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Group-theoretical origin of symmetries of hypergeometric class equations and functions

Abstract: We show that properties of hypergeometric class equations and functions become transparent if we derive them from appropriate 2nd order differential equations with constant coefficients. More precisely, properties of the hypergeometric and Gegenbauer equation can be derived from generalized symmetries of the Laplace equation in 4, resp. 3 dimension. Properties of the confluent, resp. Hermite equation can be derived from generalized symmetries of the heat equation in 2, resp. 1 dimension. Finally, the theory of the ${}_1F_1$ equation (equivalent to the Bessel equation) follows from the symmetries of the Helmholtz equation in 2 dimensions. All these symmetries become very simple when viewed on the level of the 6- or 5-dimensional ambient space.
Crucial role is played by the Lie algebra of generalized symmetries of these 2nd order PDE's, its Cartan algebra, the set of roots and the Weyl group. Standard hypergeometric class functions are special solutions of these PDE's diagonalizing the Cartan algebra. Recurrence relations of these functions correspond to the roots. Their discrete symmetries correspond to the elements of the Weyl group.

Keywords: hypergeometric equation, confluent equation, Hermite equation, Bessel equation, Lie groups, Lie algebras, conformal invariance, Laplace equation

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1 Introduction

These lecture notes are devoted to the properties of the following equations:

the *Gauss hypergeometric equation*, called also the ${}_2F_1$ *equation*,

$$\Big(w(1-w)\partial_w^2 + \big(c-(a+b+1)w\big)\partial_w - ab\Big)F(w) = 0; \tag{1.1}$$

the *Gegenbauer equation*

$$\Big((1-w^2)\partial_w^2 - (a+b+1)w\partial_w - ab\Big)F(w) = 0; \tag{1.2}$$

Kummer's confluent equation, called also the ${}_1F_1$ *equation*,

$$\Big(w\partial_w^2 + (c-w)\partial_w - a\Big)F(w) = 0; \tag{1.3}$$

the *Hermite equation*

$$\Big(\partial_w^2 - 2w\partial_w - 2a\Big)F(w) = 0; \tag{1.4}$$

and the ${}_0F_1$ *equation* (equivalent to the better known *Bessel equation*, see eg. [De])

$$\Big(w\partial_w^2 + c\partial_w - 1\Big)F(w) = 0. \tag{1.5}$$

Here, w is a complex variable, ∂_w is the differentiation with respect to w, and a, b, c are arbitrary complex parameters.

These equations are typical representatives of the so-called *hypergeometric class equations* [NU]. (Nikiforov and Uvarov call them *hypergeometric type equations*; following [SL], we prefer in this context to use the word *class*, reserving *type* for narrower families of equations). We refer the reader to Sect. 2, where we discuss the terminology concerning hypergeometric class equations and functions that we use.

The equations (1.1)–(1.5) and their solutions belong to the most natural objects of mathematics and often appear in applications [Flü, MF, WW].

The aim of these notes is to elucidate the mathematical structure of a large class of identities satisfied by hypergeometric class equations and functions. We believe that our approach brings order and transparency to this subject, usually considered to be complicated and messy.

We will restrict ourselves to *generic parameters* a, b, c. We will not discuss special properties of two distingushed classes of parameters, when additional identities are true:

(1) the *polynomial case* (which corresponds to negative integer values of a);
(2) the *degenerate case* (which corresponds to integer values of c).

The notes are to a large extent based on [De] and [DeMaj], with some additions and improvements.

1.1 From 2nd order PDE's with constant coefficients to hypergeometric class equations

In our approach, each of the equations (1.1)–(1.5) is derived from a certain *complex 2nd order PDE with constant coefficients*. The identities satisfied by this PDE and their solutions are very straightforward—they look obvious and symmetric. After an appropriate change of variables, we derive (1.1)–(1.5) and identities satisfied by their solutions. They look much more complicated and messy.

We will argue that the main source of these identities are *generalized symmetries* of the parent PDE. Let us briefly recall this concept.

Suppose that we are given an equation

$$\mathcal{K}f = 0, \tag{1.6}$$

where $\mathcal{K}$ is a linear differential operator. Let g be a Lie algebra and G a group equipped with pairs of representations

$$g \ni B \mapsto B^\flat, B^\#, \tag{1.7a}$$

$$G \ni \alpha \mapsto \alpha^\flat, \alpha^\#, \tag{1.7b}$$

where (1.7a) has its values in 1st order differential operators and (1.7b) in point transformations with multipliers. We say that (1.7a) and (1.7b) are generalized symmetries of (1.6) if

$$B^\flat \mathcal{K} = \mathcal{K} B^\#, \tag{1.8a}$$

$$\text{resp.} \qquad \alpha^\flat \mathcal{K} = \mathcal{K} \alpha^\#. \tag{1.8b}$$

Note that (1.8a), resp. (1.8b) imply that $B^\#$ and $\alpha^\#$ preserve the space of solutions of (1.6).

We will omit the word "generalized" if $B^{\#} = B^{\flat}$ and $\alpha^{\flat} = \alpha^{\#}$.

We can distinguish 3 kinds of PDE's with constant coefficients in complex domain. Below we list these PDE's, together with the Lie algebra and group of their generalized symmetries:

(1) The *Laplace equation* on $\mathbb{C}^n$

$$\Delta_n f = 0, \quad n > 2. \tag{1.9}$$

The *orthogonal Lie algebra* and *group* in $n+2$ dimensions, denoted $\mathrm{so}(n+2, \mathbb{C})$, resp. $\mathrm{O}(n+2, \mathbb{C})$, both acting conformally in n dimensions. (For $n = 1, 2$ there are additional conformal symmetries).

(2) The *heat equation* on $\mathbb{C}^{n-2} \oplus \mathbb{C}$:

$$(\Delta_{n-2} + 2\partial_s) f = 0. \tag{1.10}$$

The *Schrödinger Lie algebra* and *group* in $n-2$ dimensions, denoted $\mathrm{sch}(n-2, \mathbb{C})$, resp. $\mathrm{Sch}(n-2, \mathbb{C})$.

(3) The *Helmholtz equation* on $\mathbb{C}^{n-1}$,

$$(\Delta_{n-1} - 1) f = 0. \tag{1.11}$$

The *affine orthogonal Lie algebra* and *group* in $n-1$ dimensions, denoted $\mathrm{aso}(n-1, \mathbb{C})$, resp. $\mathrm{AO}(n-1, \mathbb{C})$.

(The reason for the strange choice of dimensions in (1.10) and (1.11) will be explained later).

The basic idea of our approach is as follows. Let us start from the equation (1.6), where $\mathcal{K}$ is appropriately chosen from among (1.9), (1.10) and (1.11). In the Lie algebra of its generalized symmetries we fix a certain maximal commutative algebra, which we will call the "Cartan algebra". Operators that are eigenvectors of the adjoint action of the "Cartan algebra" will be called "root operators".

In the group of generalized symmetries we fix a subgroup, which we call the "Weyl group". It is chosen in such a way, that its adjoint action fixes the "Cartan algebra".

Note that in some cases the Lie algebra of symmetries is simple, and then the names *Cartan algebra*, *root operators* amd *Weyl symmetries* correspond to the standard names. In other cases the Lie algebra is not semisimple, and then the names are less standard – this is the reason for the quotation marks that we use above. In the sequel we drop the quotation marks.

Let us fix a basis of the Cartan algebra $N_1, \dots, N_k$. Suppose that the dimension of the underlying space is by 1 greater than the dimension of the Cartan algebra. Then we introduce new variables, say $w, u_1, \dots, u_k$ such that $N_i = u_i \partial_{u_i}$.

Substituting a function of the form

$$f = u_1^{\alpha_1} \cdots u_k^{\alpha_k} F(w), \tag{1.12}$$

to the equation (1.6), and using

$$N_i u^{\alpha_i} = \alpha_i u^{\alpha_i} \tag{1.13}$$

we obtain the equation

$$\mathcal{F}_{\alpha_1,\dots,\alpha_k} F = 0, \tag{1.14}$$

which coincides with one of the equations (1.1)–(1.5). The eigenvalues of the Cartan operators become the parameters of this equation.

Root operators shift the Cartan elements, typically by 1 or -1 (like the well-known creation and annihilation operators). Therefore, root operators inserted into the relations (1.8a) lead to *transmutation relations* for (1.1)–(1.5).

Similarly, elements of the Weyl group permute Cartan elements or change their signs. Therefore, Weyl symmetries inserted into (1.8b) leads to *discrete symmetries* of (1.1)–(1.5).

Of course, one can apply (1.8b) to elements of G other than Weyl symmetries, obtaining interesting integral and addition identities for hypergeometric class functions. They are, however, outside of the scope of these notes.

There are five 2nd order PDE with constant coefficients where we can perform this procedure. They are all listed in the following table:

PDE	Lie algebra	dimension of Cartan algebra	discrete symmetries	equation
Δ_4	$\mathrm{so}(6,\mathbb{C})$	3	cube	${}_2F_1$;
Δ_3	$\mathrm{so}(5,\mathbb{C})$	2	square	Gegenbauer;
$\Delta_2 + 2\partial_t$	$\mathrm{sch}(2,\mathbb{C})$	2	$\mathbb{Z}_2 \times \mathbb{Z}_2$	${}_1F_1$ or ${}_2F_0$;
$\Delta_1 + 2\partial_t$	$\mathrm{sch}(1,\mathbb{C})$	1	$\mathbb{Z}_4$	Hermite;
$\Delta_2 - 1$	$\mathrm{aso}(2,\mathbb{C})$	1	$\mathbb{Z}_2$	${}_0F_1$.

Note that some other 2nd order PDE's have too few variables to be in the above list: this is the case of Δ_1 and Δ_2. Others have too many variables: one can try to perform the above procedure, however it leads to a differential equation in more than one variable.

1.2 Conformal invariance of the Laplace equation

The key tool of our approach is the conformal invariance of the Laplace equation. Let us sketch a derivation of this invariance. For simplicity we restrict our attention to the complex case, for which we do not need to distinguish between various signatures of the metric tensor.

In order to derive the conformal invariance of the Laplacian on $\mathbb{C}^n$, or on other complex manifolds with maximal conformal symmetry, it is convenient to start from the so-called *ambient space* $\mathbb{C}^{n+2}$, where the actions of $\mathrm{so}(n+2,\mathbb{C})$ and $\mathrm{O}(n+2,\mathbb{C})$ are obvious. In the next step these actions are restricted to the *null quadric*, and finally to the *projective null quadric*. Thus the dimension of the manifold goes down from $n+2$ to n. The null quadric can be viewed as a line bundle over the projective null quadric. By choosing an appropriate *section* we can identify the projective null quadric, or at least its open dense subset, with the flat space $\mathbb{C}^n$ or some other complex manifolds with a complex Riemannian structure, e.g. the product of two spheres. The Lie algebra $\mathrm{so}(n+2,\mathbb{C})$ and the group $\mathrm{O}(n+2,\mathbb{C})$ act conformally on these manifolds.

What is more interesting, the above construction leads to a definition of an invariantly defined operator, which we denote $\Delta^\diamond$, transforming functions on the null quadric homogeneous of degree $1-\frac{n}{2}$ onto functions homogeneous of degree $-1-\frac{n}{2}$. After fixing a section, this operator can be identified with the conformal Laplacian on the corresponding complex Riemannian manifold of dimension n. For instance, one obtains the Laplacian Δ_n on $\mathbb{C}^n$. The representations of $\mathrm{so}(n+2,\mathbb{C})$ and $\mathrm{O}(n+2,\mathbb{C})$ on the level of the ambient space were true symmetries of Δ_{n+2}. After the reduction to n dimensions, they become generalized symmetries of the conformal Laplacian.

The fact that conformal transformations of the Euclidean space are generalized symmetries of the Laplace equation was apparently known already to Lord Kelvin. Its explanation in terms of the null quadric first appeared in [Boc], and is discussed e.g. in [CGT]. The reduction of Δ_{n+2} to Δ_n mentioned above, is based on a beautiful idea of Dirac in [Dir], which was later rediscovered e.g. in [HH, FG]—see a discussion by Eastwood [East].

The construction indicated above gives a rather special class of (pseudo-)Riemannian manifolds—those having a conformal group of maximal dimension, see e.g. [EMN]. However, conformal invariance can be generalized to arbitrary (pseudo-)Riemannian manifolds. In fact, the Laplace-Beltrami operator plus an appropriate multiple of the scalar curvature, sometimes called the *Yamabe Laplacian*, is invariant in a generalized sense with respect to conformal maps, see e.g. [Tay, Or].

1.3 The Schrödinger Lie algebra and Lie algebra as generalized symmetries of the Heat equation

The heat equation (1.10) possesses a large Lie algebra and group of generalized symmetries, which in the complex case, as we already indicated, we denote by $\mathrm{sch}(n-2,\mathbb{C})$ and $\mathrm{Sch}(n-2,\mathbb{C})$. Apparently, they were known already to Lie [L]. They were rediscovered (in the essentially equivalent context of the free Schrödinger equation) by Schrödinger [Sch]. They were then studied e.g. in [Ha, Ni].

By adding an additional variable, one can consider the heat equation as the Laplace equation acting on functions with an exponential dependence on one of the variables. This allows us to express generalized symmetries of (1.10) by generalized symmetries of (1.9). They can be identified as a subalgebra of $\mathrm{so}(n+2,\mathbb{C})$, resp. a subgroup of $\mathrm{O}(n+2,\mathbb{C})$ consisting of elements commuting with a certain distinguished element of $\mathrm{so}(n+2,\mathbb{C})$.

1.4 Affine orthogonal group and algebra as symmetries of the Helmholtz equation

Recall that the affine orthogonal group $\mathrm{AO}(n-1,\mathbb{C})$ is generated by rotations and translations of $\mathbb{C}^{n-1}$. It is obvious that elements of $\mathrm{AO}(n-1,\mathbb{C})$ commute with the Helmholtz operator $\Delta_{n-1} - 1$. The same is true concerning the affine orthogonal Lie algebra $\mathrm{aso}(n-1,\mathbb{C})$. Therefore, they are symmetries of the Helmholtz equation (1.11).

The Helmholtz equation is conceptually simpler than that of the Laplace and heat equation, because all generalized symmetries are true symmetries.

Note that one can embed the symmetries of the Helmholtz equation in conformal symmetries of the Laplace equation, similarly as was done with the heat equation. In fact, $\mathrm{aso}(n-1,\mathbb{C})$ is a subalgebra of $\mathrm{so}(n+2,\mathbb{C})$, and $\mathrm{AO}(n-1,\mathbb{C})$ is a subgroup of $\mathrm{O}(n+2,\mathbb{C})$.

1.5 Factorization relations

Another important class of identities satisfied by hypergeometric class operators are *factorizations* [IH]. They come in pairs. They are identities of the form

$$\mathcal{F}_1 = \mathcal{A}_-\mathcal{A}_+ + c_1, \tag{1.15a}$$

$$\mathcal{F}_2 = \mathcal{A}_+\mathcal{A}_- + c_2, \tag{1.15b}$$

where $\mathcal{A}_+$, $\mathcal{A}_-$ are 1st order differential operators, c_1, c_2 are numbers and $\mathcal{F}_1$, $\mathcal{F}_2$ are operators coming from (1.1)—(1.5) with slightly shifted parameters.

The number of such factorizations is the same as the number of roots of the Lie algebra of generalized symmetries. They can be derived from certain identities in the *enveloping algebra*. They are closely related to the *Casimir operators* of its subalgebras.

Factorizations imply *transmutation relations*. In fact, it is easy to see that (1.15b) and (1.15a) imply

$$\mathcal{A}_-\mathcal{F}_2 = (\mathcal{F}_1 + c_2 - c_1)\mathcal{A}_-, \tag{1.16a}$$

$$\mathcal{A}_+\mathcal{F}_1 = (\mathcal{F}_2 + c_1 - c_2)\mathcal{A}_+. \tag{1.16b}$$

Note that (1.16a) implies that the operator $\mathcal{A}_-$ maps the kernel of $\mathcal{F}_2$ to the kernel of $\mathcal{F}_1 + c_2 - c_1$. Similarly, (1.16b) implies that the operator $\mathcal{A}_+$ maps the kernel of $\mathcal{F}_1$ to the kernel of $\mathcal{F}_2 + c_1 - c_2$. The above construction is usually called the *Darboux transformation*.

1.6 Standard solutions of hypergeometric class equations

So far we discussed only identities satisfied by the operators corresponding to the equations (1.1)—(1.5). The approach discussed in these notes is also helpful in deriving and classifying the identities for their solutions.

The equations (1.1)—(1.5) have at least 1 and at most 3 singular points on the Riemann sphere. One can typically find two solutions with a simple behavior at each of these points. We call them *standard solutions*. (If it is a regular–singular point, then the solutions are given by convergent power series, otherwise we have to use other methods to define them). The discrete symmetries map standard solutions on standard solutions. The best known example of this method of generating solutions is *Kummer's table* [Ku], which lists various possible expressions for solutions of the hypergeometric equation.

1.7 Recurrence relations of hypergeometric class functions

All transmutation relations have the form

$$\mathcal{A}\mathcal{F}_1 = \mathcal{F}_2\mathcal{A}, \tag{1.17}$$

where $\mathcal{A}$ is a first order differential operator and $\mathcal{F}_1$, $\mathcal{F}_2$ is a pair of hypergeometric class operators of the same type. Typically, some parameters of $\mathcal{F}_2$ differ from the corresponding parameters of $\mathcal{F}_1$ by ± 1. Clearly, if a function F_1 solves $\mathcal{F}_1 F_1 = 0$, then $\mathcal{A}F_1$ solves $\mathcal{F}_2\mathcal{A}F_1 = 0$.

It turns out that if F_1 is a standard solution of $\mathcal{F}_1$, then $\mathcal{A}F_1$ is proportional to one of standard solutions of $\mathcal{F}_2$, say F_2. Thus we obtain an identity

$$\mathcal{A}F_1 = aF_2, \tag{1.18}$$

called a *recurrence relation*, or a *contiguity relation*.

The recurrence relation (1.18) is fixed by the transmutation relation (1.17) except for the coefficient a. In practice it is not difficult to determine a.

1.8 From wave packets to integral representations

Hypergeometric class functions possess integral representations, where integrands are elementary functions. We show that integral representations come from certain natural solutions of the parent 2nd order PDE, which at the same time are eigenfunctions of Cartan operators. It will be convenient to have a name for this kind of solutions—we will call them *wave packets*.

Let us describe how to construct wave packets for the Laplace equation. It is easy to see that each function depending only on variables from an isotropic subspace is harmonic, that is, satisfies the Laplace equation. By assuming that the function is homogeneous in appropriate variables we can make sure that it is an eigenfunction of Cartan operators.

Unfortunately, the above class of functions is too narrow for our purposes. There is still another construction that can be applied: we can rotate a function and integrate it ("smear it out") with respect to a weight. This procedure does not destroy the harmonicity. By choosing the weight appropriately, we can make sure that the resulting wave packet is an eigenfunction of Cartan perators. (The "smearing out" is essentially a generalization of the Fourier (or Mellin) transformation to the complex domain.)

After substituting special coordinates to a wave packet, we obtain a function of the form (1.12) with F solving (1.14), and having the form of an integral of an elementary function.

Wave packets for the heat and Helmholtz equation can be derived from wave packets for the Laplace equation.

1.9 Plan of the lecture notes

In Sect. 2 we give a concise introduction to hypergeometric class equations and functions. One can view this section as an extension of the introduction, concentrated on the terminology and classification of equations and functions we consider in these notes.

The remaining sections can be divided into two categories. The first category consists of Sects 3, 4 and 7. They have a general character and are devoted to basic geometric analysis in any dimension. The most important one among them is Sect. 4, devoted to the conformal invariance of the Laplace equation. Of comparable importance is Sect. 7, where the Schrödinger Lie algebra and group are introduced. In Subsect. 3.10—3.13 we explain how to construct "wave packets". No special functions appear in Sects 3, 4 and 7. They can be read independently of the rest of the notes.

The second category consists of Sects 5, 6, 8, 9 and 10. They are devoted to a detailed analysis of equations (1.1), (1.2), (1.3), (1.4), resp. (1.5). Typically, each section starts with the ambient space corresponding to the 2nd order PDE from the left column of the table in Subsect. 1.1. In the ambient space these symmetries are very easy to describe. Then we reduce the dimension and introduce special coordinates, which leads to the equation in the right column of the table.

We made serious efforts to make Sects 5, 6, 8, 9 and 10 as parallel as possible. there is a one to one correspondence between subsections in all these 5 sections. We try to use a uniform terminology and analogous conventions. This makes our text somewhat repetitive—we believe that this is helpful to the reader. Note also that these sections are to a large extent independent of one another.

We use various (minor but helpful) ideas to make our presentation as short and transparent as possible. One of them is the use of two kinds of parameters. The parameters that appear in (1.1), (1.2), (1.3), (1.4), and (1.5), denoted a, b, c, are called *classical parameters*. They are convenient when one defines ${}_kF_m$ functions by power series. However, in most of our text we prefer to use a different set of parameters, denoted by Greek letters $\alpha, \beta, \mu, \theta, \lambda$. They are much more convenient when we describe symmetries.

Another helpful idea is a consistent use of *split coordinates* in $\mathbb{C}^n$ or $\mathbb{R}^n$. In these coordinates root operators and Weyl symmetries have an especially simple form.

The notes are full of long lists of identities. We are convinced that most of them are easy to understand and appreciate without much effort. Typically, they are highly symmetric and parallel to one another.

We hesitated whether to use the complex or real setting for these notes. The complex setting was e.g. in [DeMaj]. It offers undoubtedly some simplifications:

there is no need to consider various signatures of the scalar product. However, the complex setting can also be problematic: analytic functions are often multivalued, which causes issues with some global constructions. Therefore, in these notes, except for the introduction, we use the real setting as the basic one. At the same time we keep in mind that all our formulas have obvious analytic continuations to appropriate complex domains.

In most of our notes, we do not make explicit the signature of the scalar product in our notation for Lie algebras and groups. E.g. by writing $\mathrm{so}(n)$ we mean $\mathrm{so}(q,p)$ for some $n = q + p$ or $so(n,\mathbb{C})$. Specifying each time the signature would be overly pedantic, especially since we usually want to complexify all objects, so that the signature loses its importance.

1.10 Comparison with literature

The literature about hypergeometric class functions is enormous—after all it is one of the oldest subjects of mathematics. Let us mention e.g. the books [BE, SL, AAR, EMOT, Ho, MOS, NIST, R, WW].

The relationship of special functions to Lie groups and algebras was noticed long time ago. For instance, the papers by Weisner [We1, We2] from the 50's describe Lie algebras associated with Bessel and Hermite functions.

The idea of studying hypergeometric class equations with help of Lie algebras was developed further by Miller. His early book [M1] considers mostly small Lie algebras/Lie groups, typically $\mathrm{sl}(2,\mathbb{C})/\mathrm{SL}(2,\mathbb{C})$ and their contractions, and applies them to obtain various identities about hypergeometric class functions. These Lie algebras have 1-dimensional Cartan algebras and a single pair of roots. This kind of analysis is able to explain only a single pair of transmutation relations for each equation. To explain bigger families of transmutation relations one needs larger Lie algebras.

A Lie algebra strictly larger than $\mathrm{sl}(2,\mathbb{C})$ is $\mathrm{so}(4,\mathbb{C})$. There exists a large literature on the relation of the hypergeometric equation with $\mathrm{so}(4,\mathbb{C})$ and its real forms, see eg. [KM, KMR]. This Lie algebra is however still too small to account for all symmetries of the hypergeometric equation—its Cartan algebra is only 2-dimensional, whereas the equation has three parameters.

An explanation of symmetries of the Gegenbauer equation in terms of $\mathrm{so}(5,\mathbb{C})$ and of the hypergeometric equation in terms of $\mathrm{so}(6,\mathbb{C}) \simeq \mathrm{sl}(4,\mathbb{C})$ was first given by Miller, see [M4], and especially [M5].

Miller and Kalnins wrote a series of papers where they studied the symmetry approach to separation of variables for various 2nd order partial differential equations, such as the Laplace and wave equation, see eg. [KM1]. A large part of this research is summed up in the book by Miller [M3]. As an important consequence of this study, one obtains detailed information about symmetries of hypergeometric class equations.

The main tool that we use to describe properties of hypergeometric class functions are *generalized symmetries* of 2nd order linear PDE's. Their theory is described in another book by Miller [M2], and further developed in [M3].

A topic that is extensively treated in the literature on the relation of special functions to group theory, such as [V, Wa, M1, VK], is derivation of various addition formulas. Addition formulas say that a certain special function can be written as a sum, often infinite, of some related functions. As we mentioned above, they are outside of the scope of this text—we concentrate on the simplest identities.

The relationship of Kummer's table with the group of symmetries of a cube (which is the Weyl group of $\mathrm{so}(6,\mathbb{C})$) was discussed in [LSV]. A recent paper, where symmetries of the hypergeometric equation play an important role is [Ko].

The use of transmutation relations as a tool to derive recurrence relations for hypergeometric class functions is well known and can be found eg. in the book by Nikiforov-Uvarov [NU], in the books by Miller [M1] or in older works such as [Tr, We1, We2].

There exist various generalizations of hypergeometric class functions. Let us mention the class of $\mathcal{A}$-hypergeometric functions, which provides a natural generalization of the usual hypergeometric function to many-variable situations [Be, Bod]. Saito [Sa] considers generalized symmetries in the framework of $\mathcal{A}$-hypergeometric functions.

Another direction of generalizations of hypergeometric functions is the family of Gel'fand-Kapranov-Zelevinsky hypergeometric functions [G, GKZ]. Similar constructions were explored by Aomoto and others [A, AK, M-H]. The main idea is to generalize integral representations of hypergeometric functions, rather than hypergeometric equations. There exist also interesting confluent versions of these functions [KHT].

A systematic presentation and derivation of symmetries of hypergeometric class equations and functions from 2nd order PDE's with constant coefficients was given in [De] and [DeMaj]. These papers consistently use Lie-algebraic parameters, describe transmutation relations, discrete symmetries and factorizations. [De] describes integral representations and recurrence relations. [DeMaj] concentrates on the study of hypergeometric class operators, leaving out the properties of hypergeometric class functions.

These lecture notes are to a large extent based on [De] and [DeMaj]. There are some corrections and minor changes of conventions. There are also some additions. A systematic derivation of all integral representations from "wave packets" in higher dimensions seems to be new.

There are a number of topics related to the hypergeometric class equation that we do not touch. Let us mention the question whether hypergeometric functions can be expressed in terms of algebraic functions. This topic, in the context of $\mathcal{A}$-hypergeometric functions was considered eg. in the interesting papers [Be, Bod].

We stick to a rather limited class of equations and functions (1.1)–(1.5). They have a surprisingly rich structure, which often seems to be lost in more general classes. Nevertheless, it is natural to ask how far one can generalize the ideas of these notes to other equations and functions, such as higher hypergeometric functions, multivariable hypergeometric functions, Heun functions, q-hypergeometric functions, Painlevé equations.

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2 Hypergeometric class equations

In this short section we fix our terminology concerning hypergeometric class equations and functions.

2.1 Remarks on notation

We use ∂_w for the operator of differentiation in the variable w. We will understand that the operator ∂_w acts on the whole expression on its right:

$$\partial_w f(w)g(w) = \partial_w\big(f(w)g(w)\big). \tag{2.1}$$

If we want to restrict the action of ∂_w to the term immediately to the right, we will write $f(w)_{,w}$, or simply $f'(w)$.

We use lhs and rhs as the abbreviations for the left hand side and right hand side.

2.2 Generalized hypergeometric series

For $a \in \mathbb{C}$ and $n \in \mathbb{N}$ we define the *Pochhammer symbol*

$$(a)_j := a(a+1)\cdots(a+j-1).$$

For $a_1, \dots, a_k \in \mathbb{C}$, $c_1, \dots, c_m \in \mathbb{C}\backslash\{0, -1, -2, \dots\}$, we define the ${}_kF_m$ *generalized hypergeometric series*, or for brevity the ${}_kF_m$ *series*:

$${}_kF_m(a_1, \dots, a_k; c_1, \dots, c_m; w) := \sum_{j=0}^{\infty} \frac{(a_1)_j \cdots (a_k)_j w^j}{(c_1)_j \cdots (c_m)_j j!}. \tag{2.2}$$

By the d'Alembert criterion,

(1) if $m+1 > k$, the series (2.2) is convergent for $w \in \mathbb{C}$;
(2) if $m+1 = k$, the series (2.2) is convergent for $|w| < 1$;
(3) if $m+1 < k$, the series (2.2) is divergent, however sometimes a certain function can be naturally associated with (2.2).

The corresponding analytic function will be called the ${}_kF_m$ *function.*

The zeroth order term of the series (2.2) is 1. A different normalization of (2.2) is often useful:

$$\begin{aligned} {}_k\mathbf{F}_m(a_1, \dots, a_k; c_1, \dots, c_m; w) &:= \frac{{}_kF_m(a_1, \dots, a_k; c_1, \dots, c_m; w)}{\Gamma(c_1)\cdots\Gamma(c_m)} \\ &= \sum_{j=0}^{\infty} \frac{(a_1)_j \cdots (a_k)_j w^j}{\Gamma(c_1+j)\cdots\Gamma(c_m+j) j!}. \end{aligned} \tag{2.3}$$

In (2.3) we do not have to restrict the values of $c_1, \dots, c_m \in \mathbb{C}$.

2.3 Generalized hypergeometric equations

Theorem 2.1. *The ${}_kF_m$ function (2.2) solves the dfferential equation*

$$\begin{aligned} &(c_1 + w\partial_w)\cdots(c_m + w\partial_w)\partial_w F(a_1, \dots, a_k; c_1, \dots, c_m; w) \\ = \;&(a_1 + w\partial_w)\cdots(a_k + w\partial_w) F(a_1, \dots, a_k; c_1, \dots, c_m; w). \end{aligned} \tag{2.4}$$

Proof. We check that both the left and right hand side of (2.4) are equal to

$$a_1 \cdots a_k F(a_1+1, \dots, a_k+1; c_1, \dots, c_m; w).$$

□

We will call (2.4) the ${}_k\mathcal{F}_m$ *equation.* It has the order $\max(k, m+1)$. Below we list all ${}_kF_m$ functions with equations of the order at most 2.

– **The ${}_2F_1$ function or the Gauss hypergeometric function**

$$F(a,b;c;w) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{n!(c)_n} w^n.$$

The series is convergent for $|w| < 1$, and it extends to a multivalued function on a covering of $\mathbb{C}\backslash\{0,1\}$. It is a solution of the *Gauss hypergeometric equation* or the ${}_2\mathcal{F}_1$ *equation*

$$\big(w(1-w)\partial_w^2 + (c-(a+b+1)w)\partial_w - ab\big) f(w) = 0.$$

– **The ${}_1F_1$ function or Kummer's confluent function**

$$F(a;c;w) = \sum_{n=0}^{\infty} \frac{(a)_n}{n!(c)_n} w^n.$$

The series is convergent for all $w \in \mathbb{C}$. It is a solution of *Kummer's confluent equation* or the ${}_1\mathcal{F}_1$ *equation*

$$\big(w\partial_w^2 + (c-w)\partial_w - a\big) f(w) = 0.$$

– **The ${}_0F_1$ function**

$$F(-;c;w) = F(c;w) = \sum_{n=0}^{\infty} \frac{1}{n!(c)_n} w^n.$$

The series is convergent for all $w \in \mathbb{C}$. It is a solution of the ${}_0\mathcal{F}_1$ *equation* (related to the Bessel equation)

$$(w\partial_w^2 + c\partial_w - 1) f(w) = 0.$$

– **The ${}_2F_0$ function**
For $\arg w \neq 0$ we define

$$F(a,b;-;w) := \lim_{c\to\infty} F(a,b;c;cw).$$

It extends to an analytic function on the universal cover of $\mathbb{C}\backslash\{0\}$ with a branch point of an infinite order at 0. It has the following divergent but asymptotic expansion:

$$F(a,b;-;w) \sim \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{n!} w^n, \quad |\arg w - \pi| < \pi - \epsilon, \quad \epsilon > 0.$$

It is a solution of the ${}_2\mathcal{F}_0$ equation

$$\big(w^2\partial_w^2 + (-1+(a+b+1)w)\partial_w + ab\big) f(w) = 0.$$

By a simple transformation described in Subsect. 8.10 it is equivalent to the ${}_1\mathcal{F}_1$ equation.

– **The ${}_1F_0$ function or the power function**

$$F(a;-;w) = (1-w)^{-a} = \sum_{n=0}^{\infty} \frac{(a)_n}{n!} w^n$$

It solves

$$\big((w-1)\partial_w - a\big) f(w) = 0.$$

– **The ${}_0F_0$ function or the exponential function**

$$F(-;-;w) = \mathrm{e}^w = \sum_{n=0}^{\infty} \frac{1}{n!} w^n.$$

It solves

$$(\partial_w - 1) f(w) = 0.$$

2.4 Hypergeometric class equations

Following [NU], equations of the form

$$\big(\sigma(w)\partial_w^2 + \tau(w)\partial_w + \eta\big)\, f(w) = 0, \tag{2.5}$$

where

$$\sigma \text{ is a polynomial of degree } \leq 2, \tag{2.6a}$$

$$\tau \text{ is a polynomial of degree } \leq 1, \tag{2.6b}$$

$$\eta \text{ is a number}, \tag{2.6c}$$

will be called *hypergeometric class equations*. Solutions of (2.5) will go under the name of *hypergeometric class functions*. Operators $\sigma(w)\partial_w^2 + \tau(w)\partial_w + \eta$ with σ, τ, η satisfying (2.6) will be called *hypergeometric class operators*.

Let us review basic classes of hypergeometric class equations. We will always assume that $\sigma(w) \neq 0$. Every class will be simplified by dividing by a constant and, except for (2.14), by an affine change of the complex variable w.

The ${}_2\mathcal{F}_1$ or Gauss hypergeometric equation

$$\big(w(1-w)\partial_w^2 + (c-(a+b+1)w)\partial_w - ab\big)\, f(w) = 0. \tag{2.7}$$

The ${}_2F_0$ equation

$$\big(w^2\partial_w^2 + (-1+(1+a+b)w)\partial_w + ab\big)\, f(w) = 0. \tag{2.8}$$

The ${}_1\mathcal{F}_1$ or Kummer's confluent equation

$$(w\partial_w^2 + (c-w)\partial_w - a) f(w) = 0. \tag{2.9}$$

The ${}_0\mathcal{F}_1$ equation

$$(w\partial_w^2 + c\partial_w - 1)f(w) = 0. \tag{2.10}$$

The Hermite equation

$$(\partial_w^2 - 2w\partial_w - 2a)f(w) = 0. \tag{2.11}$$

2nd order Euler equation

$$\left(w^2\partial_w^2 + bw\partial_w + a\right) f(w) = 0. \tag{2.12}$$

1st order Euler equation for the derivative

$$(w\partial_w^2 + c\partial_w)f(w) = 0. \tag{2.13}$$

2nd order equation with constant coefficients

$$(\partial_w^2 + c\partial_w + a)f(w) = 0. \tag{2.14}$$

Note that the equations (2.12), (2.13) and (2.14) are elementary. The remaining ones (2.7), (2.8), (2.9), (2.10) and (2.11) are the subject of these lecture notes. This is why they are contained in the list (1.1)–(1.5) given at the beginning of these notes. (Actually, (2.8) is not explicitly mentioned in this list, however it is equivalent to (2.9), so that these two equations are treated together). This list contains also

The Gegenbauer equation

$$\left((1-w^2)\partial_w^2 - (a+b+1)w\partial_w - ab\right) f(w) = 0, \tag{2.15}$$

which can be reduced to a subclass of ${}_2\mathcal{F}_1$ equations by a simple affine transformation. Its distinguishing property is the invariance with respect to the reflection. The Gegenbauer equation has special properties, which justify its separate treatment.

3 (Pseudo-)Euclidean spaces

In this section we introduce basic terminology and notation related to Lie algebras and groups acting on functions on $\mathbb{R}^n$ or, more generally, on manifolds. Lie algebras will be usually represented as 1st order differential operators. Lie groups will typically act as point transformations times multipliers.

We will discuss various operators related to (pseudo-)orthogonal Lie algebras and groups. In particular, we will introduce a convenient notation to describe their Cartan algebras, root operators and Weyl groups. We will also discuss briefly the Laplacian and the Casimir operator.

We will show how to some special classes of harmonic functions—solutions of the Laplace equation. Of particular importance will be solutions that at the same time are eigenfunctions of the Cartan algebra. This construction will involve a contour integral, which can be viewed as a modification of the Fourier or Mellin transformation. These solutions will be informally called *wave packets*.

Finally, in the last subsection we will show how to construct a harmonic function in $n-1$ dimension from a harmonic function in n dimensions.

3.1 Basic notation

We will write $\mathbb{R}^\times$ for $\mathbb{R}\backslash\{0\}$, $\mathbb{R}_+$ for $]0,\infty[$ and $\mathbb{R}_-$ for $]-\infty,0[$. We write $\mathbb{C}^\times$ for $\mathbb{C}\backslash\{0\}$.

We will treat $\mathbb{R}^n$ as a (real) subspace of $\mathbb{C}^n$. If possible, we will often extend functions from real domains to holomorphic functions on complex domains.

In the following two subsections, $\Omega, \Omega_1, \Omega_2$ are open subsets of $\mathbb{R}^n$, or more generally, manifolds.

Often it is advantageous to consider a similar formalism where $\Omega, \Omega_1, \Omega_2$ are open subsets of $\mathbb{C}^n$, or more generally, complex manifolds. We will usually stick to the terminology typical for the real case. The reader can easily translate it to the complex picture, if needed.

3.2 Point transformations with multipliers

Let $\alpha : \Omega_1 \to \Omega_2$ be a diffeomorphism. The *transport* of functions by the map α will be also denoted by α.[1] More precisely, for $f \in C^\infty(\Omega_1)$ we define $\alpha f \in C^\infty(\Omega_2)$ by

$$(\alpha f)(y) := f(\alpha^{-1}(y)).$$

If $m \in C^\infty(\Omega_2)$, then we have a map $m\alpha : C^\infty(\Omega_1) \to C^\infty(\Omega_2)$ given by

$$(m\alpha f)(y) := m(y) f(\alpha^{-1}(y)). \tag{3.1}$$

1 An alternative notation used often in mathematical literature for the transport by α is α_* or $(\alpha^*)^{-1}$.

Transformations of the form (3.1) will be called *point transformations with a multiplier.*

Clearly, transformations of the form (3.1) with $\Omega = \Omega_1 = \Omega_2$ and m everywhere nonzero form a group.

3.3 1st order differential operators

A vector field X on Ω will be identified with the differential operator

$$Xf(y) = \sum_i X^i(y)\partial_{y^i} f(y), \quad f \in C^\infty(\Omega),$$

where $X^i \in C^\infty(\Omega)$, $i = 1, \dots, n$. More generally, we will often use 1st order differential operators

$$(X + M)f(y) := \sum_i X^i(y)\partial_{y^i} f(y) + M(y)f(y), \tag{3.2}$$

where $M \in C^\infty(\Omega)$. Clearly, the set of operators of the form (3.2) is a Lie algebra.

Let $\alpha : \Omega_1 \to \Omega_2$ be a diffeomorphism. If X is a vector field on Ω_1, then $\alpha(X)$ is the vector field on Ω_2 defined as

$$\alpha(X) := \alpha X \alpha^{-1}.$$

3.4 Affine linear transformations

The *general linear group* is denoted $\mathrm{GL}(\mathbb{R}^n)$. It has a natural extension $\mathrm{AGL}(\mathbb{R}^n) := \mathbb{R}^n \rtimes \mathrm{GL}(\mathbb{R}^n)$ called the *affine general linear group.* $(w, \alpha) \in \mathrm{AGL}(\mathbb{R}^n)$ acts on $\mathbb{R}^n$ by

$$\mathbb{R}^n \ni y \mapsto w + \alpha y \in \mathbb{R}^n.$$

The permutation group S_n can be naturally identified with a subgroup of $\mathrm{GL}(\mathbb{R}^n)$. If $\pi \in S_n$, then

$$(\pi y)^i := y^{\pi_i^{-1}}.$$

On the level of functions, we have

$$\pi f(y^1, \dots, y^n) = f(y^{\pi_1}, \dots, y^{\pi_n}).$$

The Lie algebra $\mathrm{gl}(\mathbb{R}^n)$ represented by vector fields on $\mathbb{R}^n$ is spanned by $y^i\partial_{y^j}$.

