asymptotics of the Laplace transform of (1.1) given by the resolvent of a certain operator Γ independent of λ commuting with PL_0P. We prove that the operator Γ is the generator of a contractive semigroup. This is the stationary counterpart to the second result of Davies.

The main results about the stationary van Hove limit involve resolvents. They have, however, easy time-dependent corollaries, which we call the smeared out weak coupling limit. By this we mean that the difference of the rescaled restricted dynamics and the approximating dynamics averaged over time with some continuous function of compact support tends to zero as the coupling constant goes to zero. This version of the result is also contained in Theorems 3.1 and 3.2.

In Theorem 3.3 we show how one can obtain pointwise (in time) van Hove limit without the first order term starting from the stationary van Hove limit.

Theorem 3.5 gives conditions when one can apply both the time-dependent method of Davies and our stationary results. We also find out that the semigroup obtained in the van Hove limit is generated by the so-called level shift operator, see Refs. 9–11 and 7.

In the physical literature one can trace back the weak coupling limit to works of Wigner–Weisskopf, Pauli and also van Hove.[18–20] First rigorous mathematical treatment of this issue comes from Davies,[4–6] who gave both its abstract theory and presented applications to open quantum systems (see also Ref. 14).

In his papers, Davies uses the time-dependent approach, i.e., he works with the restricted dynamics. The key step in this approach is the construction of the resulting semigroup by the integral formula (e.g., Theorem 3.1.33 in Ref. 3). The use of this formula induces technical assumptions which may be not easy to verify. For the convenience of the reader we describe the result of Davies in Theorem 3.4.

In our approach to the weak coupling limit, instead of working with the perturbed dynamics, we investigate the resolvent of the perturbed generator. We use some regularity assumptions for resolvents which seem easier to verify in some circumstances. In particular our regularity assumptions are closely related to the so-called limiting absorption principle which can be investigated with help of the so-called Mourre theory.[8]

We end this introduction with a description of the main physical motivation of our work—an application of the van Hove limit to open quantum systems. We essentially follow Ref. 14, see also Refs. 4 and 5. For more information, especially concerning the relationship of the van Hove limit to applications of the method of the level shift operator to the return to equilibrium, we refer the reader to Ref. 11. For related analysis of open quantum systems see also Refs. 15–17.

Let us consider a quantum mechanical system which consists of two parts—the small system $\mathcal{S}$ and the reservoir part $\mathcal{R}$. To describe $\mathcal{S}$ one chooses appropriate Hilbert space $\mathcal{H}_{\mathcal{S}}$. Then the states are given by density matrices (i.e., trace class normalized positive operators on $\mathcal{H}_{\mathcal{S}}$). The time evolution of the isolated small system $\mathcal{S}$ is implemented by the Hamiltonian $H_{\mathcal{S}}$. In a similar way we describe the reservoir part $\mathcal{R}$. We have Hilbert space $\mathcal{H}_{\mathcal{R}}$, the Liouvillean $L_{\mathcal{R}}$ (the generator of the time evolution) and density matrices on $\mathcal{H}_{\mathcal{R}}$. Let us additionally assume that there exists a stationary state, denoted $\omega_{\mathcal{R}}$, of $\mathcal{R}$ for the evolution implemented by $L_{\mathcal{R}}$.

The time evolution of whole system $\mathcal{S}+\mathcal{R}$ is given by self-adjoint operator

$$L_\lambda := H_{\mathcal{S}} \otimes \mathbf{1}_{\mathcal{R}} + \mathbf{1}_{\mathcal{S}} \otimes L_{\mathcal{R}} + \lambda V$$

acting on $\mathcal{H} := \mathcal{H}_{\mathcal{S}} \otimes \mathcal{H}_{\mathcal{R}}$ where V is some interaction operator and $\lambda \in \mathbb{R}$. Hence for any density matrix χ on $\mathcal{H}_{\mathcal{S}} \otimes \mathcal{H}_{\mathcal{R}}$ its evolution is given by

$$U_t^\lambda \chi := e^{-tiL_\lambda} \chi e^{tiL_\lambda}.$$

Assume that the initial state of $\mathcal{S}+\mathcal{R}$ is $\rho \otimes \omega_{\mathcal{R}}$ for some density matrix ρ on $\mathcal{H}_{\mathcal{S}}$. Then after time t the state of $\mathcal{S}+\mathcal{R}$ is given by $U_t^\lambda(\rho \otimes \omega_{\mathcal{R}})$. If we treat $\mathcal{R}$ just as a device which induces changes of $\mathcal{S}$ and we want only to know what happens to $\mathcal{S}$, then to obtain the state of $\mathcal{S}$ at time t we take the partial trace over the degrees of freedom of $\mathcal{R}$,

$$\rho(t) := \mathrm{tr}_{\mathcal{R}}(U_t^\lambda(\rho \otimes \omega_{\mathcal{R}})). \tag{1.2}$$

Note that the action of U_t^λ can be extended to the whole space of the trace class operators $B^1(\mathcal{H}_{\mathcal{S}} \otimes \mathcal{H}_{\mathcal{R}})$. Recall that $B^1(\mathcal{H}_{\mathcal{S}} \otimes \mathcal{H}_{\mathcal{R}})$ is a Banach space under the norm $\|\cdot\|_1 = \mathrm{tr}|\cdot|$ and U_t^λ is a one parameter strongly continuous group of isometries on $B^1(\mathcal{H}_{\mathcal{S}} \otimes \mathcal{H}_{\mathcal{R}})$. If we introduce the operator

$$P: B^1(\mathcal{H}_{\mathcal{S}} \otimes \mathcal{H}_{\mathcal{R}}) \to B^1(\mathcal{H}_{\mathcal{S}}) \otimes \omega_{\mathcal{R}} \subset B^1(\mathcal{H}_{\mathcal{S}} \otimes \mathcal{H}_{\mathcal{R}}),$$

$$PW := \mathrm{tr}_{\mathcal{R}}(W) \otimes \omega_{\mathcal{R}},$$

then P is a projection of norm one and the equation (1.2) can be rewritten

$$\rho(t) \otimes \omega_{\mathcal{R}} := PU_t^\lambda P(\rho \otimes \omega_{\mathcal{R}}).$$

Note also that $[U_t^0, P] = 0$. Therefore, we have a setup, where we can apply our results. In the weak coupling limit we obtain completely positive semigroup of contractions which is sometimes called a quantum dynamical Markov semigroup. Hence starting with a fully reversible dynamics for the whole system, we end up with an irreversible evolution of the small subsystem. Now, in the weak coupling approximation, when we study the small system we may exchange complicated object $PU_t^\lambda P$ for a semigroup and use it in order to determine physical quantities. However there is a price to be paid—the results that we get in this approximation are typically the lowest order nonvanishing corrections in the coupling constant to the real quantities.

II. PRELIMINARIES

Notation: Let $\mathcal{X}$ be a Banach space. For a linear operator L on $\mathcal{X}$, $\mathrm{sp}\, L$ denotes its spectrum and $\mathrm{Dom}\, L$ its domain. If Ξ is an isolated bounded subset of $\mathrm{sp}\, L$ then the spectral projection of L onto Ξ, defined by the usual integral formula,[13] is denoted $\mathbf{1}_\Xi(L)$. If e is an isolated point of $\mathrm{sp}\, L$, then we will write $\mathbf{1}_e(L)$ for $\mathbf{1}_{\{e\}}(L)$.

Let $\mathbf{1}^{vv}$ be a distinguished bounded projection on $\mathcal{X}$. It will be convenient to denote $\mathbf{1}^{\overline{vv}} := \mathbf{1} - \mathbf{1}^{vv}$. We also introduce closed subspaces

$$\mathcal{X}^v := \mathbf{1}^{vv}\mathcal{X}, \quad \mathcal{X}^{\overline{v}} := \mathbf{1}^{\overline{vv}}\mathcal{X} = \mathrm{Ker}\, \mathbf{1}^{vv},$$

so that $\mathcal{X}$ is decomposed into a direct sum

$$\mathcal{X} = \mathcal{X}^v \oplus \mathcal{X}^{\overline{v}}. \tag{2.1}$$

Any operator H on $\mathcal{X}$ satisfying

$$\mathrm{Dom}(H) = (\mathrm{Dom}(H) \cap \mathcal{X}^v) \oplus (\mathrm{Dom}(H) \cap \mathcal{X}^{\overline{v}})$$

can be written with respect to the decomposition (2.1) as

$$H = \begin{bmatrix} H^{vv} & H^{v\bar{v}} \\ H^{\bar{v}v} & H^{\bar{v}\bar{v}} \end{bmatrix} = H^{vv} + H^{v\bar{v}} + H^{\bar{v}v} + H^{\bar{v}\bar{v}}. \tag{2.2}$$

Obviously $H^{vv} = \mathbf{1}^{vv} H \mathbf{1}^{vv}$, etc. In particular

$$\mathbf{1} := \begin{bmatrix} \mathbf{1}^{vv} & 0 \\ 0 & \mathbf{1}^{\bar{v}\bar{v}} \end{bmatrix}.$$

For $e \in i\mathbb{R}$ and for $\alpha \geqslant 0$ we denote

$$\mathbf{Wedge}(e,\alpha) := \{z \in \mathbb{C}: \mathrm{Re}\, z > 0, |\mathrm{Im}(z-e)| \leqslant \alpha\, \mathrm{Re}\, z\}.$$

III. MAIN RESULTS

Let L_0 be the generator of a one parameter strongly continuous group of isometries $t \mapsto e^{tL_0}$ on the Banach space $\mathcal{X}$. Recall that L_0 is norm closed, norm densely defined, conservative operator (i.e., both L_0 and $-L_0$ are dissipative) and $\mathrm{sp}\, L_0 \subset i\mathbb{R}$.[3,6]

Let $\mathbf{1}^{vv}$ be a distinguished projection on $\mathcal{X}$ such that $\|\mathbf{1}^{vv}\| = 1$. Assume that $\mathbf{1}^{vv}$ commutes with L_0 or equivalently $[\mathbf{1}^{vv}, e^{tL_0}] = 0$ for all t. Then the operator L_0 written with respect to the decomposition (2.1) has the form

$$L_0 = \begin{bmatrix} L_0^{vv} & 0 \\ 0 & L_0^{\bar{v}\bar{v}} \end{bmatrix}. \tag{3.1}$$

We write for shortness $E := L_0^{vv}$. Note that E generates a one parameter strongly continuous group of isometries on $\mathcal{X}^v$.

Let Q with $\mathrm{Dom}\, L_0 \subset \mathrm{Dom}\, Q$ be another operator that we will treat as a perturbation of L_0. Fix $\lambda_0 > 0$. We assume that for $0 \leqslant \lambda < \lambda_0$ the operator

$$L_\lambda := L_0 + \lambda Q,$$

defined on $\mathrm{Dom}\, L_\lambda = \mathrm{Dom}\, L_0$ is the generator of a one parameter strongly continuous semigroup of contractions on $\mathcal{X}$.

We will assume that the off-diagonal elements of Q, i.e., $Q^{\bar{v}v}$ and $Q^{v\bar{v}}$ are bounded. We also assume that for all $0 \leqslant \lambda < \lambda_0$ operator $E + \lambda Q^{vv}$ generates a group of isometries on $\mathcal{X}^v$.

Note that if L_λ with bounded $Q^{\bar{v}v}$ and $Q^{v\bar{v}}$ generates a group of isometries then $E + \lambda Q^{vv}$ generates a group of isometries on $\mathcal{X}^v$.

A. Van Hove limit—stationary approach

In this section we discuss the van Hove limit under the assumptions involving the resolvent of L_λ. The statement of our result is similar to the statement of the results of Davies, which we recall later.

1. Van Hove limit with the first order term

Theorem 3.1: *Assume that for all* $0 \leqslant \lambda < \lambda_0$ *:*

(i) *For all* $\xi > 0$ *we have* $\xi \notin \mathrm{sp}(L_\lambda^{\bar{v}\bar{v}})$,
(ii) *there exists an operator* $\Gamma_0 \in B(\mathcal{X}^v)$ *such that, for any* $\xi > 0$,

$$\Gamma_0 := \lim_{\lambda \searrow 0} Q^{v\bar{v}} (\lambda^2 \xi \mathbf{1}^{\bar{v}\bar{v}} - L_\lambda^{\bar{v}\bar{v}})^{-1} Q^{\bar{v}v}. \tag{3.2}$$

[*Note that the right-hand side (RHS) of (3.2) may depend on* ξ. *We assume that it does not*].

Let

$$A_{\lambda,0} := E + \lambda Q^{vv} + \lambda^2 \Gamma_0. \tag{3.3}$$

Then the following holds:

(1) Γ_0 *generates a semigroup of contractions on* $\mathcal{X}^v$.
(2) $A_{\lambda,0}$ *generates a semigroup of contractions on* $\mathcal{X}^v$.
(3) *For each* $\xi > 0$,

$$\lim_{\lambda \searrow 0} (\mathbf{1}^{vv}(\xi 1 - \lambda^{-2} L_\lambda)^{-1} \mathbf{1}^{vv} - (\xi \mathbf{1}^{vv} - \lambda^{-2} A_{\lambda,0})^{-1}) = 0. \tag{3.4}$$

(4) *For any* $f \in C_0([0,\infty[)$

$$\lim_{\lambda \searrow 0} \int_0^\infty f(\sigma)(\mathbf{1}^{vv} e^{\sigma\lambda^{-2}L_\lambda} \mathbf{1}^{vv} - e^{\sigma\lambda^{-2}A_{\lambda,0}}) \mathrm{d}\sigma = 0. \tag{3.5}$$

Note that above we use the notation $(z\mathbf{1}^{\overline{vv}} - L_\lambda^{\overline{vv}})^{-1}$ for the inverse of the operator $z\mathbf{1}^{\overline{vv}} - L_\lambda^{\overline{vv}}$ restricted to $\mathcal{X}^{\overline{v}}$. We will often use a similar notation without a comment.

2. Van Hove limit without the first order term

In this section we describe two versions of the van Hove limit without the first order term. In the first we either work at the resolvents or smear out the dynamics in time. In the second, we work at the dynamics pointwise in time. The statement of the second result is essentially the same as that of Davies, however assumptions are different.

We will need the following additional assumptions.

Assumption 3.A: sp E *is a finite set.*

Note that Assumption 3.A implies that we can write

$$\mathbf{1}^{vv} = \sum_{e \in \operatorname{sp} E} \mathbf{1}_e(E).$$

Assumption 3.B: $Q^{vv} = 0$.

Theorem 3.2: *Let Assumptions 3.A and 3.B hold. Assume additionally that*

(i) *for* $0 \leq \lambda < \lambda_0$, *for each* $e \in \operatorname{sp} E$ *and for all* $\xi > 0$ *we have* $e + \xi \notin \operatorname{sp}(L_\lambda^{\overline{vv}})$,
(ii) *there exists an operator* $\Gamma \in B(\mathcal{X}^v)$ *such that, for any* $\xi > 0$,

$$\Gamma := \sum_{e \in \operatorname{sp} E} \lim_{\lambda \searrow 0} \mathbf{1}_e(E) Q^{v\overline{v}}((e + \lambda^2 \xi)\mathbf{1}^{\overline{vv}} - L_\lambda^{\overline{vv}})^{-1} Q^{\overline{v}v} \mathbf{1}_e(E). \tag{3.6}$$

[*Note that the RHS of (3.6) may depend on* ξ. *We assume that it does not*].