The Lie algebra $\mathrm{agl}(\mathbb{R}^n) := \mathbb{R}^n \rtimes \mathrm{gl}(\mathbb{R}^n)$ is spanned by $\mathrm{gl}(\mathbb{R}^n)$ and by ∂_{y_i}.

A special element of $\mathrm{gl}(\mathbb{R}^n)$ is the *generator of dilations*, known also as the *Euler vector field*,

$$A_n := \sum_{i=1}^{n} y^i \partial_{y^i}. \tag{3.3}$$

We will often use the complex versions of the above groups, with $\mathbb{R}$ replaced with $\mathbb{C}$. We will write $\mathrm{GL}(n)$ and $\mathrm{gl}(n)$, where the choice of the field follows from the context.

3.5 (Pseudo-)orthogonal group

A *pseudo-Euclidean space* is $\mathbb{R}^n$ equipped with a symmetric nondegenerate $n \times n$ matrix $g = [g_{ij}]$. g defines the *scalar product* of vectors $x, y \in \mathbb{R}^n$ and the *square* of a vector $x \in \mathbb{R}^n$:

$$\langle x|y\rangle := \sum_{ij} x^i g_{ij} y^j, \quad \langle x|x\rangle = \sum_{ij} x^i g_{ij} x^j.$$

The matrix $[g^{ij}]$ will denote the inverse of $[g_{ij}]$.

We will denote by $\mathbb{S}^{n-1}(R)$ the *sphere in* $\mathbb{R}^n$ *of squared radius* $R \in \mathbb{R}$:

$$\mathbb{S}^{n-1}(R) := \{y \in \mathbb{R}^n \ : \ \langle y|y\rangle = R\}. \tag{3.4}$$

We will write $\mathbb{S}^{n-1} := \mathbb{S}^{n-1}(1)$.

Actually, $\mathbb{S}^{n-1}$ is the usual sphere only for the Euclidean signature. For non-Euclidean spaces it is a hyperboloid. Usually we will keep a uniform notation for all signatures. Occasionally, if we want to stress that $\mathbb{S}^{n-1}$ has a specific signature, it will be denoted $\mathbb{S}^{q,p-1}$, where the signature of the ambient space is (q,p) (see (3.7)).

We also introduce the *null quadric*

$$\mathcal{V}^{n-1} := \mathbb{S}^{n-1}(0)\backslash\{0\}. \tag{3.5}$$

The *(pseudo-)orthogonal* and the *special (pseudo-)orthogonal group of* g is defined as

$$\begin{aligned} \mathrm{O}(g) &:= \{\alpha \in \mathrm{GL}(n) \ : \ \langle \alpha y|\alpha x\rangle = \langle y|x\rangle, \ y, x \in \mathbb{R}^n\}, \\ \mathrm{SO}(g) &:= \{\alpha \in \mathrm{O}(g) \ : \ \det \alpha = 1\}. \end{aligned}$$

We also have the *affine (special) orthogonal group* $\mathrm{AO}(g) := \mathbb{R}^n \rtimes \%_{00}(g)$, $\mathrm{ASO}(g) := \mathbb{R}^n \rtimes \mathrm{SO}(g)$.

It is easy to see that the *pseudo-orthogonal Lie algebra*, represented by vector fields on $\mathbb{R}^n$, can be defined by

$$\mathrm{so}(g) := \{B \in \mathrm{gl}(n) \ : \ B\langle y|y\rangle = 0\}.$$

For $i, j = 1, \dots, n$, define

$$B_{ij} := \sum_k (g_{ik} y^k \partial_{y^j} - g_{jk} y^k \partial_{y^i}).$$

$\{B_{ij} \ : \ i < j\}$ is a basis of $\mathrm{so}(g)$. Clearly, $B_{ij} = -B_{ji}$ and $B_{ii} = 0$.

The *affine pseudo-orthogonal Lie algebra* $\mathrm{aso}(g) := \mathbb{R}^n \rtimes \mathrm{so}(g)$ is spanned by ∂_{y^i} and $\mathrm{so}(g)$.

We will often use the complex versions of the above groups and Lie algebras. In the real formalism we have to distinguish between various signatures of g—in the complex formalism there is only one signature and we can drop the prefix *pseudo*.

3.6 Invariant operators

Consider a pseudo-Euclidean space $\mathbb{R}^n$. We define the Laplacian and the Casimir operator

$$\begin{aligned}
\Delta_n &:= \sum_{i,j=1}^{n} g^{ij} \partial_{y^i} \partial_{y^j}, \\
\mathcal{C}_n &:= \frac{1}{2} \sum_{i,j,k,l=1}^{n} g^{ik} g^{jl} B_{ij} B_{kl}.
\end{aligned}$$

The above definitions do not depend on the choice of a basis. Δ_n commutes with $\mathrm{AO}(g)$ and $\mathrm{aso}(g)$. $\mathcal{C}_n$ commutes with $\mathrm{O}(g)$ and $\mathrm{so}(g)$.

Note the identity

$$\langle y|y\rangle \Delta_n = A_n^2 + (n-2)A_n + \mathcal{C}_n, \tag{3.6}$$

where A_n is defined in (3.3).

3.7 Orthonormal coordinates

Suppose that $q + p = n$. Every scalar product of signature (q, p) can be brought to the form

$$\langle y|y\rangle = -\sum_{i=1}^{q} y_i^2 + \sum_{j=q+1}^{q+p} y_j^2. \tag{3.7}$$

so(g) has a basis consisting of

$$B_{ij}=-y_i\partial_{y_j}+y_j\partial_{y_i},\qquad 1\leq i<j\leq q \tag{3.8a}$$
$$B_{ij}=y_i\partial_{y_j}+y_j\partial_{y_i}\qquad 1\leq i\leq q,\quad q<j\leq n; \tag{3.8b}$$
$$B_{ij}=y_i\partial_{y_j}-y_j\partial_{y_i},\qquad q<i<j\leq n. \tag{3.8c}$$

The Laplacian and the Casimir operator are

$$\Delta_n=-\sum_{1\leq i\leq q}\partial_{y_i}^2+\sum_{q<j\leq n}\partial_{y_j}^2, \tag{3.9}$$
$$\mathcal{C}_n=\sum_{1\leq i<j\leq q}B_{ij}^2+\sum_{q<i<j\leq n}B_{ij}^2-\sum_{\substack{1\leq i\leq q,\\ q<j\leq n}}B_{ij}^2. \tag{3.10}$$

We will rarely use orthonormal coordinates.

In the context of the signature (q,p) the standard notation for the orthogoanl groups/Lie algebras is $\mathrm{O}(q,p)$, $\mathrm{AO}(q,p)$, so(q,p), aso(q,p). We will however often use the notation $\mathrm{O}(n)$, $\mathrm{AO}(n)$, so(n), aso(n), without specifying the signature of the quadratic form, and even allowing for an arbitrary choice of the field ($\mathbb{R}$ or $\mathbb{C}$).

3.8 Split coordinates

Suppose that $2m=n$. (m,m) will be called the *split signature*. If the scalar product has such a signature, we can find coordinates such that

$$\langle y|y\rangle=\sum_{i=1}^m 2y_{-i}y_i. \tag{3.11}$$

We will say that (3.11) is a scalar product in *split coordinates*.

so$(2m)$ has a basis consisting of

$$N_i:=B_{-ii}\quad=\quad -y_{-i}\partial_{y_{-i}}+y_i\partial_{y_i},\quad j=1,\dots,m, \tag{3.12a}$$
$$B_{ij}\quad=\quad y_{-i}\partial_{y_j}-y_{-j}\partial_{y_i},\quad 1\leq|i|<|j|\leq m. \tag{3.12b}$$

The subalgebra of so$(2m)$ spanned by (3.12a) is maximal commutative. It is called the *Cartan algebra* of so$(2m)$. (3.12b) are its *root operators*. They satisfy

$$[N_k,B_{ij}]\quad=\quad-(\mathrm{sgn}(i)\delta_{k,|i|}+\mathrm{sgn}(j)\delta_{k,|j|})B_{ij}.$$

The Laplacian and the Casimir operator are

$$\Delta_{2m} = \sum_{i=1}^{m} 2\partial_{y_{-i}}\partial_{y_i}, \tag{3.13}$$

$$\mathcal{C}_{2m} = \sum_{1\le|i|<|j|\le m} B_{ij}B_{-i-j} - \sum_{i=1}^{m} N_i^2. \tag{3.14}$$

Suppose now that $2m+1=n$. In this case, $(m,m+1)$ will be called the *split signature*. Every scalar product of such signature can be brought to the form

$$\langle y|y\rangle = y_0^2 + \sum_{i=1}^{m} 2y_{-i}y_i. \tag{3.15}$$

We will say that (3.15) is a scalar product in *split coordinates*.

so$(2m+1)$ has then a basis consisting of the above described basis of so$(2m)$ and

$$B_{0j} \;=\; y_0\partial_{y_j} - y_{-j}\partial_{y_0}, \quad |j| = 1,\dots,m. \tag{3.16}$$

The additional roots satisfy

$$[N_k, B_{0j}] = -\mathrm{sgn}(j)\delta_{k,|j|}B_{0j}. \tag{3.17}$$

The subalgebra spanned by (3.12a) is still maximal commutative in so$(2m+1)$. It is called a *Cartan algebra* of so$(2m+1)$.

We have

$$\Delta_{2m+1} = \partial_{y_0}^2 + \sum_{i=1}^{m} 2\partial_{y_{-i}}\partial_{y_i}, \tag{3.18}$$

$$\mathcal{C}_{2m+1} = \sum_{|i|=1}^{m} B_{0i}B_{0-i} + \sum_{1\le|i|<|j|\le m} B_{ij}B_{-i-j} - \sum_{i=1}^{m} N_i^2. \tag{3.19}$$

In the real case we will most often consider the split signature, both in even and odd dimensions. In both real and complex cases we will usually prefer split coordinates. We will often write (3.11) and (3.15) in the form

$$\langle y|y\rangle = \sum_{|i|\le m} y_{-i}y_i \tag{3.20}$$

where it is understood that $i \in \{-m,\dots,-1,1,\dots,m\}$ in the even case and $i \in \{-m,\dots,-1,0,1,\dots,m\}$ in the odd case.

3.9 Weyl group

In this subsection we introduce a certain finite subgroup of $\mathrm{O}(n)$, which will be called the *Weyl group*. We will also introduce a notation for elements of these groups. The reader is referred to Subsects 5.1 and 6.1, for examples of application of this notation. We will assume that the signature is split and split coordinates have been chosen.

Consider first dimension $2m$. Permutations of $\{-1,\dots,-m\}\cup\{1,\dots,m\}$ that preserve the pairs $\{-1,1\},\dots\{-m,m\}$ define elements of $\mathrm{O}(2m)$. They form a group, that we will call denote D_m. It is isomorphic to $\mathbb{Z}_2^m\rtimes S_m$. It is the Weyl group of $\mathrm{O}(2m)$.

The *flip* interchanging $-i,i$ will be denoted τ_i. The flips τ_i, with $i=1,\dots,m$, generate a subgroup of D_m isomorphic to $\mathbb{Z}_2^m$.

To every $\pi\in S_m$ there corresponds an element of D_m denoted σ_π, that permutes pairs $(-i,i)$. We have

$$\sigma_\pi f(y_{-1},y_1,\dots,y_{-m},y_m) \quad:=\quad f(y_{-\pi_1},y_{\pi_1},\dots,y_{-\pi_m},y_{\pi_m}). \tag{3.21}$$

Let $\epsilon=(\epsilon_1,\dots,\epsilon_m)$ and $\epsilon_1,\dots,\epsilon_m\in\{1,-1\}$. We will write $\epsilon\pi$ as the shorthand for $\epsilon_1\pi_1,\dots,\epsilon_m\pi_m$. We will use the notation

$$\sigma_{\epsilon\pi}:=\sigma_\pi\prod_{\epsilon_j=-1}\tau_j. \tag{3.22}$$

We have

$$\sigma_{\epsilon\pi}B_{ij}\sigma_{\epsilon\pi}^{-1}\ =B_{\epsilon_i\pi_i,\epsilon_j\pi_j};\quad \sigma_{\epsilon\pi}N_j\sigma_{\epsilon\pi}^{-1}\ =\epsilon_jN_{\pi_j}.$$

Using $\mathbb{R}^{2m+1}=\mathbb{R}\oplus\mathbb{R}^{2m}$, we embed D_m in $\mathrm{O}(2m+1)$. We also introduce $\tau_0\in\mathrm{O}(2m+1)$ given by

$$\tau_0 f(y_0,y_{-1},y_1,\dots,y_{-m},y_m):=f(-y_0,y_{-1},y_1,\dots,y_{-m},y_m). \tag{3.23}$$

Clearly, τ_0 commutes with D_m. The group B_m is defined as the group generated by D_m and τ_0. It is isomorphic to $\mathbb{Z}_2\times\mathbb{Z}_2^m\rtimes S_m$. It is the Weyl group of $\mathrm{O}(2m+1)$.

We set

$$\tau_{\epsilon\pi}:=\tau_0\sigma_{\epsilon\pi}.$$

We have

$$\tau_{\epsilon\pi}B_{0j}\tau_{\epsilon\pi}^{-1}=-B_{0,\epsilon_j\pi_j},\qquad \tau_{\epsilon\pi}B_{ij}\tau_{\epsilon\pi}^{-1}=B_{\epsilon_i\pi_i,\epsilon_j\pi_j},\qquad \tau_{\epsilon\pi}N_j\tau_{\epsilon\pi}^{-1}=\epsilon_jN_{\pi_j}.$$

3.10 Harmonic functions

Suppose that $\mathbb{R}^n$ is equipped with a scalar product. We say that a function F on $\mathbb{R}^n$ is *harmonic* if

$$\Delta_n F = 0. \tag{3.24}$$

Proposition 3.1. *Let $e_1, \dots e_k \in \mathbb{R}^n$ satisfy*

$$\langle e_i | e_j \rangle = 0, \quad 1 \le i, j \le k.$$

In other words, assume that $e_1, \dots, e_k$ span an isotropic subspace *of $\mathbb{R}^n$. Let f be a function of k variables. Then*

$$F(z) := f\big(\langle e_1 | z \rangle, \dots, \langle e_k | z \rangle\big)$$

is harmonic.

For instance, consider $\mathbb{R}^n$ with a split scalar product, where $n = 2m$ or $n = 2m+1$. Then any function $f(y_1, \dots, y_m)$ is harmonic, for instance

$$F_{\alpha_1, \dots \alpha_m} := y_1^{\alpha_1} \cdots y_m^{\alpha_m}, \tag{3.25}$$

which in addition satisfies

$$N_j F_{\alpha_1, \dots \alpha_m} = \alpha_j F_{\alpha_1, \dots \alpha_m}. \tag{3.26}$$

Harmonic functions satisfying in addition the eigenvalue equations (3.26) will play an important role in our approach. Unfortunately, functions of the form (3.25) constitute a rather narrow class. We need more general harmonic functions, which we will call *wave packets*. They are obtained by smearing a rotated (3.25) with an appropriate weight, so that it is an eigenfunction of Cartan operators. This construction will be explained in the Subsect. 3.11–3.13. It is essentially a version of the Fourier (or Mellin) transformation, possibly with a deformed complex contour of integration.

Note that the aim of Subsects 3.11 and 3.12 is to provide motivation, based on the concept of the Fourier transformation, for Subsect. 3.13, which contains the construction that will be used in what follows.

3.11 Eigenfunctions of angular momentum I

Suppose that $\mathbb{R}^n = \mathbb{R}^2 \oplus \mathbb{R}^{n-2}$, where we write $z = (x, y, z') \in \mathbb{R}^n$ and

$$\langle x, y, z' | x, y, z' \rangle = x^2 + y^2 + \langle z' | z' \rangle.$$

Set
$$N_1 := -\mathrm{i}(x\partial_y - y\partial_x).$$

Let $m \in \mathbb{Z}$. Consider a function $f(x, y, z')$. Then

$$F_m(x, y, z') := \frac{1}{2\pi}\int\limits_0^{2\pi} f(\cos\phi x - \sin\phi y, \sin\phi x + \cos\phi y, z')\mathrm{e}^{-\mathrm{i}m\phi}\mathrm{d}\phi, \tag{3.27}$$

$$\text{satisfies} \quad N_1 F_m(x, y, z') = mF_m(x, y, z'). \tag{3.28}$$

Note that if f is harmonic, then so is F_m. This construction is essentially the Fourier transformation.

Introduce complex coordinates

$$z_{\pm 1} := \frac{1}{\sqrt{2}}(x \pm \mathrm{i}y). \tag{3.29}$$

We will write $f(z_{-1}, z_1, z') = f(x, y, z')$, $F_m(z_{-1}, z_1, z') = F(x, y, z')$. The operator N_1 takes the familiar form

$$N_1 = -z_{-1}\partial_{z_{-1}} + z_1\partial_{z_1}, \tag{3.30}$$

and the metric becomes

$$\langle z_{-1}, z_1, z' | z_{-1}, z_1, z'\rangle = 2z_{-1}z_1 + \langle z'|z'\rangle. \tag{3.31}$$

Then (3.27) and (3.28) can be rewritten as

$$F_m(z_{-1}, z_1, z') := \frac{1}{2\pi\mathrm{i}}\int\limits_\gamma f(\tau^{-1}z_{-1}, \tau z_1, z')\tau^{-m-1}\mathrm{d}\tau, \tag{3.32}$$

$$N_1 F_m(z_{-1}, z_1, z') = mF_m(z_{-1}, z_1, z'), \tag{3.33}$$

where γ is the closed contour $[0, 2\pi[\ni \phi \mapsto \tau = \mathrm{e}^{\mathrm{i}\phi}$.

3.12 Eigenfunctions of angular momentum II

We again consider $\mathbb{R}^n = \mathbb{R}^2 \oplus \mathbb{R}^{n-2}$, but we change the signature of the metric. We assume that the scalar product is given by

$$\langle z_{-1}, z_1, z' | z_{-1}, z_1, z'\rangle = 2z_{-1}z_1 + \langle z'|z'\rangle. \tag{3.34}$$

We start from a function $f(z_{-1}, z_1, z')$. We would like to construct an eigenfunction of N_1 with a generic eigenvalue α, and not only with an integer eigenvalues as (3.32). To do this we repeat a similar procedure as in the previous subsection. Now, however, we need to integrate over a half-line, so we need conditions at the ends: we assume that

$$f(\tau^{-1}z_{-1},\tau z_1,z')\tau^{-\alpha}\Big|_{\tau=0}^{\tau=\infty}=0. \tag{3.35}$$

We set

$$F_\alpha:=\frac{1}{2\pi\mathrm{i}}\int\limits_0^\infty f(\tau^{-1}z_{-1},\tau z_1,z')\tau^{-\alpha-1}\mathrm{d}\tau. \tag{3.36}$$

Then, with N_1 given by (3.30),

$$N_1F_\alpha(z_{-1},z_1,z')=\alpha F_\alpha(z_{-1},z_1,z'). \tag{3.37}$$

Indeed,

$$\begin{aligned}
&\partial_\tau f(\tau^{-1}z_{-1},\tau z_1,z')\tau^{-\alpha}\\
=&-\alpha f(\tau^{-1}z_{-1},\tau z_1,z')\tau^{-\alpha-1}\\
&-\tau^{-2}z_{-1}\partial_1 f(\tau^{-1}z_{-1},\tau z_1,z')\tau^{-\alpha}+z_1\partial_2 f(\tau^{-1}z_{-1},\tau z_1,z')\tau^{-\alpha}\\
=&\big(-\alpha-z_{-1}\partial_{z_{-1}}+z_1\partial_{z_1}\big)f(\tau^{-1}z_{-1},\tau z_1,z')\tau^{-\alpha-1}.
\end{aligned}$$

Hence

$$0=\frac{1}{2\pi\mathrm{i}}\int\limits_0^\infty \mathrm{d}\tau\partial_\tau f(\tau^{-1}z_{-1},\tau z_1,z')\tau^{-\alpha}\;=\;(-\alpha+N_1)F_\alpha. \tag{3.38}$$

Note that F_α is the Mellin transform of $\tau\mapsto f(\tau^{-1}z_{-1},\tau z_1,z')$. If f is harmonic, then so is F_α.

3.13 Eigenfunctions of angular momentum III

Assume now that z_{-1},z_1,z' are complex variables and f is holomorphic. Then we can formulate a result that includes (3.28) and (3.37), allowing for a greater flexibility of the choice of the contour of integration:

Proposition 3.2. *Suppose that* $]0,1[\ni s\overset{\gamma}{\mapsto}\tau(s)$ *is a contour on the Riemann surface of*

$$\tau\mapsto f(\tau^{-1}z_{-1},\tau z_1,z')\tau^{-\alpha}$$

that satisfies

$$f(\tau^{-1}z_{-1},\tau z_1,z')\tau^{-\alpha}\Big|_{\tau(0)}^{\tau(1)}=0. \tag{3.39}$$

Then

$$F_\alpha := \frac{1}{2\pi\mathrm{i}} \int\limits_\gamma f(\tau^{-1}z_{-1}, \tau z_1, z')\tau^{-\alpha-1}\mathrm{d}\tau \tag{3.40}$$

solves

$$N_1 F_\alpha = \alpha F_\alpha.$$

Proof. We repeat the arguments of the previous subsection, where we replace $[0,\infty[$ with γ. □

3.14 Dimensional reduction

In this subsection we describe how to construct harmonic functions in $n-1$ dimensions out of a harmonic function in n dimensions.

Suppose that $\mathbb{R}^n$ is equipped with the scalar product

$$\langle z_{-1}, z_1, z' | z_{-1}, z_1, z'\rangle_n = 2z_{-1}z_1 + \langle z'|z'\rangle_{n-2}.$$

As usual, we write

$$N_1 = -z_{-1}\partial_{z_{-1}} + z_1\partial_{z_1}, \tag{3.41}$$

$$\Delta_n = 2\partial_{z_{-1}}\partial_{z_1} + \Delta_{n-2}. \tag{3.42}$$

Introduce new variables and the Laplacian in $n-1$ dimensions.

$$z_0 : = \sqrt{2z_{-1}z_1}, \quad u := \sqrt{\frac{z_1}{z_{-1}}}, \tag{3.43}$$

$$\Delta_{n-1} := \partial_{z_0}^2 + \Delta_{n-2}. \tag{3.44}$$

In the new variables,

$$N_1 = u\partial_u, \tag{3.45}$$

$$\Delta_n = \partial_{z_0}^2 + \frac{1}{z_0}\partial_{z_0} - \frac{1}{z_0^2}(u\partial_u)^2 + \Delta_{n-2}. \tag{3.46}$$

Consequently,

$$z_0^{\frac12}\Delta_n z_0^{-\frac12} = -\frac{1}{z_0^2}\Big(N_1 - \frac12\Big)\Big(N_1 + \frac12\Big) + \Delta_{n-1}. \tag{3.47}$$

Therefore, if we set

$$F_\pm(z_0, u, z') = u^{\pm\frac12} z_0^{-\frac12} f_\pm(z_0, z'), \tag{3.48}$$

then

$$N_1 F_\pm = \pm\frac{1}{2} F_\pm, \tag{3.49}$$

$$z_0^{\frac{1}{2}} u^{\mp\frac{1}{2}} \Delta_n F_\pm = \Delta_{n-1} f_\pm. \tag{3.50}$$

Hence, the $n-1$-dimensional Laplace equation $\Delta_{n-1} f = 0$ is essentially equivalent to the n-dimensional Laplace equation $\Delta_n F = 0$ restricted to the eigenspace of $N_1 = \pm\frac{1}{2}$.

4 Conformal invariance of the Laplacian

Conformal manifolds are manifolds equipped with a conformal stucture—a pseudo-Euclidean metric defined up to a positive multiplier. Conformal transformations are transformations that preserve the conformal structure.

The main objects of this section are *projective null quadrics*. They possess a natural conformal structure with an exceptionally large group of conformal transformations. In fact, on the $n+2$ dimensional pseudo-Euclidean *ambient space* we have the obvious action of the pseudo-orthogonal Lie algebra and group. This action is inherited by the $n+1$ dimensional *null quadric* $\mathcal{V}$, and then by its n-dimensional projectivization $\mathcal{Y}$. One can view $\mathcal{Y}$ as the base of the line bundle $\mathcal{V} \to \mathcal{Y}$. By choosing a section γ of this bundle we can equip $\mathcal{Y}$ with a pseudo-Riemannian structure. Choosing various sections defines metrics that differ only by a positive multiple—thus $\mathcal{Y}$ has a natural conformal structure. If the signature of the ambient space is $(q+1, p+1)$, then the signature of $\mathcal{Y}$ is (q, p).

We discuss a few examples of pseudo-Riemannian manifolds conformally equivalent to $\mathcal{Y}$ or to its open dense subset. The main example is the flat pseudo-Euclidean space. Another example is the product of two spheres $\mathbb{S}^q \times \mathbb{S}^p$, which is conformally equivalent to the entire $\mathcal{Y}$ of signature (q, p).

Especially simple and important are the low dimensional cases: in 1 dimension $\mathcal{Y} \simeq \mathbb{S}^1$ and in 2 dimensions $\mathcal{Y} \simeq \mathbb{S}^1 \times \mathbb{S}^1$. One should however remark that the dimensions 1 and 2 are somewhat special— in these dimensions the full conformal Lie algebra is infinite dimensional, and the above construction gives only its subalgebra.

Conformal transformations are generalized symmetries of the Laplacian. One can see this with help of a beautiful argument that goes back to Dirac. Its first step is the construction of a certain geometrically defined operator denoted $\Delta_{n+2}^\diamond$, that transforms functions on $\mathcal{V}$ homogeneous of degree $1-\frac{n}{2}$ into functions homogeneous of degree $-1-\frac{n}{2}$. After fixing a section γ of the line bundle $\mathcal{V} \to \mathcal{Y}$, we can identify the somewhat abstract operator $\Delta_{n+2}^{\diamond}$ with a concrete operator Δ_{n+2}^{γ} acting on fuctions on $\gamma(\mathcal{Y})$. This operator turns out to be the Yamabe Laplace-Beltrami operator for the corresponding pseudo-Riemannian structure.

On the $n+2$-dimensional ambient space the Laplacian Δ_{n+2} obviously commutes with the pseudo-orthogonal Lie algebra and group. On the level of $\gamma(\mathcal{Y})$ this commutation becomes a transmutation of Δ_{n+2}^{γ} with two different representations—one corresponding to the degree $1-\frac{n}{2}$, the other corresponding to the degree $-1-\frac{n}{2}$.

At the end of this section we consider in more detail the conformal action of the pseudo-orthogonal Lie algebra and group corresponding to the degree of homogeneity η on the flat pseudo-Euclidean space. In particular, we compute the representations for all elements of the pseudo-orthogonal Lie algebra. For the pseudo-orthogonal group, we compute the representations of Weyl symmetries.

4.1 Pseudo-Riemannian manifolds

We say that a manifold $\mathcal{Y}$ is *pseudo-Riemannian* if it is equipped with a nondegenerate symmetric covariant 2-tensor

$$\mathcal{Y} \ni y \mapsto g(y) = [g_{ij}(y)],$$

called the *metric tensor*. For any vector field Y it defines a function $g(Y,Y) \in C^\infty(\mathcal{Y})$:

$$\mathcal{Y} \ni y \mapsto g(Y,Y)(y) := g_{ij}(y)Y^i(y)Y^j(y).$$

Let α be a diffeomorphism of $\mathcal{Y}$. As is well known, the tensor g can be transported by α. More precisely, $\alpha^*(g)$ is defined by

$$\alpha^*(g)(Y,Y) := g\big(\alpha(Y), \alpha(Y)\big),$$

where Y is an arbitrary vector field. We say that α is *isometric* if $\alpha^* g = g$.

Let X be a vector field. The Lie derivative in the direction of X can be applied to the tensor g. More precisely, $\mathcal{L}_X g$ is defined by

$$(\mathcal{L}_X g)(Y,Y) := g\big([X,Y],Y\big) + g\big(Y,[X,Y]\big).$$

We say that a vector field X is *Killing* if $\mathcal{L}_X g = 0$.

4.2 Conformal manifolds

We say that the metric tensor g_1 is *conformally equivalent to* g if there exists a positive function $m \in C^\infty(\mathcal{Y})$ such that

$$m(y)g(y) = g_1(y).$$

Clearly, the conformal equivalence is an equivalence relation in the set of metric tensors. We say that a manifold $\mathcal{Y}$ is equipped with a *conformal structure*, if it is equipped with an equivalence class of conformally equivalent metric tensors.

We say that a diffeomorphism α is *conformal* if for some metric tensor g in the conformal class of $\mathcal{Y}$, $\alpha^* g$ is conformally equivalent to g. Clearly, this is equivalent to saying that for all g in the conformal class of $\mathcal{Y}$, $\alpha^* g$ is conformally equivalent to g.

We say that a vector field X is *conformal Killing* if for any metric tensors from the conformal class of $\mathcal{Y}$ there exists a smooth function $M \in C^\infty(\mathcal{Y})$ such that

$$\mathcal{L}_X g = Mg. \tag{4.1}$$

Clearly, if (4.1) is true for one metric tensor g from the conformal class of $\mathcal{Y}$, it is true for all metric tensors conformally equivalent to g.

4.3 Projective null quadric

Consider a pseudo-Euclidean vector space $(\mathbb{R}^{n+2}, g)$ of signature $(q+1, p+1)$, which we will call the *ambient space*. Recall that

$$\mathcal{V}^{n+1} := \{z \in \mathbb{R}^{n+2} \ : \ \langle z|z\rangle = 0, \quad z \neq 0\}.$$

is the *null quadric*. For simplicity, we will often write $\mathcal{V}$ for $\mathcal{V}^{n+1}$.

The scaling, that is the action of $\mathbb{R}^\times$, preserves $\mathcal{V}$. Let $\mathcal{Y} := \mathcal{V}/\mathbb{R}^\times$ be the *projective null quadric*. We obtain a *line bundle* $\mathcal{V} \to \mathcal{Y}$ with the base $\mathcal{Y}$ and the fiber $\mathbb{R}^\times$.

Let $\mathcal{Y}_i$ be an open subset of $\mathcal{Y}$ and $\mathcal{V}_i$ be the corresponding open subset of $\mathcal{V}$. Let

$$\mathcal{Y}_i \ni y \mapsto \gamma_i(y) \in \mathcal{V}_i$$

be a *section* of the bundle $\mathcal{V}_i \to \mathcal{Y}_i$, that is a smooth map satisfying $y = \mathbb{R}^\times \gamma_i(y)$. Let g_{γ_i} be the metric tensor g restricted to $\gamma_i(\mathcal{Y}_i)$ transported to $\mathcal{Y}_i$.

It is easy to prove the following fact:

Proposition 4.1. *Let γ_i, $i = 1, 2$, be sections of $\mathcal{V}_i \to \mathcal{Y}_i$. Then g_{γ_i} are metrics on $\mathcal{Y}_i$ of signature (q, p). The metrics g_{γ_1} and g_{γ_2} restricted to $\mathcal{Y}_1 \cap \mathcal{Y}_2$ are conformally equivalent.*

Prop. 4.1 equips $\mathcal{Y}$ with a conformal structure.

Choosing a section in the bundle $\mathcal{V} \to \mathcal{Y}$ endows $\mathcal{Y}$ with the structure of a pseudo-Riemannian manifold. For some special sections we obtain in particular various *symmetric spaces* together with an explicit description of their conformal structure. In following subsections we present a few examples of this construction.

Instead of $\mathcal{Y}$ one can consider $\tilde{\mathcal{Y}} := \mathcal{V}/\mathbb{R}_+$. We obtain a bundle $\mathcal{V} \to \tilde{\mathcal{Y}}$ with fibre $\mathbb{R}_+$, which has similar properties as the bundle $\mathcal{V} \to \mathcal{Y}$. It is a double covering of $\mathcal{Y}$, which means that we have a canonical $2-1$ surjection $\tilde{\mathcal{Y}} \to \mathcal{Y}$.

Let γ be a section of $\mathcal{V} \to \mathcal{Y}$. Every $y \in \mathcal{Y}$ equals $\mathbb{R}^\times \gamma(y)$, and hence it is the disjoint union of $\tilde{y}_+ := \mathbb{R}_+\gamma(y)$ and $\tilde{y}_- := \mathbb{R}_-\gamma(y)$. Clearly $\{\tilde{y}_+, \tilde{y}_-\} \subset \tilde{\mathcal{Y}}$ is the preimage of y under the canonical covering. Let us set

$$\tilde{\gamma}(\tilde{y}_+) := \gamma(y), \quad \tilde{\gamma}(\tilde{y}_-) := -\gamma(y). \tag{4.2}$$

Then $\tilde{\gamma}$ is a section of the bundle $\mathcal{V} \to \tilde{\mathcal{Y}}$. With help of $\tilde{\gamma}$ we can equip $\tilde{\mathcal{Y}}$ with a metric $\tilde{g}_{\tilde{\gamma}}$. Obviously, if $\mathcal{Y}$ is equipped with the metric g_γ, the canonical surjection $\tilde{\mathcal{Y}} \to \mathcal{Y}$ is isometric.

We would like to treat $\mathcal{Y}$ as the principal object, since it has a direct generalization to the complex case. However, for some purposes $\tilde{\mathcal{Y}}$ is preferable.

4.4 Projective null quadric as a compactification of a pseudo-Euclidean space

Consider a pseudo-Euclidean space $(\mathbb{R}^n, g_n)$ of signature (q, p) embedded in the pseudo-Euclidean space $(\mathbb{R}^{n+2}, g_{n+2})$ of signature $(q+1, p+1)$. We assume that the square of a vector $(z', z_-, z_+) \in \mathbb{R}^{n+2} = \mathbb{R}^n \oplus \mathbb{R}^2$ is

$$\langle z', z_-, z_+ | z', z_-, z_+\rangle_{n+2} := \langle z' | z'\rangle_n + 2z_+z_-.$$

Set

$$\mathcal{V}_0 := \{(z', z_-, z_+) \in \mathcal{V} \ : \ z_- \neq 0\}, \quad \mathcal{Y}_0 := \mathcal{V}_0/\mathbb{R}^\times.$$

$\mathcal{Y}_0$ is dense and open in $\mathcal{Y}$.

We have a bijection and a section

$$\mathcal{Y}_0 \ni \mathbb{R}^\times \begin{bmatrix} y \\ 1 \\ -\frac{\langle y|y\rangle_n}{2} \end{bmatrix} \leftrightarrow \underset{\mathbb{R}^n}{\overset{y}{\Cap}} \mapsto \begin{bmatrix} y \\ 1 \\ -\frac{\langle y|y\rangle_n}{2} \end{bmatrix} \in \mathcal{V}_0. \tag{4.3}$$

Thus $\mathbb{R}^n$ is identified with $\mathcal{Y}_0$. The metric on $\mathcal{Y}_0$ given by the above section coincides with the original metric on $\mathbb{R}^n$. We have thus embedded $\mathbb{R}^n$ with its conformal structure as a dense open subset of $\mathcal{Y}$.

4.5 Projective null quadric as a sphere/compactification of a hyperboloid

Consider a Euclidean space $(\mathbb{R}^{n+1}, g_{n+1})$ embedded in a pseudo-Euclidean space $(\mathbb{R}^{n+2}, g_{n+2})$ of signature $(1, n+1)$. We assume that the square of a vector $(z', z_0) \in \mathbb{R}^{n+1} \oplus \mathbb{R} = \mathbb{R}^{n+2}$ is

$$\langle z', z_0 | z', z_0 \rangle_{n+2} = \langle z' | z' \rangle_{n+1} - z_0^2.$$

Recall that

$$\mathbb{S}^n := \{\omega \in \mathbb{R}^{n+1} \ : \ \langle \omega | \omega \rangle = 1\}$$

is the unit sphere of dimension n.

We have a bijection and a section

$$\mathcal{Y} \ni \mathbb{R}^\times \begin{bmatrix} \omega \\ 1 \end{bmatrix} \leftrightarrow \underset{\mathbb{S}^n}{\overset{y}{\pitchfork}} \mapsto \begin{bmatrix} \omega \\ 1 \end{bmatrix} \in \mathcal{V}. \tag{4.4}$$

Thus $\mathbb{S}^n$ is identified with $\mathcal{Y}$. The metric on $\mathcal{Y}$ given by the above section coincides with the usual metric on $\mathbb{S}^n$.

$\tilde{\mathcal{Y}}$ is in this case simply the disjoint sum of two copies of $\mathbb{S}^n$.

The above construction can be repeated with minor changes for a general signature. Indeed, let the signature of $(\mathbb{R}^{n+1}, g_{n+1})$ be $(q, p+1)$, so that the signature of $(\mathbb{R}^{n+2}, g_{n+2})$ is $(q+1, p+1)$. Set

$$\mathcal{V}_0 := \{(z', z_0) \in \mathcal{V} \ : \ z_0 \neq 0\}, \quad \mathcal{Y}_0 := \mathcal{V}_0 / \mathbb{R}^\times.$$

We have then the bijection and section

$$\mathcal{Y}_0 \ni \mathbb{R}^\times \begin{bmatrix} \omega \\ 1 \end{bmatrix} \leftrightarrow \underset{\mathbb{S}^{q,p}}{\overset{\omega}{\pitchfork}} \mapsto \begin{bmatrix} \omega \\ 1 \end{bmatrix} \in \mathcal{V}_0. \tag{4.5}$$

Note that now instead of the unit Euclidean sphere we have the unit hyperboloid of signature (q, p), which has been identified with $\mathcal{Y}_0$, a dense open subset of $\mathcal{Y}$.

4.6 Projective null quadric as the Cartesian product of spheres

Consider now the space $\mathbb{R}^{n+2}$ of signature $(q+1,p+1)$. The square of a vector $(\vec{t},\vec{x}) = (t_0,\dots,t_q,x_0,\dots,x_p)$ is defined as

$$\langle\vec{t},\vec{x}|\vec{t},\vec{x}\rangle := -t_0^2 - \cdots - t_q^2 + x_0^2 + \cdots + x_p^2. \tag{4.6}$$

Note that $\mathbb{S}^q \times \mathbb{S}^p$ is contained in $\mathcal{V}$. It is easy to see that the map

$$\mathcal{Y} \ni \mathbb{R}^\times(\vec{\rho},\vec{\omega}) \hookleftarrow (\vec{\rho},\vec{\omega}) \in \mathbb{S}^q \times \mathbb{S}^p \subset \mathcal{V}. \tag{4.7}$$

is a double covering. Indeed, we easily see that the map is onto and

$$\mathbb{R}^\times(\vec{\rho},\vec{\omega}) = \mathbb{R}^\times(-\vec{\rho},-\vec{\omega}).$$

Thus

$$\mathcal{Y} \simeq \mathbb{S}^q \times \mathbb{S}^p/\mathbb{Z}_2, \qquad \tilde{\mathcal{Y}} \simeq \mathbb{S}^q \times \mathbb{S}^p.$$

The map (4.7) can be interpreted as a section of $\mathcal{V} \to \tilde{\mathcal{Y}}$. The corresponding metric tensor on $\mathcal{Y}$ is minus the standard metric tensor on $\mathbb{S}^q$ plus the standard metric tensor on $\mathbb{S}^p$. Its signature is (q,p).

Again, similarly as in the previous subsection, the above construction can be generalized. Indeed, replace (4.6) with

$$\begin{aligned}\langle\vec{t},\vec{x}|\vec{t},\vec{x}\rangle := &- t_0^2 - \cdots - t_{q_1}^2 + t_{q_1+1}^2 + \cdots + t_{q_1+p_1}^2\\ &+ x_0^2 + \cdots + x_{p_1}^2 - x_{p_1+1}^2 - \cdots x_{p_2+q_2}^2.\end{aligned}$$

We then obtain a map

$$\mathcal{Y} \ni \mathbb{R}^\times(\vec{\rho},\vec{\omega}) \hookleftarrow (\vec{\rho},\vec{\omega}) \in \mathbb{S}^{p_1,q_1} \times \mathbb{S}^{q_2,p_2} \subset \mathcal{V}. \tag{4.8}$$

Unlike (4.7), the map (4.8) is in general not onto—it doubly covers only an open dense subset of $\mathcal{Y}$.