(iii) *For any* $e, e' \in \operatorname{sp} E$, $e \neq e'$ *and* $\xi > 0$,

$$\lim_{\lambda \searrow 0} \lambda \mathbf{1}_e(E) Q^{v\overline{v}}((e + \lambda^2 \xi)\mathbf{1}^{\overline{vv}} - L_\lambda^{\overline{vv}})^{-1} Q^{\overline{v}v} \mathbf{1}_{e'}(E) = 0,$$

$$\lim_{\lambda \searrow 0} \lambda \mathbf{1}_{e'}(E) Q^{v\overline{v}}((e + \lambda^2 \xi)\mathbf{1}^{\overline{vv}} - L_\lambda^{\overline{vv}})^{-1} Q^{\overline{v}v} \mathbf{1}_e(E) = 0.$$

Then the following holds:

(1) Γ *generates semigroup of contractions on* $\mathcal{X}^v$,
(2) $[E, \Gamma] = 0$,
(3) *for each* $\xi > 0$ *we have*

$$\lim_{\lambda \searrow 0} \sum_{e \in \operatorname{sp} E} \mathbf{1}_e(E) \mathbf{1}^{vv}(\xi 1 - \lambda^{-2}(L_\lambda - e1))^{-1} \mathbf{1}^{vv} = (\xi \mathbf{1}^{vv} - \Gamma)^{-1}, \tag{3.7}$$

(4) *for any* $f \in C_0([0,\infty[)$,

$$\lim_{\lambda\searrow 0}\int_0^\infty f(\sigma)(e^{-\sigma\lambda^{-2}E}\mathbf{1}^{vv}e^{\sigma\lambda^{-2}L_\lambda}\mathbf{1}^{vv}-e^{\sigma\Gamma})\mathrm{d}\sigma=0. \tag{3.8}$$

Theorem 3.3: *Let assumptions 3.A and 3.B hold. Assume additionally that*

(i) *for* $0\leqslant\lambda<\lambda_0$ *and all* $\xi\in\mathbb{C}$, $\mathrm{Re}\,\xi>0$, *we have* $\xi\notin\mathrm{sp}(L_\lambda^{\overline{vv}})$,

(ii) *for all* $\alpha_0\geqslant 0$ *and for any* $\xi\in\mathrm{Wedge}(0,\alpha_0)$, *there exists an operator* $\Gamma\in B(\mathcal{X}^v)$ *such that,*

$$\Gamma := \sum_{e\in\mathrm{sp}\,E}\lim_{\lambda\searrow 0}\mathbf{1}_e(E)Q^{v\overline{v}}((e+\lambda^2\xi)\mathbf{1}^{\overline{vv}}-L_\lambda^{\overline{vv}})^{-1}Q^{\overline{v}v}\mathbf{1}_e(E), \tag{3.9}$$

[*Note that the RHS of (3.9) may depend on* ξ. *We assume that it does not*].

(iii) *For each* $e\in\mathrm{sp}\,E$ *we have*

$$\sup_{\mathrm{Re}\,\xi>0;0\leqslant\lambda<\lambda_0}\|Q^{v\overline{v}}((e+\lambda^2\xi)\mathbf{1}^{\overline{vv}}-L_\lambda^{\overline{vv}})^{-1}Q^{\overline{v}v}\|<\infty.$$

Then,

(1) *all statemants of the Theorem 3.2 hold. Besides (3) holds in a stronger form, for* $\alpha_0\geqslant 0$ *the formula (3.7) is valid for each* $\xi\in\mathrm{Wedge}(0,\alpha_0)$.

(2) *Let* $\psi\in\mathcal{X}^v$. *For* $\sigma\geqslant 0$ *we have*

$$\lim_{\lambda\searrow 0}(e^{-\sigma\lambda^{-2}E}\mathbf{1}^{vv}e^{\sigma\lambda^{-2}L_\lambda}\mathbf{1}^{vv}-e^{\sigma\Gamma})\psi=0 \tag{3.10}$$

uniformly for $\sigma\in[\tau_0,\tau_1]$ *for any fixed* $0<\tau_0\leqslant\tau_1<\infty$.

B. Van Hove limit—time-dependent approach

In this section we discuss the van Hove limit under the assumptions involving the dynamics. In Theorem 3.4 we recall the original approach to the van Hove limit due to Davies.[4–6] (Strictly speaking, Davies assumed that the perturbation Q is bounded. In Theorem 3.4 we impose slightly less restrictive assumptions, which can be handled by an essentially the same proof.)

Let L_0, Q, and $\mathbf{1}^{vv}$ be the same as before. Clearly, $L_\lambda^{\overline{vv}}$ generates a semigroup on $\mathcal{X}^{\overline{v}}$. Therefore, we can define the operator

$$K(\lambda,\tau) := \int_{x=0}^{\lambda^{-2}\tau} e^{-xL_\lambda^{vv}}Q^{v\overline{v}}e^{xL_\lambda^{\overline{vv}}}Q^{\overline{v}v}\,\mathrm{d}x. \tag{3.11}$$

The following theorem describes the van Hove limit for the dynamics in both cases—with and without the first order term.

Theorem 3.4: *Assume additionally that*

(i) *for all* $\tau_1>0$ *there is a constant* $C>0$ *such that* $\|K(\lambda,\tau)\|<C$ *for* $|\lambda|\leqslant 1$ *and* $0\leqslant\tau\leqslant\tau_1$.

(ii) *There exists bounded operator K on* $\mathcal{X}^v$ *such that if* $0<\tau_0\leqslant\tau_1<\infty$ *then*

$$\lim_{\lambda\searrow 0}(\sup_{\tau_0\leqslant\tau\leqslant\tau_1}\|K(\lambda,\tau)-K\|)=0.$$

Then the following holds:

(1)

$$\lim_{\lambda\searrow 0}\left(\sup_{0\leqslant t\leqslant\tau_1}\|\mathbf{1}^{vv}e^{t\lambda^{-2}L_\lambda}\mathbf{1}^{vv}-e^{t\lambda^{-2}(E+\lambda Q^{vv}+\lambda^2K)}\|\right)=0. \tag{3.12}$$

(2) *If additionally Assumptions 3.A and 3.B hold then*

$$\lim_{\lambda\searrow 0}\left(\sup_{0\le t\le\tau_1}\|e^{-t\lambda^{-2}E}\mathbf{1}^{vv}e^{t\lambda^{-2}L_\lambda}\mathbf{1}^{vv}-e^{tK^\natural}\|\right)=0, \tag{3.13}$$

where

$$K^\natural := \sum_{e\in \mathrm{sp}\, E}\mathbf{1}_e(E)K\mathbf{1}_e(E)=\lim_{a\to\infty}\frac{1}{2a}\int_{-a}^{a}e^{tE}Ke^{-tE}\,\mathrm{d}t.$$

Let us recall how the operators $K(\lambda,\tau)$ are motivated. If we treat the off-diagonal elements of L_λ as a (bounded) perturbation of the diagonal part of L_λ then, by a well-known formula, we have

$$e^{tL_\lambda}=e^{t(L_\lambda^{vv}+L_\lambda^{\overline{vv}})}+\lambda\int_{s=0}^{t}e^{(t-s)(L_\lambda^{vv}+L_\lambda^{\overline{vv}})}(Q^{v\overline{v}}+Q^{\overline{v}v})e^{sL_\lambda}\,\mathrm{d}s.$$

Using this formula one gets

$$\mathbf{1}^{vv}e^{\lambda^{-2}\tau L_\lambda}\mathbf{1}^{vv}=\mathbf{1}^{vv}e^{\lambda^{-2}\tau L_\lambda^{vv}}+\int_{\sigma=0}^{\tau}\mathbf{1}^{vv}e^{\lambda^{-2}(\tau-\sigma)L_\lambda^{vv}}K(\lambda,\tau-\sigma)\mathbf{1}^{vv}e^{\lambda^{-2}\sigma L_\lambda}\mathbf{1}^{vv}\,\mathrm{d}\sigma.$$

Now we discuss how one can obtain the van Hove limit for the resolvents under time-dependent assumptions. In fact we show when one can use both stationary and time-dependent approaches. We will concentrate on the case without the first order term.