4.7 Dimension $n = 1$

Consider now the dimension $n = 1$ in more detail. The ambient space is $\mathbb{R}^3$ with the split scalar product

$$\langle z|z\rangle = z_0^2 + 2z_{-1}z_{+1}.$$

The 1-dimensional projective quadric is isomorphic to $\mathbb{S}^1$ or, what is the same, the 1-dimensional projective space:

$$\mathcal{Y}^1 \simeq \mathbb{S}^1 \simeq \mathbb{R}\cup\{\infty\} = P^1\mathbb{R}.$$

Indeed, it is easy to see that

$$\phi : \mathbb{R} \cup \{\infty\} \to \mathcal{Y}^1$$

defined by

$$\begin{aligned} \phi(s) &:= \Big(s, 1, -\frac{1}{2}s^2\Big)\mathbb{R}^\times, \quad s \in \mathbb{R}; \\ \phi(\infty) &:= (1,0,0)\mathbb{R}^\times \end{aligned}$$

is a homeomorphism.

The group $\mathrm{O}(1,2)$ acts on $P^1\mathbb{R}$ by homographies (Möbius transformations).

The Lie algebra $\mathrm{so}(1,2)$ is spanned by

$$B_{0,1},\ B_{0,-1},\ N_1,$$

with the commutation relations

$$\begin{aligned} [B_{0,1}, B_{0,-1}] &= N_1, \\ [B_{0,1}, N_1] &= B_{0,1}, \\ [B_{0,-1}, N_1] &= -B_{0,-1}. \end{aligned}$$

Appying (3.19) with $m = 1$ we obtain its Casimir operator:

$$\mathcal{C}_3 = 2B_{0,1}B_{0,-1} - N_1^2 - N_1 \tag{4.9a}$$
$$= 2B_{0,-1}B_{0,1} - N_1^2 + N_1. \tag{4.9b}$$

4.8 Dimension $n = 2$

Consider finally the dimension $n = 2$ in the signature $(1,1)$. The ambient space is $\mathbb{R}^4$ with the split scalar product

$$\langle z|z\rangle = 2z_{-1}z_{+1} + 2z_{-2}z_{+2}.$$

The 2-dimensional projective quadric is isomorphic to the product of two circles:

$$\mathcal{Y}^2 \simeq P^1\mathbb{R} \times P^1\mathbb{R}.$$

Indeed, define

$$\phi : \big(\mathbb{R} \cup \{\infty\}\big) \times \big(\mathbb{R} \cup \{\infty\}\big) \to \mathcal{Y}^2$$

by

$$\begin{aligned}
\phi(t,s) &:= (-ts,1,t,s)\mathbb{R}^\times, &(4.10a)\\
\phi(\infty,s) &:= (-s,0,1,0)\mathbb{R}^\times, &(4.10b)\\
\phi(t,\infty) &:= (-t,0,0,1)\mathbb{R}^\times, &(4.10c)\\
\phi(\infty,\infty) &:= (-1,0,0,0)\mathbb{R}^\times, &(4.10d)
\end{aligned}$$

where $t,s\in\mathbb{R}$. We easily check that ϕ is a homeomorphism. In fact, rewriting (4.10a) as

$$\begin{aligned}
\phi(t,s) &= \Big(-s,\frac{1}{t},1,\frac{s}{t}\Big)\mathbb{R}^\times\\
&= \Big(-t,\frac{1}{s},\frac{t}{s},1\Big)\mathbb{R}^\times\\
&= \Big(-1,\frac{1}{ts},\frac{1}{s},\frac{1}{t})\mathbb{R}^\times,
\end{aligned}$$

we see the continuity of ϕ at (4.10b), (4.10c), resp. (4.10d).

The Lie algebra $\mathrm{so}(2,2)$ is spanned by

$$N_1,\ N_2,\ B_{1,2},\ B_{1,-2},\ B_{-1,2},\ B_{-1,-2}.$$

Appying (3.19) with $m=2$ we obtain its Casimir operator:

$$\mathcal{C}_4 = 2B_{1,2}B_{-1,-2} + 2B_{1,-2}B_{-1,2} - N_1^2 - N_2^2 - 2N_1.$$

As is well known, $\mathrm{so}(2,2)$ decomposes into a direct sum of two copies of $\mathrm{so}(1,2)$. Concretely,

$$\mathrm{so}(2,2) = \mathrm{so}^+(1,2)\oplus\mathrm{so}^-(1,2),$$

where $\mathrm{so}^+(1,2)$, resp. $\mathrm{so}^-(1,2)$, both isomorphic to $\mathrm{so}(1,2)$, are spanned by

$$B_{1,2},\ B_{-1,-2},\ N_1+N_2;\quad \text{resp.}\quad B_{1,-2},\ B_{-1,2},\ N_1-N_2.$$

They have the commutation relations

$$\begin{aligned}
\Big[\frac{B_{1,2}}{\sqrt{2}},\frac{B_{-1,-2}}{\sqrt{2}}\Big] &= \frac{N_1+N_2}{2}, & \Big[\frac{B_{1,-2}}{\sqrt{2}},\frac{B_{-1,2}}{\sqrt{2}}\Big] &= \frac{N_1-N_2}{2},\\
\Big[\frac{N_1+N_2}{2},\frac{B_{-1,-2}}{\sqrt{2}}\Big] &= \frac{B_{-1,-2}}{\sqrt{2}}, & \Big[\frac{N_1-N_2}{2},\frac{B_{-1,2}}{\sqrt{2}}\Big] &= \frac{B_{-1,2}}{\sqrt{2}},\\
\Big[\frac{N_1+N_2}{2},\frac{B_{1,2}}{\sqrt{2}}] &= -\frac{B_{1,2}}{\sqrt{2}}; & \Big[\frac{N_1-N_2}{2},\frac{B_{1,-2}}{\sqrt{2}}\Big] &= -2\frac{B_{1,-2}}{\sqrt{2}}.
\end{aligned}$$

The corresponding Casimir operators are

$$\begin{aligned}
\mathcal{C}_3^+ &= B_{1,2}B_{-1,-2} - \frac{1}{4}(N_1+N_2)^2 - \frac{1}{2}N_1 - \frac{1}{2}N_2 \\
&= B_{-1,-2}B_{1,2} - \frac{1}{4}(N_1+N_2)^2 + \frac{1}{2}N_1 + \frac{1}{2}N_2, \\
\mathcal{C}_3^- &= B_{1,-2}B_{-1,2} - \frac{1}{4}(N_1-N_2)^2 - \frac{1}{2}N_1 + \frac{1}{2}N_2 \\
&= B_{-1,2}B_{1,-2} - \frac{1}{4}(N_1-N_2)^2 + \frac{1}{2}N_1 - \frac{1}{2}N_2.
\end{aligned}$$

Thus

$$\mathcal{C}_4 = 2\mathcal{C}_3^+ + 2\mathcal{C}_3^-.$$

In the enveloping algebra of so$(2,2)$ the operators $\mathcal{C}_3^+$ and $\mathcal{C}_3^-$ are distinct. They satisfy $\alpha(\mathcal{C}_-) = \mathcal{C}_+$ for $\alpha \in \mathrm{O}(2,2)\backslash\mathrm{SO}(2,2)$, for instance for $\alpha = \tau_i$, $i = 1,2$.

However, inside the associative algebra of differential operators on $\mathbb{R}^4$ we have the identity

$$B_{1,2}B_{-1,-2} - B_{-1,2}B_{1,-2} = N_1N_2 + N_1,$$

which implies

$$\mathcal{C}_3^+ = \mathcal{C}_3^-$$

inside this algebra. Therefore, represented in the algebra of differential operators we have

$$\begin{aligned}
\mathcal{C}_4 &= 4B_{1,2}B_{-1,-2} - (N_1+N_2)^2 - 2N_1 - 2N_2 && (4.11a) \\
&= 4B_{-1,-2}B_{1,2} - (N_1+N_2)^2 + 2N_1 + 2N_2 && (4.11b) \\
&= 4B_{1,-2}B_{-1,2} - (N_1-N_2)^2 - 2N_1 + 2N_2 && (4.11c) \\
&= 4B_{-1,2}B_{1,-2} - (N_1-N_2)^2 + 2N_1 - 2N_2. && (4.11d)
\end{aligned}$$

4.9 Conformal invariance of the projective null quadric

Obviously, $\mathrm{O}(n+2)$ and so$(n+2)$ preserve $\mathcal{V}$. They commute with the scaling (the action of $\mathbb{R}^\times$). Therefore, we obtain the action on $\mathcal{Y} = \mathcal{V}/\mathbb{R}^\times$, which we denote as follows:

$$\begin{aligned}
\mathrm{so}(n+2) \ni B &\mapsto B^\diamond, && (4.12a) \\
\mathrm{O}(n+2) \ni \alpha &\mapsto \alpha^\diamond. && (4.12b)
\end{aligned}$$

Clearly, the vector fields $B^\diamond$ are conformal Killing and the diffeomorphisms $\alpha^\diamond$ are conformal.

Let $\eta \in \mathbb{C}$. We define $\Lambda_+^\eta(\mathcal{V})$ to be the set of smooth functions on $\mathcal{V}$ (positively) homogeneous of degree η, that is, satisfying

$$f(ty) = t^\eta f(y), \quad t > 0, \quad y \in \mathcal{V}.$$

Clearly, $B \in \mathrm{so}(n+2)$ and $\alpha \in \mathrm{O}(n+2)$ preserve $\Lambda_+^\eta(\mathcal{V})$. We will denote by $B^{\diamond,\eta}$, resp. $\alpha^{\diamond,\eta}$ the restriction of B, resp. α to $\Lambda_+^\eta(\mathcal{V})$. Thus we have representations

$$\mathrm{so}(n+2) \ni B \quad \mapsto \quad B^{\diamond,\eta}, \tag{4.13a}$$

$$\mathrm{O}(n+2) \ni \alpha \quad \mapsto \quad \alpha^{\diamond,\eta}, \tag{4.13b}$$

acting on $\Lambda_+^\eta(\mathcal{Y})$.

Clearly, $\Lambda_+^0(\mathcal{V})$ can be identified with $C^\infty(\tilde{\mathcal{Y}})$. Moreover, (4.12a), resp. (4.12b) coincide with (4.13a), resp. (4.13b) for $\eta = 0$.

If $\eta \in \mathbb{Z}$ one can use another concept of homogeneity. We define $\Lambda^\eta(\mathcal{V})$ to be the set of smooth functions on $\mathcal{V}$ satisfying

$$f(ty) = t^\eta f(y), \quad t \neq 0, \quad y \in \mathcal{V}.$$

The properties of $\Lambda^\eta(\mathcal{V})$ are similar to $\Lambda_+^\eta(\mathcal{V})$, except that $\Lambda^0(\mathcal{V})$ can be identified with $C^\infty(\mathcal{Y})$.

4.10 Laplacian on homogeneous functions

The following theorem according to Eastwood [East] goes back to Dirac [Dir]. We find it curious because it allows in some situations to restrict a *second order* differential operator to a submanifold.

Theorem 4.2. *Let $\Omega \subset \mathbb{R}^{n+2}$ be an open conical set. Let $K \in C^\infty(\Omega)$ be homogeneous of degree $1 - \frac{n}{2}$ such that*

$$K\Big|_{\mathcal{V}\cap\Omega} = 0.$$

Then

$$\Delta_{n+2} K\Big|_{\mathcal{V}\cap\Omega} = 0.$$

Before we give two proofs of this theorem, let us describe some of its consequences.

Let $k \in \Lambda_{+}^{1-\frac{n}{2}}(\mathcal{V})$. We can always find Ω, a conical neighborhood of $\mathcal{V}$, and $K \in \mathcal{A}(\Omega)$ homogeneous of degree $1 - \frac{n}{2}$ such that

$$k = K\Big|_{\mathcal{V}}.$$

Note that $\Delta_{n+2}K$ is homogeneous of degree $-1 - \frac{n}{2}$. We set

$$\Delta_{n+2}^{\diamond} k := \Delta_{n+2}K\Big|_{\mathcal{V}}. \tag{4.14}$$

By Theorem 4.2, the above definition (4.14) does not depend on the choice of Ω and K. We have thus defined a map

$$\Delta_{n+2}^{\diamond} : \Lambda_{+}^{1-\frac{n}{2}}(\mathcal{V}) \to \Lambda_{+}^{-1-\frac{n}{2}}(\mathcal{V}). \tag{4.15}$$

Obviously,

$$B\Delta_{n+2} = \Delta_{n+2}B, \quad B \in \mathrm{so}(n+2), \tag{4.16a}$$

$$\alpha\Delta_{n+2} = \Delta_{n+2}\alpha, \quad \alpha \in \mathrm{O}(n+2). \tag{4.16b}$$

Restricting (4.16) to $\Lambda_{+}^{1-\frac{n}{2}}(\mathcal{V})$ we obtain

$$B^{\diamond,-1-\frac{n}{2}}\Delta_{n+2}^{\diamond} = \Delta_{n+2}^{\diamond}B^{\diamond,1-\frac{n}{2}}, \quad B \in \mathrm{so}(n+2), \tag{4.17a}$$

$$\alpha^{\diamond,-1-\frac{n}{2}}\Delta_{n+2}^{\diamond} = \Delta_{n+2}^{\diamond}\alpha^{\diamond,1-\frac{n}{2}}, \quad \alpha \in \mathrm{O}(n+2). \tag{4.17b}$$

1st proof of Thm 4.2. We use the decomposition $\mathbb{R}^{n+2} = \mathbb{R}^n \oplus \mathbb{R}^2$ described in Subsect. 4.4, with the distinguished coordinates denoted z_-, z_+. We denote the square of a vector, the Laplacian, the Casimir, resp. the generator of dilations on $\mathbb{R}^{n+2}$ by R_{n+2}, Δ_{n+2}, $\mathcal{C}_{n+2}$, resp. A_{n+2}. Similarly, we denote the square of a vector, the Laplacian, the Casimir, resp. the generator of dilations on $\mathbb{R}^n$ by R_n, Δ_n, $\mathcal{C}_n$ resp. A_n. We will also write

$$N_{m+1} := z_+\partial_{z_+} - z_-\partial_{z_-}.$$

We have

$$\begin{aligned}
R_{n+2} &= R_n + 2z_+z_-, \\
\Delta_{n+2} &= \Delta_n + 2\partial_{z_+}\partial_{z_-}, \\
A_{n+2} &= A_n + z_+\partial_{z_+} + z_-\partial_{z_-}.
\end{aligned}$$

The following identity is a consequence of (3.6):

$$\begin{aligned} R_n\Delta_{n+2} &= R_n\Delta_n + \big(R_{n+2}-2z_+z_-\big)2\partial_{z_+}\partial_{z_-} \\ &= \mathcal{C}_n + \Big(A_n - 1 + \frac{n}{2}\Big)^2 - \Big(\frac{n}{2}-1\Big)^2 \\ &\quad + R_{n+2}2\partial_{z_+}\partial_{z_-} - (z_+\partial_{z_+} + z_-\partial_{z_-})^2 + N_{m+1}^2 \\ &= R_{n+2}2\partial_{z_+}\partial_{z_-} \\ &\quad + \Big(A_n - 1 + \frac{n}{2} - z_+\partial_{z_+} - z_-\partial_{z_-}\Big)\Big(A_{n+2} - 1 + \frac{n}{2}\Big) \\ &\quad - \Big(\frac{n}{2}-1\Big)^2 + \mathcal{C}_n + N_{m+1}^2. \end{aligned} \tag{4.18}$$

$\left(\frac{n}{2}-1\right)^2$ is a scalar. $\mathcal{C}_n$ and N_{m+1}^2 are polynomials in elements of $\mathrm{so}(n+2)$, which are tangent to $\mathcal{V}$. Therefore, all operators in the last line of (4.18) can be restricted to $\mathcal{V}$. The operator $A_{n+2} - 1 + \frac{n}{2}$ vanishes on functions in $\Lambda_+^{1-\frac{n}{2}}(\Omega)$. The operator $R_{n+2}2\partial_{z_+}\partial_{z_-}$ is zero when restricted to $\mathcal{V}$ (because R_{n+2} vanishes on $\mathcal{V}$).

Therefore, if K is homogeneous of degree $1-\frac{n}{2}$ vanishing on $\mathcal{V}$, then $R_n\Delta_{n+2}K$ vanishes on $\mathcal{V}$. We are free to choose different coordinates which give different R_n's. Therefore we can conclude that $\Delta_{n+2}K$ vanishes on $\mathcal{V}$. □

Corollary 4.3. *Using the operator $\Delta_{n+2}^\diamond$, we can write*

$$R_n\Delta_{n+2}^\diamond = -\Big(\frac{n}{2}-1\Big)^2 + \mathcal{C}_n^{\diamond,1-\frac{n}{2}} + \big(N_{m+1}^{\diamond,1-\frac{n}{2}}\big)^2. \tag{4.19}$$

2nd proof of Thm 4.2. We use the decomposition $\mathbb{R}^{n+2} = \mathbb{R}^{n+1}\oplus\mathbb{R}$ with the distinguished variable denoted by z_0, as in Subsect. 4.5. We denote the square of a vector, the Laplacian, the Casimir, resp. the generator of dilations on $\mathbb{R}^{n+1}$ by R_{n+1}, Δ_{n+1}, $\mathcal{C}_{n+1}$, resp. A_{n+1}. We have

$$\begin{aligned} R_{n+2} &= R_{n+1} + z_0^2, \\ A_{n+2} &= A_{n+1} + z_0\partial_{z_0}, \\ \Delta_{n+2} &= \Delta_{n+1} + \partial_{z_0}^2. \end{aligned}$$

We have the following identity

$$\begin{aligned}
R_{n+1}\Delta_{n+2} &= R_{n+1}\Delta_{n+1} + \big(R_{n+2} - z_0^2\big)\partial_{z_0}^2 \\
&= \mathcal{C}_{n+1} + \Big(A_{n+1} + \frac{n-1}{2}\Big)^2 - \Big(\frac{n-1}{2}\Big)^2 \\
&\quad + R_{n+2}\partial_{z_0}^2 - \Big(z_0\partial_{z_0} - \frac{1}{2}\Big)^2 + \Big(\frac{1}{2}\Big)^2 \\
&= R_{n+2}\partial_{z_0}^2 + \Big(A_{n+1} + \frac{n}{2} - z_0\partial_{z_0}\Big)\Big(A_{n+2} + \frac{n}{2} - 1\Big) \\
&\quad - \Big(\frac{n}{2} - 1\Big)\frac{n}{2} + \mathcal{C}_{n+1}.
\end{aligned} \tag{4.20}$$

Then we argue similarly as in the 1st proof. □

Corollary 4.4. *Using the operator $\Delta_{n+2}^{\diamond}$, we can write*

$$R_{n+1}\Delta_{n+2}^{\diamond} = -\Big(\frac{n}{2} - 1\Big)\frac{n}{2} + \mathcal{C}_{n+1}^{\diamond, 1-\frac{n}{2}}. \tag{4.21}$$

4.11 Fixing a section

For nonzero η, in order to identify functions from $\Lambda_+^{\eta}(\mathcal{V})$ with functions on $\tilde{\mathcal{Y}}$ we need to fix a section of the line bundle $\mathcal{V} \to \tilde{\mathcal{Y}}$. Let us describe this in detail.

Let $\mathcal{V}_0$ be an open homogeneous subset of $\mathcal{V}$ and $\tilde{\mathcal{Y}}_0 := \mathcal{V}_0/\mathbb{R}_+$. Consider a section $\gamma : \tilde{\mathcal{Y}}_0 \to \mathcal{V}_0$. We then have the obvious identification $\psi^{\gamma,\eta} : \Lambda_+^{\eta}(\mathcal{V}_0) \to C^{\infty}(\tilde{\mathcal{Y}}_0)$: for $k \in \Lambda_+^{\eta}(\mathcal{V}_0)$ we set

$$\big(\psi^{\gamma,\eta}k\big)(y) := k\big(\gamma(y)\big), \quad y \in \tilde{\mathcal{Y}}_0. \tag{4.22}$$

The map $\psi^{\gamma,\eta}$ is bijective and we can introduce its inverse, denoted $\phi^{\gamma,\eta}$, defined for any $f \in C^{\infty}(\tilde{\mathcal{Y}}_0)$ by

$$\big(\phi^{\gamma,\eta}f\big)\big(s\gamma(y)\big) = s^{\eta}f(y), \quad s \in \mathbb{R}_+, \quad y \in \tilde{\mathcal{Y}}_0. \tag{4.23}$$

Let $B \in \mathrm{so}(n+2)$ and $\alpha \in \mathrm{O}(n+2)$. As usual, B and α are interpreted as transformations acting on functions on $\mathbb{R}^{n+2}$. Both B and α preserve $\Lambda_+^{\eta}(\mathcal{V}_0)$. Therefore, we can define

$$B^{\gamma,\eta} := \psi^{\gamma,\eta}B\phi^{\gamma,\eta}, \tag{4.24a}$$

$$\alpha^{\gamma,\eta} := \psi^{\gamma,\eta}\alpha\phi^{\gamma,\eta}. \tag{4.24b}$$

$B^{\gamma,\eta}$ is a 1st order differential operator on $\tilde{\mathcal{Y}}_0$. $\alpha^{\gamma,\eta}$ maps $C^{\infty}\big(\tilde{\mathcal{Y}}_0 \cap (\alpha^{\diamond})^{-1}(\tilde{\mathcal{Y}}_0)\big)$ onto $C^{\infty}\big(\tilde{\mathcal{Y}}_0 \cap \alpha^{\diamond}(\tilde{\mathcal{Y}}_0)\big)$.

It is easy to see that for any $B \in \mathrm{so}(n+2)$ and $\alpha \in \mathrm{O}(n+2)$ there exist $M_B \in C^\infty(\tilde{\mathcal{Y}}_0)$ and $m_\alpha \in C^\infty\big(\tilde{\mathcal{Y}}_0 \cap \alpha^\diamond(\tilde{\mathcal{Y}}_0)\big)$ such that

$$B^{\diamond,\eta} f(y) = B^\diamond f(y) + \eta M_B(y) f(y), \tag{4.25a}$$

$$\alpha^{\diamond,\eta} f(y) = m_\alpha^\eta(y) \alpha^\diamond f(y). \tag{4.25b}$$

We define also

$$\Delta_{n+2}^\gamma := \psi^{\gamma,-1-\frac{n}{2}} \Delta_{n+2}^\diamond \phi^{\gamma,1-\frac{n}{2}}. \tag{4.26}$$

This is a second order differential operator on $\tilde{\mathcal{Y}}_0$. It satisfies

$$B^{\gamma,-1-\frac{n}{2}} \Delta_{n+2}^\gamma \quad = \quad \Delta_{n+2}^\gamma B^{\gamma,1-\frac{n}{2}}, \quad B \in \mathrm{so}(n+2), \tag{4.27a}$$

$$\alpha^{\gamma,-1-\frac{n}{2}} \Delta_{n+2}^\gamma \quad = \quad \Delta_{n+2}^\gamma \alpha^{\gamma,1-\frac{n}{2}}, \quad \alpha \in \mathrm{O}(n+2). \tag{4.27b}$$

Note that for even n the numbers $\pm 1 - \frac{n}{2}$ are integers. Therefore, $\Lambda^{\pm 1-\frac{n}{2}}(\mathcal{V})$ are well defined. In the above construction, we can then use $\mathcal{Y}$ instead of its double cover $\tilde{\mathcal{Y}}$. We also do not have problems in the complex case.

For odd n the numbers $\pm 1 - \frac{n}{2}$ are not integers, and so $\Lambda^{\pm 1-\frac{n}{2}}(\mathcal{V})$ are ill defined. Therefore, we have to use $\Lambda_+^{\pm 1-\frac{n}{2}}(\mathcal{V})$ and $\tilde{\mathcal{Y}}$.

4.12 Conformal invariance of the flat Laplacian

In this subsection we illustrate the somewhat abstract theory of the previous subsections with the example of the *flat section* described in (4.3). Recall that the flat section identifies an open subset of $\mathcal{Y}$ with $\mathbb{R}^n$. Therefore we obtain an action of $\mathrm{so}(n+2)$ and $\mathrm{O}(n+2)$ on $\mathbb{R}^n$. As a result we will obtain the invariance of the Laplacian on the flat pseudo-Euclidean space with respect to conformal transformations. The results of this subsection will be needed for our discussion of symmetries of the heat equation.

We will use the notation of (4.24a) and (4.24b), where instead of γ we write "fl", for the *flat section*. We will describe conformal symmetries on two levels:

(a) the ambient space $\mathbb{R}^{n+2}$

(b) the space $\mathbb{R}^n$.

We will use the split coordinates, that is, $z \in \mathbb{R}^{n+2}$ and $y \in \mathbb{R}^n$ have the square

$$\langle z|z\rangle \quad = \quad \sum_{|j|\le m+1} z_{-j} z_j, \tag{4.28a}$$

$$\langle y|y\rangle \quad = \quad \sum_{|j|\le m} y_{-j} y_j. \tag{4.28b}$$

As a rule, if a given operator does not depend on η, we omit the subscript η. Derivation of all the following identities will be sketched in Subsect. 4.13.

Cartan algebra of $\mathrm{so}(n+2)$

Cartan operators of $\mathrm{so}(n)$, $i = 1, \dots, m$:

$$\begin{aligned} N_i &= -z_{-i}\partial_{z_{-i}} + z_i\partial_{z_i}, &(4.29a)\\ N_i^{\mathrm{fl}} &= -y_{-i}\partial_{y_{-i}} + y_i\partial_{y_i}. &(4.29b)\end{aligned}$$

Generator of dilations:

$$\begin{aligned} N_{m+1} &= -z_{-m-1}\partial_{z_{-m-1}} + z_{m+1}\partial_{z_{m+1}}, &(4.30a)\\ N_{m+1}^{\mathrm{fl},\eta} &= \sum_{|i|\le m} y_i\partial_{y_i} - \eta \;=\; A_n - \eta. &(4.30b)\end{aligned}$$

Root operators

Roots of $\mathrm{so}(n)$, $|i| < |j| \le m$:

$$\begin{aligned} B_{i,j} &= z_{-i}\partial_{z_j} - z_{-j}\partial_{z_i}, &(4.31a)\\ B_{i,j}^{\mathrm{fl}} &= y_{-i}\partial_{y_j} - y_{-j}\partial_{y_i}. &(4.31b)\end{aligned}$$

Generators of translations, $|j| \le m$:

$$\begin{aligned} B_{m+1,j} &= z_{-m-1}\partial_{z_j} - z_{-j}\partial_{z_{m+1}}, &(4.32a)\\ B_{m+1,j}^{\mathrm{fl}} &= \partial_{y_j}. &(4.32b)\end{aligned}$$

Generators of special conformal transformations, $|j| \le m$:

$$\begin{aligned} B_{-m-1,j} &= z_{m+1}\partial_{z_j} - z_{-j}\partial_{z_{-m-1}}, &(4.33a)\\ B_{-m-1,j}^{\mathrm{fl},\eta} &= -\frac{1}{2}\langle y|y\rangle\partial_{y_j} + y_{-j}\sum_{|i|\le m} y_i\partial_{y_i} - \eta y_{-j}. &(4.33b)\end{aligned}$$

Weyl symmetries

We will write K for a function on $\mathbb{R}^{n+2}$ and f for a function on $\mathbb{R}^n$. We only give some typical elements that generate the whole Weyl group.

Reflection in the 0th coordinate (for odd n):

$$\begin{aligned} \tau_0 K(z_0, \dots) &= K(-z_0, \dots), &(4.34a)\\ \tau_0^{\mathrm{fl}} f(y_0, \dots) &= f(-y_0, \dots). &(4.34b)\end{aligned}$$

Flips, $j=1,\dots,m$:

$$\begin{aligned}\tau_j K(\dots,z_{-j},z_j,\dots,z_{-m-1},z_{m+1})&\\ =K(\dots,z_j,z_{-j},\dots,z_{-m-1},z_{m+1}),&\qquad (4.35a)\\ \tau_j^{\mathrm{fl}} f(\dots,y_{-j},y_j,\dots)=f(\dots y_j,y_{-j},\dots).&\qquad (4.35b)\end{aligned}$$

Inversion:

$$\tau_{m+1}K(\dots,z_{-m-1},z_{m+1})=K(\dots,z_{m+1},z_{-m-1}),\qquad (4.36a)$$

$$\tau_{m+1}^{\mathrm{fl},\eta}f(y)=\Big(-\frac{\langle y|y\rangle}{2}\Big)^{\eta}f\Big(-\frac{2y}{\langle y|y\rangle}\Big).\qquad (4.36b)$$

Permutations, $\pi\in S_m$:

$$\begin{aligned}\sigma_\pi K(\dots,z_{-j},z_j,\dots,z_{-m-1},z_{m+1})&\\ =K(\dots,z_{-\pi_j},z_{\pi_j},\dots,z_{-m-1},z_{m+1}),&\qquad (4.37a)\\ \sigma_\pi^{\mathrm{fl}} f(\dots,y_{-j},y_j,\dots)=\ f(\dots y_{-\pi_j},y_{\pi_j},\dots).&\qquad (4.37b)\end{aligned}$$

Special conformal transformations, $j=1,\dots,m$:

$$\begin{aligned}\sigma_{(j,m+1)}K(z_{-1},z_1,\dots,z_{-j},z_j,\dots,z_{-m-1},z_{m+1})&\\ =K(z_{-1},z_1,\dots,z_{-m-1},z_{m+1},\dots,z_{-j},z_j),&\qquad (4.38a)\\ \sigma_{(j,m+1)}^{\mathrm{fl},\eta}f(y_{-1},y_1,\dots,y_{-j},y_j,\dots)&\\ =\quad y_{-j}^{\eta}f\Big(\frac{y_{-1}}{y_{-j}},\frac{y_1}{y_{-j}},\dots,\frac{1}{y_{-j}},-\frac{\langle y|y\rangle}{2y_{-j}}\dots\Big).&\qquad (4.38b)\end{aligned}$$

Laplacian

$$\begin{aligned}\Delta_{n+2}&=\sum_{|i|\le m+1}\partial_{z_i}\partial_{z_{-i}}, &(4.39a)\\ \Delta_{n+2}^{\mathrm{fl}}&=\sum_{|i|\le m}\partial_{y_i}\partial_{y_{-i}}\ =\ \Delta_n. &(4.39b)\end{aligned}$$

We have the representations on functions on $\mathbb{R}^n$:

$$\begin{aligned}\mathrm{so}(n+2)\ni B&\mapsto B^{\mathrm{fl},\eta}, &(4.40a)\\ \mathrm{O}(n+2)\ni\alpha&\mapsto\alpha^{\mathrm{fl},\eta}. &(4.40b)\end{aligned}$$

They yield generalized symmetries:

$$\begin{aligned}B^{\mathrm{fl},\frac{-2-n}{2}}\Delta_n&=\Delta_nB^{\mathrm{fl},\frac{2-n}{2}},\quad B\in\mathrm{so}(n+2), &(4.41a)\\ \alpha^{\mathrm{fl},\frac{-2-n}{2}}\Delta_n&=\Delta_n\alpha^{\mathrm{fl},\frac{2-n}{2}},\quad \alpha\in\mathrm{O}(n+2). &(4.41b)\end{aligned}$$

4.13 Computations

Below we sketch explicit computations that lead to the formulas on from the previous subsection. Consider $\mathbb{R}^n \times \mathbb{R}^\times \times \mathbb{R}$ (defined by $z_{-m-1} \neq 0$), which is an open dense subset of $\mathbb{R}^{n+2}$. Clearly, $\mathcal{V}_0$ is contained in $\mathbb{R}^n \times \mathbb{R}^\times \times \mathbb{R}$.

We will write $\Lambda^\eta(\mathbb{R}^n \times \mathbb{R}^\times \times \mathbb{R})$ for the space of functions *homogeneous* of degree η on $\mathbb{R}^n \times \mathbb{R}^\times \times \mathbb{R}$.

Instead of using the maps $\phi^{\mathrm{fl},\eta}$ and $\psi^{\mathrm{fl},\eta}$, as in (4.23) and (4.22), we will prefer $\Phi^{\mathrm{fl},\eta} : C^\infty(\mathbb{R}^n) \to \Lambda^\eta(\mathbb{R}^n \times \mathbb{R}^\times \times \mathbb{R})$ and $\Psi^{\mathrm{fl},\eta} : \Lambda^\eta(\mathbb{R}^n \times \mathbb{R}^\times \times \mathbb{R}) \to C^\infty(\mathbb{R}^n)$ defined below.

For $K \in \Lambda^\eta\big(\mathbb{R}^n \times \mathbb{R}^\times \times \mathbb{R}\big)$, we define $\Psi^{\mathrm{fl},\eta}K \in C^\infty(\mathbb{R}^n)$ by

$$\big(\Psi^{\mathrm{fl},\eta}K\big)(y) = K\Big(y, 1, -\frac{\langle y|y\rangle}{2}\Big), \quad y \in \mathbb{R}^n.$$

Let $f \in C^\infty(\mathbb{R}^n)$. Then there exists a unique function in $\Lambda^\eta\big(\mathbb{R}^n \times \mathbb{R}^\times \times \mathbb{R}\big)$ that extends f and does not depend on z_{m+1}. It is given by

$$\big(\Phi^{\mathrm{fl},\eta}f\big)(z, z_{-m-1}, z_{m+1}) := z_{-m-1}^\eta f\Big(\frac{z}{z_{-m-1}}\Big), \quad z \in \mathbb{R}^n.$$

The map $\Psi^{\mathrm{fl},\eta}$ is a left inverse of $\Phi^{\mathrm{fl},\eta}$:

$$\Psi^{\mathrm{fl},\eta}\Phi^{\mathrm{fl},\eta} = \iota,$$

where ι denotes the identity. Clearly,

$$\begin{aligned} \Phi^{\mathrm{fl},\eta}f\Big|_{\mathcal{V}_0} &= \phi^{\mathrm{fl},\eta}f, \\ \Psi^{\mathrm{fl},\eta}K &= \psi^{\mathrm{fl},\eta}\Big(K\Big|_{\mathcal{V}_0}\Big). \end{aligned}$$

Moreover, functions in $\Lambda^\eta(\mathbb{R}^n \times \mathbb{R}^\times \times \mathbb{R})$ restricted to $\mathcal{V}_0$ are in $\Lambda^\eta(\mathcal{V}_0)$. Therefore,

$$\begin{aligned} B^{\mathrm{fl},\eta} &= \Psi^{\mathrm{fl},\eta}B\Phi^{\mathrm{fl},\eta}, \quad B \in \mathrm{so}(\mathbb{R}^{n+2}), \\ \alpha^{\mathrm{fl},\eta} &= \Psi^{\mathrm{fl},\eta}\alpha\Phi^{\mathrm{fl},\eta}, \quad \alpha \in \mathrm{O}(\mathbb{R}^{n+2}). \end{aligned}$$

(Note that α, B preserve $\Lambda^\eta(\mathbb{R}^n \times \mathbb{R}^\times \times \mathbb{R})$). Note also that

$$\Delta_{n+2}^{\mathrm{fl}} = \Psi^{\mathrm{fl},\eta}\Delta_{n+2}\Phi^{\mathrm{fl},\eta} = \Delta_n.$$

In practice, the above idea can be implemented by the following change of coordinates on $\mathbb{R}^{n+2}$:

$$\begin{aligned} y_i &:= \tfrac{z_i}{z_{-m-1}}, \ |i| \leq m, \\ R &:= \sum_{|i|\leq m+1} z_i z_{-i}, \\ p &:= z_{-m-1}. \end{aligned}$$

The inverse transformation is

$$\begin{aligned} z_i &= p y_i, \ |i| \leq m, \\ z_{m+1} &= \tfrac{1}{2}\big(\tfrac{R}{p} - p \sum_{|i|\leq m} y_i y_{-i}\big), \\ z_{-m-1} &= p \end{aligned}$$

The derivatives are equal to

$$\begin{aligned} \partial_{z_i} &= z_{-m-1}^{-1}\partial_{y_i} + 2z_{-i}\partial_R, \ |i| \leq m, \\ \partial_{z_{m+1}} &= 2z_{-m-1}\partial_R, \\ \partial_{z_{-m-1}} &= \partial_p - z_{-m-1}^{-2}\sum_{|i|\leq m} z_i\partial_{y_i} + 2z_{m+1}\partial_R. \end{aligned}$$

Note that these coordinates are defined on $\mathbb{R}^n \times \mathbb{R}^\times \times \mathbb{R}$. The set $\mathcal{V}_0$ is given by the condition $R = 0$. The flat section is given by $p = 1$.

For a function $y \mapsto f(y)$ we have

$$\big(\Phi^{\mathrm{fl},\eta} f\big)(y, R, p) = p^\eta f(y).$$

For a function $(y, R, p) \mapsto K(y, R, p)$ we have

$$\big(\Psi^{\mathrm{fl},\eta} K\big)(y) = K(y, 1, 0).$$

Note also that on $\Lambda^\eta(\mathbb{R}^n \times \mathbb{R}^\times \times \mathbb{R})$ we have

$$p\partial_p + 2R\partial_R = \eta.$$

5 Laplacian in 4 dimensions and the hypergeometric equation

The goal of this section is to derive the ${}_2\mathcal{F}_1$ equation together with its symmetries from the Laplacian in 4 dimensions, or actually from the Laplacian in 6 dimensions, if one takes into account the ambient space. Let us describe the main steps of this derivation:

(1) We start from the $4 + 2 = 6$ dimensional ambient space, with the obvious representations of so(6) and O(6), and the Laplacian Δ_6.

(2) As explained in Subsect. 4.9, we introduce the representations $\mathrm{so}(6) \ni B \mapsto B^{\diamond,\eta}$ and $\mathrm{O}(6) \ni \alpha \mapsto \alpha^{\diamond,\eta}$. Besides, as explained in Subsect. 4.10, we obtain the reduced Laplacian $\Delta_6^\diamond$. The most relevant values of η are $1 - \frac{4}{2} = -1$ and $-1 - \frac{4}{2} = -3$, which yield generalized symmetries of $\Delta_6^\diamond$.

(3) We fix a section γ of the null quadric. It allows us to construct the representations $B^{\gamma,\eta}$, $\alpha^{\gamma,\eta}$ and the operator Δ_6^γ, acting on a 4 dimensional manifold whose pseudo-Riemannian structure depends on γ.
(4) We choose coordinates w, u_1, u_2, u_3, so that the Cartan operators are expressed in terms of u_1, u_2, u_3. We compute Δ_6^γ, $B^{\gamma,\eta}$, and $\alpha^{\gamma,\eta}$ in the new coordinates.
(5) We make an ansatz that diagonalizes the Cartan operators, whose eigenvalues, denoted by α, β, μ, become parameters. Δ_6^γ, $B^{\gamma,\eta}$, and $\alpha^{\gamma,\eta}$ involve now only the single variable w. Δ_6^γ turns out to be the ${}_2\mathcal{F}_1$ hypergeometric operator. The generalized symmetries of Δ_6^γ yield transmutation relations and discrete symmetries of the ${}_2\mathcal{F}_1$ operator.

Step 1 is described in Subsect. 5.1.

We have a considerable freedom in the choice of the section γ of Step 3. For instance, it can be the flat section, which we described in Subsects 4.4 and 4.12. However, to simplify computations we prefer to choose a different section, which we call the spherical section. (Both approaches are described in [DeMaj]).

We perform Steps 2, 3 and 4 at once. They are described jointly in Subsect. 5.2. We choose coordinates w, r, p, u_1, u_2, u_3 in 6 dimensions, so that the null quadric, the spherical section and the homogeneity of functions are expressed in a simple way. In these coordinates, after the reductions of Steps 2 and 3, the variables r, p disappear. We are left with the variables w, u_1, u_2, u_3, and we are ready for Step 5.