Theorem 3.5: *Let Assumptions 3.A and 3.B hold. Assume additionally that*

(i) *for* $0\le\lambda<\lambda_0$ *and all* $z\in\mathbb{C}$, $\mathrm{Re}\, z>0$ *we have* $z\notin \mathrm{sp}(L_\lambda^{\overline{vv}})$,

(ii) $\int_0^\infty \sup_{0\le\lambda<\lambda_0}\|Q^{v\overline{v}}e^{sL_\lambda^{\overline{vv}}}Q^{\overline{v}v}\|\mathrm{d}s<\infty$,

(iii) *for any* $s>0$, $\lim_{\lambda\searrow 0}Q^{v\overline{v}}e^{sL_\lambda^{\overline{vv}}}Q^{\overline{v}v}=Q^{v\overline{v}}e^{sL_0^{\overline{vv}}}Q^{\overline{v}v}$.

Then

(1) *The assumptions of both Theorem 3.2 and Theorem 3.4 hold. Moreover, we have*

$$K^\natural=\Gamma.$$

(2) *The following limits exist, coincide and equal to* Γ:

$$\lim_{\epsilon\searrow 0}\sum_{e\in\mathrm{sp}\, E}\mathbf{1}_e(E)Q^{v\overline{v}}((e+\epsilon)\mathbf{1}^{\overline{vv}}-L_0^{\overline{vv}})^{-1}Q^{\overline{v}v}\mathbf{1}_e(E)$$
$$=\lim_{\epsilon\searrow 0}\sum_{e\in\mathrm{sp}\, E}\int_0^\infty e^{-\epsilon s}\mathbf{1}_e(E)Qe^{sL_0}Qe^{-sL_0}\mathbf{1}_e(E)\mathrm{d}s. \tag{3.14}$$

Note that the assumptions of Theorem 3.5 are stronger than that of Theorem 3.3 and Theorem 3.4 (2).

The operator (3.14) is often called the level shift operator. It is used to describe the second order shift of eigenvalues of L_λ.[7,9–11]

IV. PROOFS

Lemma 4.1: For $0\le\lambda<\lambda_0$ *the operator*

$$\widetilde{L}_\lambda=\begin{bmatrix}0 & \lambda Q^{v\overline{v}}\\ \lambda Q^{\overline{v}v} & L_0^{\overline{vv}}+\lambda Q^{\overline{vv}}\end{bmatrix}$$

defined on $\mathrm{Dom}(\widetilde{L}_\lambda)=\mathcal{X}^v\oplus\mathrm{Dom}(L_0^{\overline{vv}})$ *generates a semigroup of contractions on* $\mathcal{X}$.

Proof: Let $0\le\lambda<\lambda_0$. The operator $\widetilde{L}_\lambda$ is densely defined. By the Lumer-Phillips theorem

(Theorem 3.1.16 in Ref. 3) it is sufficient to show that (I) $\tilde{L}_\lambda$ is dissipative, (II) for some $\epsilon > 0$ we have $\epsilon \notin \mathrm{sp}(\tilde{L}_\lambda)$.

Step (I): For $0 \leq \lambda$ the operator,

$$Z := L_\lambda - L_\lambda^{vv}$$

with the domain $\mathrm{Dom}(Z) = \mathrm{Dom}(E) \oplus \mathrm{Dom}(L_0^{\overline{vv}})$ is densely defined and dissipative. Hence, by Proposition 3.1.15 in Ref. 3, Z is closable and its closure is dissipative. But the closure of Z coincides with $\tilde{L}_\lambda$.

Step (II): For $0 \leq \lambda < \lambda_0$ the operator $L_\lambda - \lambda(Q^{v\bar{v}} + Q^{\bar{v}v})$ generates a semigroup such that for $\mathrm{Re}\, z > \|\lambda(Q^{v\bar{v}} + Q^{\bar{v}v})\|$ we have $z \notin \mathrm{sp}(L_\lambda - \lambda(Q^{v\bar{v}} + Q^{\bar{v}v}))$ and

$$\|(z\mathbf{1} - (L_\lambda - \lambda(Q^{v\bar{v}} + Q^{\bar{v}v})))^{-1}\| \leq (\mathrm{Re}\, z - \|\lambda(Q^{v\bar{v}} + Q^{\bar{v}v})\|)^{-1}.$$

Hence, for all $\epsilon > \epsilon_0$ for some ϵ_0 large enough,

$$\|(\epsilon\mathbf{1} - (\tilde{L}_\lambda - \lambda(Q^{v\bar{v}} + Q^{\bar{v}v})))^{-1}(\lambda(Q^{v\bar{v}} + Q^{\bar{v}v}))\| < 1.$$

Hence, by the Neumann theorem, $\epsilon\mathbf{1} - \tilde{L}_\lambda$ is invertible for all $\epsilon > \epsilon_0$, and so $\epsilon \notin \mathrm{sp}(\tilde{L}_\lambda)$. □

By the Feshbach projection method,[1,2,7,8] if $z \notin \mathrm{sp}(L_\lambda) \cup \mathrm{sp}(L_\lambda^{\overline{vv}})$ then the restricted resolvent is given by

$$\mathbf{1}^{vv}(z\mathbf{1} - L_\lambda)^{-1}\mathbf{1}^{vv} = G_v^{-1}(z),$$

where

$$G_v(z) = z\mathbf{1}^{vv} - E - \lambda Q^{vv} - \lambda^2 Q^{v\bar{v}}(z\mathbf{1}^{\overline{vv}} - L_\lambda^{\overline{vv}})^{-1}Q^{\bar{v}v}.$$

Hence

$$\mathbf{1}^{vv}(\xi - \lambda^{-2}L_\lambda)^{-1}\mathbf{1}^{vv} = \lambda^2 G_v^{-1}(\lambda^2\xi).$$

In what follows we will use these facts without a comment. (See Refs. 4 and 15–17.)

Proof of Theorem 3.1:

(1) By Lemma 4.1, the operator $\tilde{L}_\lambda$ generates a semigroup of contractions, which for $\xi > 0$ implies

$$\|(\xi\mathbf{1}^{vv} - \lambda^2 Q^{v\bar{v}}(\xi\mathbf{1}^{\overline{vv}} - L_\lambda^{\overline{vv}})^{-1}Q^{\bar{v}v})^{-1}\| = \|\mathbf{1}^{vv}(\xi\mathbf{1} - \tilde{L}_\lambda)^{-1}\mathbf{1}^{vv}\| \leq \xi^{-1}.$$

Hence, for all $\xi > \xi_0 > 0$, the operator $(\xi\mathbf{1}^{vv} - Q^{v\bar{v}}(\lambda^2\xi\mathbf{1}^{\overline{vv}} - L_\lambda^{\overline{vv}})^{-1}Q^{\bar{v}v})^{-1}$ is uniformly bounded. We know that for any $\xi > 0$,

$$\lim_{\lambda \searrow 0}(\xi\mathbf{1}^{\overline{vv}} - Q^{v\bar{v}}(\xi\lambda^2\mathbf{1}^{\overline{vv}} - L_\lambda^{\overline{vv}})^{-1}Q^{\bar{v}v}) = \xi\mathbf{1}^{\overline{vv}} - \Gamma_0. \tag{4.1}$$

Therefore, for all $\xi > \xi_0$, the operator $\xi\mathbf{1}^{vv} - \Gamma_0$ is invertible on $\mathcal{X}^v$ and

$$\|(\xi\mathbf{1}^{vv} - \Gamma_0)^{-1}\| \leq \xi^{-1}.$$

Hence Γ_0 generates a semigroup of contractions on $\mathcal{X}^v$ (Theorem 2.21 and Corollary 2.22 in Ref. 6).

(2) Let $0 \leq \lambda < \lambda_0$. Since $E + \lambda Q^{vv}$ generates a group of isometries and Γ_0 is bounded and dissipative then the result follows from Theorem 3.1.32 in Ref. 3.