Step 5 is described in Subsects 5.3 and 5.4.

Subsects 5.5 and 5.6 are devoted to factorizations of the ${}_2\mathcal{F}_1$ operator. Again, we see that the additional dimensions make all the formulas more symmetric. The role of factorizations is explained in Subsect. 1.5.

Subsects 5.4 and 5.6 contain long lists of identities for the hypergeometric operator. We think that it is easy to appreciate and understand them at a glance, without studying them line by line. Actually, the analogous lists of identities in the next sections, corresponding to other types of equations, are shorter but in a sense more complicated, because they correspond to “less symmetric” groups.

All the material so far has been devoted to the ${}_2\mathcal{F}_1$ operator and its multidimensional “parents”. Starting with Subsect. 5.7 we discuss the ${}_2F_1$ function and, more generally, distinguished solutions of the ${}_2\mathcal{F}_1$ equation. The symmetries of the ${}_2\mathcal{F}_1$ operator are helpful in deriving and organizing the identities concerning these solutions.

Subsects 5.10, 5.11, 5.12 are devoted to integral representations of solutions of the ${}_2\mathcal{F}_1$ equation. In particular, Subsect. 5.10 shows that these representation are disguised “wave packets” solving the Laplace equation and diagonalizing Cartan operators.

In Subsect. 5.13 we derive connection formulas, where we use the pairs of solutions with a simple behavior at 0 and at ∞ as two bases of solutions. The connection formulas follow easily from integral representations. These identities look symmetric when expressed in terms of the Lie-algebraic parameters.

5.1 $\mathrm{so}(6)$ in 6 dimensions

We consider $\mathbb{R}^6$ with the split coordinates

$$z_{-1}, z_1, z_{-2}, z_2, z_{-3}, z_3 \tag{5.1}$$

and the scalar product given by

$$\langle z|z\rangle = 2z_{-1}z_1 + 2z_{-2}z_2 + 2z_{-3}z_3. \tag{5.2}$$

The Lie algebra so(6) acts naturally on $\mathbb{R}^6$. Below we describe its natural basis. Then we consider its Weyl group, D_3, acting on functions on $\mathbb{R}^6$. For brevity, we list only elements from its subgroup $D_3 \cap \mathrm{SO}(6)$. Finally, we write down the Laplacian.

Lie algebra so(6)**.** Cartan algebra

$$N_1 = -z_{-1}\partial_{z_{-1}} + z_1\partial_{z_1}, \tag{5.3a}$$
$$N_2 = -z_{-2}\partial_{z_{-2}} + z_2\partial_{z_2}, \tag{5.3b}$$
$$N_3 = -z_{-3}\partial_{z_{-3}} + z_3\partial_{z_3}. \tag{5.3c}$$

Root operators

$$B_{-2,-1} = z_2\partial_{z_{-1}} - z_1\partial_{z_{-2}}, \tag{5.4a}$$
$$B_{2,1} = z_{-2}\partial_{z_1} - z_{-1}\partial_{z_2}, \tag{5.4b}$$
$$B_{2,-1} = z_{-2}\partial_{z_{-1}} - z_1\partial_{z_2}, \tag{5.4c}$$
$$B_{-2,1} = z_2\partial_{z_1} - z_{-1}\partial_{z_{-2}}; \tag{5.4d}$$

$$B_{-3,-2} = z_3\partial_{z_{-2}} - z_2\partial_{z_{-3}}, \tag{5.4e}$$
$$B_{3,2} = z_{-3}\partial_{z_2} - z_{-2}\partial_{z_3}, \tag{5.4f}$$
$$B_{3,-2} = z_{-3}\partial_{z_{-2}} - z_2\partial_{z_3}, \tag{5.4g}$$
$$B_{-3,2} = z_3\partial_{z_2} - z_{-2}\partial_{z_{-3}}; \tag{5.4h}$$

$$B_{-3,-1} = z_3\partial_{z_{-1}} - z_1\partial_{z_{-3}}, \tag{5.4i}$$
$$B_{3,1} = z_{-3}\partial_{z_1} - z_{-1}\partial_{z_3}, \tag{5.4j}$$
$$B_{3,-1} = z_{-3}\partial_{z_{-1}} - z_1\partial_{z_3}, \tag{5.4k}$$
$$B_{-3,1} = z_3\partial_{z_1} - z_{-1}\partial_{z_{-3}}. \tag{5.4l}$$

Weyl symmetries

$$
\begin{aligned}
\sigma_{123}K(z_{-1},z_1,z_{-2},z_2,z_{-3},z_3) &= K(z_{-1},z_1,z_{-2},z_2,z_{-3},z_3), &(5.5a)\\
\sigma_{-12-3}K(z_{-1},z_1,z_{-2},z_2,z_{-3},z_3) &= K(z_1,z_{-1},z_{-2},z_2,z_3,z_{-3}), &(5.5b)\\
\sigma_{1-2-3}K(z_{-1},z_1,z_{-2},z_2,z_{-3},z_3) &= K(z_{-1},z_1,z_2,z_{-2},z_3,z_{-3}), &(5.5c)\\
\sigma_{-1-23}K(z_{-1},z_1,z_{-2},z_2,z_{-3},z_3) &= K(z_1,z_{-1},z_2,z_{-2},z_{-3},z_3); &(5.5d)
\end{aligned}
$$

$$
\begin{aligned}
\sigma_{213}K(z_{-1},z_1,z_{-2},z_2,z_{-3},z_3) &= K(z_{-2},z_2,z_{-1},z_1,z_{-3},z_3), &(5.5e)\\
\sigma_{-21-3}K(z_{-1},z_1,z_{-2},z_2,z_{-3},z_3) &= K(z_2,z_{-2},z_{-1},z_1,z_3,z_{-3}), &(5.5f)\\
\sigma_{2-1-3}K(z_{-1},z_1,z_{-2},z_2,z_{-3},z_3) &= K(z_{-2},z_2,z_1,z_{-1},z_3,z_{-3}), &(5.5g)\\
\sigma_{-2-13}K(z_{-1},z_1,z_{-2},z_2,z_{-3},z_3) &= K(z_2,z_{-2},z_1,z_{-1},z_{-3},z_3); &(5.5h)
\end{aligned}
$$

$$
\begin{aligned}
\sigma_{321}K(z_{-1},z_1,z_{-2},z_2,z_{-3},z_3) &= K(z_{-3},z_3,z_{-2},z_2,z_{-1},z_1), &(5.5i)\\
\sigma_{-32-1}K(z_{-1},z_1,z_{-2},z_2,z_{-3},z_3) &= K(z_3,z_{-3},z_{-2},z_2,z_1,z_{-1}), &(5.5j)\\
\sigma_{3-2-1}K(z_{-1},z_1,z_{-2},z_2,z_{-3},z_3) &= K(z_{-3},z_3,z_2,z_{-2},z_1,z_{-1}), &(5.5k)\\
\sigma_{-3-21}K(z_{-1},z_1,z_{-2},z_2,z_{-3},z_3) &= K(z_3,z_{-3},z_2,z_{-2},z_{-1},z_1); &(5.5l)
\end{aligned}
$$

$$
\begin{aligned}
\sigma_{312}K(z_{-1},z_1,z_{-2},z_2,z_{-3},z_3) &= K(z_{-3},z_3,z_{-1},z_1,z_{-2},z_2), &(5.5m)\\
\sigma_{-31-2}K(z_{-1},z_1,z_{-2},z_2,z_{-3},z_3) &= K(z_3,z_{-3},z_{-1},z_1,z_2,z_{-2}), &(5.5n)\\
\sigma_{3-1-2}K(z_{-1},z_1,z_{-2},z_2,z_{-3},z_3) &= K(z_{-3},z_3,z_1,z_{-1},z_2,z_{-2}), &(5.5o)\\
\sigma_{-3-12}K(z_{-1},z_1,z_{-2},z_2,z_{-3},z_3) &= K(z_3,z_{-3},z_1,z_{-1},z_{-2},z_2); &(5.5p)
\end{aligned}
$$

$$
\begin{aligned}
\sigma_{231}K(z_{-1},z_1,z_{-2},z_2,z_{-3},z_3) &= K(z_{-2},z_2,z_{-3},z_3,z_{-1},z_1), &(5.5q)\\
\sigma_{-23-1}K(z_{-1},z_1,z_{-2},z_2,z_{-3},z_3) &= K(z_2,z_{-2},z_{-3},z_3,z_1,z_{-1}), &(5.5r)\\
\sigma_{2-3-1}K(z_{-1},z_1,z_{-2},z_2,z_{-3},z_3) &= K(z_{-2},z_2,z_3,z_{-3},z_1,z_{-1}), &(5.5s)\\
\sigma_{-2-31}K(z_{-1},z_1,z_{-2},z_2,z_{-3},z_3) &= K(z_2,z_{-2},z_3,z_{-3},z_{-1},z_1); &(5.5t)
\end{aligned}
$$

$$
\begin{aligned}
\sigma_{132}K(z_{-1},z_1,z_{-2},z_2,z_{-3},z_3) &= K(z_{-1},z_1,z_{-3},z_3,z_{-2},z_2), &(5.5u)\\
\sigma_{-13-2}K(z_{-1},z_1,z_{-2},z_2,z_{-3},z_3) &= K(z_1,z_{-1},z_{-3},z_3,z_2,z_{-2}), &(5.5v)\\
\sigma_{1-3-2}K(z_{-1},z_1,z_{-2},z_2,z_{-3},z_3) &= K(z_{-1},z_1,z_3,z_{-3},z_2,z_{-2}), &(5.5w)\\
\sigma_{-1-32}K(z_{-1},z_1,z_{-2},z_2,z_{-3},z_3) &= K(z_1,z_{-1},z_3,z_{-3},z_{-2},z_2). &(5.5x)
\end{aligned}
$$

Laplacian

$$\Delta_6 = 2\partial_{z_{-1}}\partial_{z_1} + 2\partial_{z_{-2}}\partial_{z_2} + 2\partial_{z_{-3}}\partial_{z_3}. \tag{5.6}$$

5.2 $\mathrm{so}(6)$ on the spherical section

In this subsection we perform Steps 2, 3 and 4 described in the introduction to this section. Recall that in Step 2 we use the null quadric

$$\mathcal{V}^5 := \{z \in \mathbb{R}^6\backslash\{0\} \ : \ 2z_{-1}z_1 + 2z_{-2}z_2 + 2z_{-3}z_3 = 0\}.$$

Then, in Step 3, we fix a section of the null quadric. We choose the section given by the equations

$$4 = 2\,(z_{-1}z_1 + z_{-2}z_2) = -2z_3z_{-3}.$$

We will call it the *spherical section*, because it coincides with $\mathbb{S}^3(4) \times \mathbb{S}^1(-4)$. The superscript used for this section will be "sph" for spherical.

In Step 4 we introduce the coordinates

$$r = \sqrt{2\,(z_{-1}z_1 + z_{-2}z_2)}\,, \qquad w = \frac{z_{-1}z_1}{z_{-1}z_1 + z_{-2}z_2}\,, \tag{5.7a}$$

$$u_1 = \frac{z_1}{\sqrt{z_{-1}z_1 + z_{-2}z_2}}\,, \qquad u_2 = \frac{z_2}{\sqrt{z_{-1}z_1 + z_{-2}z_2}}\,, \tag{5.7b}$$

$$p = \sqrt{-2z_3z_{-3}}\,, \qquad u_3 = \sqrt{-\frac{z_3}{z_{-3}}}. \tag{5.7c}$$

with the inverse transformation

$$z_{-1} = \frac{rw}{\sqrt{2}u_1}, \qquad z_1 = \frac{u_1 r}{\sqrt{2}}, \tag{5.8a}$$

$$z_{-2} = \frac{r(1-w)}{\sqrt{2}u_2}, \qquad z_2 = \frac{u_2 r}{\sqrt{2}}, \tag{5.8b}$$

$$z_{-3} = -\frac{p}{\sqrt{2}u_3}, \qquad z_3 = \frac{pu_3}{\sqrt{2}}. \tag{5.8c}$$

The null quadric in these coordinates is given by $r^2 = p^2$. We will restrict ourselves to the sheet $r = p$. The generator of dilations is

$$A_6 = r\,\partial_r + p\,\partial_p.$$

The spherical section is given by the condition $r^2 = 4$.

All the objects of the previous subsection will be now presented in the above coordinates after the reduction to the spherical section. This reduction allows us to eliminate the variables r, p. We omit the superscript η, whenever there is no dependence on this parameter.

Lie algebra $\mathrm{so}(6)$**.** Cartan operators:

$$N_1^{\mathrm{sph}} = u_1\,\partial_{u_1}\,,$$
$$N_2^{\mathrm{sph}} = u_2\,\partial_{u_2}\,,$$
$$N_3^{\mathrm{sph}} = u_3\,\partial_{u_3}\,.$$

Roots:

$$
\begin{aligned}
B^{\mathrm{sph}}_{-2,-1} &= u_1u_2\partial_w,\\
B^{\mathrm{sph}}_{2,1} &= \frac{1}{u_1u_2}\left((1-w)w\partial_w+(1-w)u_1\partial_{u_1}-wu_2\partial_{u_2}\right),\\
B^{\mathrm{sph}}_{2,-1} &= \frac{u_1}{u_2}\left((1-w)\partial_w-u_2\partial_{u_2}\right),\\
B^{\mathrm{sph}}_{-2,1} &= \frac{u_2}{u_1}\left(w\partial_w+u_1\partial_{u_1}\right);
\end{aligned}
$$

$$
\begin{aligned}
B^{\mathrm{sph},\eta}_{-3,-2} &= -u_2u_3\left(w\partial_w+\frac{1}{2}\big(u_1\partial_{u_1}+u_2\partial_{u_2}+u_3\partial_{u_3}-\eta\big)\right),\\
B^{\mathrm{sph},\eta}_{3,2} &= -\frac{1}{u_2u_3}\left(w(w-1)\partial_w+\frac{(w-1)}{2}(u_1\partial_{u_1}+u_2\partial_{u_2}-u_3\partial_{u_3}-\eta)+u_2\partial_{u_2}\right),\\
B^{\mathrm{sph},\eta}_{3,-2} &= \frac{u_2}{u_3}\left(w\partial_w+\frac{1}{2}\big(u_1\partial_{u_1}+u_2\partial_{u_2}-u_3\partial_{u_3}-\eta\big)\right),\\
B^{\mathrm{sph},\eta}_{-3,2} &= \frac{u_3}{u_2}\left(w(w-1)\partial_w+\frac{(w-1)}{2}(u_1\partial_{u_1}+u_2\partial_{u_2}+u_3\partial_{u_3}-\eta)+u_2\partial_{u_2}\right);
\end{aligned}
$$

$$
\begin{aligned}
B^{\mathrm{sph},\eta}_{-3,-1} &= -u_1u_3\left((w-1)\partial_w+\frac{1}{2}\big(u_1\partial_{u_1}+u_2\partial_{u_2}+u_3\partial_{u_3}-\eta\big)\right),\\
B^{\mathrm{sph},\eta}_{3,1} &= \frac{1}{u_1u_3}\left(w(w-1)\partial_w+\frac{w}{2}\big(u_1\partial_{u_1}+u_2\partial_{u_2}-u_3\partial_{u_3}-\eta\big)-u_1\partial_{u_1}\right),\\
B^{\mathrm{sph},\eta}_{3,-1} &= \frac{u_1}{u_3}\left((w-1)\partial_w+\frac{1}{2}\big(u_1\partial_{u_1}+u_2\partial_{u_2}-u_3\partial_{u_3}-\eta\big)\right);\\
B^{\mathrm{sph},\eta}_{-3,1} &= -\frac{u_3}{u_1}\left(w(w-1)\partial_w+\frac{w}{2}\big(u_1\partial_{u_1}+u_2\partial_{u_2}+u_3\partial_{u_3}-\eta\big)-u_1\partial_{u_1}\right).
\end{aligned}
$$

Weyl symmetries

$$
\begin{aligned}
\sigma^{\mathrm{sph},\eta}_{123}f(w,u_1,u_2,u_3) &= f\left(w,u_1,u_2,u_3\right),\\
\sigma^{\mathrm{sph},\eta}_{-12-3}f(w,u_1,u_2,u_3) &= f\left(w,\frac{w}{u_1},u_2,\frac{1}{u_3}\right),\\
\sigma^{\mathrm{sph},\eta}_{1-2-3}f(w,u_1,u_2,u_3) &= f\left(w,u_1,\frac{1-w}{u_2},\frac{1}{u_3}\right),\\
\sigma^{\mathrm{sph},\eta}_{-1-23}f(w,u_1,u_2,u_3) &= f\left(w,\frac{w}{u_1},\frac{1-w}{u_2},u_3\right);
\end{aligned}
$$

$$
\begin{aligned}
\sigma_{213}^{\mathrm{sph},\eta} f(w,u_1,u_2,u_3) &= f\left(1-w,u_2,u_1,u_3\right),\\
\sigma_{-21-3}^{\mathrm{sph},\eta} f(w,u_1,u_2,u_3) &= f\left(1-w,\frac{1-w}{u_2},u_1,\frac{1}{u_3}\right),\\
\sigma_{2-1-3}^{\mathrm{sph},\eta} f(w,u_1,u_2,u_3) &= f\left(1-w,u_2,\frac{w}{u_1},\frac{1}{u_3}\right),\\
\sigma_{-2-13}^{\mathrm{sph},\eta} f(w,u_1,u_2,u_3) &= f\left(1-w,\frac{1-w}{u_2},\frac{w}{u_1},u_3\right);
\end{aligned}
$$

$$
\begin{aligned}
\sigma_{321}^{\mathrm{sph},\eta} f(w,u_1,u_2,u_3) &= \left(\sqrt{-w}\right)^{\eta} f\left(\frac{1}{w},\frac{u_3}{\sqrt{-w}},\frac{u_2}{\sqrt{-w}},\frac{u_1}{\sqrt{-w}}\right),\\
\sigma_{-32-1}^{\mathrm{sph},\eta} f(w,u_1,u_2,u_3) &= \left(\sqrt{-w}\right)^{\eta} f\left(\frac{1}{w},\frac{1}{\sqrt{-w}u_3},\frac{u_2}{\sqrt{-w}},\frac{\sqrt{-w}}{u_1}\right),\\
\sigma_{3-2-1}^{\mathrm{sph},\eta} f(w,u_1,u_2,u_3) &= \left(\sqrt{-w}\right)^{\eta} f\left(\frac{1}{w},\frac{u_3}{\sqrt{-w}},\frac{(w-1)}{\sqrt{-w}u_2},\frac{\sqrt{-w}}{u_1}\right),\\
\sigma_{-3-21}^{\mathrm{sph},\eta} f(w,u_1,u_2,u_3) &= \left(\sqrt{-w}\right)^{\eta} f\left(\frac{1}{w},\frac{1}{\sqrt{-w}u_3},\frac{(w-1)}{\sqrt{-w}u_2},\frac{u_1}{\sqrt{-w}}\right);
\end{aligned}
$$

$$
\begin{aligned}
\sigma_{312}^{\mathrm{sph},\eta} f(w,u_1,u_2,u_3) &= \left(\sqrt{w-1}\right)^{\eta} f\left(\frac{1}{1-w},\frac{u_3}{\sqrt{w-1}},\frac{u_1}{\sqrt{w-1}},\frac{u_2}{\sqrt{w-1}}\right),\\
\sigma_{-31-2}^{\mathrm{sph},\eta} f(w,u_1,u_2,u_3) &= \left(\sqrt{w-1}\right)^{\eta} f\left(\frac{1}{1-w},\frac{1}{\sqrt{w-1}u_3},\frac{u_1}{\sqrt{w-1}},\frac{\sqrt{w-1}}{u_2}\right),\\
\sigma_{3-1-2}^{\mathrm{sph},\eta} f(w,u_1,u_2,u_3) &= \left(\sqrt{w-1}\right)^{\eta} f\left(\frac{1}{1-w},\frac{u_3}{\sqrt{w-1}},\frac{w}{\sqrt{w-1}u_1},\frac{\sqrt{w-1}}{u_2}\right),\\
\sigma_{-3-12}^{\mathrm{sph},\eta} f(w,u_1,u_2,u_3) &= \left(\sqrt{w-1}\right)^{\eta} f\left(\frac{1}{1-w},\frac{1}{\sqrt{w-1}u_3},\frac{w}{\sqrt{w-1}u_1},\frac{u_2}{\sqrt{w-1}}\right);
\end{aligned}
$$

$$
\begin{aligned}
\sigma_{231}^{\mathrm{sph},\eta} f(w,u_1,u_2,u_3) &= \left(\sqrt{-w}\right)^{\eta} f\left(\frac{w-1}{w},\frac{u_2}{\sqrt{-w}},\frac{u_3}{\sqrt{-w}},\frac{u_1}{\sqrt{-w}}\right),\\
\sigma_{-23-1}^{\mathrm{sph},\eta} f(w,u_1,u_2,u_3) &= \left(\sqrt{-w}\right)^{\eta} f\left(\frac{w-1}{w},\frac{(w-1)}{\sqrt{-w}u_2},\frac{u_3}{\sqrt{-w}},\frac{\sqrt{-w}}{u_1}\right),\\
\sigma_{2-3-1}^{\mathrm{sph},\eta} f(w,u_1,u_2,u_3) &= \left(\sqrt{-w}\right)^{\eta} f\left(\frac{w-1}{w},\frac{u_2}{\sqrt{-w}},\frac{1}{\sqrt{-w}u_3},\frac{\sqrt{-w}}{u_1}\right),\\
\sigma_{-2-31}^{\mathrm{sph},\eta} f(w,u_1,u_2,u_3) &= \left(\sqrt{-w}\right)^{\eta} f\left(\frac{w-1}{w},\frac{(w-1)}{\sqrt{-w}u_2},\frac{1}{\sqrt{-w}u_3},\frac{u_1}{\sqrt{-w}}\right);
\end{aligned}
$$

$$\sigma_{132}^{\mathrm{sph},\eta} f(w,u_1,u_2,u_3) = \big(\sqrt{w-1}\big)^{\eta} f\left(\frac{w}{w-1}, \frac{u_1}{\sqrt{w-1}}, \frac{u_3}{\sqrt{w-1}}, \frac{u_2}{\sqrt{w-1}}\right),$$

$$\sigma_{-13-2}^{\mathrm{sph},\eta} f(w,u_1,u_2,u_3) = \big(\sqrt{w-1}\big)^{\eta} f\left(\frac{w}{w-1}, \frac{w}{\sqrt{w-1}u_1}, \frac{u_3}{\sqrt{w-1}}, \frac{\sqrt{w-1}}{u_2}\right),$$

$$\sigma_{1-3-2}^{\mathrm{sph},\eta} f(w,u_1,u_2,u_3) = \big(\sqrt{w-1}\big)^{\eta} f\left(\frac{w}{w-1}, \frac{u_1}{\sqrt{w-1}}, \frac{1}{\sqrt{w-1}u_3}, \frac{\sqrt{w-1}}{u_2}\right),$$

$$\sigma_{-1-32}^{\mathrm{sph},\eta} f(w,u_1,u_2,u_3) = \big(\sqrt{w-1}\big)^{\eta} f\left(\frac{w}{w-1}, \frac{w}{\sqrt{w-1}u_1}, \frac{1}{\sqrt{w-1}u_3}, \frac{u_2}{\sqrt{w-1}}\right).$$

Laplacian

$$\begin{aligned}\Delta_6^{\mathrm{sph}} = {} & w(1-w)\partial_w^2 - \big((1+u_1\partial_{u_1})(w-1) + (1+u_2\partial_{u_2})w\big)\partial_w \\ & -\frac{1}{4}(u_1\partial_{u_1} + u_2\partial_{u_2} + 1)^2 + \frac{1}{4}(u_3\partial_{u_3})^2. \end{aligned} \tag{5.9}$$

Let us give the computations that yield (5.9). Using

$$\begin{aligned}
\partial_{z_{-1}} &= \frac{u_1}{\sqrt{2}r}\Big(-u_1\partial_{u_1} - u_2\partial_{u_2} + r\partial_r + 2(1-w)\partial_w\Big),\\
\partial_{z_1} &= \frac{\sqrt{2}}{ru_1}\Big(\big(1-\frac{w}{2}\big)u_1\partial_{u_1} - \frac{w}{2}u_2\partial_{u_2} + \frac{w}{2}r\partial_r + w(1-w)\partial_w\Big),\\
\partial_{z_{-2}} &= \frac{u_2}{\sqrt{2}r}\Big(-u_1\partial_{u_1} - u_2\partial_{u_2} + r\partial_r - 2w\partial_w\Big),\\
\partial_{z_2} &= \frac{\sqrt{2}}{ru_2}\Big(\frac{(w-1)}{2}u_1\partial_{u_1} + \frac{(w+1)}{2}u_2\partial_{u_2} + \frac{(1-w)}{2}r\partial_r + w(w-1)\partial_w\Big),\\
\partial_{z_{-3}} &= \frac{u_3}{\sqrt{2}p}\Big(u_3\partial_{u_3} - p\partial_p\Big),\\
\partial_{z_3} &= \frac{1}{\sqrt{2}pu_3}\Big(u_3\partial_{u_3} + p\partial_p\Big),
\end{aligned}$$

we compute the Laplacian in coordinates (5.7):

$$\begin{aligned}\Delta_6 = {} & \frac{1}{r^2}\Big(4w(1-w)\partial_w^2 - 4\big((1+u_1\partial_{u_1})(w-1) + (1+u_2\partial_{u_2})w\big)\partial_w \\ & -(u_1\partial_{u_1} + u_2\partial_{u_2} + 1)^2 + (r\partial_r)^2 + 2r\partial_r + 1\Big) \\ & + \frac{1}{p^2}\Big((u_3\partial_{u_3})^2 - (p\partial_p)^2\Big). \end{aligned} \tag{5.10}$$

Next we note that

$$\frac{1}{r^2}\Big((r\partial_r)^2 + 2r\partial_r - (p\partial_p)^2 + 1\Big) = \frac{1}{r^2}\big(r\partial_r - p\partial_p + 1\big)\big(r\partial_r + p\partial_p + 1\big). \tag{5.11}$$

Using $p^2 = r^2$ and $r\partial_r + p\partial_p = -1$, we see that (5.11) is zero on functions of degree -1. Thus we obtain

$$\begin{aligned}\Delta_6^\diamond = \frac{4}{r^2}\Big(& w(1-w)\partial_w^2 - \big((1+u_1\partial_{u_1})(w-1) + (1+u_2\partial_{u_2})w\big)\partial_w \\ & - \frac{1}{4}(u_1\partial_{u_1} + u_2\partial_{u_2} + 1)^2 + \frac{1}{4}(u_3\partial_{u_3})^2\Big). \end{aligned} \tag{5.12}$$

To convert $\Delta_6^\diamond$ into the Δ_6^{sph}, we simply remove the prefactor $\frac{4}{r^2}$.

5.3 Hypergeometric equation

Let us make the ansatz

$$f(u_1, u_2, u_3, w) = u_1^\alpha u_2^\beta u_3^\mu F(w). \tag{5.13}$$

Clearly,

$$\begin{aligned} N_1^{\mathrm{sph}} f &= \alpha f, && (5.14\mathrm{a})\\ N_2^{\mathrm{sph}} f &= \beta f, && (5.14\mathrm{b})\\ N_3^{\mathrm{sph}} f &= \mu f, && (5.14\mathrm{c})\\ u_1^{-\alpha} u_2^{-\beta} u_3^{-\mu} \Delta_6^{\mathrm{sph}} f &= \mathcal{F}_{\alpha,\beta,\mu}(w, \partial_w) F(w), && (5.14\mathrm{d}) \end{aligned}$$

where

$$\begin{aligned}\mathcal{F}_{\alpha,\beta,\mu}(w,\partial_w) :=& w(1-w)\partial_w^2 - \big((1+\alpha)(w-1) + (1+\beta)w\big)\partial_w \\ & - \frac{1}{4}(\alpha+\beta+1)^2 + \frac{1}{4}\mu^2, \end{aligned} \tag{5.15}$$

which is the ${}_2\mathcal{F}_1$ *hypergeometric operator* in the Lie-algebraic parameters.

Traditionally, the *hypergeometric equation* is given by the operator

$$\mathcal{F}(a,b;c;w,\partial_w) := w(1-w)\partial_w^2 + \big(c - (a+b+1)w\big)\partial_w - ab, \tag{5.16}$$

where $a, b, c \in \mathbb{C}$ will be called the *classical parameters*. Here is the relationship between the Lie-algebraic and classical parameters:

$$\alpha := c - 1, \quad \beta := a + b - c, \qquad \mu := a - b; \tag{5.17a}$$

$$a = \frac{1+\alpha+\beta+\mu}{2}, \quad b = \frac{1+\alpha+\beta-\mu}{2}, \quad c = 1 + \alpha. \tag{5.17b}$$

Note that the Lie-algebraic parameters α, β, μ are differences of the indices of the singular points $0, 1, \infty$. For many purposes, they are more convenient than the traditional parameters a, b, c. They are used e.g. in Subsect. 2.7.2 of [BE], where they are called λ, ν, μ. In the standard notation for Jacobi Polynomials $P_n^{\alpha,\beta}$, the parameters α, β correspond to our α, β (where the singular points have been moved from $0, 1$ to $-1, 1$).

5.4 Transmutation relations and discrete symmetries

By (4.17), we have the following generalized symmetries

$$B^{\mathrm{sph},-3}\Delta_6^{\mathrm{sph}} = \Delta_6^{\mathrm{sph}} B^{\mathrm{sph},-1}, \quad B \in \mathrm{so}(6), \tag{5.18a}$$

$$\alpha^{\mathrm{sph},-3}\Delta_6^{\mathrm{sph}} = \Delta_6^{\mathrm{sph}} \alpha^{\mathrm{sph},-1}, \quad \alpha \in \mathrm{O}(6). \tag{5.18b}$$

Applying (5.18a) to the roots of so(6) we obtain the *transmutation relations for the hypergeometric operator*:

$$\begin{aligned}
&\partial_w \mathcal{F}_{\alpha,\beta,\mu} \\
&\quad = \mathcal{F}_{\alpha+1,\beta+1,\mu}\partial_w, \\
&\big(w(1-w)\partial_w + (1-w)\alpha - w\beta\big)\mathcal{F}_{\alpha,\beta,\mu} \\
&\quad = \mathcal{F}_{\alpha-1,\beta-1,\mu}\big(w(1-w)\partial_w + (1-w)\alpha - w\beta\big), \\
&\big((1-w)\partial_w - \beta\big)\mathcal{F}_{\alpha,\beta,\mu} \\
&\quad = \mathcal{F}_{\alpha+1,\beta-1,\mu}\big((1-w)\partial_w - \beta\big), \\
&(w\partial_w + \alpha)\mathcal{F}_{\alpha,\beta,\mu} \\
&\quad = \mathcal{F}_{\alpha-1,\beta+1,\mu}(w\partial_w + \alpha);
\end{aligned}$$

$$\begin{aligned}
&\Big(w\partial_w + \frac{1}{2}(\alpha+\beta+\mu+1)\Big)w\mathcal{F}_{\alpha,\beta,\mu} \\
&\quad = w\mathcal{F}_{\alpha,\beta+1,\mu+1}\Big(w\partial_w + \frac{1}{2}(\alpha+\beta+\mu+1)\Big), \\
&\Big(w(w-1)\partial_w + \frac{1}{2}(w-1)(\alpha+\beta-\mu+1) - \beta\Big)w\mathcal{F}_{\alpha,\beta,\mu} \\
&\quad = w\mathcal{F}_{\alpha,\beta-1,\mu-1}\Big(w(w-1)\partial_w + \frac{1}{2}(w-1)(\alpha+\beta-\mu+1) - \beta\Big), \\
&\Big(w\partial_w + \frac{1}{2}(\alpha+\beta-\mu+1)\Big)w\mathcal{F}_{\alpha,\beta,\mu} \\
&\quad = w\mathcal{F}_{\alpha,\beta+1,\mu-1}\Big(w\partial_w + \frac{1}{2}(\alpha+\beta-\mu+1\Big), \\
&\Big(w(w-1)\partial_w - \frac{1}{2}(1-w)(\alpha+\beta+\mu+1) + \beta\Big)w\mathcal{F}_{\alpha,\beta,\mu} \\
&\quad = w\mathcal{F}_{\alpha,\beta-1,\mu+1}\Big(w(w-1)\partial_w - \frac{1}{2}(1-w)(\alpha+\beta+\mu+1) + \beta\Big);
\end{aligned}$$

$$\Big((w-1)\partial_w+\frac12(\alpha+\beta+\mu+1)\Big)(1-w)\mathcal{F}_{\alpha,\beta,\mu}$$
$$=(1-w)\mathcal{F}_{\alpha+1,\beta,\mu+1}\Big((w-1)\partial_w+\frac12(\alpha+\beta+\mu+1)\Big),$$
$$\Big(w(w{-}1)\partial_w{+}\frac12 w(\alpha{+}\beta{-}\mu{+}1){+}\alpha\Big)(1-w)\mathcal{F}_{\alpha,\beta,\mu}$$
$$=(1-w)\mathcal{F}_{\alpha-1,\beta,\mu-1}\Big(w(w{-}1)\partial_w{+}\frac12 w(\alpha{+}\beta{-}\mu{+}1){+}\alpha\Big),$$
$$\Big((w-1)\partial_w+\frac12(\alpha+\beta-\mu+1)\Big)(1-w)\mathcal{F}_{\alpha,\beta,\mu}$$
$$=(1-w)\mathcal{F}_{\alpha+1,\beta,\mu-1}\Big((w-1)\partial_w+\frac12(\alpha+\beta-\mu+1)\Big),$$
$$\Big(w(w{-}1)\partial_w{+}\frac12 w(\alpha{+}\beta{+}\mu{+}1)-\alpha\Big)(1-w)\mathcal{F}_{\alpha,\beta,\mu}$$
$$=(1-w)\mathcal{F}_{\alpha-1,\beta,\mu+1}\Big(w(w{-}1)\partial_w{+}\frac12 w(\alpha{+}\beta{+}\mu{+}1){-}\alpha\Big).$$

Applying (5.18b) to the Weyl group D_3 we obtain the *discrete symmetries of the hypergeometric operator*. We describe them below, restricting ourselves to $D_3\cap\mathrm{SO}(6)$.

All the operators below equal $\mathcal{F}_{\alpha,\beta,\mu}(w,\partial_w)$ for the corresponding w:

$$\begin{array}{rlll}
w=v: & & \mathcal{F}_{\alpha,\beta,\mu}(v,\partial_v), & \\
& (-v)^{-\alpha}(v-1)^{-\beta} & \mathcal{F}_{-\alpha,-\beta,\mu}(v,\partial_v) & (-v)^{\alpha}(v-1)^{\beta} \\
& (v-1)^{-\beta} & \mathcal{F}_{\alpha,-\beta,-\mu}(v,\partial_v) & (v-1)^{\beta}, \\
& (-v)^{-\alpha} & \mathcal{F}_{-\alpha,\beta,-\mu}(v,\partial_v) & (-v)^{\alpha};
\end{array}$$

$$\begin{array}{rlll}
w=1-v: & & \mathcal{F}_{\beta,\alpha,\mu}(v,\partial_v), & \\
& (v-1)^{-\alpha}(-v)^{-\beta} & \mathcal{F}_{-\beta,-\alpha,\mu}(v,\partial_v) & (v-1)^{\alpha}(-v)^{\beta}, \\
& (v-1)^{-\alpha} & \mathcal{F}_{\beta,-\alpha,-\mu}(v,\partial_v) & (v-1)^{\alpha}, \\
& (-v)^{-\beta} & \mathcal{F}_{-\beta,\alpha,-\mu}(v,\partial_v) & (-v)^{\beta};
\end{array}$$

$$\begin{array}{rll}
w=\frac1v:\quad (-v)^{\frac{\alpha+\beta+\mu+1}{2}} & (-v)\mathcal{F}_{\mu,\beta,\alpha}(v,\partial_v) & (-v)^{\frac{-\alpha-\beta-\mu-1}{2}}, \\
(-v)^{\frac{\alpha+\beta-\mu+1}{2}}(v-1)^{-\beta} & (-v)\mathcal{F}_{-\mu,-\beta,\alpha}(v,\partial_v) & (-v)^{\frac{-\alpha-\beta+\mu-1}{2}}(v-1)^{\beta}, \\
(-v)^{\frac{\alpha+\beta+\mu+1}{2}}(v-1)^{-\beta} & (-v)\mathcal{F}_{\mu,-\beta,-\alpha}(v,\partial_v) & (-v)^{\frac{-\alpha-\beta-\mu-1}{2}}(v-1)^{\beta}, \\
(-v)^{\frac{\alpha+\beta-\mu+1}{2}} & (-v)\mathcal{F}_{-\mu,\beta,-\alpha}(v,\partial_v) & (-v)^{\frac{-\alpha-\beta+\mu-1}{2}};
\end{array}$$

$$\begin{array}{rll}
w=\frac{v-1}{v}:\quad (-v)^{\frac{\alpha+\beta+\mu+1}{2}} & (-v)\mathcal{F}_{\mu,\alpha,\beta}(v,\partial_v) & (-v)^{\frac{-\alpha-\beta-\mu-1}{2}}, \\
(-v)^{\frac{\alpha+\beta-\mu+1}{2}}(v-1)^{-\alpha} & (-v)\mathcal{F}_{-\mu,-\alpha,\beta}(v,\partial_v) & (-v)^{\frac{-\alpha-\beta+\mu-1}{2}}(v-1)^{\alpha}, \\
(-v)^{\frac{\alpha+\beta+\mu+1}{2}}(v-1)^{-\alpha} & (-v)\mathcal{F}_{\mu,-\alpha,-\beta}(v,\partial_v) & (-v)^{\frac{-\alpha-\beta-\mu-1}{2}}(v-1)^{\alpha}, \\
(-v)^{\frac{\alpha+\beta-\mu+1}{2}} & (-v)\mathcal{F}_{-\mu,\alpha,-\beta}(v,\partial_v) & (-v)^{\frac{-\alpha-\beta+\mu-1}{2}};
\end{array}$$

$$\begin{array}{rcl}
w{=}\frac{1}{1-v}\ :\ (v{-}1)^{\frac{\alpha+\beta+\mu+1}{2}} & (v{-}1)\mathcal{F}_{\beta,\mu,\alpha}(v,\partial_v) & (v{-}1)^{\frac{-\alpha-\beta-\mu-1}{2}},\\
(-v)^{-\beta}(v{-}1)^{\frac{\alpha+\beta-\mu+1}{2}} & (v{-}1)\mathcal{F}_{-\beta,-\mu,\alpha}(v,\partial_v) & (-v)^{\beta}(v{-}1)^{\frac{-\alpha-\beta+\mu-1}{2}},\\
(v{-}1)^{\frac{\alpha+\beta-\mu+1}{2}} & (v{-}1)\mathcal{F}_{\beta,-\mu,-\alpha}(v,\partial_v) & (v{-}1)^{\frac{-\alpha-\beta+\mu-1}{2}},\\
(-v)^{-\beta}(v{-}1)^{\frac{\alpha+\beta+\mu+1}{2}} & (v{-}1)\mathcal{F}_{-\beta,\mu,-\alpha}(v,\partial_v) & (-v)^{\beta}(v{-}1)^{\frac{-\alpha-\beta-\mu-1}{2}};\\
w{=}\frac{v}{v-1}\ :\ (v{-}1)^{\frac{\alpha+\beta+\mu+1}{2}} & (v{-}1)\mathcal{F}_{\alpha,\mu,\beta}(v,\partial_v) & (v{-}1)^{\frac{-\alpha-\beta-\mu-1}{2}},\\
(-v)^{-\alpha}(v{-}1)^{\frac{\alpha+\beta-\mu+1}{2}} & (v{-}1)\mathcal{F}_{-\alpha,-\mu,\beta}(v,\partial_v) & (-v)^{\alpha}(v{-}1)^{\frac{-\alpha-\beta+\mu-1}{2}},\\
(v{-}1)^{\frac{\alpha+\beta-\mu+1}{2}} & (v{-}1)\mathcal{F}_{\alpha,-\mu,-\beta}(v,\partial_v) & (v{-}1)^{\frac{-\alpha-\beta+\mu-1}{2}},\\
(-v)^{-\alpha}(v{-}1)^{\frac{\alpha+\beta+\mu+1}{2}} & (v{-}1)\mathcal{F}_{-\alpha,\mu,-\beta}(v,\partial_v) & (-v)^{\alpha}(v{-}1)^{\frac{-\alpha-\beta-\mu-1}{2}}.
\end{array}$$

5.5 Factorizations of the Laplacian

In the Lie algebra so(6) represented on $\mathbb{R}^6$ we have 3 distinguished Lie subalgebras isomorphic to so(4):

$$\mathrm{so}_{12}(4),\quad \mathrm{so}_{23}(4),\quad \mathrm{so}_{13}(4), \tag{5.19}$$

where we use a hopefully obvious notation. By (4.11), the corresponding Casimir operators are

$$\begin{aligned}
\mathcal{C}_{12} &= 4B_{1,2}B_{-1,-2} - (N_1+N_2+1)^2+1 && (5.20a)\\
&= 4B_{-1,-2}B_{1,2} - (N_1+N_2-1)^2+1 && (5.20b)\\
&= 4B_{1,-2}B_{-1,2} - (N_1-N_2+1)^2+1 && (5.20c)\\
&= 4B_{-1,2}B_{1,-2} - (N_1-N_2-1)^2+1; && (5.20d)\\
\mathcal{C}_{23} &= 4B_{2,3}B_{-2,-3} - (N_2+N_3+1)^2+1 && (5.20e)\\
&= 4B_{-2,-3}B_{2,3} - (N_2+N_3-1)^2+1 && (5.20f)\\
&= 4B_{2,-3}B_{-2,3} - (N_2-N_3+1)^2+1 && (5.20g)\\
&= 4B_{-2,3}B_{2,-3} - (N_2-N_3-1)^2+1; && (5.20h)\\
\mathcal{C}_{13} &= 4B_{1,3}B_{-1,-3} - (N_1+N_3+1)^2+1 && (5.20i)\\
&= 4B_{-1,-3}B_{1,3} - (N_1+N_3-1)^2+1 && (5.20j)\\
&= 4B_{1,-3}B_{-1,3} - (N_1-N_3+1)^2+1 && (5.20k)\\
&= 4B_{-1,3}B_{1,-3} - (N_1-N_3-1)^2+1. && (5.20l)
\end{aligned}$$

Of course, for any η we can append the superscript $^{\diamond,\eta}$ to all the operators in (5.20).

After the reduction described in (4.19), we obtain the identities

$$(2z_{-1}z_1 + 2z_{-2}z_2)\Delta_6^{\diamond} = -1 + \mathcal{C}_{12}^{\diamond,-1} + (N_3^{\diamond,-1})^2, \tag{5.21a}$$

$$(2z_{-2}z_2 + 2z_{-3}z_3)\Delta_6^{\diamond} = -1 + \mathcal{C}_{23}^{\diamond,-1} + (N_1^{\diamond,-1})^2, \tag{5.21b}$$

$$(2z_{-1}z_1 + 2z_{-3}z_3)\Delta_6^{\diamond} = -1 + \mathcal{C}_{13}^{\diamond,-1} + (N_2^{\diamond,-1})^2. \tag{5.21c}$$

We insert (5.20) with superscript $^{\diamond,-1}$ to (5.21), obtaining

$$\begin{aligned}
&(2z_{-1}z_1+2z_{-2}z_2)\Delta_6^\diamond\\
=&4B_{1,2}B_{-1,-2}-(N_1+N_2+N_3+1)(N_1+N_2-N_3+1) && (5.22a)\\
=&4B_{-1,-2}B_{1,2}-(N_1+N_2+N_3-1)(N_1+N_2-N_3-1) && (5.22b)\\
=&4B_{1,-2}B_{-1,2}-(N_1-N_2+N_3+1)(N_1-N_2-N_3+1) && (5.22c)\\
=&4B_{-1,2}B_{1,-2}-(N_1-N_2+N_3-1)(N_1-N_2-N_3-1); && (5.22d)\\
&(2z_{-2}z_2+2z_{-3}z_3)\Delta_6^\diamond\\
=&4B_{2,3}B_{-2,-3}-(N_1+N_2+N_3+1)(-N_1+N_2+N_3+1) && (5.22e)\\
=&4B_{-2,-3}B_{2,3}-(N_1+N_2+N_3-1)(-N_1+N_2+N_3-1) && (5.22f)\\
=&4B_{2,-3}B_{-2,3}-(N_1+N_2-N_3+1)(-N_1+N_2-N_3+1) && (5.22g)\\
=&4B_{-2,3}B_{2,-3}-(N_1+N_2-N_3-1)(-N_1+N_2-N_3-1); && (5.22h)\\
&(2z_{-1}z_1+2z_{-3}z_3)\Delta_6^\diamond\\
=&4B_{1,3}B_{-1,-3}-(N_1+N_2+N_3+1)(N_1-N_2+N_3+1) && (5.22i)\\
=&4B_{-1,-3}B_{1,3}-(N_1+N_2+N_3-1)(N_1-N_2+N_3-1) && (5.22j)\\
=&4B_{1,-3}B_{-1,3}-(N_1+N_2-N_3+1)(N_1-N_2-N_3+1) && (5.22k)\\
=&4B_{-1,3}B_{1,-3}-(N_1+N_2-N_3-1)(N_1-N_2-N_3-1); && (5.22l)
\end{aligned}$$

where for typographical reasons we omitted the superscript $^{\diamond,-1}$ at all the operators B and N.

If we use the coordinates (5.7) and the spherical section, then we have to rewrite (5.22) by making the replacements

$$\begin{aligned}
2z_{-1}z_1+2z_{-2}z_2 &\to \quad 1, && (5.23a)\\
2z_{-2}z_2+2z_{-3}z_3 &\to \quad -w, && (5.23b)\\
2z_{-1}z_1+2z_{-3}z_3 &\to \quad w-1, && (5.23c)
\end{aligned}$$

as well as replacing the superscript $^\diamond$ with $^{\mathrm{sph}}$.

5.6 Factorizations of the hypergeometric operator

The factorizations of Δ_6^{sph} described in Subsect. 5.5 yield the following factorizations of the hypergeometric operator:

$$\begin{aligned}
&\mathcal{F}_{\alpha,\beta,\mu}\\
=&\Big(w(1-w)\partial_w+\big((1+\alpha)(1-w)-(1+\beta)w\big)\Big)\partial_w\\
&-\frac14(\alpha+\beta+\mu+1)(\alpha+\beta-\mu+1)\\
=&\partial_w\Big(w(1-w)\partial_w+\big(\alpha(1-w)-\beta w\big)\Big)\\
&-\frac14(\alpha+\beta+\mu-1)(\alpha+\beta-\mu-1)\\
=&\Big(w\partial_w+\alpha+1\Big)\Big((1-w)\partial_w-\beta\Big)\\
&-\frac14(\alpha-\beta+\mu+1)(\alpha-\beta-\mu+1)\\
=&\Big((1-w)\partial_w-\beta-1\Big)\Big(w\partial_w+\alpha\Big)\\
&-\frac14(\alpha-\beta+\mu-1)(\alpha-\beta-\mu-1);
\end{aligned}$$

$$\begin{aligned}
&w\mathcal{F}_{\alpha,\beta,\mu}\\
=&\Big(w\partial_w+\frac12(\alpha+\beta+\mu-1)\Big)\Big(w(1-w)\partial_w+\frac12(1-w)(\alpha+\beta-\mu+1)-\beta\Big)\\
&-\frac14(\alpha+\beta+\mu-1)(\alpha-\beta-\mu+1)\\
=&\Big(w(1-w)\partial_w+\frac12(1-w)(\alpha+\beta-\mu+1)-\beta-1\Big)\Big(w\partial_w+\frac12(\alpha+\beta+\mu+1)\Big)\\
&-\frac14(\alpha+\beta+\mu+1)(\alpha-\beta-\mu-1)\\
=&\Big(w\partial_w+\frac12(\alpha+\beta-\mu-1)\Big)\Big(w(1-w)\partial_w+\frac12(1-w)(\alpha+\beta+\mu+1)-\beta\Big)\\
&-\frac14(\alpha+\beta-\mu-1)(\alpha-\beta+\mu+1)\\
=&\Big(w(1-w)\partial_w+\frac12(1-w)(\alpha+\beta+\mu+1)-\beta-1\Big)\Big(w\partial_w+\frac12(\alpha+\beta-\mu+1)\Big)\\
&-\frac14(\alpha+\beta-\mu+1)(\alpha-\beta+\mu-1);
\end{aligned}$$

$$
\begin{aligned}
&(w-1)\mathcal{F}_{\alpha,\beta,\mu}\\
=&\Big(w(w-1)\partial_w+\frac12 w(\alpha+\beta-\mu+1)-\alpha-1\Big)\Big((w-1)\partial_w+\frac12(\alpha+\beta+\mu+1)\Big)\\
&-\frac14(\alpha+\beta+\mu+1)(\alpha-\beta+\mu+1)\\
=&\Big((w-1)\partial_w+\frac12(\alpha+\beta+\mu-1)\Big)\Big(w(w-1)\partial_w+\frac12 w(\alpha+\beta-\mu+1)-\alpha\Big)\\
&-\frac14(\alpha+\beta+\mu-1)(\alpha-\beta+\mu-1)\\
=&\Big(w(w-1)\partial_w+\frac12 w(\alpha+\beta+\mu+1)-\alpha-1\Big)\Big((w-1)\partial_w+\frac12(\alpha+\beta-\mu+1)\Big)\\
&-\frac14(\alpha+\beta-\mu+1)(\alpha-\beta-\mu+1)\\
=&\Big((w-1)\partial_w+\frac12(\alpha+\beta-\mu-1)\Big)\Big(w(w-1)\partial_w+\frac12 w(\alpha+\beta+\mu+1)-\alpha\Big)\\
&-\frac14(\alpha+\beta-\mu-1)(\alpha-\beta-\mu-1).
\end{aligned}
$$

5.7 The ${}_2F_1$ hypergeometric function

0 is a regular singular point of the ${}_2\mathcal{F}_1$ hypergeometric equation. Its indices are 0 and $1-c$. For $c\neq 0,-1,-2,\dots$ the Frobenius method yields the unique solution of the hypergeometric equation equal to 1 at 0, given by the series

$$F(a,b;c;w)=\sum_{j=0}^{\infty}\frac{(a)_j(b)_j}{(c)_j}\frac{w^j}{j!}$$

convergent for $|w|<1$. The function extends to the whole complex plane cut at $[1,\infty[$ and is called the *hypergeometric function*. Sometimes it is more convenient to consider the function

$$\mathbf{F}(a,b;c;w):=\frac{F(a,b,c,w)}{\Gamma(c)}=\sum_{j=0}^{\infty}\frac{(a)_j(b)_j}{\Gamma(c+j)}\frac{w^j}{j!}$$

defined for all $a,b,c\in\mathbb{C}$. Another useful function proportional to F is

$$\mathbf{F}^{\mathrm{I}}(a,b;c;w):=\frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)}F(a,b;c;w)=\sum_{j=0}^{\infty}\frac{\Gamma(b+j)\Gamma(c-b)(a)_j}{\Gamma(c+j)}\frac{w^j}{j!}.$$

We will usually prefer to parametrize all varieties of the hypergeometric function with the Lie-algebraic parameters:

$$
\begin{aligned}
F_{\alpha,\beta,\mu}(w) &= F\Big(\frac{1+\alpha+\beta+\mu}{2}, \frac{1+\alpha+\beta-\mu}{2}; 1+\alpha; w\Big),\\
\mathbf{F}_{\alpha,\beta,\mu}(w) &= \mathbf{F}\Big(\frac{1+\alpha+\beta+\mu}{2}, \frac{1+\alpha+\beta-\mu}{2}; 1+\alpha; w\Big)\\
&= \frac{1}{\Gamma(\alpha+1)} F_{\alpha,\beta,\mu}(w),\\
\mathbf{F}^{\mathrm{I}}_{\alpha,\beta,\mu}(w) &= \mathbf{F}^{\mathrm{I}}\Big(\frac{1+\alpha+\beta+\mu}{2}, \frac{1+\alpha+\beta-\mu}{2}; 1+\alpha; w\Big)\\
&= \frac{\Gamma\big(\frac{1+\alpha+\beta-\mu}{2}\big)\Gamma\big(\frac{1+\alpha-\beta+\mu}{2}\big)}{\Gamma(\alpha+1)} F_{\alpha,\beta,\mu}(w).
\end{aligned}
$$

5.8 Standard solutions

The hypergeometric equation has 3 singular points. With each of them we can associate two solutions with a simple behavior. Therefore, we obtain 6 standard solutions.

Applying the discrete symmetries from $D_3 \cap \mathrm{SO}(6)$ to the hypergeometric function, we obtain 24 expressions for solutions of the hypergeometric equation, which go under the name of *Kummer's table*. Some of them coincide as functions, so that we obtain 6 standard solutions, each expressed in 4 ways:

$$
\begin{aligned}
\text{Solution} \sim 1 \text{ at } 0:\quad & F_{\alpha,\beta,\mu}(w)\\
&= (1-w)^{-\beta} F_{\alpha,-\beta,-\mu}(w)\\
&= (1-w)^{\frac{-1-\alpha-\beta+\mu}{2}} F_{\alpha,-\mu,-\beta}\Big(\frac{w}{w-1}\Big)\\
&= (1-w)^{\frac{-1-\alpha-\beta-\mu}{2}} F_{\alpha,\mu,\beta}\Big(\frac{w}{w-1}\Big);
\end{aligned}
$$

$$
\begin{aligned}
\text{Solution} \sim w^{-\alpha} \text{ at } 0:\quad & w^{-\alpha} F_{-\alpha,\beta,-\mu}(w)\\
&= w^{-\alpha}(1-w)^{-\beta} F_{-\alpha,-\beta,\mu}(w)\\
&= w^{-\alpha}(1-w)^{\frac{-1+\alpha-\beta+\mu}{2}} F_{-\alpha,-\mu,\beta}\Big(\frac{w}{w-1}\Big)\\
&= w^{-\alpha}(1-w)^{\frac{-1+\alpha-\beta-\mu}{2}} F_{-\alpha,\mu,-\beta}\Big(\frac{w}{w-1}\Big);
\end{aligned}
$$

$$\begin{aligned}
\text{Solution} \sim 1 \text{ at } 1: \quad & F_{\beta,\alpha,\mu}(1-w) \\
=& w^{-\alpha} F_{\beta,-\alpha,-\mu}(1-w) \\
=& w^{\frac{-1-\alpha-\beta+\mu}{2}} F_{\beta,-\mu,-\alpha}(1-w^{-1}) \\
=& w^{\frac{-1-\alpha-\beta-\mu}{2}} F_{\beta,\mu,\alpha}(1-w^{-1});
\end{aligned}$$

$$\begin{aligned}
\text{Solution} \sim (1-w)^{-\beta} \text{ at } 1: \quad & (1-w)^{-\beta} F_{-\beta,\alpha,-\mu}(1-w) \\
=& w^{-\alpha}(1-w)^{-\beta} F_{-\beta,-\alpha,\mu}(1-w) \\
=& w^{\frac{-1-\alpha+\beta-\mu}{2}} (1-w)^{-\beta} F_{-\beta,\mu,-\alpha}(1-w^{-1}) \\
=& w^{\frac{-1-\alpha+\beta+\mu}{2}} (1-w)^{-\beta} F_{-\beta,-\mu,\alpha}(1-w^{-1});
\end{aligned}$$

$$\begin{aligned}
\text{Solution} \sim w^{-a} \text{ at } \infty: \quad & (-w)^{\frac{-1-\alpha-\beta-\mu}{2}} F_{\mu,\beta,\alpha}(w^{-1}) \\
=& (-w)^{\frac{-1-\alpha+\beta-\mu}{2}} (1-w)^{-\beta} F_{\mu,-\beta,-\alpha}(w^{-1}) \\
=& (1-w)^{\frac{-1-\alpha-\beta-\mu}{2}} F_{\mu,\alpha,\beta}((1-w)^{-1}) \\
=& (-w)^{-\alpha}(1-w)^{\frac{-1+\alpha-\beta-\mu}{2}} F_{\mu,-\alpha,-\beta}((1-w)^{-1});
\end{aligned}$$

$$\begin{aligned}
\text{Solution} \sim w^{-b} \text{ at } \infty: \quad & (-w)^{\frac{-1-\alpha-\beta+\mu}{2}} F_{-\mu,\beta,-\alpha}(w^{-1}) \\
=& (-w)^{\frac{-1-\alpha+\beta+\mu}{2}} (1-w)^{-\beta} F_{-\mu,-\beta,\alpha}(w^{-1}) \\
=& (1-w)^{\frac{-1-\alpha-\beta+\mu}{2}} F_{-\mu,\alpha,-\beta}((1-w)^{-1}) \\
=& (-w)^{-\alpha}(1-w)^{\frac{-1+\alpha-\beta+\mu}{2}} F_{-\mu,-\alpha,\beta}((1-w)^{-1}).
\end{aligned}$$

5.9 Recurrence relations

To each root of so(6) there corresponds a recurrence relation:

$$\begin{aligned}
\partial_w \mathbf{F}^{\mathrm{I}}_{\alpha,\beta,\mu}(w) &= \frac{1+\alpha+\beta+\mu}{2} \mathbf{F}^{\mathrm{I}}_{\alpha+1,\beta+1,\mu}(w), \\
-(w(1-w)\partial_w + \alpha(1-w) - \beta w)\mathbf{F}^{\mathrm{I}}_{\alpha,\beta,\mu}(w) &= \frac{1-\alpha-\beta+\mu}{2} \mathbf{F}^{\mathrm{I}}_{\alpha-1,\beta-1,\mu}(w), \\
((1-w)\partial_w - \beta)\mathbf{F}^{\mathrm{I}}_{\alpha,\beta,\mu}(w) &= \frac{1+\alpha-\beta-\mu}{2} \mathbf{F}^{\mathrm{I}}_{\alpha+1,\beta-1,\mu}(w), \\
-(w\partial_w + \alpha)\mathbf{F}^{\mathrm{I}}_{\alpha,\beta,\mu}(w) &= \frac{1-\alpha+\beta-\mu}{2} \mathbf{F}^{\mathrm{I}}_{\alpha-1,\beta+1,\mu}(w);
\end{aligned}$$

$$\left(w\partial_w + \frac{1+\alpha+\beta+\mu}{2}\right)\mathbf{F}^{\mathrm{I}}_{\alpha,\beta,\mu}(w) = \frac{1+\alpha+\beta+\mu}{2}\mathbf{F}^{\mathrm{I}}_{\alpha,\beta+1,\mu+1}(w),$$
$$-\left(w(w-1)\partial_w+\beta+\frac{1+\alpha+\beta-\mu}{2}(w-1)\right)\mathbf{F}^{\mathrm{I}}_{\alpha,\beta,\mu}(w) = \frac{1+\alpha-\beta-\mu}{2}\mathbf{F}^{\mathrm{I}}_{\alpha,\beta-1,\mu-1}(w),$$
$$-\left(w\partial_w + \frac{1+\alpha+\beta-\mu}{2}\right)\mathbf{F}^{\mathrm{I}}_{\alpha,\beta,\mu}(w) = \frac{1-\alpha+\beta-\mu}{2}\mathbf{F}^{\mathrm{I}}_{\alpha,\beta+1,\mu-1}(w),$$
$$\left(w(w-1)\partial_w+\beta+\frac{1+\alpha+\beta+\mu}{2}(w-1)\right)\mathbf{F}^{\mathrm{I}}_{\alpha,\beta,\mu}(w) = \frac{1-\alpha-\beta+\mu}{2}\mathbf{F}^{\mathrm{I}}_{\alpha,\beta-1,\mu+1}(w);$$

$$\left((w-1)\partial_w + \frac{1+\alpha+\beta+\mu}{2}\right)\mathbf{F}^{\mathrm{I}}_{\alpha,\beta,\mu}(w) = \frac{1+\alpha+\beta+\mu}{2}\mathbf{F}^{\mathrm{I}}_{\alpha+1,\beta,\mu+1}(w),$$
$$\left(w(w-1)\partial_w-\alpha+\frac{1+\alpha+\beta-\mu}{2}w\right)\mathbf{F}^{\mathrm{I}}_{\alpha,\beta,\mu}(w) = \frac{1-\alpha+\beta-\mu}{2}\mathbf{F}^{\mathrm{I}}_{\alpha-1,\beta,\mu-1}(w),$$
$$\left((w-1)\partial_w + \frac{1+\alpha+\beta-\mu}{2}\right)\mathbf{F}^{\mathrm{I}}_{\alpha,\beta,\mu}(w) = \frac{1+\alpha-\beta-\mu}{2}\mathbf{F}^{\mathrm{I}}_{\alpha+1,\beta,\mu-1}(w),$$
$$\left(w(w-1)\partial_w-\alpha+\frac{1+\alpha+\beta+\mu}{2}w\right)\mathbf{F}^{\mathrm{I}}_{\alpha,\beta,\mu}(w) = \frac{1-\alpha-\beta+\mu}{2}\mathbf{F}^{\mathrm{I}}_{\alpha-1,\beta,\mu+1}(w).$$

The recurrence relations are essentially fixed by the transmutation relations. The only missing piece of information is the coefficient on the right hand side, which can be derived by analyzing the behavior of both sides around zero. Another way to obtain these coefficients is to use the integral representations described in the following subsections.

5.10 Wave packets in 6 dimensions

We start with the following easy fact:

Lemma 5.1. *For any τ, the following function is harmonic on $\mathbb{R}^6$:*

$$(z_1-\tau^{-1}z_{-2})^{\alpha+\nu}(z_2+\tau^{-1}z_{-1})^{\beta+\nu}z_3^{\mu} \tag{5.24}$$

Proof. Set $e_1 := (1,0,0,-\tau^{-1})$, $e_2 := (0,\tau^{-1},1,0)$. Then

$$\langle e_1|e_1\rangle = \langle e_2|e_2\rangle = \langle e_2|e_1\rangle = 0.$$

Hence, (5.24) is harmonic by Prop. 3.1. □

Let us make a wave packet out of (5.24), which is an eigenfunction of the Cartan operators:

$$
\begin{aligned}
&K_{\alpha,\beta,\mu,\nu}(z_{-1},z_1,z_{-2},z_2,z_{-3},z_3)\\
:=&\int_\gamma (z_1-\tau^{-1}z_{-2})^{\alpha+\nu}(z_2+\tau^{-1}z_{-1})^{\beta+\nu}z_3^{\mu}\tau^{\nu-1}\frac{\mathrm{d}\tau}{2\pi\mathrm{i}}.
\end{aligned}
\tag{5.25}
$$

Proposition 5.2. *Let the contour* $]0,1[\ni s\overset{\gamma}{\mapsto}\tau(s)$ *satisfy*

$$
(z_1-\tau^{-1}z_{-2})^{\alpha+\nu}(z_2+\tau^{-1}z_{-1})^{\beta+\nu}\tau^{\nu-1}\Big|_{\tau(0)}^{\tau(1)}=0. \tag{5.26}
$$

Then $K_{\alpha,\beta,\mu,\nu}$ *is harmonic and*

$$
N_1K_{\alpha,\beta,\mu,\nu}=\alpha K_{\alpha,\beta,\mu,\nu}, \tag{5.27a}
$$

$$
N_2K_{\alpha,\beta,\mu,\nu}=\beta K_{\alpha,\beta,\mu,\nu}, \tag{5.27b}
$$

$$
N_3K_{\alpha,\beta,\mu,\nu}=\mu K_{\alpha,\beta,\mu,\nu}. \tag{5.27c}
$$

Proof. $K_{\alpha,\beta,\mu,\nu}$ is harmonic by Lemma 5.1. Writing

$$
K_{\alpha,\beta,\mu,\nu}(z)=\int_\gamma(\tau z_1-z_{-2})^{\alpha+\nu}(z_2+\tau^{-1}z_{-1})^{\beta+\nu}z_3^{\mu}\tau^{-\alpha-1}\frac{\mathrm{d}\tau}{2\pi\mathrm{i}} \tag{5.28a}
$$

$$
=\int_\gamma(z_1-\tau^{-1}z_{-2})^{\alpha+\nu}(\tau z_2+z_{-1})^{\beta+\nu}z_3^{\mu}\tau^{-\beta-1}\frac{\mathrm{d}\tau}{2\pi\mathrm{i}}, \tag{5.28b}
$$

we see that (5.27a) and (5.27b) follow from assumption (5.26) by Prop. 3.2. (5.27c) is obvious. □

Proposition 5.3. *If in addition to (5.26) we assume that*

$$
(z_1-\tau^{-1}z_{-2})^{\alpha+\nu}(z_2+\tau^{-1}z_{-1})^{\beta+\nu}\tau^{\nu}\Big|_{\tau(0)}^{\tau(1)}=0, \tag{5.29}
$$

and that we are allowed to differentiate under the integral sign, we obtain the recurrence relations

$$B_{-12}K_{\alpha,\beta,\mu,\nu} = (\beta+\nu)K_{\alpha+1,\beta-1,\mu,\nu}, \tag{5.30a}$$
$$B_{1-2}K_{\alpha,\beta,\mu,\nu} = -(\alpha+\nu)K_{\alpha-1,\beta+1,\mu,\nu}, \tag{5.30b}$$
$$B_{12}K_{\alpha,\beta,\mu,\nu} = (\nu+1)K_{\alpha-1,\beta-1,\mu,\nu+1}, \tag{5.30c}$$
$$B_{-1-2}K_{\alpha,\beta,\mu,\nu} = -(\alpha+\beta+\nu+1)K_{\alpha+1,\beta+1,\mu,\nu-1}, \tag{5.30d}$$
$$B_{1-3}K_{\alpha,\beta,\mu,\nu} = -(\alpha+\nu)K_{\alpha-1,\beta,\mu+1,\nu}, \tag{5.30e}$$
$$B_{-1-3}K_{\alpha,\beta,\mu,\nu} = -(\beta+\nu)K_{\alpha+1,\beta,\mu+1,\nu-1}, \tag{5.30f}$$
$$B_{2-3}K_{\alpha,\beta,\mu,\nu} = -(\beta+\nu)K_{\alpha,\beta-1,\mu+1,\nu}, \tag{5.30g}$$
$$B_{-2-3}K_{\alpha,\beta,\mu,\nu} = (\alpha+\nu)K_{\alpha,\beta+1,\mu+1,\nu-1}. \tag{5.30h}$$

Proof. Relations (5.30a), (5.30b), (5.30e), (5.30f), (5.30g) and (5.30h) are elementary. They follow by simple differentiation under the integral sign and do not need assumptions (5.29) and (5.26).
Relations (5.30c) and (5.30d) require assumption (5.26) and follow by the following computations:

$$\begin{aligned} &B_{12}(z_1-\tau^{-1}z_{-2})^{\alpha+\nu}(z_2+\tau^{-1}z_{-1})^{\beta+\nu}\tau^{\nu+1} \\ =&\,\partial_{\tau^{-1}}(z_1-\tau^{-1}z_{-2})^{\alpha+\nu}(z_2+\tau^{-1}z_{-1})^{\beta+\nu}\tau^{\nu+1} \\ &+(\nu+1)(z_1-\tau^{-1}z_{-2})^{\alpha+\nu}(z_2+\tau^{-1}z_{-1})^{\beta+\nu}\tau^{\nu}, \end{aligned} \tag{5.31}$$

$$\begin{aligned} &B_{-1-2}(\tau z_1-z_{-2})^{\alpha+\nu}(\tau z_2+z_{-1})^{\beta+\nu}\tau^{-\alpha-\beta-\nu-1} \\ =&-\partial_{\tau}(\tau z_1-z_{-2})^{\alpha+\nu}(\tau z_2+z_{-1})^{\beta+\nu}\tau^{-\alpha-\beta-\nu-1} \\ &-(\alpha+\beta+\nu+1)(\tau z_1-z_{-2})^{\alpha+\nu}(\tau z_2+z_{-1})^{\beta+\nu}\tau^{-\alpha-\beta-\nu-2}, \end{aligned} \tag{5.32}$$

where in (5.32) we used yet another representation:

$$K_{\alpha,\beta,\mu,\nu}(z) := \int\limits_{\gamma}(\tau z_1-z_{-2})^{\alpha+\nu}(\tau z_2+z_{-1})^{\beta+\nu}z_3^{\mu}\tau^{-\alpha-\beta-\nu-1}\frac{\mathrm{d}\tau}{2\pi\mathrm{i}}. \tag{5.33}$$

□

If in addition

$$\nu = \frac{-\alpha-\beta-\mu-1}{2},$$

then (5.25) is homogeneous of degree -1, so that we can reduce it to 4 dimensions. Let us substitute the coordinates (5.7), and then set $\tau = \frac{s}{u_1u_2}$, $s = t-w$:

$$K_{\alpha,\beta,\mu,\nu}(u_1,u_2,u_3,r,p,w) = 2^{\frac{1}{2}}u_1^{\alpha}u_2^{\beta}u_3^{\mu}r^{-\mu-1}p^{\mu}F(w), \tag{5.34}$$

$$
\begin{aligned}
F(w) &= \int\limits_{\gamma} (s-1+w)^{\frac{\alpha-\beta-\mu-1}{2}}(s+w)^{\frac{-\alpha+\beta-\mu-1}{2}} s^{\frac{-\alpha-\beta+\mu-1}{2}} \mathrm{d}s \\
&= \int\limits_{\gamma} (t-1)^{\frac{\alpha-\beta-\mu-1}{2}} t^{\frac{-\alpha+\beta-\mu-1}{2}} (t-w)^{\frac{-\alpha-\beta+\mu-1}{2}} \mathrm{d}t. \qquad (5.35)
\end{aligned}
$$

On the spherical section we can remove r and p. Therefore, the function F given by (5.35) satisfies the hypergeometric equation:

$$\mathcal{F}_{\alpha,\beta,\mu}(w,\partial_w)F(w) = 0. \qquad (5.36)$$

From (5.30) we can also easily obtain the recurrence relations for F. Note that in this list the recurrence relations corresponding to $B_{1,3}$, $B_{-1,3}$, $B_{2,3}$ and $B_{-2,3}$ are missing. However, they can be obtained after the reduction to 4 dimensions by an application of the factorization formulas.

5.11 Integral representations

Below we independently prove (5.36), without going through the additional variables. We will use the classical parameters.

Theorem 5.4. *Let* $[0,1] \ni \tau \overset{\gamma}{\mapsto} t(\tau)$ *satisfy*

$$t^{a-c+1}(1-t)^{c-b}(t-w)^{-a-1}\Big|_{t(0)}^{t(1)} = 0.$$

Then

$$\mathcal{F}(a,b;c;w,\partial_w) \int\limits_{\gamma} t^{a-c}(1-t)^{c-b-1}(t-w)^{-a}\mathrm{d}t = 0. \qquad (5.37)$$

Proof. We check that for any contour γ

$$\text{lhs of (5.37)} = -a \int\limits_{\gamma} \mathrm{d}t\, \partial_t t^{a-c+1}(1-t)^{c-b}(t-w)^{-a-1}.$$

□

Analogous (and nonequivalent) integral representations can be obtained by interchanging a and b in Theorem 5.4.

The hypergeometric function with the type I normalization has the integral representation

$$\int_1^\infty t^{a-c}(t-1)^{c-b-1}(t-w)^{-a}\mathrm{d}t \tag{5.38}$$
$$= \mathbf{F}^{\mathrm{I}}(a,b;c;w), \qquad \mathrm{Re}(c-b)>0,\ \mathrm{Re}b>0, \quad w\notin[1,\infty[.$$

Indeed, by Theorem 5.4 the left hand side of (5.38) is annihilated by the hypergeometric operator (5.16). Besides, by Euler's identity it equals $\frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)}$ at 0. So does the right hand side. Therefore, (5.38) follows by the uniqueness of the solution by the Frobenius method.

5.12 Integral representations of standard solutions

The integrand of (5.37) has four singularities: $\{0,1,\infty,w\}$. It is natural to chose γ as the interval joining a pair of singularities. This choice leads to 6 standard solutions with the I-type normalization:

$$\begin{aligned}
\sim 1 \text{ at } 0:&\quad [1,\infty];\\
\sim w^{-\alpha} \text{ at } 0:&\quad [0,w];\\
\sim 1 \text{ at } 1:&\quad [0,\infty];\\
\sim (1-w)^{-\beta} \text{ at } 1:&\quad [1,w];\\
\sim w^{-a} \text{ at } \infty:&\quad [w,\infty];\\
\sim w^{-b} \text{ at } \infty:&\quad [0,1].
\end{aligned}$$

Below we give explicit formulas. To highlight their symmetry, we use Lie-algebraic parameters.

$$\mathrm{Re}(1+\alpha)>|\mathrm{Re}(\beta-\mu)|: \tag{5.39}$$
$$\int_1^\infty t^{\frac{-1-\alpha+\beta+\mu}{2}}(t-1)^{\frac{-1+\alpha-\beta+\mu}{2}}(t-w)^{\frac{-1-\alpha-\beta-\mu}{2}}\mathrm{d}t$$
$$= \mathbf{F}^{\mathrm{I}}_{\alpha,\beta,\mu}(w), \quad w\notin[1,\infty[;$$

$$\mathrm{Re}(1-\alpha) > |\mathrm{Re}(\beta-\mu)| : \tag{5.40}$$

$$\int_0^w t^{\frac{-1-\alpha+\beta+\mu}{2}} (1-t)^{\frac{-1+\alpha-\beta+\mu}{2}} (w-t)^{\frac{-1-\alpha-\beta-\mu}{2}} \mathrm{d}t$$

$$= w^{-\alpha} \mathbf{F}^{\mathrm{I}}_{-\alpha,\beta,-\mu}(w), \quad w \notin]-\infty, 0] \cup [1, \infty[,$$

$$\int_w^0 (-t)^{\frac{-1-\alpha+\beta+\mu}{2}} (1-t)^{\frac{-1+\alpha-\beta+\mu}{2}} (t-w)^{\frac{-1-\alpha-\beta-\mu}{2}} \mathrm{d}t$$

$$= (-w)^{-\alpha} \mathbf{F}^{\mathrm{I}}_{-\alpha,\beta,-\mu}(w), \quad w \notin [0, \infty[;$$

$$\mathrm{Re}(1+\beta) > |\mathrm{Re}(\alpha-\mu)| : \tag{5.41}$$

$$\int_{-\infty}^0 (-t)^{\frac{-1-\alpha+\beta+\mu}{2}} (1-t)^{\frac{-1+\alpha-\beta+\mu}{2}} (w-t)^{\frac{-1-\alpha-\beta-\mu}{2}} \mathrm{d}t$$

$$= \mathbf{F}^{\mathrm{I}}_{\beta,\alpha,\mu}(1-w), \quad w \notin]-\infty, 0];$$

$$\mathrm{Re}(1-\beta) > |\mathrm{Re}(\alpha+\mu)| : \tag{5.42}$$

$$\int_w^1 t^{\frac{-1-\alpha+\beta+\mu}{2}} (1-t)^{\frac{-1+\alpha-\beta+\mu}{2}} (t-w)^{\frac{-1-\alpha-\beta-\mu}{2}} \mathrm{d}t$$

$$= (1-w)^{-\beta} \mathbf{F}^{\mathrm{I}}_{-\beta,\alpha,-\mu}(1-w), \quad w \notin]-\infty, 0] \cup [1, \infty[,$$

$$\int_1^w t^{\frac{-1-\alpha+\beta+\mu}{2}} (t-1)^{\frac{-1+\alpha-\beta+\mu}{2}} (w-t)^{\frac{-1-\alpha-\beta-\mu}{2}} \mathrm{d}t$$

$$= (w-1)^{-\beta} \mathbf{F}^{\mathrm{I}}_{-\beta,\alpha,-\mu}(1-w), \quad w \notin]-\infty, 1];$$

$$\mathrm{Re}(1-\mu) > |\mathrm{Re}(\alpha+\beta)| : \tag{5.43}$$

$$\int_w^\infty t^{\frac{-1-\alpha+\beta+\mu}{2}} (t-1)^{\frac{-1+\alpha-\beta+\mu}{2}} (t-w)^{\frac{-1-\alpha-\beta-\mu}{2}} \mathrm{d}t$$

$$= w^{\frac{-1-\alpha-\beta+\mu}{2}} \mathbf{F}^{\mathrm{I}}_{-\mu,\beta,-\alpha}(w^{-1}), \qquad w \notin]-\infty, 1],$$

$$\int_{-\infty}^w (-t)^{\frac{-1-\alpha+\beta+\mu}{2}} (1-t)^{\frac{-1+\alpha-\beta+\mu}{2}} (w-t)^{\frac{-1-\alpha-\beta-\mu}{2}} \mathrm{d}t$$

$$= (-w)^{\frac{-1-\alpha-\beta+\mu}{2}} \mathbf{F}^{\mathrm{I}}_{-\mu,\beta,-\alpha}(w^{-1}), \quad w \notin]0, \infty];$$

$$\operatorname{Re}(1+\mu) > |\operatorname{Re}(\alpha-\beta)| : \tag{5.44}$$

$$\begin{aligned}
&\int_0^1 t^{\frac{-1-\alpha+\beta-\mu}{2}}(1-t)^{\frac{-1+\alpha-\beta+\mu}{2}}(t-w)^{\frac{-1-\alpha-\beta-\mu}{2}}\mathrm{d}t \\
&\qquad = (-w)^{\frac{-1-\alpha-\beta-\mu}{2}}\mathbf{F}^{\mathrm{I}}_{\mu,\beta,\alpha}(w^{-1}), \quad w \notin [0,\infty[, \\
&\int_0^1 t^{\frac{-1-\alpha+\beta-\mu}{2}}(1-t)^{\frac{-1+\alpha-\beta+\mu}{2}}(w-t)^{\frac{-1-\alpha-\beta-\mu}{2}}\mathrm{d}t \\
&\qquad = w^{\frac{-1-\alpha-\beta-\mu}{2}}\mathbf{F}^{\mathrm{I}}_{\mu,\beta,\alpha}(w^{-1}), \quad w \notin [-\infty,1[.
\end{aligned}$$

5.13 Connection formulas

Generically, each pair of standard solution is a basis of solutions to the hypergeometric equation. For instance, we can use the pair of solutions ~ 1 and $\sim w^{-\alpha}$ at 0 as one basis, and the pair $\sim w^{-a}$ and $\sim w^{-b}$ as another basis. We also assume that $w \notin [0,\infty[$.