(3) Let $0 < \lambda < \lambda_0$. Recall that for $\xi > 0$,

$$\|\lambda^2 G_v^{-1}(\lambda^2\xi)\| \leq \xi^{-1}, \tag{4.2}$$

$$\|(\xi \mathbf{1}^{vv} - \lambda^{-2} A_{\lambda,0})^{-1}\| \leq \xi^{-1}, \tag{4.3}$$

$$\lim_{\lambda \searrow 0} Q^{v\bar{v}}(\lambda^2 \xi \mathbf{1}^{\overline{vv}} - L_\lambda^{\overline{vv}})^{-1} Q^{\bar{v}v} = \Gamma_0. \tag{4.4}$$

For any $\xi > 0$ we have

$$\|\mathbf{1}^{vv}(\xi \mathbf{1} - \lambda^{-2} L_\lambda)^{-1} \mathbf{1}^{vv} - (\xi \mathbf{1}^{vv} - \lambda^{-2} A_{\lambda,0})^{-1}\|$$
$$= \|\lambda^2 G_v^{-1}(\lambda^2 \xi)(Q^{v\bar{v}}(\lambda^2 \xi \mathbf{1}^{\overline{vv}} - L_\lambda^{\overline{vv}})^{-1} Q^{\bar{v}v} - \Gamma_0)(\xi \mathbf{1}^{vv} - \lambda^{-2} A_{\lambda,0})^{-1}\|.$$

Hence by (4.2)–(4.4) the RHS of the above expression tends to zero as λ tends to zero.

(4) For $\xi > 0$, by the Laplace transform, we have

$$\mathbf{1}^{vv}(\xi \mathbf{1} - \lambda^{-2} L_\lambda)^{-1} \mathbf{1}^{vv} - (\xi \mathbf{1}^{vv} - \lambda^{-2} A_{\lambda,0})^{-1} = \int_0^\infty e^{-\xi\sigma}(\mathbf{1}^{vv} e^{\lambda^{-2}\sigma L_\lambda} \mathbf{1}^{vv} - e^{\lambda^{-2}\sigma A_{\lambda,0}}) \mathrm{d}\sigma.$$

Hence by (3.4) we get

$$\lim_{\lambda \searrow 0} \int_0^\infty e^{-\xi\sigma}(\mathbf{1}^{vv} e^{\lambda^{-2}\sigma L_\lambda} \mathbf{1}^{vv} - e^{\lambda^{-2}\sigma A_{\lambda,0}}) \mathrm{d}\sigma = 0. \tag{4.5}$$

By the Stone–Weierstrass theorem, the family of functions

$$[0,\infty[\ni \sigma \mapsto e^{-\xi\sigma} \in \mathbb{R}, \quad \xi > 0$$

forms an algebra which is dense in continuous functions of compact support on $[0,\infty[$. This fact together with (4.5) implies (3.5). □

Lemma 4.2: Let E be the generator of a group of isometries. Let e be an isolated point in $\mathrm{sp}(E)$. *Then e is a semisimple eigenvalue which means* $E\mathbf{1}_e(E) = e\mathbf{1}_e(E)$ *and* $\|\mathbf{1}_e(E)\| = 1$.

Proof: Let e be an isolated point in $\mathrm{sp}(E)$. Then for any $\epsilon > 0$,

$$\|((e+\epsilon)\mathbf{1}^{vv} - E)^{-1}\| \leq \epsilon^{-1}. \tag{4.6}$$

So for $z \in \{z \in \mathbb{C} : \mathrm{dist}(e,z) < \delta\} \setminus \{e\}$ for some $\delta > 0$ we have

$$(z\mathbf{1}^{vv} - E)^{-1} = \mathbf{1}_e(E)(z-e)^{-1} + h(z), \tag{4.7}$$

where h is an analytic function on $\{z \in \mathbb{C} : \mathrm{dist}(e,z) < \delta\}$. Hence e is semisimple. But (4.7) also implies that

$$\lim_{\epsilon \searrow 0} \epsilon((e+\epsilon)\mathbf{1}^{vv} - E)^{-1} = \mathbf{1}_e(E)$$

and hence, by (4.6), we get $\|\mathbf{1}_e(E)\| = 1$. □

For an isolated point $e \in \mathrm{sp}\, E$ let us write for shortness

$$\mathbf{1}^{ee} := \mathbf{1}_e(E), \quad \mathbf{1}^{\underline{ee}} := \mathbf{1}^{vv} - \mathbf{1}_e(E),$$

$$\mathcal{X}^e := \mathrm{Ran}\, \mathbf{1}_e(E), \quad \mathcal{X}^{\underline{e}} := \mathrm{Ran}\, \mathbf{1}^{\underline{ee}}$$

then

$$\mathcal{X}^v = \mathcal{X}^e \oplus \mathcal{X}^{\underline{e}}. \tag{4.8}$$

If $e', e \in \mathrm{sp}\, E$ and $A \in B(\mathcal{X}^v)$ then we denote $A^{e'e} := \mathbf{1}^{e'e'} A \mathbf{1}^{ee}$.

Proof of Theorem 3.2: (2) follows immediately if we note that Lemma 4.2 implies $E = \sum_{e \in \mathrm{sp}\, E} e\mathbf{1}_e(E)$ and that we have $\Gamma := \sum_{e \in \mathrm{sp}\, E} \mathbf{1}_e(E) \Gamma \mathbf{1}_e(E)$.

(3) Let $e \in \mathrm{sp}\, E$. Set

$$G_\lambda(\xi,e) := \xi\mathbf{1}^{vv} + \lambda^{-2}(e\mathbf{1}^{vv} - E) - Q^{v\overline{v}}((\lambda^2\xi + e)\mathbf{1}^{\overline{vv}} - L_\lambda^{\overline{vv}})^{-1}Q^{\overline{v}v}.$$

For $\xi > 0$ we have

$$G_\lambda(\xi,e)^{-1} = \mathbf{1}^{vv}(\xi + \lambda^{-2}(e - L_\lambda))^{-1}\mathbf{1}^{vv}.$$

This and the dissipativity of L_λ implies the bound

$$\|G_\lambda(\xi,e)^{-1}\| \leq \xi^{-1}. \tag{4.9}$$

Write for shortness G instead of $G_\lambda(\xi,e)$.

Decompose $G = G_{\rm diag} + G_{\rm off}$ into its diagonal and off-diagonal part,

$$G_{\rm diag} := \sum_{e' \in {\rm sp}\, E} \mathbf{1}^{e'e'}G\mathbf{1}^{e'e'},$$

$$G_{\rm off} := \sum_{e' \in {\rm sp}\, E} \mathbf{1}^{e'e'}G\mathbf{1}^{\underline{e}'\underline{e}'} = \sum_{e' \in {\rm sp}\, E} \mathbf{1}^{\underline{e}'\underline{e}'}G\mathbf{1}^{e'e'}.$$

First we would like to show that for $\xi > 0$ and small enough λ, $G_{\rm diag}$ is invertible. By the Neumann theorem, it is easy to see that $\mathbf{1}^{\underline{ee}}G_{\rm diag}$ is invertible on Ran $\mathbf{1}^{\underline{ee}}$ for small enough λ. Moreover, we have

$$\|\mathbf{1}^{\underline{ee}}G_{\rm diag}^{-1}\| \leq c\lambda^2. \tag{4.10}$$

It is more complicated to prove that $\mathbf{1}^{ee}G_{\rm diag}$ is invertible on Ran $\mathbf{1}^{ee}$.

We fix $\xi > 0$. We know that G is invertible and $\|G^{-1}\| \leq \xi^{-1}$. Hence we can write

$$G_{\rm diag}G^{-1} = 1 - G_{\rm off}G^{-1}.$$

Therefore

$$\mathbf{1}^{ee}G_{\rm diag}G^{-1} = \mathbf{1}^{ee} - \mathbf{1}^{ee}G_{\rm off}\mathbf{1}^{\underline{ee}}G^{-1},$$

$$\mathbf{1}^{\underline{ee}}G_{\rm diag}G^{-1} = \mathbf{1}^{\underline{ee}} - \mathbf{1}^{\underline{ee}}G_{\rm off}G^{-1}. \tag{4.11}$$

The latter identity can be for small enough λ transformed into

$$\mathbf{1}^{\underline{ee}}G^{-1} = G_{\rm diag}^{-1}\mathbf{1}^{\underline{ee}} - G_{\rm diag}^{-1}\mathbf{1}^{\underline{ee}}G_{\rm off}G^{-1}. \tag{4.12}$$

We insert (4.12) into the first identity of (4.11) to obtain

$$\mathbf{1}^{ee}G_{\rm diag}G^{-1} = \mathbf{1}^{ee} - \mathbf{1}^{ee}G_{\rm off}\mathbf{1}^{\underline{ee}}G_{\rm diag}^{-1} + \mathbf{1}^{ee}G_{\rm off}\mathbf{1}^{\underline{ee}}G_{\rm diag}^{-1}G_{\rm off}G^{-1}. \tag{4.13}$$

We multiply (4.13) from the right by $\mathbf{1}^{ee}$ to get

$$\mathbf{1}^{ee}G_{\rm diag}\mathbf{1}^{ee}G^{-1}\mathbf{1}^{ee} = \mathbf{1}^{ee} + \mathbf{1}^{ee}G_{\rm off}\mathbf{1}^{\underline{ee}}G_{\rm diag}^{-1}G_{\rm off}G^{-1}\mathbf{1}^{ee}. \tag{4.14}$$

Now, using

$$\lim_{\lambda\searrow 0}\|\lambda G_{\rm off}\| = 0, \tag{4.15}$$

(4.9) and (4.10) we obtain

$$\lim_{\lambda\searrow 0} \mathbf{1}^{ee}G_{\rm off}\mathbf{1}^{\underline{ee}}G_{\rm diag}^{-1}G_{\rm off}G^{-1}\mathbf{1}^{ee} = 0.$$

Thus, for small enough λ,

$$\mathbf{1}^{ee}G_{\text{diag}}B_1 = \mathbf{1}^{ee},$$

where

$$B_1 := \mathbf{1}^{ee}G^{-1}\mathbf{1}^{ee}(\mathbf{1}^{ee} + \mathbf{1}^{ee}G_{\text{off}}\mathbf{1}^{\underline{ee}}G_{\text{diag}}^{-1}G_{\text{off}}G^{-1}\mathbf{1}^{ee})^{-1}.$$

Similarly, for small enough λ, we find B_2 such that

$$B_2\mathbf{1}^{ee}G_{\text{diag}} = \mathbf{1}^{ee}.$$

This implies that $\mathbf{1}^{ee}G_{\text{diag}}$ is invertible on Ran $\mathbf{1}^{ee}$.

Next, we can write

$$G^{-1} = G_{\text{diag}}^{-1} - G_{\text{diag}}^{-1}G_{\text{off}}G_{\text{diag}}^{-1} + G_{\text{diag}}^{-1}G_{\text{off}}G_{\text{diag}}^{-1}G_{\text{off}}G^{-1}.$$

Hence

$$\mathbf{1}^{ee}G^{-1} = \mathbf{1}^{ee}G_{\text{diag}}^{-1}(\mathbf{1} - G_{\text{off}}\mathbf{1}^{\underline{ee}}G_{\text{diag}}^{-1} + G_{\text{off}}\mathbf{1}^{\underline{ee}}G_{\text{diag}}^{-1}G_{\text{off}}G^{-1}). \tag{4.16}$$

Therefore, for a fixed ξ, by (4.9), (4.10), and (4.15), we see that as $\lambda \searrow 0$ we have

$$-G_{\text{off}}\mathbf{1}^{\underline{ee}}G_{\text{diag}}^{-1} + G_{\text{off}}\mathbf{1}^{\underline{ee}}G_{\text{diag}}^{-1}G_{\text{off}}G^{-1} \to 0.$$

Therefore, for small enough λ, we can invert the expression in the prentheses of (4.16). Consequently,

$$\mathbf{1}^{ee}(G_{\text{diag}}^{-1} - G^{-1}) = \mathbf{1}^{ee}G^{-1}(1 - G_{\text{off}}\mathbf{1}^{\underline{ee}}G_{\text{diag}}^{-1} + G_{\text{off}}\mathbf{1}^{\underline{ee}}G_{\text{diag}}^{-1}G_{\text{off}}G^{-1})^{-1} \times (G_{\text{off}}\mathbf{1}^{\underline{ee}}G_{\text{diag}}^{-1} - G_{\text{off}}\mathbf{1}^{\underline{ee}}G_{\text{diag}}^{-1}G_{\text{off}}G^{-1}). \tag{4.17}$$

Therefore, for a fixed ξ, by (4.9), (4.10), and (4.15), we see that as $\lambda \searrow 0$ we have

$$\mathbf{1}^{ee}(G_{\text{diag}}^{-1} - G^{-1}) \to 0. \tag{4.18}$$

Equations (4.9) and (4.18) imply that $\mathbf{1}^{ee}G_{\text{diag}}^{-1}$ is uniformly bounded as $\lambda \searrow 0$. We know that

$$\mathbf{1}^{ee}G_{\text{diag}} \to \mathbf{1}^{ee}\xi - \mathbf{1}^{ee}\Gamma. \tag{4.19}$$

Therefore, $\xi\mathbf{1}^{ee} - \mathbf{1}^{ee}\Gamma$ is invertible on Ran $\mathbf{1}^{ee}$ and

$$\mathbf{1}^{ee}G_{\text{diag}}^{-1} \to (\mathbf{1}^{ee}\xi - \mathbf{1}^{ee}\Gamma)^{-1}.$$

Using again (4.18) we see that

$$\mathbf{1}^{ee}G^{-1} \to (\mathbf{1}^{ee}\xi - \mathbf{1}^{ee}\Gamma)^{-1}. \tag{4.20}$$

Summing up (4.20) over e we obtain

$$\sum_{e \in \text{sp } E} \mathbf{1}^{ee}G_\lambda(\xi,e)^{-1} \to (\xi\mathbf{1}^{vv} - \Gamma)^{-1}, \tag{4.21}$$

which ends the proof of (3).

(1) We have

$$\sum_{e \in \text{sp } E} \mathbf{1}^{ee}G_\lambda(\xi,e)^{-1} = \sum_{e \in \text{sp } E} \int_0^\infty e^{-t(\xi+\lambda^{-2}e)}\mathbf{1}^{ee}e^{tL_\lambda/\lambda^2}\mathbf{1}^{vv}\,\mathrm{d}t = \int_0^\infty e^{-t\xi}e^{-tE/\lambda^2}\mathbf{1}^{vv}e^{tL_\lambda/\lambda^2}\mathbf{1}^{vv}\,\mathrm{d}t. \tag{4.22}$$

Clearly, $\|e^{-tE/\lambda^2}\mathbf{1}^{vv}e^{tL_\lambda/\lambda^2}\mathbf{1}^{vv}\| \leq 1$. Therefore,

$$\left\| \sum_{e \in \mathrm{sp}\, E} \mathbf{1}^{ee} G_\lambda(\xi,e)^{-1} \right\| \leq \xi^{-1}.$$

Hence, by (4.21),

$$\|(\xi \mathbf{1}^{vv} - \Gamma)^{-1}\| \leq \xi^{-1},$$

which means that Γ is the generator of a semigroup of contractions.

(4) To prove this we repeat the argument used in the proof of Theorem 3.1.

Proof of Theorem 3.3:

(1) Follows by a simple modification of the argument used in Theorem 3.2.