Introduce the matrix

$$A_{\alpha,\beta,\mu} := \frac{\pi}{\sin(\pi\mu)}\begin{bmatrix} \frac{-1}{\Gamma(\frac{1+\alpha+\beta-\mu}{2})\Gamma(\frac{1+\alpha-\beta-\mu}{2})} & \frac{1}{\Gamma(\frac{1+\alpha+\beta+\mu}{2})\Gamma(\frac{1+\alpha-\beta+\mu}{2})} \\ \frac{-1}{\Gamma(\frac{1-\alpha-\beta-\mu}{2})\Gamma(\frac{1-\alpha+\beta-\mu}{2})} & \frac{1}{\Gamma(\frac{1-\alpha-\beta+\mu}{2})\Gamma(\frac{1-\alpha+\beta+\mu}{2})} \end{bmatrix}.$$

Then

$$\begin{aligned}
&\begin{bmatrix} \mathbf{F}_{\alpha,\beta,\mu}(w) \\ (-w)^{-\alpha}\mathbf{F}_{-\alpha,\beta,-\mu}(w) \end{bmatrix} \\
&= \quad A_{\alpha,\beta,\mu}\begin{bmatrix} (-w)^{\frac{-1-\alpha-\beta-\mu}{2}}\mathbf{F}_{\mu,\beta,\alpha}(w^{-1}) \\ (-w)^{\frac{-1-\alpha-\beta+\mu}{2}}\mathbf{F}_{-\mu,\beta,-\alpha}(w^{-1}) \end{bmatrix}.
\end{aligned} \tag{5.45}$$

Note that in the Lie-algebraic parameters the matrix $A_{\alpha,\beta,\mu}$ has a very symmetric form. Here are some of its properties:

$$A_{\alpha,\beta,\mu} = A_{\alpha,-\beta,\mu} = -\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} A_{-\alpha,\beta,-\mu}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = A^{-1}_{\mu,\beta,\alpha}, \tag{5.46}$$

$$\det A_{\alpha,\beta,\mu} = -\frac{\sin(\pi\alpha)}{\sin(\pi\mu)}. \tag{5.47}$$

Relation (5.45) can be derived from the integral representations. Indeed, consider $\operatorname{Im} w < 0$. Take the branches of the powers of $-t$ and $1-t$ and $w-t$

continued from the left clockwise onto the upper halfplane. Then (under some conditions on α, β, μ) we can write

$$\Big(\int_{-\infty}^{0} + \int_{0}^{1} + \int_{1}^{+\infty}\Big)(-t)^{\frac{-1-\alpha+\beta\pm\mu}{2}}(1-t)^{\frac{-1+\alpha-\beta\pm\mu}{2}}(w-t)^{\frac{-1-\alpha-\beta\mp\mu}{2}}\mathrm{d}t = 0.$$

We obtain

$$\mathbf{F}^{\mathrm{I}}_{\beta,\alpha,\pm\mu}(1-w) - \mathrm{e}^{\mathrm{i}\pi\alpha}(-w)^{\frac{-1-\alpha-\beta\mp\mu}{2}}\mathbf{F}^{\mathrm{I}}_{\pm\mu,\beta,\alpha}(w^{-1}) - \mathrm{i}\mathrm{e}^{\mathrm{i}\pi\frac{\alpha+\beta\mp\mu}{2}}\mathbf{F}^{\mathrm{I}}_{\alpha,\beta,\pm\mu}(w) = 0.$$

Using

$$\mathbf{F}^{\mathrm{I}}_{\alpha,\beta,\mu}(w) = \Gamma\Big(\frac{1+\alpha+\beta-\mu}{2}\Big)\Gamma\Big(\frac{1+\alpha-\beta+\mu}{2}\Big)\mathbf{F}_{\alpha,\beta,\mu}(w),$$

we express everything in terms of $\mathbf{F}$. We eliminate $\mathbf{F}_{\beta,\alpha,\mu}(1-w) = \mathbf{F}_{\beta,\alpha,-\mu}(1-w)$. We find

$$\begin{aligned}\mathbf{F}_{\alpha,\beta,\mu}(w) &= -\frac{\pi(-w)^{\frac{-1-\alpha-\beta-\mu}{2}}\mathbf{F}_{\mu,\beta,\alpha}(w^{-1})}{\sin(\pi\mu)\Gamma\left(\frac{1+\alpha+\beta-\mu}{2}\right)\Gamma\left(\frac{1+\alpha-\beta-\mu}{2}\right)} \\ &\quad + \frac{\pi(-w)^{\frac{-1-\alpha-\beta+\mu}{2}}\mathbf{F}_{-\mu,\beta,-\alpha}(w^{-1})}{\sin(\pi\mu)\Gamma\left(\frac{1+\alpha+\beta+\mu}{2}\right)\Gamma\left(\frac{1+\alpha-\beta+\mu}{2}\right)},\end{aligned}$$

which is the first line of (5.45). A similar argument, starting with the integral $\int_{-\infty}^{0} + \int_{0}^{w} + \int_{w}^{+\infty}$, yields the second line of (5.45).

6 Laplacian in 3 dimensions and the Gegenbauer equation

The Gegenbauer equation is equivalent to a subclass of the ${}_2\mathcal{F}_1$ equation. Nevertheless, not all its symmetries are directly inherited from the symmetries of the ${}_2\mathcal{F}_1$ equation. Therefore it deserves a separate treatment, which is given in this section. We start from the Laplacian in 5 dimensions, pass through 3 dimensions, and eventually we derive the Gegenbauer equation.

This section is to a large extent parallel to the previous one, devoted to the ${}_2\mathcal{F}_1$ equation. The number of symmetries, parameters, etc. is now smaller than in the previous section, since we are in lower dimensions. Nevertheless, some things are here more complicated and less symmetric. This is related to the fact that the number of dimensions is odd, which corresponds to a less symmetric orthogonal group and Lie algebra.

Let us describe the main steps of our derivation of the Gegenbauer equation, even though they are almost the same as for the ${}_2\mathcal{F}_1$ equation.

(1) We start from the $3+2=5$ dimensional ambient space, with the obvious representation of so(5) and O(5), and the Laplacian Δ_5.
(2) We go to the representations $\mathrm{so}(5) \ni B \mapsto B^{\diamond,\eta}$ and $\mathrm{O}(5) \ni \alpha \mapsto \alpha^{\diamond,\eta}$ and to the reduced Laplacian $\Delta_5^{\diamond}$. The most relevant values of η are $1-\frac{3}{2}=-\frac{1}{2}$ and $-1-\frac{3}{2}=-\frac{5}{2}$.
(3) We fix a section γ of the null quadric, obtaining the representations $B^{\gamma,\eta}$ and $\alpha^{\gamma,\eta}$, as well as the operator Δ_5^{γ}, acting on an appropriate pseudo-Riemannian 3 dimensional manifold.
(4) We choose coordinates w, u_2, u_3, so that the Cartan elements can be expressed in terms of u_2, u_3. We compute $B^{\gamma,\eta}$, $\alpha^{\gamma,\eta}$ and Δ_5^{γ} in the new coordinates.
(5) We make an ansatz that diagonalizes the Cartan elements. The eigenvalues, denoted by α, λ, become parameters. $B^{\gamma,\eta}$, $\alpha^{\gamma,\eta}$ and Δ_5^{γ} involve now only the single variable w. Δ_5^{γ} turns out to be the Gegenbauer operator. We obtain its transmutation relations and discrete symmetries.

Again, we choose a special section which makes computations relatively easy. We perform Steps 2, 3 and 4 at once, by choosing convenient coordinates w, r, p, u_2, u_3 in 5 dimensions. After the reductions of Steps 2 and 3, we are left with the variables w, u_2, u_3, and we can perform Step 5.

The remaining material of this section is parallel to the analogous material of the previous section except for Subsect. 6.4, which describes a quadratic relation reducing the Gegenbauer equation to the ${}_2\mathcal{F}_1$ equation. We describe a derivation of this relation starting from the level of the ambient space.

6.1 so(5) in 5 dimensions

We consider $\mathbb{R}^5$ with the coordinates

$$z_0, z_{-2}, z_2, z_{-3}, z_3 \tag{6.1}$$

and the scalar product given by

$$\langle z|z\rangle = z_0^2 + 2z_{-2}z_2 + 2z_{-3}z_3. \tag{6.2}$$

Note that we omit the indices $-1, 1$; this makes it easier to compare $\mathbb{R}^5$ with $\mathbb{R}^6$.

The Lie algebra so(5) acts naturally on $\mathbb{R}^5$. Below we describe its natural basis. Then we consider the Weyl group B_2 acting on functions on $\mathbb{R}^5$. For brevity, we list only elements from its subgroup $B_2 \cap \mathrm{SO}(5)$. Finally, we write down the Laplacian.

Lie algebra so(5)**.** Cartan algebra

$$N_2 = -z_{-2}\partial_{z_{-2}} + z_2\partial_{z_2}, \tag{6.3a}$$
$$N_3 = -z_{-3}\partial_{z_{-3}} + z_3\partial_{z_3}. \tag{6.3b}$$

Root operators

$$B_{0,-2} = z_0\partial_{z_{-2}} - z_2\partial_{z_0}, \tag{6.4a}$$
$$B_{0,2} = z_0\partial_{z_2} - z_{-2}\partial_{z_0}, \tag{6.4b}$$
$$B_{0,-3} = z_0\partial_{z_{-3}} - z_3\partial_{z_0}, \tag{6.4c}$$
$$B_{0,3} = z_0\partial_{z_3} - z_{-3}\partial_{z_0}; \tag{6.4d}$$

$$B_{-3,-2} = z_3\partial_{z_{-2}} - z_2\partial_{z_{-3}}, \tag{6.4e}$$
$$B_{3,2} = z_{-3}\partial_{z_2} - z_{-2}\partial_{z_3}, \tag{6.4f}$$
$$B_{3,-2} = z_{-3}\partial_{z_{-2}} - z_2\partial_{z_3}, \tag{6.4g}$$
$$B_{-3,2} = z_3\partial_{z_2} - z_{-2}\partial_{z_{-3}}. \tag{6.4h}$$

Weyl symmetries

$$\sigma_{23}K(z_0, z_{-2}, z_2, z_{-3}, z_3) = K(z_0, z_{-2}, z_2, z_{-3}, z_3), \tag{6.5a}$$
$$\tau_{2-3}K(z_0, z_{-2}, z_2, z_{-3}, z_3) = K(-z_0, z_{-2}, z_2, z_3, z_{-3}), \tag{6.5b}$$
$$\sigma_{-2-3}K(z_0, z_{-2}, z_2, z_{-3}, z_3) = K(z_0, z_2, z_{-2}, z_3, z_{-3}), \tag{6.5c}$$
$$\tau_{-23}K(z_0, z_{-2}, z_2, z_{-3}, z_3) = K(-z_0, z_2, z_{-2}, z_{-3}, z_3); \tag{6.5d}$$

$$\sigma_{32}K(z_0, z_{-2}, z_2, z_{-3}, z_3) = K(z_0, z_{-3}, z_3, z_{-2}, z_2), \tag{6.5e}$$
$$\tau_{3-2}K(z_0, z_{-2}, z_2, z_{-3}, z_3) = K(-z_0, z_{-3}, z_3, z_2, z_{-2}), \tag{6.5f}$$
$$\sigma_{-3-2}K(z_0, z_{-2}, z_2, z_{-3}, z_3) = K(z_0, z_3, z_{-3}, z_2, z_{-2}), \tag{6.5g}$$
$$\tau_{-32}K(z_0, z_{-2}, z_2, z_{-3}, z_3) = K(-z_0, z_3, z_{-3}, z_{-2}, z_2). \tag{6.5h}$$

Laplacian

$$\Delta_5 = \partial_{z_0}^2 + 2\partial_{z_{-2}}\partial_{z_2} + 2\partial_{z_{-3}}\partial_{z_3}. \tag{6.6}$$

6.2 so(5) on the spherical section

In this subsection we perform Steps 2, 3 and 4, as described in the introduction to this section. Recall that Step 2 involves restricting to the null quadric

$$\mathcal{V}^4 := \{z \in \mathbb{R}^5 \ : \ z_0^2 + 2z_{-2}z_2 + 2z_{-3}z_3 = 0\}.$$

To perform Step 3 we need to fix a section of this quadric. We choose the section given by the equations

$$1 = z_0^2 + 2z_{-2}z_2 = -2z_3z_{-3}.$$

We will call it the *spherical section*, because it is $\mathbb{S}^2(1) \times \mathbb{S}^1(-1)$. The superscript used for this section will be "sph" for spherical.

We introduce the coordinates w, r, p, u_2, u_3:

$$r = \sqrt{z_0^2 + 2z_{-2}z_2}\,, \tag{6.7a}$$

$$w = \frac{z_0}{\sqrt{2z_{-2}z_2 + z_0^2}}\,, \qquad u_2 = \frac{\sqrt{2}z_2}{\sqrt{z_0^2 + 2z_{-2}z_2}}\,, \tag{6.7b}$$

$$p = \sqrt{-2z_3z_{-3}}\,, \qquad u_3 = \sqrt{-\frac{z_3}{z_{-3}}}\,. \tag{6.7c}$$

Here is the inverse transformation:

$$z_0 = wr, \qquad z_{-2} = \frac{r(1-w^2)}{\sqrt{2}u_2}, \qquad z_2 = \frac{u_2 r}{\sqrt{2}}, \tag{6.8a}$$

$$z_{-3} = -\frac{p}{\sqrt{2}u_3}, \qquad z_3 = \frac{pu_3}{\sqrt{2}}. \tag{6.8b}$$

Similarly as in the previous section, the null quadric in these coordinates is given by $r^2 = p^2$. We choose the sheet $r = p$. The generator of dilations is

$$A_5 = r\,\partial_r + p\,\partial_p.$$

The spherical section is given by the condition $r^2 = 1$.

Lie algebra so(5)**.** Cartan operators

$$N_2^{\mathrm{sph}} = u_2\,\partial_{u_2}\,,$$
$$N_3^{\mathrm{sph}} = u_3\,\partial_{u_3}\,.$$

Roots

$$B_{0,-2}^{\mathrm{sph}} = -\frac{u_2}{\sqrt{2}}\partial_w,$$
$$B_{0,2}^{\mathrm{sph}} = \frac{1}{\sqrt{2}u_2}\Big((w^2-1)\partial_w + 2wu_2\partial_{u_2}\Big),$$
$$B_{0,-3}^{\mathrm{sph},\eta} = \frac{u_3}{\sqrt{2}}\Big((w^2-1)\partial_w + wu_2\partial_{u_2} + wu_3\partial_{u_3} - w\eta\Big),$$
$$B_{0,3}^{\mathrm{sph},\eta} = \frac{1}{\sqrt{2}u_3}\Big((1-w^2)\partial_w - wu_2\partial_{u_2} + wu_3\partial_{u_3} + w\eta\Big);$$

$$
\begin{aligned}
B^{\mathrm{sph},\eta}_{-3,-2} &= \frac{u_2 u_3}{2}\left(-w\partial_w - u_2\partial_{u_2} - u_3\partial_{u_3} + \eta\right),\\
B^{\mathrm{sph},\eta}_{3,2} &= \frac{1}{2u_2u_3}\Big(w(1-w^2)\partial_w - (1+w^2)u_2\partial_{u_2} + (w^2-1)u_3\partial_{u_3} + (w^2-1)\eta\Big),\\
B^{\mathrm{sph},\eta}_{3,-2} &= \frac{u_2}{2u_3}\left(w\partial_w + u_2\partial_{u_2} - u_3\partial_{u_3} - \eta\right),\\
B^{\mathrm{sph},\eta}_{-3,2} &= \frac{u_3}{2u_2}\Big(w(w^2-1)\partial_w + (1+w^2)u_2\partial_{u_2} + (w^2-1)u_3\partial_{u_3} + (1-w^2)\eta\Big).
\end{aligned}
$$

Weyl symmetries

$$
\begin{aligned}
\sigma^{\mathrm{sph},\eta}_{23} f(w,u_2,u_3) =& f(w,u_2,u_3),\\
\tau^{\mathrm{sph},\eta}_{2-3} f(w,u_2,u_3) =& f\Big(-w, u_2, \frac{1}{u_3}\Big),\\
\sigma^{\mathrm{sph},\eta}_{-2-3} f(w,u_2,u_3) =& f\Big(w, \frac{1-w^2}{u_2}, \frac{1}{u_3}\Big),\\
\tau^{\mathrm{sph},\eta}_{-23} f(w,u_2,u_3) =& f\Big(-w, \frac{1-w^2}{u_2}, u_3\Big);
\end{aligned}
$$

$$
\begin{aligned}
\sigma^{\mathrm{sph},\eta}_{32} f(w,u_2,u_3) =& (w^2-1)^{\frac{\eta}{2}} f\Big(\frac{w}{\sqrt{w^2-1}}, \frac{u_3}{\sqrt{w^2-1}}, \frac{u_2}{\sqrt{w^2-1}}\Big),\\
\tau^{\mathrm{sph},\eta}_{3-2} f(w,u_2,u_3) =& (w^2-1)^{\frac{\eta}{2}} f\Big(\frac{-w}{\sqrt{w^2-1}}, \frac{u_3}{\sqrt{w^2-1}}, \frac{\sqrt{w^2-1}}{u_2}\Big),\\
\sigma^{\mathrm{sph},\eta}_{-3-2} f(w,u_2,u_3) =& (w^2-1)^{\frac{\eta}{2}} f\Big(\frac{w}{\sqrt{w^2-1}}, \frac{-1}{u_3\sqrt{w^2-1}}, \frac{\sqrt{w^2-1}}{u_2}\Big),\\
\tau^{\mathrm{sph},\eta}_{-32} f(w,u_2,u_3) =& (w^2-1)^{\frac{\eta}{2}} f\Big(\frac{-w}{\sqrt{w^2-1}}, \frac{-1}{u_3\sqrt{w^2-1}}, \frac{u_2}{\sqrt{w^2-1}}\Big).
\end{aligned}
$$

Laplacian

$$
\Delta^{\mathrm{sph}}_5 = (1-w^2)\partial_w^2 - 2(1+u_2\partial_{u_2})w\partial_w - \Big(u_2\partial_{u_2} + \frac{1}{2}\Big)^2 + (u_3\partial_{u_3})^2.
$$

Let us sketch the computations that lead to (6.9). Using

$$
\begin{aligned}
\partial_{z_0} &= \frac{1}{r}\Big(wr\partial_r - wu_2\partial_{u_2} + (1-w^2)\partial_w\Big),\\
\partial_{z_{-2}} &= \frac{u_2}{\sqrt{2}r}\Big(r\partial_r - u_2\partial_{u_2} - w\partial_w\Big),\\
\partial_{z_2} &= \frac{1}{\sqrt{2}ru_2}\Big((1-w^2)r\partial_r + (1+w^2)u_2\partial_{u_2} + (w^2-1)w\partial_w\Big),\\
\partial_{z_{-3}} &= \frac{u_3}{\sqrt{2}p}\Big(u_3\partial_{u_3} - p\partial_p\Big),\\
\partial_{z_3} &= \frac{1}{\sqrt{2}pu_3}\Big(u_3\partial_{u_3} + p\partial_p\Big).
\end{aligned}
$$

we change the variables in the Laplacian:

$$\begin{aligned}\Delta_5 =& \frac{1}{r^2}\Big((1-w^2)\partial_w^2 - 2(1+u_2\partial_{u_2})w\partial_w - (u_2\partial_{u_2})^2 - u_2\partial_{u_2} \\ &+ (r\partial_r)^2 + r\partial_r\Big) + \frac{1}{p^2}\Big(-(p\partial_p)^2 + (u_3\partial_{u_3})^2\Big).\end{aligned} \tag{6.9}$$

Now,

$$\begin{aligned}(r\,\partial_r)^2 + r\,\partial_r - \frac{r^2}{p^2}(p\,\partial_p)^2 &= \Big(r\,\partial_r - p\partial_p + \frac{1}{2}\Big)\Big(r\,\partial_r + p\partial_p + \frac{1}{2}\Big) \\ &\quad + \Big(1 - \frac{r^2}{p^2}\Big)(p\partial_p)^2 - \frac{1}{4}.\end{aligned}$$

Therefore, using $r^2 = p^2$, $r\partial_r + p\partial_p = -\frac{1}{2}$, we obtain

$$\begin{aligned}\Delta_5^{\diamond} = \frac{1}{r^2}\Big(&(1-w^2)^2\partial_w^2 - 2(1+u_2\partial_{u_2})w\partial_w \\ &- \big(u_2\partial_{u_2} + \frac{1}{2}\big)^2 + (u_3\partial_{u_3})^2\Big).\end{aligned} \tag{6.10}$$

To obtain the Laplacian at the spherical section we drop $\frac{1}{r^2}$.

6.3 The Gegenbauer operator

Let us make the ansatz

$$f(u_2, u_3, w) = u_2^{\alpha} u_3^{\lambda} S(w). \tag{6.11}$$

Clearly,

$$N_2^{\mathrm{sph}} f = \alpha f, \tag{6.12a}$$

$$N_3^{\mathrm{sph}} f = \lambda f, \tag{6.12b}$$

$$u_2^{-\alpha} u_3^{-\lambda} \Delta_5^{\mathrm{sph}} f = \mathcal{S}_{\alpha,\lambda}(w, \partial_w) S(w), \tag{6.12c}$$

where

$$\mathcal{S}_{\alpha,\lambda}(w,\partial_w) \quad := \quad (1-w^2)\partial_w^2 - 2(1+\alpha)w\partial_w + \lambda^2 - \Big(\alpha + \frac{1}{2}\Big)^2 \tag{6.13}$$

is the *Gegenbauer operator*. Here is another parametrization of the Gegenbauer operator, which we call *classical*:

$$\mathcal{S}(a,b;w,\partial_w) := (1-w^2)\partial_w^2 - (a+b+1)w\partial_w - ab. \tag{6.14}$$

Here is the relationship between the classical and Lie-algebraic parameters:

$$\alpha := \frac{a+b-1}{2}, \qquad \lambda := \frac{b-a}{2}, \tag{6.15a}$$

$$a = \frac{1}{2} + \alpha - \lambda, \qquad b = \frac{1}{2} + \alpha + \lambda. \tag{6.15b}$$

The Gegenbauer operator is the ${}_2\mathcal{F}_1$ operator with its finite singular points moved to -1 and 1, which in addition is reflection invariant. Because of the reflection invariance, the third classical parameter can be obtained from the first two: $c = \frac{a+b+1}{2}$. Therefore, we use only $a, b \in \mathbb{C}$ as the (classical) parameters of the Gegenbauer equation.

We can reduce the Gegenbauer equation to the ${}_2\mathcal{F}_1$ equation by two affine transformations. They move the singular points from -1, 1 to 0, 1 or 1, 0:

$$\mathcal{S}(a, b; w, \partial_w) = \mathcal{F}\big(a, b; \tfrac{a+b+1}{2}; v, \partial_v\big), \tag{6.16}$$

where

$$v = \frac{1-w}{2}, \quad w = 1 - 2v, \tag{6.17a}$$

$$\text{or} \qquad v = \frac{1+w}{2}, \quad w = -1 + 2v. \tag{6.17b}$$

In the Lie-algebraic parameters

$$\mathcal{S}_{\alpha,\lambda}(w, \partial_w) = \mathcal{F}_{\alpha,\alpha,2\lambda}(v, \partial_v). \tag{6.18}$$

6.4 Quadratic transformation

Let us go back to 6 dimensions and the Laplacian

$$\Delta_6 = 2\partial_{z_{-1}}\partial_{z_1} + 2\partial_{z_{-2}}\partial_{z_2} + 2\partial_{z_{-3}}\partial_{z_3}. \tag{6.19}$$

Let us use the reduction described in Subsect. 3.14. Introduce new variables

$$z_0 := \sqrt{2z_{-1}z_1}, \quad u := \sqrt{\frac{z_1}{z_{-1}}}. \tag{6.20}$$

In the new variables,

$$N_1 = u\partial_u, \tag{6.21}$$

$$\begin{aligned}\Delta_6 = &\Big(\partial_{z_0} + \frac{1}{2z_0}\Big)^2 - \frac{1}{z_0^2}\Big(u\partial_u - \frac{1}{2}\Big)\Big(u\partial_u + \frac{1}{2}\Big)\\ &+ 2\partial_{z_{-2}}\partial_{z_2} + 2\partial_{z_{-3}}\partial_{z_3}.\end{aligned} \tag{6.22}$$

Therefore,

$$(uz_0)^{\frac{1}{2}}\Delta_6(uz_0)^{-\frac{1}{2}} = -\frac{1}{z_0^2}N_1\Big(N_1-1\Big)+\Delta_5, \tag{6.23a}$$

$$(u^{-1}z_0)^{\frac{1}{2}}\Delta_6(u^{-1}z_0)^{-\frac{1}{2}} = -\frac{1}{z_0^2}N_1\Big(N_1+1\Big)+\Delta_5. \tag{6.23b}$$

Compare the coordinates the coordinates (5.7) for 6 dimensions and (6.7) for 5 dimensions. The coordinates p, u_3 are the same. Taking into account $z_0 := \sqrt{2z_{-1}z_1}$, the coordinates r, u_2 also coincide. This is not the case of w, so let us rename w from (6.7) as v. We then have $w = v^2$. We also have

$$uz_0 = \sqrt{2}z_1 = u_1 r, \quad u^{-1}z_0 = \sqrt{2}z_{-1} = rwu_1^{-1}.$$

Hence on functions that do not depend on u we obtain

$$r^{\frac{1}{2}}u_1^{\frac{1}{2}}\Delta_6 r^{-\frac{1}{2}}u_1^{-\frac{1}{2}} = \Delta_5, \tag{6.24a}$$

$$r^{\frac{1}{2}}u_1^{-\frac{1}{2}}v\Delta_6 r^{-\frac{1}{2}}u_1^{\frac{1}{2}}v^{-1} = \Delta_5. \tag{6.24b}$$

This implies that a quadratic substitution transforms the ${}_2\mathcal{F}_1$ operator with $\alpha = \pm\frac{1}{2}$ into the Gegenbauer operator. Explicitly, if

$$w = v^2, \quad v = \sqrt{w};$$

then in the classical parameters

$$\mathcal{S}(a,b;v,\partial_v) = 4\mathcal{F}\Big(\frac{a}{2},\frac{b}{2};\frac{1}{2};w,\partial_w\Big), \tag{6.25a}$$

$$v^{-1}\mathcal{S}(a,b;v,\partial_v)v = 4\mathcal{F}\Big(\frac{a+1}{2},\frac{b+1}{2};\frac{3}{2};w,\partial_w\Big), \tag{6.25b}$$

and in the Lie-algebraic parameters

$$\mathcal{S}_{\alpha,\lambda}(v,\partial_v) = 4\mathcal{F}_{-\frac{1}{2},\alpha,\lambda}(w,\partial_w), \tag{6.26a}$$

$$v^{-1}\mathcal{S}_{\alpha,\lambda}(v,\partial_v)v = 4\mathcal{F}_{\frac{1}{2},\alpha,\lambda}(w,\partial_w). \tag{6.26b}$$

6.5 Transmutation relations and discrete symmetries

We have the following generalized symmetries:

$$B^{\mathrm{sph},-\frac{5}{2}}\Delta_5^{\mathrm{sph}} = \Delta_5^{\mathrm{sph}}B^{\mathrm{sph},-\frac{1}{2}}, \quad B\in \mathrm{so}(5); \tag{6.27a}$$

$$\alpha^{\mathrm{sph},-\frac{5}{2}}\Delta_5^{\mathrm{sph}} = \Delta_5^{\mathrm{sph}}\alpha^{\mathrm{sph},-\frac{1}{2}}, \quad \alpha\in \mathrm{O}(5). \tag{6.27b}$$

Equality (6.27a) applied to the roots of so(5) yield the following transmutation relations:

$$
\begin{aligned}
\partial_w & \quad \mathcal{S}_{\alpha,\lambda}\\
= \quad \mathcal{S}_{\alpha+1,\lambda} & \quad \partial_w,\\
((1-w^2)\partial_w - 2\alpha w) & \quad \mathcal{S}_{\alpha,\lambda}\\
= \quad \mathcal{S}_{\alpha-1,\lambda} & \quad ((1-w^2)\partial_w - 2\alpha w),\\
((1-w^2)\partial_w - (\alpha+\lambda+\tfrac12)w) & \quad (1-w^2)\mathcal{S}_{\alpha,\lambda}\\
= \quad (1-w^2)\mathcal{S}_{\alpha,\lambda+1} & \quad ((1-w^2)\partial_w - (\alpha+\lambda+\tfrac12)w),\\
((1-w^2)\partial_w - (\alpha-\lambda+\tfrac12)w) & \quad (1-w^2)\mathcal{S}_{\alpha,\lambda}\\
= \quad (1-w^2)\mathcal{S}_{\alpha,\lambda-1} & \quad ((1-w^2)\partial_w - (\alpha-\lambda+\tfrac12)w);\\
(w\partial_w + \alpha - \lambda + \tfrac12) & \quad w^2\mathcal{S}_{\alpha,\lambda}\\
= \quad w^2\mathcal{S}_{\alpha+1,\lambda-1} & \quad (w\partial_w + \alpha - \lambda + \tfrac12),\\
(w(1-w^2)\partial_w - \alpha - \lambda + \tfrac12 - (\alpha-\lambda+\tfrac12)w^2) & \quad w^2\mathcal{S}_{\alpha,\lambda}\\
= \quad w^2\mathcal{S}_{\alpha-1,\lambda+1} & \quad (w(1-w^2)\partial_w - \alpha - \lambda + \tfrac12 - (\alpha-\lambda+\tfrac12)w^2),\\
(w\partial_w + \alpha - \lambda + \tfrac12) & \quad w^2\mathcal{S}_{\alpha,\lambda}\\
= \quad w^2\mathcal{S}_{\alpha+1,\lambda+1} & \quad (w\partial_w + \alpha - \lambda + \tfrac12),\\
(w(1-w^2)\partial_w - \alpha + \lambda + \tfrac12 - (\alpha+\lambda+\tfrac12)w^2) & \quad w^2\mathcal{S}_{\alpha,\lambda}\\
= \quad w^2\mathcal{S}_{\alpha-1,\lambda-1} & \quad (w(1-w^2)\partial_w - \alpha + \lambda + \tfrac12 - (\alpha+\lambda+\tfrac12)w^2).
\end{aligned}
$$

Next we describe discrete symmetries of the Gegenbauer operator, which follow from Relation (6.27b) applied to Weyl symmetries. All the operators below equal $\mathcal{S}_{\alpha,\lambda}(w,\partial_w)$ for the appropriate w:

$$
\begin{array}{rrccc}
w = \pm v: & & \mathcal{S}_{\alpha,\pm\lambda}(v,\partial_v), & \\
w = \pm v: & (v^2-1)^{-\alpha} & \mathcal{S}_{-\alpha,\mp\lambda}(v,\partial_v) & (v^2-1)^{\alpha},\\
w = \frac{\pm v}{(v^2-1)^{\frac12}}: & (v^2-1)^{\frac12(\alpha+\lambda+\frac52)} & \mathcal{S}_{\lambda,\pm\alpha}(v,\partial_v) & (v^2-1)^{\frac12(-\alpha-\lambda-\frac12)},\\
w = \frac{\pm v}{(v^2-1)^{\frac12}}: & (v^2-1)^{\frac12(\alpha-\lambda+\frac52)} & \mathcal{S}_{-\lambda,\mp\alpha}(v,\partial_v) & (v^2-1)^{\frac12(-\alpha+\lambda-\frac12)}.
\end{array}
$$

Note that we use $\pm$ to describe two symmetries at once. Therefore, the above list has all $2\times4=8$ symmetries corresponding to the lists of Weyl symmetries (6.5).

6.6 Factorizations of the Laplacian

In the Lie algebra so(5) represented on $\mathbb{R}^5$ we have 3 distinguished Lie subalgebras: two isomorphic to so(3) and one isomorphic to so(4):

$$\mathrm{so}_{02}(3), \quad \mathrm{so}_{03}(3), \quad \mathrm{so}_{23}(4), \tag{6.28}$$

where we use an obvious notation. By (4.9) and (4.11), the corresponding Casimir operators are

$$\begin{aligned}
\mathcal{C}_{02} &= 2B_{0,-2}B_{0,2} - \left(N_2 - \frac{1}{2}\right)^2 + \frac{1}{4} && \text{(6.29a)}\\
&= 2B_{0,2}B_{0,-2} - \left(N_2 + \frac{1}{2}\right)^2 + \frac{1}{4}, && \text{(6.29b)}\\
\mathcal{C}_{03} &= 2B_{0,-3}B_{0,3} - \left(N_3 - \frac{1}{2}\right)^2 + \frac{1}{4} && \text{(6.29c)}\\
&= 2B_{0,3}B_{0,-3} - \left(N_3 + \frac{1}{2}\right)^2 + \frac{1}{4}, && \text{(6.29d)}\\
\mathcal{C}_{23} &= 4B_{2,3}B_{-2,-3} - (N_2 + N_3 + 1)^2 + 1 && \text{(6.29e)}\\
&= 4B_{-2,-3}B_{2,3} - (N_2 + N_3 - 1)^2 + 1 && \text{(6.29f)}\\
&= 4B_{2,-3}B_{-2,3} - (N_2 - N_3 + 1)^2 + 1 && \text{(6.29g)}\\
&= 4B_{-2,3}B_{2,-3} - (N_2 - N_3 - 1)^2 + 1. && \text{(6.29h)}
\end{aligned}$$

After the reduction described in (4.21) and (4.19), we obtain the identities

$$\begin{aligned}
(z_0^2 + 2z_{-2}z_2)\Delta_5^\diamond &= -\frac{1}{4} + \mathcal{C}_{02}^{\diamond,-\frac{1}{2}} + (N_3^{\diamond,-\frac{1}{2}})^2, && \text{(6.30a)}\\
(z_0^2 + 2z_{-3}z_3)\Delta_5^\diamond &= -\frac{1}{4} + \mathcal{C}_{03}^{\diamond,-\frac{1}{2}} + (N_2^{\diamond,-\frac{1}{2}})^2, && \text{(6.30b)}\\
(2z_{-2}z_2 + 2z_{-3}z_3)\Delta_5^\diamond &= -\frac{3}{4} + \mathcal{C}_{23}^{\diamond,-\frac{1}{2}}. && \text{(6.30c)}
\end{aligned}$$

Inserting (6.29) into (6.30), we obtain

$$
\begin{aligned}
&(z_0^2 + 2z_{-2}z_2)\Delta_5^\diamond \\
=&2B_{0,-2}B_{0,2} - \Big(N_2 + N_3 - \frac{1}{2}\Big)\Big(N_2 - N_3 - \frac{1}{2}\Big) && (6.31a)\\
=&2B_{0,2}B_{0,-2} - \Big(N_2 + N_3 + \frac{1}{2}\Big)\Big(N_2 - N_3 + \frac{1}{2}\Big), && (6.31b)\\
&(z_0^2 + 2z_{-3}z_3)\Delta_5^\diamond \\
=&2B_{0,-3}B_{0,3} - \Big(N_2 + N_3 - \frac{1}{2}\Big)\Big(-N_2 + N_3 - \frac{1}{2}\Big) && (6.31c)\\
=&2B_{0,3}B_{0,-3} - \Big(N_2 + N_3 + \frac{1}{2}\Big)\Big(-N_2 + N_3 + \frac{1}{2}\Big), && (6.31d)\\
&(2z_{-2}z_2 + 2z_{-3}z_3)\Delta_5^\diamond \\
=&4B_{2,3}B_{-2,-3} - \Big(N_2 + N_3 + \frac{3}{2}\Big)\Big(N_2 + N_3 + \frac{1}{2}\Big) && (6.31e)\\
=&4B_{-2,-3}B_{2,3} - \Big(N_2 + N_3 - \frac{3}{2}\Big)\Big(N_2 + N_3 - \frac{1}{2}\Big) && (6.31f)\\
=&4B_{2,-3}B_{-2,3} - \Big(N_2 - N_3 + \frac{3}{2}\Big)\Big(N_2 - N_3 + \frac{1}{2}\Big) && (6.31g)\\
=&4B_{-2,3}B_{2,-3} - \Big(N_2 - N_3 - \frac{3}{2}\Big)\Big(N_2 - N_3 - \frac{1}{2}\Big), && (6.31h)
\end{aligned}
$$

where all the B and N operators need to have the superscript ${}^{\diamond,-\frac{1}{2}}$.

If we use the spherical section, we need to make the replacements

$$
\begin{aligned}
z_0^2 + 2z_{-2}z_2 &\to 1, && (6.32a)\\
z_0^2 + 2z_{-3}z_3 &\to w^2 - 1, && (6.32b)\\
2z_{-2}z_2 + 2z_{-3}z_3 &\to -w^2, && (6.32c)
\end{aligned}
$$

and replace the superscript ${}^\diamond$ with ${}^{\mathrm{sph}}$.

6.7 Factorizations of the Gegenbauer equation

The factorizations of Δ_5^{sph} of Subsect. 6.6 yield the following factorizations of the Gegenbauer operator:

$$\begin{aligned}
\mathcal{S}_{\alpha,\lambda} =&\partial_w\Big((1-w^2)\partial_w-2\alpha w\Big)\\
&+\Big(\alpha+\lambda-\frac12\Big)\Big(-\alpha+\lambda+\frac12\Big)\\
=&\Big((1-w^2)\partial_w-2(1+\alpha)w\Big)\partial_w\\
&+\Big(\alpha+\lambda+\frac12\Big)\Big(-\alpha+\lambda-\frac12\Big),\\
(1-w^2)\mathcal{S}_{\alpha,\lambda} =&\Big((1-w^2)\partial_w-\big(\alpha+\lambda-\frac12\big)w\Big)\Big((1-w^2)\partial_w-\big(\alpha-\lambda+\frac12\big)w\Big)\\
&+\Big(\alpha+\lambda-\frac12\Big)\Big(\alpha-\lambda+\frac12\Big)\\
=&\Big((1-w^2)\partial_w-\big(\alpha-\lambda-\frac12\big)w\Big)\Big((1-w^2)\partial_w-\big(\alpha+\lambda+\frac12\big)w\Big)\\
&+\Big(\alpha+\lambda+\frac12\Big)\Big(\alpha-\lambda-\frac12\Big),
\end{aligned}$$

$$\begin{aligned}
w^2\mathcal{S}_{\alpha,\lambda} =&\Big(w(1-w^2)\partial_w-\alpha-\lambda-\frac32+\big(-\alpha+\lambda-\frac12\big)w^2\Big)\Big(w\partial_w+\alpha+\lambda+\frac12\Big)\\
&+\Big(\alpha+\lambda+\frac12\Big)\Big(\alpha+\lambda+\frac32\Big)\\
=&\Big(w\partial_w+\alpha+\lambda-\frac32\Big)\Big(w(1-w^2)\partial_w-\alpha-\lambda+\frac12+\big(-\alpha+\lambda-\frac12\big)w^2\Big)\\
&+\Big(\alpha+\lambda-\frac12\Big)\Big(\alpha+\lambda-\frac32\Big)\\
=&\Big(w(1-w^2)\partial_w-\alpha+\lambda-\frac32+\big(-\alpha-\lambda-\frac12\big)w^2\Big)\Big(w\partial_w+\alpha-\lambda+\frac12\Big)\\
&+\Big(\alpha-\lambda+\frac12\Big)\Big(\alpha-\lambda+\frac32\Big)\\
=&\Big(w\partial_w+\alpha-\lambda-\frac32\Big)\Big(w(1-w^2)\partial_w-\alpha+\lambda+\frac12+\big(-\alpha-\lambda-\frac12\big)w^2\Big)\\
&+\Big(\alpha-\lambda-\frac12\Big)\Big(\alpha-\lambda-\frac32\Big).
\end{aligned}$$

6.8 Standard solutions

As usual, by standard solutions we mean solutions with a simple behavior around singular points. The singular points of the Gegenbauer equation are $\{1,-1,\infty\}$. The discussion of the point -1 can be easily reduced to that of 1. Therefore, it is enough to discuss $2\times2=4$ solutions corresponding to two indices at 1 and ∞.