(2) For $e \in \mathrm{sp}\, E$ and for $\mathrm{Re}\, \xi > 0$ we denote

$$G_\lambda(\xi,e) \coloneqq \xi \mathbf{1}^{vv} + \lambda^{-2}(e\mathbf{1}^{vv} - E) - Q^{v\overline{v}}((\lambda^2 \xi + e)\mathbf{1}^{\overline{vv}} - L_\lambda^{\overline{vv}})^{-1} Q^{\overline{v}v}.$$

Obviously

$$G_\lambda(\xi,e)^{-1} = \mathbf{1}^{vv}(\xi + \lambda^{-2}(e - L_\lambda))^{-1}\mathbf{1}^{vv}$$

and

$$\|G_\lambda(\xi,e)^{-1}\| \leq \mathrm{Re}\, \xi^{-1}. \tag{4.23}$$

Let $\psi \in \mathcal{X}^v$. Let $\omega_0 > 0$ and $\sigma \geq 0$. By the inverse Laplace transform (Ref. 12 Chap. XI) and by the proof of Theorem 3.2, uniformly for $0 < \tau_0 \leq \sigma \leq \tau_1$, we get

$$\begin{aligned}
&(e^{-\sigma\lambda^{-2}E}\mathbf{1}^{vv} e^{\sigma\lambda^{-2}L_\lambda}\mathbf{1}^{vv} - e^{\sigma\Gamma})\psi \\
&= \frac{1}{2\pi i} \lim_{R\to\infty} \int_{\omega_0 + i[-R,R]} e^{\sigma\xi} \sum_{e \in \mathrm{sp}\, E} (\mathbf{1}^{ee}((\xi + \lambda^{-2}e)\mathbf{1} - \lambda^{-2}L_\lambda)^{-1}\mathbf{1}^{vv} - (\xi\mathbf{1}^{ee} - \Gamma^{ee})^{-1})\psi \, \mathrm{d}\xi \\
&= \frac{1}{2\pi i} \sum_{e \in \mathrm{sp}\, E} \lim_{R\to\infty} \int_{-R}^{R} f_\lambda(y,e) i \, \mathrm{d}y,
\end{aligned}$$

where

$$f_\lambda(y,e) \coloneqq e^{\sigma(\omega_0 + iy)}(\mathbf{1}^{ee} G_\lambda(\omega_0 + iy, e)^{-1} - ((\omega_0 + iy)\mathbf{1}^{ee} - \Gamma^{ee})^{-1})\psi.$$

For each $e \in \mathrm{sp}\, E$,

$$\left\| \frac{d}{dy} f_\lambda(y,e) \right\| \leq e^{\sigma\omega_0} 2(\sigma\omega_0^{-1} + \omega_0^{-2})\|\psi\|.$$

This shows that the family $f_\lambda(y,e)$ is equicontinuous as $\lambda \to 0$.

For any fixed $R > 0$ if only $\alpha_0 = (R+1)/\omega_0$ then $\xi = \omega_0 + iy \in \mathrm{Wedge}(0,\alpha_0)$ hence, by (1), we get the pointwise convergence $f_\lambda(y,e) \to 0$ as $\lambda \searrow 0$ for $y \in [-R,R]$. Finally pointwise convergence together with equicontinuity implies uniform convergence on compacts, so for any fixed $R > 0$,

$$\lim_{\lambda\searrow 0} \int_{-R}^{R} f_\lambda(y,e)\psi i \, \mathrm{d}y = \int_{-R}^{R} \lim_{\lambda\searrow 0} f_\lambda(y,e)\psi i \, \mathrm{d}y = 0.$$

To end the proof it is sufficient to show that for each $e \in \mathrm{sp}\, E$,

$$\lim_{R\to\infty} \sup_{0\leq\lambda<\lambda_0} \left\| \int_{|y|>R} f_\lambda(y,e) i \, \mathrm{d}y \right\| = 0.$$

Since Γ^{ee} is independent of λ we need only to show that

$$\lim_{R\to\infty}\sup_{0\leqslant\lambda<\lambda_0}\left\|\int_{|y|>R}e^{\sigma(\omega_0+iy)}\mathbf{1}^{ee}G_\lambda(\omega_0+iy,e)^{-1}\psi\,\mathrm{d}y\right\|=0. \tag{4.24}$$

Decompose $G_\lambda(\xi,e)$ into its diagonal and off-diagonal part,

$$G_{\text{diag}}:=\sum_{e'\in\text{sp }E}\mathbf{1}^{e'e'}G_\lambda(\xi,e)\mathbf{1}^{e'e'}=\xi\mathbf{1}^{ee}-T_\lambda(\xi,e)^{ee}+\sum_{e'\in\text{sp }E;e'\neq e}((\xi-\lambda^{-2}(e'-e))\mathbf{1}^{e'e'}$$
$$-T_\lambda(\xi,e)^{e'e'}),$$

$$G_{\text{off}}:=\sum_{e'\in\text{sp }E}\mathbf{1}^{e'e'}G_\lambda(\xi,e)\mathbf{1}^{\underline{e}'\underline{e}'}=\sum_{e'\in\text{sp }E}\mathbf{1}^{\underline{e}'\underline{e}'}G_\lambda(\xi,e)\mathbf{1}^{e'e'}=\sum_{e'\in\text{sp }E}T_\lambda(\xi,e)^{e'\underline{e}'},$$

where

$$T_\lambda(\xi,e):=Q^{v\overline{v}}((\lambda^2\xi+e)\mathbf{1}^{\overline{vv}}-L_0^{\overline{vv}}-\lambda Q^{\overline{vv}})^{-1}Q^{\overline{v}v}.$$

By the assumption for each $e\in\text{sp }E$

$$\sup_{\text{Re }\xi>0;0\leqslant\lambda<\lambda_0}\|T_\lambda(\xi,e)\|<C<\infty. \tag{4.25}$$

In the rest of the proof we write for shortness $G=G_\lambda(\xi,e)$ and $T=T_\lambda(\xi,e)$.

Fix $e\in\text{sp }E$ and let $\xi:=\omega_0+iy$. Fix $\omega_0>C+1$. Then, by the Neumann theorem, the operator $\mathbf{1}^{ee}G_{\text{diag}}=\xi\mathbf{1}^{ee}-T^{ee}$ is invertible on $\mathcal{X}^e$ and we have

$$\|(\xi\mathbf{1}^{ee}-T^{ee})^{-1}\|\leqslant(|y|-C)^{-1}\quad\text{for }|y|>C. \tag{4.26}$$

Note that for each $e'\in\text{sp }E$, $e'\neq e$ the operator $\lambda^{-2}(e'-e)\mathbf{1}^{e'e'}$ generates a group of isometries on $\mathcal{X}^{e'}$. Hence the operator $\mathbf{1}^{e'e'}G_{\text{diag}}=(\xi-\lambda^{-2}(e'-e))\mathbf{1}^{e'e'}-T^{e'e'}$ is invertible on $\mathcal{X}^{e'}$ and we have

$$\|((\xi-\lambda^{-2}(e'-e))\mathbf{1}^{e'e'}-T^{e'e'})^{-1}\|\leqslant(\omega_0-C)^{-1}<1. \tag{4.27}$$

This shows that G_{diag} is invertible on $\mathcal{X}^v$. We have

$$\mathbf{1}^{ee}G^{-1}=\mathbf{1}^{ee}G_{\text{diag}}^{-1}(\mathbf{1}^{ee}-G_{\text{off}}\mathbf{1}^{\underline{ee}}G^{-1})=\xi^{-1}(\mathbf{1}^{ee}+T^{ee}(\xi\mathbf{1}^{ee}-T^{ee})^{-1})(\mathbf{1}^{ee}-G_{\text{off}}\mathbf{1}^{\underline{ee}}G^{-1})$$
$$=\xi^{-1}(\mathbf{1}^{ee}+T^{ee}(\xi\mathbf{1}^{ee}-T^{ee})^{-1})(\mathbf{1}^{ee}-G_{\text{off}}\mathbf{1}^{\underline{ee}}G_{\text{diag}}^{-1}+G_{\text{off}}\mathbf{1}^{\underline{ee}}G_{\text{diag}}^{-1}G_{\text{off}}\mathbf{1}^{ee}G^{-1}).$$

Now, using (4.23) and (4.25)–(4.27), we get for $|y|>C$,

$$\|\xi^{-1}T^{ee}(\xi\mathbf{1}^{ee}-T^{ee})^{-1}(\mathbf{1}^{ee}-G_{\text{off}}\mathbf{1}^{\underline{ee}}G^{-1})\|\leqslant D_1(\omega_0^2+y^2)^{-1/2}(|y|-C)^{-1}$$

$$\|\xi^{-1}G_{\text{off}}\mathbf{1}^{\underline{ee}}G_{\text{diag}}^{-1}G_{\text{off}}\mathbf{1}^{ee}G^{-1}\|\leqslant D_1(\omega_0^2+y^2)^{-1/2}(|y|-C)^{-1}$$

for some $D_1>0$ independent of λ. Hence to prove (4.24) if suffices to show that

$$\lim_{R\to\infty}\sup_{|\lambda|<\lambda_0}\left\|\int_{|y|>R}e^{\sigma(\omega_0+iy)}(\omega_0+iy)^{-1}(\mathbf{1}^{ee}-G_{\text{off}}\mathbf{1}^{\underline{ee}}G_{\text{diag}}^{-1})\psi\,\mathrm{d}y\right\|=0.$$