The standard solutions can be expressed in terms of the function

$$S_{\alpha,\lambda}(w) = S(a,b;w) := F\Big(a,b;\frac{a+b+1}{2};\frac{1-w}{2}\Big) = F\Big(\frac{a}{2},\frac{b}{2};\frac{a+b+1}{2};1-w^2\Big). \tag{6.33}$$

Here are the 4 standard solutions. We consistently use the Lie-algebraic parameters.

$$\sim 1 \text{ at } 1: \qquad S_{\alpha,\lambda}(w) = F_{\alpha,\alpha,2\lambda}\Big(\frac{1-w}{2}\Big) = F_{\alpha,-\frac{1}{2},\lambda}(1-w^2),$$

$$\sim \frac{1}{2^\alpha(1-w)^\alpha} \text{ at } 1: \qquad (1-w^2)^{-\alpha}S_{-\alpha,-\lambda}(w) = 2^{-\alpha}(1-w)^{-\alpha}F_{-\alpha,\alpha,-2\lambda}\Big(\frac{1-w}{2}\Big) = (1-w^2)^{-\alpha}F_{-\alpha,-\frac{1}{2},-\lambda}(1-w^2),$$

$$\sim w^{-a} \text{ at } \infty: \qquad (w^2-1)^{\frac{-1-2\alpha+2\lambda}{4}}S_{-\lambda,-\alpha}\Big(\frac{w}{\sqrt{w^2-1}}\Big) = (1+w)^{-\frac{1}{2}-\alpha+\lambda}F_{-2\lambda,\alpha,-\alpha}\Big(\frac{2}{1+w}\Big) = w^{-\frac{1}{2}-\alpha+\lambda}F_{-\lambda,\alpha,\frac{1}{2}}(w^{-2}),$$

$$\sim w^{-b} \text{ at } \infty: \qquad (w^2-1)^{\frac{-1-2\alpha-2\lambda}{4}}S_{\lambda,\alpha}\Big(\frac{w}{\sqrt{w^2-1}}\Big) = (1+w)^{-\frac{1}{2}-\alpha-\lambda}F_{2\lambda,\alpha,\alpha}\Big(\frac{2}{1+w}\Big) = w^{-\frac{1}{2}-\alpha-\lambda}F_{\lambda,\alpha,\frac{1}{2}}(w^{-2}).$$

6.9 Recurrence relations

We will use the following normalization to express recurrence relations:

$$\mathbf{S}_{\alpha,\lambda}(w) := \frac{1}{\Gamma(\alpha+1)}S_{\alpha,\lambda}(w) = \frac{1}{\Gamma(\frac{a+b+1}{2})}F\Big(a,b;\frac{a+b+1}{2};\frac{1-w}{2}\Big) = \mathbf{F}_{\alpha,\alpha,2\lambda}\Big(\frac{1-w}{2}\Big). \tag{6.34}$$

To each root of so(5) there corresponds a recurrence relation:

$$\partial_w \mathbf{S}_{\alpha,\lambda}(w) = -\frac{1}{2}\Big(\frac{1}{2}+\alpha-\lambda\Big)\Big(\frac{1}{2}+\alpha+\lambda\Big)\mathbf{S}_{\alpha+1,\lambda}(w),$$
$$\big((1-w^2)\partial_w - 2\alpha w\big)\,\mathbf{S}_{\alpha,\lambda}(w) = -2\mathbf{S}_{\alpha-1,\lambda}(w),$$

$$\bigg((1-w^2)\partial_w - \Big(\frac{1}{2}+\alpha+\lambda\Big)w\bigg)\,\mathbf{S}_{\alpha,\lambda}(w) = -\Big(\frac{1}{2}+\alpha+\lambda\Big)\mathbf{S}_{\alpha,\lambda+1}(w),$$
$$\bigg((1-w^2)\partial_w - \Big(\frac{1}{2}+\alpha-\lambda\Big)w\bigg)\,\mathbf{S}_{\alpha,\lambda}(w) = -\Big(\frac{1}{2}+\alpha-\lambda\Big)\mathbf{S}_{\alpha,\lambda-1}(w);$$

$$\bigg(w\partial_w + \frac{1}{2}+\alpha-\lambda\bigg)\,\mathbf{S}_{\alpha,\lambda}(w) = \frac{1}{2}\Big(\frac{1}{2}+\alpha-\lambda\Big)\Big(\frac{3}{2}+\alpha-\lambda\Big)\mathbf{S}_{\alpha+1,\lambda-1}(w),$$
$$\bigg(w(1-w^2)\partial_w+\Big(\frac{1}{2}-\alpha+\lambda\Big)(1-w^2)-2\alpha w^2\bigg)\,\mathbf{S}_{\alpha,\lambda}(w) = -2\mathbf{S}_{\alpha-1,\lambda+1}(w),$$

$$\bigg(w\partial_w + \frac{1}{2}+\alpha+\lambda\bigg)\,\mathbf{S}_{\alpha,\lambda}(w) = \frac{1}{2}\Big(\frac{1}{2}+\alpha+\lambda\Big)\Big(\frac{3}{2}+\alpha+\lambda\Big)\mathbf{S}_{\alpha+1,\lambda+1}(w),$$
$$\bigg(w(1-w^2)\partial_w+\Big(\frac{1}{2}-\alpha-\lambda\Big)(1-w^2)-2\alpha w^2\bigg)\,\mathbf{S}_{\alpha,\lambda}(w) = -2\mathbf{S}_{\alpha-1,\lambda-1}(w).$$

6.10 Wave packets in 5 dimensions

We easily check the following lemma:

Lemma 6.1. *For any τ, the function $z_2^{\alpha}\big(\sqrt{2}z_0 - \tau^{-1}z_{-3} + \tau z_3\big)^{\nu}$ is harmonic.*

Let us make a wave packet from the above functions.

Proposition 6.2. *Let the contour $]0,1[\ni s \overset{\gamma}{\mapsto} \tau(s)$ satisfy*

$$\big(\sqrt{2}z_0 - \tau^{-1}z_{-3} + \tau z_3\big)^{\nu}\tau^{-\lambda}\Big|_{\tau(0)}^{\tau(1)} = 0. \tag{6.35}$$

Then the function

$$K_{\alpha,\nu,\lambda}(z_0, z_{-2}, z_2, z_{-3}, z_3) := \int_{\gamma} z_2^{\alpha}\big(\sqrt{2}z_0 - \tau^{-1}z_{-3} + \tau z_3\big)^{\nu}\tau^{-\lambda-1}\mathrm{d}\tau$$

is harmonic and

$$N_2 K_{\alpha,\nu,\lambda} = \alpha K_{\alpha,\nu,\lambda}, \tag{6.36a}$$
$$N_3 K_{\alpha,\nu,\lambda} = \lambda K_{\alpha,\nu,\lambda}. \tag{6.36b}$$

Proof. (6.36a) is obvious. To obtain (6.36b) we use Prop. 3.2. □

If in addition
$$\nu=-\alpha-\frac{1}{2},$$
then $K_{\alpha,\nu,\lambda}$ is homogeneous of degree $-\frac{1}{2}$. Therefore, we can reduce it to dimension 3. Let us express it in the coordinates w,r,p,u_2,u_3:

$$\begin{aligned}K(w,r,p,u_2,u_3)&=\int u_2^\alpha r^\alpha\Big(wr\sqrt{2}+\frac{p}{\tau u_3\sqrt{2}}+\frac{\tau p u_3}{\sqrt{2}}\Big)^{-\alpha-\frac{1}{2}}\tau^{-\lambda-1}\mathrm{d}\tau\\&=(\sqrt{2})^{\alpha+\frac{1}{2}}u_2^\alpha u_3^\lambda r^{-\frac{1}{2}}\int\Big(2w\sigma+(1+\sigma^2)\frac{p}{r}\Big)^{-\alpha-\frac{1}{2}}\sigma^{\alpha-\lambda-\frac{1}{2}}\mathrm{d}\sigma,\end{aligned}$$

where we set $\sigma:=u_3\tau$. Noting that on the spherical section $p=r$, we see that

$$S(w):=\int\big(2w\sigma+1+\sigma^2\big)^{-\alpha-\frac{1}{2}}\sigma^{\alpha-\lambda-\frac{1}{2}}\mathrm{d}\sigma \tag{6.37}$$

satisfies the Gegenbauer equation.

6.11 Integral representations

In this subsection we describe two kinds of integral representations for solutions to the Gegenbauer equation. The first is essentially inherited from the ${}_2\mathcal{F}_1$ equation. The second was derived using additional variables in the previous subsection. Here we give independent derivations. We will use classical parameters.

Theorem 6.3. *a) Let* $[0,1]\ni\tau\overset{\gamma}{\mapsto}t(\tau)$ *satisfy*
$$(t^2-1)^{\frac{b-a+1}{2}}(t-w)^{-b-1}\Big|_{t(0)}^{t(1)}=0.$$
Then
$$\mathcal{S}(a,b;w,\partial_w)\int_\gamma(t^2-1)^{\frac{b-a-1}{2}}(t-w)^{-b}\mathrm{d}t=0. \tag{6.38}$$

b) Let $[0,1]\ni\tau\overset{\gamma}{\mapsto}t(\tau)$ *satisfy*
$$(t^2+2tw+1)^{\frac{-b-a}{2}+1}t^{b-2}\Big|_{t(0)}^{t(1)}=0.$$
Then
$$\mathcal{S}(a,b;w,\partial_w)\int_\gamma(t^2+2tw+1)^{\frac{-b-a}{2}}t^{b-1}\mathrm{d}t=0. \tag{6.39}$$

Proof. For any contour γ we have

$$\begin{aligned}
\text{lhs of (6.38)} &= a\int_\gamma \mathrm{d}t\,\partial_t (t^2-1)^{\frac{b-a+1}{2}}(t-w)^{-b-1},\\
\text{lhs of (6.39)} &= \int_\gamma \mathrm{d}t\,\partial_t (t^2+2tw+1)^{\frac{-b-a}{2}+1}t^{b-2}.
\end{aligned}$$

□

Note that in the above theorem we can interchange a and b. Thus we obtain four kinds of integral representations.

6.12 Integral representations of the standard solutions

As described in Thm 6.3, we have two types of integral representations of solutions of Gegenbauer equations: a) and b). It is natural to use singular points of the integrands as the endpoints of the contours of integration. For the representations of type a) we have singular points at $\infty, -1, 1, w$. For representations of type b) singular points are at $\infty, 0$ and the two roots of $t^2+2tw+1=0$. Choosing an appropriate contour we obtain all standard solutions with both types of representations with some special normalizations. It is convenient to introduce special notation for these normalizations:

$$\begin{aligned}
\mathbf{S}^{\mathrm{I}}_{\alpha,\lambda}(w) &:= 2^{-\frac12-\alpha-\lambda}\frac{\Gamma(\frac{1+2\alpha+2\lambda}{2})\Gamma(\frac{1-2\lambda}{2})}{\Gamma(\alpha+1)}S_{\alpha,\lambda}(w) \\
&= 2^{-b}\frac{\Gamma(b)\Gamma(\frac{a-b+1}{2})}{\Gamma(\frac{a+b+1}{2})}F\Big(a,b;\frac{a+b+1}{2};\frac{1-w}{2}\Big) \\
&= 2^{-\frac12-\alpha-\lambda}\mathbf{F}^{\mathrm{I}}_{\alpha,\alpha,2\lambda}\Big(\frac{1-w}{2}\Big),
\end{aligned} \tag{6.40}$$

$$\begin{aligned}
\mathbf{S}^{\mathrm{II}}_{\alpha,\lambda}(w) &:= \frac{\Gamma(\frac{1+2\alpha-2\lambda}{2})\Gamma(\frac{1+2\alpha+2\lambda}{2})}{\Gamma(2\alpha+1)}S_{\alpha,\lambda}(w) \\
&= \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}F\Big(a,b;\frac{a+b+1}{2};\frac{1-w}{2}\Big),
\end{aligned} \tag{6.41}$$

$$\begin{aligned}
\mathbf{S}^{0}_{\alpha,\lambda}(w) &:= \sqrt{\pi}\frac{\Gamma(\frac{1+2\alpha}{2})}{\Gamma(\alpha+1)}S_{\alpha,\lambda}(w) \\
&= \sqrt{\pi}\frac{\Gamma(\frac{1+2\alpha}{2})}{\Gamma(\alpha+1)}F\Big(a,b;\frac{a+b+1}{2};\frac{1-w}{2}\Big).
\end{aligned} \tag{6.42}$$

In the following table we list all standard solutions together with the contours of integration and the corresponding normalizations.

	a)	b)
~ 1 **at** 1**:**	$]-\infty,-1]$, I;	$[0,\infty[$, II;
$\sim \frac{1}{2^{\alpha}(1-w)^{\alpha}}$ **at** 1**:**	$]-1,w]$, I;	$[-\mathrm{i}\sqrt{1-w^2}-w,\mathrm{i}\sqrt{1-w^2}-w]$, 0;
$\sim w^{-a}$ **at** ∞**:**	$]-1,1]$, 0;	$[\sqrt{w^2-1}-w,0[$, I;
$\sim w^{-b}$ **at** ∞**:**	$]w,\infty]$, II;	$]-\infty,-\sqrt{w^2-1}-w]$, I.

Here are representations of type a):

$$\frac{1}{2}>\mathrm{Re}\lambda>-\frac{1}{2}-\mathrm{Re}\alpha: \tag{6.43}$$

$$\int\limits_{-\infty}^{-1}(t^2-1)^{-\frac{1}{2}-\lambda}(w-t)^{-\frac{1}{2}-\alpha+\lambda}\mathrm{d}t$$

$$=\mathbf{S}^{\mathrm{I}}_{\alpha,\lambda}(w),\quad w\notin]-\infty,-1];$$

$$\frac{1}{2}>\mathrm{Re}\lambda>-\frac{1}{2}+\mathrm{Re}\alpha: \tag{6.44}$$

$$\int\limits_{w}^{1}(1-t^2)^{-\frac{1}{2}-\lambda}(w-t)^{-\frac{1}{2}-\alpha+\lambda}\mathrm{d}t$$

$$=(1-w^2)^{-\alpha}\mathbf{S}^{\mathrm{I}}_{-\alpha,-\lambda}(w),\quad w\notin]-\infty,-1]\cup[1,\infty[;$$

$$\frac{1}{2}>\mathrm{Re}\lambda: \tag{6.45}$$

$$\int\limits_{-1}^{1}(1-t^2)^{-\frac{1}{2}-\lambda}(w-t)^{-\frac{1}{2}-\alpha+\lambda}\mathrm{d}t$$

$$=(w^2-1)^{\frac{-1-2\alpha+2\lambda}{4}}\mathbf{S}^{0}_{-\lambda,\alpha}\Big(\frac{w}{\sqrt{w^2-1}}\Big),\quad w\notin]-\infty,1];$$

$$\mathrm{Re}\lambda + \frac{1}{2} > |\mathrm{Re}\alpha| : \tag{6.46}$$

$$\int_w^\infty (t^2-1)^{-\frac{1}{2}-\lambda}(t-w)^{-\frac{1}{2}-\alpha+\lambda}\mathrm{d}t$$
$$= (w^2-1)^{\frac{-1-2\alpha-2\lambda}{4}}\mathbf{S}^{\mathrm{II}}_{\lambda,\alpha}\Big(\frac{w}{\sqrt{w^2-1}}\Big), \quad w \notin]-\infty,1].$$

Next we list representations of type b):

$$\mathrm{Re}\alpha + \frac{1}{2} > |\mathrm{Re}\lambda| : \tag{6.47}$$

$$\int_0^\infty (t^2+2tw+1)^{-\alpha-\frac{1}{2}}t^{-\frac{1}{2}+\alpha+\lambda}\mathrm{d}t$$
$$= \mathbf{S}^{\mathrm{II}}_{\alpha,\lambda}(w) \quad w \notin]-\infty,-1];$$

$$\frac{1}{2} > \mathrm{Re}\alpha : \tag{6.48}$$

$$\int_{-\mathrm{i}\sqrt{1-w^2}-w}^{\mathrm{i}\sqrt{1-w^2}-w} (t^2+2tw+1)^{-\alpha-\frac{1}{2}}(-t)^{-\frac{1}{2}+\alpha+\lambda}\mathrm{d}t$$
$$= \mathrm{i}(1-w^2)^{-\alpha}\mathbf{S}^{0}_{-\alpha,-\lambda}(w), \quad w \notin]-\infty,-1] \cup [1,\infty[;$$

$$-\mathrm{Re}\lambda + \frac{1}{2} > -\mathrm{Re}\alpha > -\frac{1}{2} : \tag{6.49}$$

$$\int_{\sqrt{w^2-1}-w}^{0} (t^2+2tw+1)^{-\alpha-\frac{1}{2}}(-t)^{-\frac{1}{2}+\alpha-\lambda}\mathrm{d}t$$
$$= (w^2-1)^{\frac{-1-2\alpha+2\lambda}{4}}\mathbf{S}^{\mathrm{I}}_{-\lambda,\alpha}\Big(\frac{w}{\sqrt{w^2-1}}\Big), \quad w \notin]-\infty,1];$$

$$\mathrm{Re}\lambda + \frac{1}{2} > -\mathrm{Re}\alpha > -\frac{1}{2} : \tag{6.50}$$

$$\int_{-\infty}^{-\sqrt{w^2-1}-w} (t^2+2tw+1)^{-\alpha-\frac{1}{2}}(-t)^{-\frac{1}{2}+\alpha-\lambda}\mathrm{d}t$$
$$= (w^2-1)^{-\frac{1}{4}-\frac{\alpha}{2}-\frac{\lambda}{2}}\mathbf{S}^{\mathrm{I}}_{\lambda,\alpha}\Big(\frac{w}{\sqrt{w^2-1}}\Big), \quad w \notin]-\infty,1].$$

7 The Schrödinger Lie algebra and the heat equation

By the *heat equation* on $\mathbb{R}^n \oplus \mathbb{R}$ we mean the equation given by the *heat operator*

$$\mathcal{L}_n := \Delta_n + 2\partial_t. \tag{7.1}$$

This operator has a large family of generalized symmetries, the so-called *Schrödinger Lie algebra* and *group*. They can be derived from conformal symmetries of the Laplace equation. In this section we describe this derivation.

In order to be consistent with Sect. 4, it is convenient to consider $\mathcal{L}_{n-2}$ instead of $\mathcal{L}_n$. Then the starting point, just as in Sect. 4, is the $n+2$-dimensional ambient space. The Schrödinger Lie algebra and group are naturally contained in the pseudo-orhogonal Lie algebra and group for $n+2$ dimensions. Then, as described in Sect. 4.12, we descend to the (flat) n dimensional space and the corresponding Laplacian Δ_n. We assume that our functions depend on y_m only through the factor e^{y_m}. The variable y_{-m} is renamed to t (the "time"). The Schrödinger Lie algebra and group respects functions of that form. The Laplacian Δ_n on such fuctions becomes the heat operator $\mathcal{L}_{n-2}$. From the generalized symmetries of Δ_n we obtain generalized symmetries of $\mathcal{L}_{n-2}$.

7.1 $\mathrm{sch}(n{-}2)$ as a subalgebra of $\mathrm{so}(n{+}2)$

We consider again the space $\mathbb{R}^{n+2}$ with the split scalar product. A special role will be played by the operator

$$B_{m+1,m} = z_{-m-1}\partial_{z_m} - z_{-m}\partial_{z_{m+1}} \in \mathrm{so}(n+2).$$

We define the *Schrödinger Lie algebra* and the *Schrödinger group* as the commutants (centralizers) of this element:

$$\mathrm{sch}(n-2) := \{B \in \mathrm{so}(n+2)\ :\ [B, B_{m+1,m}] = 0\}, \tag{7.2a}$$

$$\mathrm{Sch}(n-2) := \{\alpha \in \mathrm{O}(n+2)\ :\ \alpha B_{m+1,m} = B_{m+1,m}\alpha\}. \tag{7.2b}$$

7.2 Structure of $\mathrm{sch}(n{-}2)$

Let us describe the structure of $\mathrm{sch}(n{-}2)$.

We will use our usual notation for elements of $\mathrm{so}(n{+}2)$ and $\mathrm{O}(n{+}2)$. In particular,

$$N_m = -z_{-m}\partial_{z_{-m}} + z_m\partial_{z_m}, \qquad N_{m+1} = -z_{-m-1}\partial_{z_{-m-1}} + z_{m+1}\partial_{z_{m+1}}.$$

Define

$$M := -N_m + N_{m+1}. \tag{7.3}$$

Note that M belongs to $\mathrm{sch}(n-2)$ and commutes with $\mathrm{so}(n-2)$, which is naturally embedded in $\mathrm{sch}(n-2)$.

The Lie algebra $\mathrm{sch}(n-2)$ is spanned by the following operators:

(1) $B_{m+1,m}$, which spans the center of $\mathrm{sch}(n-2)$.

(2) $B_{m,j}$, $B_{m+1,j}$, $|j| = 1, \dots, m-1$, which have the following nonzero commutator:

$$[B_{m,j}, B_{m+1,-j}] = B_{m+1,m}. \tag{7.4}$$

(3) $B_{m+1,-m}$, $B_{-m-1,m}$, M, which have the usual commutation relations of $\mathrm{sl}(2) \simeq \mathrm{so}(3)$:

$$[B_{m+1,-m}, B_{-m-1,m}] = M, \tag{7.5a}$$

$$[M, B_{m+1,-m}] = -2B_{m+1,-m}, \tag{7.5b}$$

$$[M, B_{-m-1,m}] = 2B_{-m-1,m}. \tag{7.5c}$$

(4) $B_{i,j}$, $|i| < |j| \le m-1$, N_i, $i = 1, \dots, m-1$, with the usual commutation relations of $\mathrm{so}(n-2)$.

The span of (2) can be identified with $\mathbb{R}^{n-2} \oplus \mathbb{R}^{n-2} \simeq \mathbb{R}^2 \otimes \mathbb{R}^{n-2}$, which has a natural structure of a symplectic space. The span of (1) and (2) is the central extension of the abelian algebra $\mathbb{R}^2 \otimes \mathbb{R}^{n-2}$ by (7.4). Such a Lie algebra is usually called the *Heisenberg Lie algebra over* $\mathbb{R}^2 \otimes \mathbb{R}^{n-2}$ and can be denoted by

$$\mathrm{heis}(2(n-2)) = \mathbb{R} \rtimes (\mathbb{R}^2 \otimes \mathbb{R}^{n-2}). \tag{7.6}$$

Lie algebras $\mathrm{sl}(2)$ and $\mathrm{so}(n-2)$ act in the obvious way on $\mathbb{R}^2$, resp. $\mathbb{R}^{n-2}$. Thus $\mathrm{sl}(2) \oplus \mathrm{so}(n-2)$ acts on $\mathbb{R}^2 \otimes \mathbb{R}^{n-2}$. Thus

$$\mathrm{sch}(n-2) \quad \simeq \quad \mathbb{R} \rtimes (\mathbb{R}^2 \otimes \mathbb{R}^{n-2}) \rtimes (\mathrm{sl}(2) \oplus \mathrm{so}(n-2)). \tag{7.7}$$

Note, in particular, that $\mathrm{sch}(n-2)$ is not semisimple.

The subalgebra spanned by the usual Cartan algebra of $\mathrm{so}(n-2)$, M and $B_{-m-1,m}$ is a maximal commutative subalgebra of $\mathrm{sch}(n-2)$. It will be called the *Cartan algebra* of $\mathrm{sch}(n-2)$.

Let us introduce $\kappa \in \mathrm{SO}(n+2)$:

$$\kappa(\dots, z_{-m}, z_m, z_{-m-1}, z_{m+1}) := (\dots, z_{-m-1}, z_{m+1}, -z_{-m}, -z_m). \tag{7.8}$$

Note that $\kappa^4 = \iota$ and $\kappa \in \mathrm{Sch}(n-2)$. On the level of functions

$$\kappa K(\dots, z_{-m}, z_m, z_{-m-1}, z_{m+1}) := K(\dots, -z_{-m-1}, -z_{m+1}, z_{-m}, z_m). \tag{7.9}$$

The subgroup of $\mathrm{Sch}(n-2)$ generated by the Weyl group of $\mathrm{O}(n-2)$ and κ will be called the *Weyl group* of $\mathrm{sch}(n-2)$.

7.3 $\mathrm{sch}(n+2)$ in n dimensions

Recall from Subsect. 4.12 that using the decomposition $\mathbb{R}^{n+2} = \mathbb{R}^n \oplus \mathbb{R}^2$ we obtain the representations

$$\mathrm{so}(n+2) \ni B \quad \mapsto \quad B^{\mathrm{fl},\eta}, \tag{7.10a}$$
$$\mathrm{O}(n+2) \ni \alpha \quad \mapsto \quad \alpha^{\mathrm{fl},\eta} \tag{7.10b}$$

acting on functions on $\mathbb{R}^n$. The Laplacian Δ_{n+2} becomes the Laplacian Δ_n and it satisfies the generalized symmetry

$$B^{\mathrm{fl},\frac{-2-n}{2}}\Delta_n \quad = \quad \Delta_n B^{\mathrm{fl},\frac{2-n}{2}}, \quad B \in \mathrm{so}(n+2), \tag{7.11a}$$
$$\alpha^{\mathrm{fl},\frac{-2-n}{2}}\Delta_n \quad = \quad \Delta_n \alpha^{\mathrm{fl},\frac{2-n}{2}}, \quad \alpha \in \mathrm{O}(n+2). \tag{7.11b}$$

The operator $B_{m+1,m}$ becomes

$$B_{m+1,m}^{\mathrm{fl},\eta} = \partial_{y_m}. \tag{7.12}$$

Therefore, all elements of $\mathrm{sch}(n-2)$ in the representation (7.10a) and all elements of $\mathrm{Sch}(n-2)$ in the representation (7.10b) have the form

$$B^{\mathrm{fl},\eta} = C + D\partial_{y_m}, \tag{7.13a}$$
$$\alpha^{\mathrm{fl},\eta} f(\dots, y_{-m}, y_m) = \beta f\big(\dots, y_{-m}, y_m + d(\dots, y_{-m})\big), \tag{7.13b}$$

where C, D, β, d, do not involve the variable y_m.

7.4 $\mathrm{sch}(n-2)$ in $(n-2)+1$ dimensions

We consider now the space $\mathbb{R}^{n-2} \oplus \mathbb{R}$ with the generic variables $(y, t) = (\dots, y_{m-1}, t)$. Note that t should be understood as the new name for y_{-m}, and we keep the old names for the first $n-2$ coordinates.

We define the map $\theta : C^\infty(\mathbb{R}^{n-2} \oplus \mathbb{R}) \to C^\infty(\mathbb{R}^n)$ by setting

$$(\theta h)(\dots, y_{m-1}, y_{-m}, y_m) := h(\dots, y_{m-1}, y_{-m})\mathrm{e}^{y_m}. \tag{7.14}$$

We also define $\zeta : C^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^{n-2} \oplus \mathbb{R})$

$$(\zeta f)(\dots, y_{m-1}, t) := f(\dots, y_{m-1}, t, 0). \tag{7.15}$$

Clearly, ζ is a left inverse of θ:

$$\zeta \circ \theta = \iota. \tag{7.16}$$

Therefore, $\theta \circ \zeta = \iota$ is true on the range of θ.

The heat operator in $n-2$ spatial dimensions can be obtained from the Laplacian in n dimension:

$$\mathcal{L}_{n-2} := \Delta_{n-2} + 2\partial_t = \zeta\Delta_n\theta. \tag{7.17}$$

For $B \in \mathrm{sch}(n-2) \subset \mathrm{so}(n+2)$ and $\alpha \in \mathrm{Sch}(n-2) \subset \mathrm{O}(n+2)$ we define

$$B^{\mathrm{sch},\eta} := \zeta B^{\mathrm{fl},\eta}\theta, \tag{7.18a}$$
$$\alpha^{\mathrm{sch},\eta} := \zeta\alpha^{\mathrm{fl},\eta}\theta. \tag{7.18b}$$

It is easy to see, using (7.13), that $\mathrm{sch}(n-2)$, $\mathrm{Sch}(n-2)$ and Δ_n preserve the range of θ. Therefore, for any η we obtain representations

$$\mathrm{sch}(n-2) \ni B \quad \mapsto \quad B^{\mathrm{sch},\eta}, \tag{7.19a}$$
$$\mathrm{Sch}(n-2) \ni \alpha \quad \mapsto \quad \alpha^{\mathrm{sch},\eta} \tag{7.19b}$$

acting on functions on $\mathbb{R}^{n-2} \oplus \mathbb{R}$. By (7.11), we also have generalized symmetries:

$$B^{\mathrm{sch},\frac{-2-n}{2}}\mathcal{L}_{n-2} = \mathcal{L}_{n-2}B^{\mathrm{sch},\frac{2-n}{2}}, \qquad B \in \mathrm{sch}(n-2), \tag{7.20a}$$
$$\alpha^{\mathrm{sch},\frac{-2-n}{2}}\mathcal{L}_{n-2} = \mathcal{L}_{n-2}\alpha^{\mathrm{sch},\frac{2-n}{2}}, \qquad \alpha \in \mathrm{Sch}(n-2). \tag{7.20b}$$

7.5 Schrödinger symmetries in coordinates

In this subsection we sum up information about Schrödinger symmetries on 3 levels described in the previous subsections.

We start with generic names of the variables and the corresponding squares:

$$z \in \mathbb{R}^{n+2}, \quad \langle z|z\rangle_{n+2} = \sum_{|j|\le m+1} z_{-j}z_j, \tag{7.21a}$$
$$y \in \mathbb{R}^{n}, \quad \langle y|y\rangle_{n} = \sum_{|j|\le m} y_{-j}y_j, \tag{7.21b}$$
$$(y,t) \in \mathbb{R}^{n-2}\oplus\mathbb{R}, \quad \langle y|y\rangle_{n-2} = \sum_{|j|\le m-1} y_{-j}y_j. \tag{7.21c}$$

Cartan algebra of $\mathrm{sch}(n-2)$**.** Central element

$$B_{m+1,m} = z_{-m-1}\partial_{z_m} - z_{-m}\partial_{z_{m+1}}, \tag{7.22a}$$
$$B^{\mathrm{fl}}_{m+1,m} = \partial_{y_m}, \tag{7.22b}$$
$$B^{\mathrm{sch}}_{m+1,m} = 1. \tag{7.22c}$$

Cartan algebra of so$(n{-}2)$, $j = 1, \dots, m-1$,

$$N_j = -z_{-j}\partial_{z_{-j}} + z_j\partial_{z_j}, \tag{7.23a}$$
$$N_j^{\mathrm{fl}} = -y_{-j}\partial_{y_{-j}} + y_j\partial_{y_j}, \tag{7.23b}$$
$$N_j^{\mathrm{sch}} = -y_{-j}\partial_{y_{-j}} + y_j\partial_{y_j}. \tag{7.23c}$$

Generator of scaling

$$M = z_{-m}\partial_{z_{-m}} - z_m\partial_{z_m} - z_{-m-1}\partial_{z_{-m-1}} + z_{m+1}\partial_{z_{m+1}}, \tag{7.24a}$$
$$M^{\mathrm{fl},\eta} = \sum_{|j|\leq m-1} y_j\partial_{y_j} + 2y_{-m}\partial_{y_{-m}} - \eta, \tag{7.24b}$$
$$M^{\mathrm{sch},\eta} = \sum_{|j|\leq m-1} y_j\partial_{y_j} + 2t\partial_t - \eta. \tag{7.24c}$$

Root operators of sch$(n{-}2)$**.** Roots of so$(n{-}2)$, $|i| < |j| \leq m-1$,

$$B_{i,j} = z_{-i}\partial_{z_j} - z_{-j}\partial_{z_i}, \tag{7.25a}$$
$$B_{i,j}^{\mathrm{fl}} = y_{-i}\partial_{y_j} - y_{-j}\partial_{y_i}, \tag{7.25b}$$
$$B_{i,j}^{\mathrm{sch}} = y_{-i}\partial_{y_j} - y_{-j}\partial_{y_i}. \tag{7.25c}$$

Space translations, $|j| \leq m-1$,

$$B_{m+1,j} = z_{-m-1}\partial_{z_j} - z_{-j}\partial_{z_{m+1}}, \tag{7.26a}$$
$$B_{m+1,j}^{\mathrm{fl}} = \partial_{y_j}, \tag{7.26b}$$
$$B_{m+1,j}^{\mathrm{sch}} = \partial_{y_j}. \tag{7.26c}$$

Time translation

$$B_{m+1,-m} = z_{-m-1}\partial_{z_{-m}} - z_m\partial_{z_{m+1}}, \tag{7.27a}$$
$$B_{m+1,-m}^{\mathrm{fl}} = \partial_{y_{-m}}, \tag{7.27b}$$
$$B_{m+1,-m}^{\mathrm{sch}} = \partial_t. \tag{7.27c}$$

Additional roots, $|j| \leq m-1$,

$$B_{m,j} = z_{-m}\partial_{z_j} - z_{-j}\partial_{z_m}, \tag{7.28a}$$
$$B_{m,j}^{\mathrm{fl}} = y_{-m}\partial_{y_j} - y_{-j}\partial_{y_m}, \tag{7.28b}$$
$$B_{m,j}^{\mathrm{sch}} = t\partial_{y_j} - y_{-j}; \tag{7.28c}$$

$$B_{-m-1,m} = z_{m+1}\partial_{z_m} - z_{-m}\partial_{z_{-m-1}}, \tag{7.29a}$$

$$\begin{aligned} B^{\mathrm{fl},\eta}_{-m-1,m} &= y_{-m}\Big(\sum_{|j|\le m-1} y_j\partial_{y_j} + y_{-m}\partial_{y_{-m}} - \eta\Big) \\ &\quad -\frac{1}{2}\sum_{|j|\le m-1} y_{-j}y_j\partial_{y_m}, \end{aligned} \tag{7.29b}$$

$$\begin{aligned} B^{\mathrm{sch},\eta}_{-m-1,m} &= t\Big(\sum_{|j|\le m-1} y_j\partial_{y_j} + t\partial_t - \eta\Big) \\ &\quad -\frac{1}{2}\sum_{|j|\le m-1} y_{-j}y_j. \end{aligned} \tag{7.29c}$$

Weyl symmetries. We present a representative selection of elements of the Weyl group of $\mathrm{Sch}(n-2)$. We will write K for a function on $\mathbb{R}^{n+2}$, f for a function on $\mathbb{R}^n$, h for a function on $\mathbb{R}^{n-2}\oplus\mathbb{R}$ in the coordinates $\big(\dots,y_{m-1},t\big)$.

Reflection (for odd n)

$$\begin{aligned} &\tau_0 K(z_0,\dots,\dots,z_{-m},z_m,z_{-m-1},z_{m+1}) \\ &\qquad = K(-z_0,\dots,z_{-m},z_m,z_{-m-1},z_{m+1}), \end{aligned} \tag{7.30a}$$

$$\begin{aligned} &\tau_0^{\mathrm{fl}} f(y_0,\dots,y_{-m},y_m) \\ &\qquad = f(-y_0,\dots,y_{-m},y_m), \end{aligned} \tag{7.30b}$$

$$\tau_0^{\mathrm{sch}} h(y_0,\dots,t) = h(-y_0,\dots,t). \tag{7.30c}$$

Flips, $j = 1,\dots,m-1$,

$$\begin{aligned} &\tau_j K(\dots,z_{-j},z_j,\dots,z_{-m},z_m,z_{-m-1},z_{m+1}) \\ &\qquad = K(\dots,z_j,z_{-j},\dots,z_{-m},z_m,z_{-m-1},z_{m+1}), \end{aligned} \tag{7.31a}$$

$$\begin{aligned} &\tau_j^{\mathrm{fl}} f(\dots,y_{-j},y_j,\dots,y_{-m},y_m) \\ &\qquad = f(\dots,y_j,y_{-j},\dots,y_{-m},y_m), \end{aligned} \tag{7.31b}$$

$$\tau_j^{\mathrm{sch}} h(\dots,y_{-j},y_j,\dots,t) = h(\dots,y_j,y_{-j},\dots,t). \tag{7.31c}$$

Permutations, $\pi \in S_{m-1}$,

$$\begin{aligned} &\sigma_\pi K(\dots,z_{-m+1},z_{m-1},z_{-m},z_m,z_{-m-1},z_{m+1}) \\ &\qquad = K(\dots,z_{-\pi_{m-1}},z_{\pi_{m-1}},z_{-m},z_m,z_{-m-1},z_{m+1}), \end{aligned} \tag{7.32a}$$

$$\begin{aligned} &\sigma_\pi^{\mathrm{fl}} f(\dots,y_{-m+1},y_{m-1},y_{-m},y_m) \\ &\qquad = f(\dots,y_{-\pi_{m-1}},y_{\pi_{m-1}},y_{-m},y_m), \end{aligned} \tag{7.32b}$$

$$\begin{aligned} &\sigma_\pi^{\mathrm{sch}} h(\dots,y_{-m+1},y_{m-1},t) \\ &\qquad = h(\dots,y_{-\pi_{m-1}},y_{\pi_{m-1}},t). \end{aligned} \tag{7.32c}$$

Special transformation κ

$$\begin{aligned}
&\kappa K(\dots, z_{m-1}, z_{-m}, z_m, z_{-m-1}, z_{m+1}) \\
&\qquad = K(\dots, z_{m-1}, -z_{-m-1}, -z_{m+1}, z_{-m}, z_m),
\end{aligned} \tag{7.33a}$$

$$\begin{aligned}
&\kappa^{\mathrm{fl},\eta} f(\dots, y_{m-1}, y_{-m}, y_m) \\
&= y_{-m}^{\eta} f\Big(\dots, \frac{y_{m-1}}{y_{-m}}, -\frac{1}{y_{-m}}, \frac{1}{2y_{-m}} \sum_{|j|\leq m} y_{-j} y_j\Big),
\end{aligned} \tag{7.33b}$$

$$\begin{aligned}
&\kappa^{\mathrm{sch},\eta} h(\dots, y_{m-1}, t) \\
&= t^{\eta} \exp\Big(\frac{1}{2t} \sum_{|j|\leq m-1} y_{-j} y_j\Big) h\Big(\dots, \frac{y_{m-1}}{t}, -\frac{1}{t}\Big).
\end{aligned} \tag{7.33c}$$

Square of κ

$$\begin{aligned}
&\kappa^2 K(\dots, z_{m-1}, z_{-m}, z_m, z_{-m-1}, z_{m+1}) \\
&\qquad = K(\dots, z_{m-1}, -z_{-m}, -z_m, -z_{-m-1}, -z_{m+1}),
\end{aligned} \tag{7.34a}$$

$$\begin{aligned}
&(\kappa^{\mathrm{fl},\eta})^2 f(\dots, y_{m-1}, y_{-m}, y_m) \\
&\qquad = f(\dots, -y_{m-1}, y_{-m}, y_m),
\end{aligned} \tag{7.34b}$$

$$(\kappa^{\mathrm{sch},\eta})^2 h(\dots, y_{m-1}, t) = h(\dots, -y_{m-1}, t). \tag{7.34c}$$

Laplacian/Laplacian / Heat operator

$$\Delta_{n+2} = \sum_{|j|\leq m+1} \partial_{z_{-j}} \partial_{z_j}, \tag{7.35a}$$

$$\Delta_n = \sum_{|j|\leq m} \partial_{y_{-j}} \partial_{y_j}, \tag{7.35b}$$

$$\mathcal{L}_{n-2} = \sum_{|j|\leq m-1} \partial_{y_{-j}} \partial_{y_j} + 2\partial_t. \tag{7.35c}$$

7.6 Special solutions of the heat equation

Let us describe how to obtain solutions of the heat equation from solutions of the Laplace equation.

Consider first a function on the level of $\mathbb{R}^{n+2}$

$$K(z) = z_{-m}^{1-\frac{n}{2}} g\Big(\frac{z_1}{z_{-m}}, \dots, \frac{z_{m-1}}{z_{-m}}\Big) \exp\Big(-\frac{z_{m+1}}{z_{-m}}\Big), \tag{7.36}$$

where g is a harmonic function on $\mathbb{R}^{n-2}$. It is easy to see that K is harmonic and satisfies

$$B_{m+1,m} K = K. \tag{7.37}$$

Besides, K is homogeneous of degree $1-\frac{n}{2}$. Therefore, we can descend on the level of dimension n, obtaining the function

$$k(y) = y_{-m}^{1-\frac{n}{2}} g\Big(\frac{y_1}{y_{-m}}, \dots, \frac{y_{m-1}}{y_{-m}}\Big) \exp\Big(\sum_{|i|\le m-1} \frac{y_{-i}y_i}{y_{-m}} + y_m\Big). \tag{7.38}$$

It is harmonic and satisfies

$$B^{\mathrm{fl}}_{m+1,m} k = k. \tag{7.39}$$

Descending on the level of $\mathbb{R}^{n-2} \oplus \mathbb{R}$ we obtain

$$h(y,t) = t^{1-\frac{n}{2}} g\Big(\frac{y_1}{t}, \dots, \frac{y_{m-1}}{t}\Big) \exp\Big(\sum_{|i|\le m-1} \frac{y_{-i}y_i}{t}\Big). \tag{7.40}$$

which solves the heat equation:

$$\mathcal{L}_{n-2} h = 0. \tag{7.41}$$

7.7 Wave packets for the heat equation

Let us use the coordinates $(y,t) \in \mathbb{R}^{n-2} \oplus \mathbb{R}$. Recall that

$$M^{\mathrm{sch},\eta} = \sum_{|j|\le m-1} y_j \partial_{y_j} + 2t\partial_t - \eta. \tag{7.42}$$

The following proposition is proven by analogous arguments as Prop. 3.2. It allows us to form wave packets that are eigenfunctions of M:

Proposition 7.1. *Suppose that* $]0,1[\ni s \overset{\gamma}{\mapsto} \tau(s)$ *is a contour satisfying*

$$f(\tau y, \tau^2 t)\tau^{-\nu}\Big|_{\tau(0)}^{\tau(1)} = 0. \tag{7.43}$$

Set

$$h_\nu(y,t) := \int_\gamma f(\tau y, \tau^2 t)\tau^{-1-\nu} \mathrm{d}\tau. \tag{7.44}$$

Then

$$M^{\mathrm{sch},\eta} h_\nu = (\nu - \eta) h_\nu. \tag{7.45}$$

8 Heat equation in 2 dimensions and the confluent equation

The goal of this section is to derive the ${}_1\mathcal{F}_1$ equation together with its symmetries from the heat equation in 2 dimensions, which in turn comes from the Laplace equation in 6 and 4 dimensions. Let us describe the main steps of this derivation:

(1) We start from the Schrödinger Lie algebra sch(2) and group Sch(2) considered as a subalgebra of so(6), resp. a subgroup of O(6), acting in 6 dimensions. The main initial operator is the Laplacian Δ_6.