The first term in the above expression is independent of λ hence we need only to consider the second term. We have

$$\mathbf{1}^{\underline{ee}}G_{\text{diag}}^{-1}=\sum_{e'\in\text{sp }E;e'\neq e}(\xi-\lambda^{-2}(e'-e))^{-1}(\mathbf{1}^{e'e'}+T^{e'e'}((\xi-\lambda^{-2}(e'-e))\mathbf{1}^{e'e'}-T^{e'e'})^{-1}).$$

Hence, by (4.25) and (4.27), we get

$$\|e^{\sigma(\omega_0+iy)}(\omega_0+iy)^{-1}G_{\rm off}\mathbf{1}^{\underline{ee}}G_{\rm diag}^{-1}\| \leqslant D_2 g(y)g(y-\lambda^{-1}\,{\rm Im}(e'-e)) \tag{4.28}$$

for some $D_2>0$ independent of λ, where

$$\mathbb{R}\ni y\mapsto g(y):=(\omega_0^2+y^2)^{-1/2}\in\mathbb{R}.$$

By the Hölder inequality

$$\int_{|y|>R} g(y)g(y-\lambda^{-1}\,{\rm Im}(e'-e)){\rm d}y \leqslant \|g\|_{L^2(]-\infty,-R]\cup[R,\infty[,{\rm d}x)}\|g\|_{L^2(\mathbb{R},{\rm d}x)}\to 0. \tag{4.29}$$

Now, by (4.28) and (4.29), we get

$$\left\|\int_{|y|>R} e^{\sigma(\omega_0+iy)}(\omega_0+iy)^{-1}G_{\rm off}\mathbf{1}^{\underline{ee}}G_{\rm diag}^{-1}\psi\,{\rm d}y\right\|\to 0$$

independently of λ. This ends the proof of (3.10). □

Proof of Theorem 3.5: Set

$$f(s):=\sup_{|\lambda|<\lambda_0}\|Q^{v\bar v}e^{sL_\lambda^{\overline{vv}}}Q^{\overline{vv}}\|.$$

We know that $f(t)$ is integrable.

For any $e\in i\mathbb{R}$ and $\xi\geqslant 0$ we can dominate the integrand in the integral,

$$F_\lambda(e,\xi):=\int_0^\infty Q^{v\bar v}e^{sL_\lambda^{\overline{vv}}}Q^{\overline{vv}}e^{-(e+\lambda^2\xi)s}\,{\rm d}s = Q^{v\bar v}(\mathbf{1}^{\overline{vv}}(e+\lambda^2\xi)-L_\lambda^{\overline{vv}})^{-1}Q^{\overline{vv}} \tag{4.30}$$

by $f(s)$. Hence, using the dominated convergence theorem we see that $F_\lambda(e,\xi)$ is continuous at $\lambda=0$ and $\xi\geqslant 0$. But

$$\sum_{e\in{\rm sp}\,E}\mathbf{1}_e(E)F_0(e,0)\mathbf{1}_e(E)=\sum_{e\in{\rm sp}\,E}\lim_{\lambda\to 0}\mathbf{1}_e(E)Q^{v\bar v}(\mathbf{1}^{\overline{vv}}(e+\lambda^2\xi)-L_\lambda^{\overline{vv}})^{-1}Q^{\overline{vv}}\mathbf{1}_e(E)=\Gamma.$$

Recall (3.11), the definition of $K(\lambda,t)$,

$$K(\lambda,t):=\int_0^{\lambda^{-2}t}e^{-sE}Q^{v\bar v}e^{sL_\lambda^{\overline{vv}}}Q^{\overline{vv}}\,{\rm d}s.$$

Its integrand can also be dominated by $f(s)$. Hence, using again the dominated convergence theorem, we see that, for $\lambda\to 0$, $K(\lambda,t)$ is convergent to

$$K=\int_0^\infty e^{-sE}Q^{v\bar v}e^{sL_0^{\overline{vv}}}Q^{\overline{vv}}\,{\rm d}s.$$

Therefore,

$$K^\natural=\sum_{e\in{\rm sp}\,E}\mathbf{1}_e(E)\int_0^\infty e^{-es}Q^{v\bar v}e^{sL_0^{\overline{vv}}}Q^{v\bar v}{\rm d}s\,\mathbf{1}_e(E)=\sum_{e\in{\rm sp}\,E}1_e(E)F_0(e,0)\mathbf{1}_e(E).$$

ACKNOWLEDGMENTS

Both authors were partly supported by the Postdoctorial Training Program HPRN-CT-2002-0277 and by the Komitet Badań Naukowych (the grants SPUB127 and 2 P03A 027 25).

[1] Bach, V., Fröhlich, J., and Sigal, I., "Convergent renormalization group analysis for non-selfadjoint operators on Fock space," Adv. Math. **137**, 205 (1998).
[2] Bach, V., Fröhlich, J., and Sigal, I., "Spectral analysis for systems of atoms and molecules coupled to the quantized radiation field," Commun. Math. Phys. **207**, 249 (1999).
[3] Brattelli, O., and Robinson, D. W., *Operator Algebras and Quantum Statistical Mechanics*, 2nd ed. (Springer-Verlag, Berlin, 1987), Vol. 1.
[4] Davies, E. B., "Markovian master equations," Commun. Math. Phys. **39**, 91 (1974).
[5] Davies, E. B., "Markovian master equations II," Math. Ann. **219**, 147 (1976).
[6] Davies, E. B., *One Parameter Semigroups* (Academic, London, 1980).
[7] Dereziński, J., and Früboes, R., "Level shift operator and 2nd order perturbation theory," J. Math. Phys. (to be published).
[8] Dereziński, J., and Jakšić, V., "Spectral theory of Pauli-Fierz operators," J. Funct. Anal. **180**, 243 (2001).
[9] Dereziński, J., and Jakšić, V., "Return to equilibrium for Pauli-Fierz systems," Ann. Henri Poincare **4**, 739 (2003).
[10] Dereziński, J., and Jakšić, V., "On the nature of the Fermi Golden Rule for open quantum systems," J. Stat. Phys. **116**, 411 (2004).
[11] Dereziński, J., and Früboes, R., Fermi Golden Rule and open quantum systems (unpublished).
[12] Hille, E., and Phillips, R. S., *Functional Analysis and Semigroups* (American Mathematical Society, Providence RI, 1957).
[13] Kato, T., *Perturbation Theory for Linear Operators*, 2nd ed. (Springer-Verlag, Berlin, 1976).
[14] Lebowitz, J., and Spohn, H., "Irreversible thermodynamics for quantum systems weakly coupled to thermal reservoirs," Adv. Chem. Phys. **39**, 109 (1978).
[15] Jakšić, V., and Pillet, C.-A., "On a model for quantum friction II: Fermi's golden rule and dynamics at positive temperature," Commun. Math. Phys. **176**, 619 (1996).
[16] Jakšić, V., and Pillet, C.-A., "On a model for quantum friction III: Ergodic properties of the spin-boson system," Commun. Math. Phys. **178**, 627 (1996).
[17] Jakšić, V., and Pillet, C.-A., "Mathematical theory of non-equilibrium quantum statistical mechanics," J. Stat. Phys. **108**, 787 (2002).
[18] van Hove, L., "Quantum-mechanical perturbations giving rise to a statistical transport equation," Physica (Utrecht) **21**, 517 (1954).
[19] van Hove, L., "The approach to equilibrium in quantum statistics," Physica (Utrecht) **23**, 441 (1957).
[20] van Hove, L., "Master equations and approach to equilibrium for quantum systems," in *Fundamental Problems in Statistical Mechanics*, combined by E. G. D. Cohen (North-Holland, Amsterdam, 1962).