(2) We descend onto 4 dimensions. The 6-dimensional Laplacian Δ_6 becomes the 4-dimensional Laplacian Δ_4.

(3) We assume that the variable y_2 appears only in the exponential e^{y_2} and the variable y_{-2} is renamed t. The Laplacian Δ_4 becomes the heat operator $\mathcal{L}_2$. The representations $B^{\mathrm{sch},\eta}$ and $\alpha^{\mathrm{sch},\eta}$ preserve our class of functions. With $\eta = -1$ and $\eta = -3$ they are generalized symmetries of the heat operator.

(4) We choose coordinates w, s, u_1, so that the Cartan operators are expressed in terms of s, u_1. We compute $\mathcal{L}_2$, $B^{\mathrm{sch},\eta}$, and $\alpha^{\mathrm{sch},\eta}$ in the new coordinates.

(5) We make an ansatz that diagonalizes the Cartan operators, whose eigenvalues, denoted by $-\theta$ and α, become parameters. The operators $\mathcal{L}_2$, $B^{\mathrm{sch},\eta}$, and $\alpha^{\mathrm{sch},\eta}$ involve now only the single variable w. The operator $\frac{s^2}{2}\mathcal{L}_2$ becomes the ${}_1\mathcal{F}_1$ operator. Generalized symmetries of $\mathcal{L}_2$ yield transmutation relations and discrete symmetries of the ${}_1\mathcal{F}_1$ operator.

The first part of this section is devoted to a description of the above steps, except for Step 2, discussed in detail in Sect. 7.

The remaining part of this section is devoted to the theory of the ${}_1\mathcal{F}_1$ equation and its solutions. Its organization is parallel to that of Sect. 5 on the ${}_2\mathcal{F}_1$ equation. The main additional complication is the fact that besides the ${}_1\mathcal{F}_1$ equation and the ${}_1F_1$ function, it is useful to discuss the closely related ${}_2\mathcal{F}_0$ equation and the ${}_2F_0$ function. In fact, some of the standard solutions of the ${}_1\mathcal{F}_1$ equation are expressed in terms of the ${}_1F_1$ function, others in terms of the ${}_2F_0$ function.

8.1 sch(2) in 6 dimensions

We again consider $\mathbb{R}^6$ with the coordinates (5.1) and the product given by (5.2):

$$\langle z|z\rangle = 2z_{-1}z_1 + 2z_{-2}z_2 + 2z_{-3}z_3.$$

We describe various object related to the Lie algebra sch(2) treated as a subalgebra of so(6). We also list a few typical Weyl symmetries of Sch(2).

Lie algebra sch(2)**.** Cartan algebra

$$\begin{aligned} M &= z_{-2}\partial_{z_{-2}} - z_2\partial_{z_2} - z_{-3}\partial_{z_{-3}} + z_3\partial_{z_3}, &(8.1a)\\ N_1 &= -z_{-1}\partial_{z_{-1}} + z_1\partial_{z_1}, &(8.1b)\\ B_{3,2} &= z_{-3}\partial_{z_2} - z_{-2}\partial_{z_3}. &(8.1c)\end{aligned}$$

Root operators

$$\begin{aligned} B_{3,-1} &= z_{-3}\partial_{z_{-1}} - z_1\partial_{z_3}, &(8.2a)\\ B_{2,1} &= z_{-2}\partial_{z_1} - z_{-1}\partial_{z_2}, &(8.2b)\\ B_{3,1} &= z_{-3}\partial_{z_1} - z_{-1}\partial_{z_3}, &(8.2c)\\ B_{2,-1} &= z_{-2}\partial_{z_{-1}} - z_1\partial_{z_2}, &(8.2d)\\ B_{3,-2} &= z_{-3}\partial_{z_{-2}} - z_2\partial_{z_3}, &(8.2e)\\ B_{-3,2} &= z_3\partial_{z_2} - z_{-2}\partial_{z_{-3}}. &(8.2f)\end{aligned}$$

Weyl symmetries

$$\begin{aligned} \iota K(z_{-1}, z_1, z_{-2}, z_2, z_{-3}, z_3) &= K(z_{-1}, z_1, z_{-2}, z_2, z_{-3}, z_3), &(8.3a)\\ \tau_1 K(z_{-1}, z_1, z_{-2}, z_2, z_{-3}, z_3) &= K(z_1, z_{-1}, z_{-2}, z_2, z_{-3}, z_3), &(8.3b)\\ \kappa K(z_{-1}, z_1, z_{-2}, z_2, z_{-3}, z_3) &= K(z_{-1}, z_1, -z_{-3}, -z_3, z_{-2}, z_2), &(8.3c)\\ \tau_1\kappa K(z_{-1}, z_1, z_{-2}, z_2, z_{-3}, z_3) &= K(z_1, z_{-1}, -z_{-3}, -z_3, z_{-2}, z_2). &(8.3d)\end{aligned}$$

Laplacian

$$\Delta_6 = 2\partial_{z_{-1}}\partial_{z_1} + 2\partial_{z_{-2}}\partial_{z_2} + 2\partial_{z_{-3}}\partial_{z_3}. \tag{8.4}$$

8.2 sch(2) in 4 dimensions

We descend on the level of $\mathbb{R}^4$, with the coordinates $(y_{-1}, y_1, y_{-2}, y_2)$ and the scalar product given by

$$\langle y|y\rangle = 2y_{-1}y_1 + 2y_{-2}y_2.$$

Lie algebra sch(2)**.** Cartan algebra

$$\begin{aligned} M^{\mathrm{fl},\eta} &= y_{-1}\partial_{y_{-1}} + y_1\partial_{y_1} + 2y_{-2}\partial_{y_{-2}} - \eta,\\ N_1^{\mathrm{fl}} &= -y_{-1}\partial_{y_-1} + y_1\partial_{y_1},\\ B_{3,2}^{\mathrm{fl}} &= \partial_{y_2}.\end{aligned}$$

Root operators

$$\begin{aligned}
B^{\mathrm{fl}}_{3,-1} &= \partial_{y_{-1}},\\
B^{\mathrm{fl}}_{2,1} &= y_{-2}\partial_{y_1} - y_1\partial_{y_2},\\
B^{\mathrm{fl}}_{3,1} &= \partial_{y_1},\\
B^{\mathrm{fl}}_{2,-1} &= y_{-2}\partial_{y_{-1}} - y_1\partial_{y_2},\\
B^{\mathrm{fl}}_{3,-2} &= \partial_{y_{-2}},\\
B^{\mathrm{fl},\eta}_{-3,2} &= y_{-2}(y_{-1}\partial_{y_{-1}} + y_1\partial_{y_1} + y_{-2}\partial_{y_{-2}} - \eta) - y_{-1}y_1\partial_{y_2}.
\end{aligned}$$

Weyl symmetries

$$\begin{aligned}
\iota f(y_{-1},y_1,y_{-2},y_2) &= f(y_{-1},y_1,y_{-2},y_2),\\
\tau_1^{\mathrm{fl}} f(y_{-1},y_1,y_{-2},y_2) &= f(y_1,y_{-1},y_{-2},y_2),\\
\kappa^{\mathrm{fl},\eta} f(y_{-1},y_1,y_{-2},y_2) &= y_{-2}^{\eta} f\Big(\frac{y_{-1}}{y_{-2}},\frac{y_1}{y_{-2}},-\frac{1}{y_{-2}},\frac{y_{-1}y_1+y_{-2}y_2}{y_{-2}}\Big),\\
\tau_1\kappa^{\mathrm{fl},\eta} f(y_{-1},y_1,y_{-2},y_2) &= y_{-2}^{\eta} f\Big(\frac{y_1}{y_{-2}},\frac{y_{-1}}{y_{-2}},-\frac{1}{y_{-2}},\frac{y_{-1}y_1+y_{-2}y_2}{y_{-2}}\Big).
\end{aligned}$$

8.3 $\mathrm{sch}(2)$ in $2+1$ dimensions

We apply the ansatz involving the exponential e^{y_2}. We rename y_{-2} to t.
Lie algebra $\mathrm{sch}(2)$**.** Cartan algebra

$$\begin{aligned}
M^{\mathrm{sch},\eta} &= y_{-1}\partial_{y_{-1}} + y_1\partial_{y_1} + 2t\partial_t - \eta, &(8.8a)\\
N_1^{\mathrm{sch}} &= -y_{-1}\partial_{y_{-1}} + y_1\partial_{y_1}, &(8.8b)\\
B_{32}^{\mathrm{sch}} &= 1. &(8.8c)
\end{aligned}$$

Root operators

$$\begin{aligned}
B^{\mathrm{sch}}_{3,-1} &= \partial_{y_{-1}}, &(8.9a)\\
B^{\mathrm{sch}}_{2,1} &= t\partial_{y_1} - y_{-1}, &(8.9b)\\
B^{\mathrm{sch}}_{3,1} &= \partial_{y_1}, &(8.9c)\\
B^{\mathrm{sch}}_{2,-1} &= t\partial_{y_{-1}} - y_1, &(8.9d)\\
B^{\mathrm{sch}}_{3,-2} &= \partial_t, &(8.9e)\\
B^{\mathrm{sch},\eta}_{-3,2} &= t(y_{-1}\partial_{y_{-1}} + y_1\partial_{y_1} + t\partial_t - \eta) - y_{-1}y_1. &(8.9f)
\end{aligned}$$

Weyl symmetries

$$
\begin{aligned}
\iota g(y_{-1}, y_1, t) &= g(y_{-1}, y_1, t), && (8.10a)\\
\tau_1^{\mathrm{sch}} h(y_{-1}, y_1, t) &= h(y_1, y_{-1}, t), && (8.10b)\\
\kappa^{\mathrm{sch},\eta} h(y_{-1}, y_1, t) &= t^{\eta} \exp\Big(\frac{y_{-1}y_1}{t}\Big) h\Big(\frac{y_{-1}}{t}, \frac{y_1}{t}, -\frac{1}{t}\Big), && (8.10c)\\
\tau_1\kappa^{\mathrm{sch},\eta} h(y_{-1}, y_1, t) &= t^{\eta} \exp\Big(\frac{y_{-1}y_1}{t}\Big) h\Big(\frac{y_1}{t}, \frac{y_{-1}}{t}, -\frac{1}{t}\Big). && (8.10d)
\end{aligned}
$$

Heat operator

$$\mathcal{L}_2 = 2\partial_{y_{-1}}\partial_{y_1} + 2\partial_t. \tag{8.11}$$

8.4 sch(2) in the coordinates w, s, u_1

We introduce new coordinates w, s, u_1

$$w = \frac{y_{-1}y_1}{t}\,, \quad u_1 = \frac{y_1}{\sqrt{t}}\,, \quad s = \sqrt{t}\,, \tag{8.12}$$

with the reverse transformations

$$y_{-1} = \frac{sw}{u_1}\,, \quad y_1 = u_1 s\,, \quad t = s^2\,. \tag{8.13}$$

Lie algebra sch(2)**.** Cartan algebra

$$
\begin{aligned}
M^{\mathrm{sch},\eta} &= s\,\partial_s - \eta,\\
N_1^{\mathrm{sch}} &= u_1\partial_{u_1},\\
B_{32}^{\mathrm{sch}} &= 1.
\end{aligned}
$$

Root operators

$$
\begin{aligned}
B_{3,-1}^{\mathrm{sch}} &= \frac{u_1}{s}\partial_w,\\
B_{2,1}^{\mathrm{sch}} &= \frac{s}{u_1}(w\partial_w + u_1\partial_{u_1} - w),\\
B_{3,1}^{\mathrm{sch}} &= \frac{1}{u_1 s}(w\partial_w + u_1\partial_{u_1}),\\
B_{2,-1}^{\mathrm{sch}} &= su_1(\partial_w - 1),\\
B_{3,-2}^{\mathrm{sch}} &= \frac{1}{s^2}\big(-w\partial_w - \frac{1}{2}u_1\partial_{u_1} + \frac{1}{2}s\partial_s\big),\\
B_{-3,2}^{\mathrm{sch},\eta} &= s^2\big(w\partial_w + \frac{1}{2}u_1\partial_{u_1} + \frac{1}{2}s\partial_s - w - \eta\big).
\end{aligned}
$$

Weyl symmetries

$$\begin{aligned}
\iota h(w,u_1,s) &= h(w,u_1,s),\\
\tau_1^{\mathrm{sch}} h\Big(w,u_1,s\Big) &= h\Big(w,\frac{w}{u_1},s\Big),\\
\kappa^{\mathrm{sch},\eta} h(w,u_1,s) &= s^{2\eta}\mathrm{e}^{w} h\Big(-w,-\mathrm{i}u_1,\frac{\mathrm{i}}{s}\Big),\\
\tau_1\kappa^{\mathrm{sch},\eta} h(w,u_1,s) &= s^{2\eta}\mathrm{e}^{w} h\Big(-w,-\frac{\mathrm{i}w}{u_1},\frac{\mathrm{i}}{s}\Big).
\end{aligned}$$

Heat operator

$$\mathcal{L}_2 = \frac{2}{s^2}\Big(w\partial_w^2 + (u_1\partial_{u_1}+1-w)\partial_w + \frac{1}{2}(-u_1\partial_{u_1}+s\partial_s)\Big). \tag{8.17}$$

8.5 Confluent operator

Let us make the ansatz

$$h(w,u_1,s) = {u_1}^{\alpha} s^{-\theta-1}F(w). \tag{8.18}$$

Clearly,

$$M^{\mathrm{sch},-1}h \quad=\quad -\theta h, \tag{8.19a}$$
$$N_1^{\mathrm{sch}}h \quad=\quad \alpha h, \tag{8.19b}$$
$$u_1^{-\alpha}s^{\theta+1}\frac{s^2}{2}\mathcal{L}_2 h \quad=\quad \mathcal{F}_{\theta,\alpha}(w,\partial_w)F(w), \tag{8.19c}$$

where we have introduced the ${}_1\mathcal{F}_1$ *operator*

$$\mathcal{F}_{\theta,\alpha}(w,\partial_w) \quad=\quad w\partial_w^2 + (1+\alpha-w)\partial_w - \frac{1}{2}(1+\theta+\alpha). \tag{8.20}$$

Let us also define the closely related ${}_2\mathcal{F}_0$ *operator*

$$\tilde{\mathcal{F}}_{\theta,\alpha}(w,\partial_w) \quad=\quad w^2\partial_w^2 + (-1+(2+\theta)w)\partial_w + \frac{1}{4}(1+\theta)^2 - \frac{1}{4}\alpha^2. \tag{8.21}$$

It is equivalent to the ${}_1\mathcal{F}_1$ operator. In fact, if $z=-w^{-1}$, then

$$(-z)^{\frac{3+\alpha+\theta}{2}}\tilde{\mathcal{F}}_{\theta,\alpha}(z,\partial_z)(-z)^{-\frac{1+\alpha+\theta}{2}} = \mathcal{F}_{\theta,\alpha}(w,\partial_w). \tag{8.22}$$

We will treat $\mathcal{F}_{\theta,\alpha}(w,\partial_w)$ as the principal operator.

Traditionally, one uses the *classical parameters* a,b,c:

$$\alpha := c-1 = a-b, \qquad \theta := -c+2a = -1+a+b; \tag{8.23a}$$
$$a = \frac{1+\alpha+\theta}{2}, \qquad b = \frac{1-\alpha+\theta}{2}, \qquad c = 1+\alpha. \tag{8.23b}$$

Here are the traditional forms of the ${}_1\mathcal{F}_1$ and ${}_2\mathcal{F}_0$ operators:

$$\mathcal{F}(a;c;w,\partial_w) := w\partial_w^2 + (c-w)\partial_w - a, \tag{8.24}$$

$$\mathcal{F}(a,b;-;w,\partial_w) := w^2\partial_w^2 + (-1+(1+a+b)w)\partial_w + ab. \tag{8.25}$$

8.6 Transmutation relations and discrete symmetries

The heat operator satisfies the following generalized symmetries:

$$B^{\mathrm{sch},-3}\mathcal{L}_2 = \mathcal{L}_2 B^{\mathrm{sch},-1}, \quad B \in \mathrm{sch}(2), \tag{8.26a}$$

$$\alpha^{\mathrm{sch},-3}\mathcal{L}_2 = \mathcal{L}_2 \alpha^{\mathrm{sch},-1}, \quad \alpha \in \mathrm{Sch}(2). \tag{8.26b}$$

Applying (8.26a) to the roots of sch(2) we obtain the following transmutation relations of the confluent operator:

$$\begin{array}{rl}
\partial_w & \mathcal{F}_{\theta,\alpha} \\
= \quad \mathcal{F}_{\theta+1,\alpha+1} & \partial_w, \\
(w\partial_w + \alpha - w) & \mathcal{F}_{\theta,\alpha} \\
= \quad \mathcal{F}_{\theta-1,\alpha-1} & (w\partial_w + \alpha - w), \\
(w\partial_w + \alpha) & \mathcal{F}_{\theta,\alpha} \\
= \quad \mathcal{F}_{\theta+1,\alpha-1} & (w\partial_w + \alpha), \\
(\partial_w - 1) & \mathcal{F}_{\theta,\alpha}, \\
= \quad \mathcal{F}_{\theta-1,\alpha+1} & (\partial_w - 1); \\
\big(w\partial_w + \frac{1}{2}(\theta+\alpha+1)\big) & w\mathcal{F}_{\theta,\alpha} \\
= \quad w\mathcal{F}_{\theta+2,\alpha} & \big(w\partial_w + \frac{1}{2}(\theta+\alpha+1)\big), \\
\big(w\partial_w + \frac{1}{2}(-\theta+\alpha+1) - w\big) & w\mathcal{F}_{\theta,\alpha} \\
= \quad w\mathcal{F}_{\theta-2,\alpha} & \big(w\partial_w + \frac{1}{2}(-\theta+\alpha+1) - w\big).
\end{array}$$

Applying (8.26b) to the Weyl symmetries of sch(2) yields discrete symmetries of the confluent operator, described below.

The following operators equal $\mathcal{F}_{\theta,\alpha}(w,\partial_w)$ for the appropriate w:

$$\begin{array}{lrcl}
w = v: & & \mathcal{F}_{\theta,\alpha}(v,\partial_v), & \\
 & v^{-\alpha} & \mathcal{F}_{\theta,-\alpha}(v,\partial_v) & v^{\alpha}, \\
w = -v: & -\mathrm{e}^{-v} & \mathcal{F}_{-\theta,\alpha}(v,\partial_v) & \mathrm{e}^{v}, \\
 & -\mathrm{e}^{-v}v^{-\alpha} & \mathcal{F}_{-\theta,-\alpha}(v,\partial_v) & \mathrm{e}^{v}v^{\alpha}.
\end{array}$$

The third symmetry is sometimes called the *1st Kummer transformation*.

8.7 Factorizations of of the heat operator

Special role is played by three distinguished subalgebras in sch(2): two isomorphic to heis(2) and one isomorphic to so(3).

First note the commutation relations

$$[B_{2,-1}, B_{3,1}] = [B_{2,1}\, , B_{3,-1}] = B_{3,2}. \tag{8.27}$$

Therefore, the following subalgebras in sch(5) are isomorphic to heis(2):

$$\mathrm{heis}_-(2) \quad \text{spanned by} \quad B_{2,-1},\ B_{3,1},\ B_{3,2}, \tag{8.28a}$$

$$\mathrm{heis}_+(2) \quad \text{spanned by} \quad B_{2,1},\ B_{3,-1},\ B_{3,2}. \tag{8.28b}$$

Note that the flip of $(1,-1)$, denoted τ_1, belongs to Sch(5) and satisfies

$$\tau_1 B_{2,-1}\tau_1 = B_{2,1}, \quad \tau_1 B_{3,1}\tau_1 = B_{3,-1}, \quad \tau_1 B_{3,2}\tau_1 = B_{3,2}. \tag{8.29}$$

Hence,

$$\tau_1 \mathrm{heis}_-(2)\tau_1 = \mathrm{heis}_+(2). \tag{8.30}$$

Let us define

$$\mathcal{C}_- = 2\, B_{2,-1}B_{3,1} + M + N_1 - B_{3,2} \tag{8.31a}$$

$$= 2\, B_{3,1}B_{2,-1} + M + N_1 + B_{3,2}, \tag{8.31b}$$

$$\mathcal{C}_+ = 2\, B_{2,1}B_{3,-1} + M - N_1 - B_{3,2} \tag{8.31c}$$

$$= 2\, B_{3,-1}B_{2,1} + M - N_1 + B_{3,2}. \tag{8.31d}$$

$\mathcal{C}_+$ and $\mathcal{C}_-$ can be viewed as the Casimir operators for $\mathrm{heis}_+(2)$, resp. for $\mathrm{heis}_-(2)$. Indeed, $\mathcal{C}_+$, resp. $\mathcal{C}_-$ commute with all operators in $\mathrm{heis}_+(2)$, resp. $\mathrm{heis}_-(2)$. We also have

$$\tau_1 \mathcal{C}_- \tau_1 = \mathcal{C}_+. \tag{8.32}$$

On the level of $\mathbb{R}^2 \oplus \mathbb{R}$, the two operators $\mathcal{C}_+$ and $\mathcal{C}_-$ coincide. Indeed, a direct calculation yields

$$\mathcal{C}_+^{\mathrm{sch},\eta} = \mathcal{C}_-^{\mathrm{sch},\eta} = 2t(\partial_{y_{-1}}\partial_{y_1} + \partial_t) - \eta - 1. \tag{8.33}$$

Second, note the commutation relations

$$[B_{-3,2}, B_{3,-2}] = N_2 - N_3 = -M. \tag{8.34}$$

Therefore, the following of sch(2) is isomorphic to so(3):

$$\mathrm{so}_{23}(3) \quad \text{spanned by} \quad B_{-3,2}, B_{3,-2}, M. \tag{8.35}$$

The Casimir operator for $\mathrm{so}_{23}(3)$ is

$$\mathcal{C}_{23} = 4\,B_{3,-2}B_{-3,2} - (M-1)^2 + 1 \tag{8.36a}$$
$$= 4\,B_{-3,2}B_{3,-2} - (M+1)^2 + 1. \tag{8.36b}$$

By (4.21) we have

$$(2z_{-2}z_2 + 2z_{-3}z_3)\Delta_6^\diamond = -1 + \mathcal{C}_{23}^{\diamond,-1} + (N_1^{\diamond,-1})^2. \tag{8.37}$$

Inserting (8.36) into (8.37) we obtain

$$(2z_{-2}z_2 + 2z_{-3}z_3)\Delta_6^\diamond$$
$$=4B_{2,-3}B_{-2,3} - (N_1+M+1)(-N_1+M+1) \tag{8.38a}$$
$$=4B_{-2,3}B_{2,-3} - (N_1+M-1)(-N_1+M-1), \tag{8.38b}$$

where the B, N_1 and M operators should be equipped with the superscript ${}^{\diamond,-1}$.

Let us sum up the factorizations in the variables $y_{-1}y_1, t$ obtained with the help of the three subalgebras:

$$t\mathcal{L}_2 = 2\,B_{2,-1}B_{3,1} + M + N_1 - 1 \tag{8.39a}$$
$$= 2\,B_{3,1}B_{2,-1} + M + N_1 + 1 \tag{8.39b}$$
$$= 2\,B_{2,1}B_{3,-1} + M - N_1 - 1 \tag{8.39c}$$
$$= 2\,B_{3,-1}B_{2,1} + M - N_1 + 1, \tag{8.39d}$$
$$2y_{-1}y_1\mathcal{L}_2 = -4B_{2,-3}B_{-2,3} - (N_1+M+1)(N_1-M-1) \tag{8.39e}$$
$$= -4B_{-2,3}B_{2,-3} - (N_1+M-1)(N_1-M+1), \tag{8.39f}$$

where the B, N_1 and M operators should be equipped with the superscript ${}^{\mathrm{sch},-1}$.

Indeed, to obtain (8.39a)–(8.39d) we insert (8.31) into (8.33). To obtain (8.39e)–(8.39f) we rewrite (8.38), multiplying it by -1.

In the variables w, u, s, we need to make the replacements

$$y_{-1}y_1 \quad\to\quad ws^2, \tag{8.40a}$$
$$t \quad\to\quad s^2. \tag{8.40b}$$

8.8 Factorizations of the confluent operator

Factorizations of $\mathcal{L}_2$ described in Subsect. 8.7 yield the following factorizations of the confluent operator:

$$\begin{aligned}
\mathcal{F}_{\theta,\alpha} &= \Big(\partial_w - 1\Big)\Big(w\partial_w + \alpha\Big) - \frac{1}{2}(\theta - \alpha + 1) \\
&= \Big(w\partial_w + 1 + \alpha\Big)\Big(\partial_w - 1\Big) - \frac{1}{2}(\theta - \alpha - 1) \\
&= \partial_w\Big(w\partial_w + \alpha - w\Big) - \frac{1}{2}(\theta + \alpha - 1) \\
&= \Big(w\partial_w + 1 + \alpha - w\Big)\partial_w - \frac{1}{2}(\theta + \alpha + 1),
\end{aligned}$$

$$\begin{aligned}
w\mathcal{F}_{\theta,\alpha} &= \Big(w\partial_w + \frac{1}{2}(-\theta + \alpha - 1) - w\Big)\Big(w\partial_w + \frac{1}{2}(\theta + \alpha + 1)\Big) \\
&\quad -\frac{1}{4}(-\theta + \alpha - 1)(\theta + \alpha + 1) \\
&= \Big(w\partial_w + \frac{1}{2}(\theta + \alpha - 1)\Big)\Big(w\partial_w + \frac{1}{2}(-\theta + \alpha + 1) - w\Big) \\
&\quad -\frac{1}{4}(-\theta + \alpha + 1)(\theta + \alpha - 1).
\end{aligned}$$

8.9 The ${}_1F_1$ function

The ${}_1\mathcal{F}_1$ equation (8.24) has a regular singular point at 0. Its indices at 0 are equal to 0, $1 - c$. For $c \neq 0, -1, -2, \ldots$, the unique solution of the confluent equation analytic at 0 and equal to 1 at 0 is called the ${}_1F_1$ *function* or *Kummer's confluent function*. It is equal to

$$F(a; c; w) := \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n} \frac{w^n}{n!}.$$

It is defined for $c \neq 0, -1, -2, \ldots$. Sometimes it is more convenient to consider the functions

$$\begin{aligned}
\mathbf{F}(a; c; w) &:= \frac{F(a; c; w)}{\Gamma(c)} = \sum_{n=0}^{\infty} \frac{(a)_n}{\Gamma(c+n)} \frac{w^n}{n!}, \\
\mathbf{F}^{\mathrm{I}}(a; c; w) &:= \frac{\Gamma(a)\Gamma(c-a)}{\Gamma(c)} F(a; c; w).
\end{aligned}$$

In the Lie-algebraic parameters:

$$\begin{aligned}
F_{\theta,\alpha}(w) &:= F\Big(\frac{1+\alpha+\theta}{2}; 1+\alpha; w\Big), \\
\mathbf{F}_{\theta,\alpha}(w) &:= \mathbf{F}\Big(\frac{1+\alpha+\theta}{2}; 1+\alpha; w\Big) = \frac{F_{\theta,\alpha}(w)}{\Gamma(\alpha+1)}, \\
\mathbf{F}^{\mathrm{I}}_{\theta,\alpha}(w) &:= \mathbf{F}^{\mathrm{I}}\Big(\frac{1+\alpha+\theta}{2}; 1+\alpha; w\Big) = \frac{\Gamma(\frac{1+\alpha+\theta}{2})\Gamma(\frac{1+\alpha-\theta}{2})F_{\theta,\alpha}(w)}{\Gamma(\alpha+1)}.
\end{aligned}$$

8.10 The ${}_2F_0$ function

Recall from (8.22) that in parallel with the ${}_1\mathcal{F}_1$ operator it is useful to consider the ${}_2\mathcal{F}_0$ operator. The ${}_2\mathcal{F}_0$ operator does not have a regular singular point at zero, hence to construct its solutions having a simple behavior at zero we cannot use the Frobenius method. One of such solutions is the ${}_2F_0$ *function.* For $w \in \mathbb{C}\backslash[0, +\infty[$ it can be defined by

$$F(a, b; -; w) := \lim_{c\to\infty} F(a, b; c; cw),$$

where $|\arg c - \pi| < \pi - \epsilon$, $\epsilon > 0$. It extends to an analytic function on the universal cover of $\mathbb{C}\backslash\{0\}$ with a branch point of an infinite order at 0. It has the following asymptotic expansion:

$$F(a, b; -; w) \sim \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{n!} w^n, \quad |\arg w - \pi| < \pi - \epsilon.$$

Sometimes instead of ${}_2F_0$ it is useful to consider the function

$$F^{\mathrm{I}}(a, b; -; w) \quad := \quad \Gamma(a)F(a, b; -; w).$$

When we use the Lie-algebraic parameters, we denote the ${}_2F_0$ function by $\tilde{F}$ and $\tilde{F}^{\mathrm{I}}$. The tilde is needed to avoid the confusion with the ${}_1F_1$ functions:

$$\begin{aligned}
\tilde{F}_{\theta,\alpha}(w) \quad &:= \quad F\Big(\frac{1+\alpha+\theta}{2}, \frac{1-\alpha+\theta}{2}; -; w\Big), \\
\tilde{F}^{\mathrm{I}}_{\theta,\alpha}(w) \quad &:= \quad F^{\mathrm{I}}\Big(\frac{1+\alpha+\theta}{2}, \frac{1-\alpha+\theta}{2}; -; w\Big) \;=\; \Gamma\Big(\frac{1-\alpha+\theta}{2}\Big)\tilde{F}_{\theta,\alpha}(w).
\end{aligned}$$

8.11 Standard solutions

The ${}_1F_1$ equation has two singular points. 0 is a regular singular point and with each of its two indices we can associate the corresponding solution. ∞ is not a regular singular point. However we can define two solutions with a simple behavior around ∞. Therefore, we obtain 4 *standard solutions.*

The solutions that have a simple behavior at zero are expressed in terms of the function $F_{\theta,\alpha}$. Using 4 discrete symmetries yields 4 distinct expressions. Taking into account Kummer's identity we obtain 2 pairs of standard solutions.

The solutions with a simple behavior at $\pm\infty$ are expressed in terms of $\tilde{F}_{\theta,\alpha}$. Again, 4 discrete symmetries yield 4 distinct expressions. Taking into account the trivial identity $\tilde{F}_{\theta,\alpha} = \tilde{F}_{\theta,-\alpha}$ we obtain 2 pairs of standard solutions.

$$\begin{aligned}
\sim 1 \text{ at } 0: \quad & F_{\theta,\alpha}(w) \\
&= \mathrm{e}^{w} F_{-\theta,\alpha}(-w); \\
\sim w^{-\alpha} \text{ at } 0: \quad & w^{-\alpha} F_{\theta,-\alpha}(w) \\
&= w^{-\alpha} \mathrm{e}^{w} F_{-\theta,-\alpha}(-w);
\end{aligned}$$

$$\begin{aligned}
\sim w^{-a} \text{ at } +\infty: \quad & w^{\frac{-1-\theta-\alpha}{2}} \tilde{F}_{\theta,\alpha}(-w^{-1}) \\
&= w^{\frac{-1-\theta-\alpha}{2}} \tilde{F}_{\theta,-\alpha}(-w^{-1}); \\
\sim (-w)^{b-1}\mathrm{e}^{w} \text{ at } -\infty: \quad & \mathrm{e}^{w}(-w)^{\frac{-1+\theta-\alpha}{2}} \tilde{F}_{-\theta,\alpha}(w^{-1}) \\
&= \mathrm{e}^{w}(-w)^{\frac{-1+\theta-\alpha}{2}} \tilde{F}_{-\theta,-\alpha}(w^{-1}).
\end{aligned}$$

The solution $\sim w^{-a}$ at $+\infty$ is often called *Tricomi's confluent function.*

8.12 Recurrence relations

Recurrence relations for the confluent function correspond to roots of the Lie algebra sch(2):

$$\begin{aligned}
\partial_w \mathbf{F}_{\theta,\alpha}(w) &= \frac{1+\theta+\alpha}{2} \mathbf{F}_{\theta+1,\alpha+1}(w), \\
(w\partial_w + \alpha - w)\, \mathbf{F}_{\theta,\alpha}(w) &= \mathbf{F}_{\theta-1,\alpha-1}(w), \\
(w\partial_w + \alpha)\, \mathbf{F}_{\theta,\alpha}(w) &= \mathbf{F}_{\theta+1,\alpha-1}(w), \\
(\partial_w - 1)\, \mathbf{F}_{\theta,\alpha}(w) &= \frac{-1+\theta-\alpha}{2} \mathbf{F}_{\theta-1,\alpha+1}(w), \\
\left(w\partial_w + \frac{1+\theta+\alpha}{2}\right) \mathbf{F}_{\theta,\alpha}(w) &= \frac{1+\theta+\alpha}{2} \mathbf{F}_{\theta+2,\alpha}(w), \\
\left(w\partial_w + \frac{1-\theta+\alpha}{2} - w\right) \mathbf{F}_{\theta,\alpha}(w) &= \frac{1-\theta+\alpha}{2} \mathbf{F}_{\theta-2,\alpha}(w).
\end{aligned}$$

8.13 Wave packets for the heat equation in 2 dimensions

Consider the space $\mathbb{R}^2 \oplus \mathbb{R}$ and the heat equation given by the operator $\mathcal{L}_2 = 2\partial_{y_{-1}}\partial_{y_1} + 2\partial_t$. Recall that

$$\begin{aligned}
M^{\mathrm{sch},-1} &= y_{-1}\partial_{y_{-1}} + y_1\partial_{y_1} + 2t\partial_t + 1, \\
N_1^{\mathrm{sch}} &= -y_{-1}\partial_{y_{-1}} + y_1\partial_{y_1}.
\end{aligned}$$

Set

$$G^a_{\theta,\alpha}(y_{-1},y_1,t) := \int\limits_{\gamma^a} \tau^{-\alpha-1} t^{\frac{-1-\theta+\alpha}{2}} (\tau^{-1}y_{-1}-1)^{\frac{-1+\theta-\alpha}{2}} \exp\left(\frac{(y_{-1}-\tau)y_1}{t}\right)\mathrm{d}\tau, \tag{8.41a}$$

$$G^b_{\theta,\alpha}(y_{-1},y_1,t) := \int\limits_{\gamma^b} \tau^{-\alpha-1} t^{\frac{-1-\theta-\alpha}{2}} (\tau y_1-1)^{\frac{-1+\theta+\alpha}{2}} \exp\left(\frac{y_{-1}(y_1-\tau^{-1})}{t}\right)\mathrm{d}\tau. \tag{8.41b}$$

(The superscripts a and b denote two kinds of wave packets, and not parameters a, b).

Proposition 8.1. *If the contours γ^a and γ^b are appropriately chosen, then*

$$\mathcal{L}_2 G^a_{\theta,\alpha} = 0, \qquad \mathcal{L}_2 G^b_{\theta,\alpha} = 0, \tag{8.42}$$

$$M^{\mathrm{sch},-1} G^a_{\theta,\alpha} = -\theta G^a_{\theta,\alpha}, \qquad M^{\mathrm{sch},-1} G^b_{\theta,\alpha} = -\theta G^b_{\theta,\alpha}, \tag{8.43}$$

$$N_1 G^a_{\theta,\alpha} = \alpha G^a_{\theta,\alpha}, \qquad N_1 G^b_{\theta,\alpha} = \alpha G^b_{\theta,\alpha}. \tag{8.44}$$

Proof. By the analysis of Subsect. 7.6, the following functions

$$g^a_\nu(y_{-1},y_1,t) := t^{-1-\nu} y_{-1}^\nu \exp\left(\frac{y_{-1}y_1}{t}\right), \tag{8.45a}$$

$$g^b_\nu(y_{-1},y_1,t) := t^{-1-\nu} y_1^\nu \exp\left(\frac{y_{-1}y_1}{t}\right) \tag{8.45b}$$

solve the heat equation. They still solve the heat equation after translating and rotating. Therefore,

$$G^a_{\theta,\alpha}(y_{-1},y_1,t) = \int\limits_{\gamma^a} g^a_{\frac{-1+\theta-\alpha}{2}}(\tau^{-1}(y_{-1}-1),\tau y_1,t)\tau^{-\alpha-1}\mathrm{d}\tau, \tag{8.46a}$$

$$G^b_{\theta,\alpha}(y_{-1},y_1,t) = \int\limits_{\gamma^b} g^b_{\frac{-1+\theta+\alpha}{2}}(\tau^{-1}y_{-1},\tau(y_1-1),t)\tau^{-\alpha-1}\mathrm{d}\tau \tag{8.46b}$$

also solve the heat equation. This proves (8.42).
If the contours satisfy the requirements of Prop. 3.2, then (8.46) imply (8.44).

We can rewrite (8.46) in a somewhat different way:

$$(8.46a) = \int\limits_{\gamma^a} g^a_{\frac{-1+\theta-\alpha}{2}}(\tau^{-1}(y_{-1}-1),\tau^{-1}y_1,\tau^{-2}t)(\tau^{-1})^\theta \mathrm{d}(\tau^{-1}), \tag{8.47a}$$

$$(8.46b) = \int\limits_{\gamma^b} g^b_{\frac{-1+\theta+\alpha}{2}}(\tau y_{-1},\tau(y_1-1),\tau^2 t)\tau^\theta \mathrm{d}\tau. \tag{8.47b}$$

If the contours satisfy the requirements of Prop. 7.1, then (8.47) imply (8.43). □

Now we express the above wave packets in the coordinates w, s, u_1:

$$(8.47a) = \int s^{-1-\theta+\alpha}\Big(\frac{ws}{\tau u_1}-1\Big)^{\frac{-1+\theta-\alpha}{2}}\exp\Big(w-\frac{\tau u_1}{s}\Big)\tau^{-\alpha-1}\mathrm{d}\tau, \tag{8.48a}$$

$$(8.47b) = \int s^{-1-\theta-\alpha}(\tau u_1 s-1)^{\frac{-1+\theta+\alpha}{2}}\exp\Big(w\big(1-\frac{1}{\tau u_1 s}\big)\Big)\tau^{-\alpha-1}\mathrm{d}\tau. \tag{8.48b}$$

In (8.48a) we make the substitution $\sigma := w - \frac{\tau u_1}{s}$, or $\tau = \frac{s}{u_1}(w-\sigma)$. In (8.48b) we make the substitution $\sigma := \frac{1}{1-\frac{1}{\tau u_1 s}}$, or $\tau = \frac{\sigma}{u_1 s(\sigma-1)}$. We obtain

$$G^a_{\theta,\alpha}(w,s,u_1) = s^{-1-\theta}{u_1}^{\alpha}F^a_{\theta,\alpha}(w), \tag{8.49a}$$

$$G^b_{\theta,\alpha}(w,s,u_1) = s^{-1-\theta}{u_1}^{\alpha}F^b_{\theta,\alpha}(w), \tag{8.49b}$$

where

$$F^a_{\theta,\alpha}(w) := \int\limits_{\gamma^a}\sigma^{\frac{-\alpha+\theta-1}{2}}(w-\sigma)^{\frac{-\alpha-\theta-1}{2}}\mathrm{e}^{\sigma}\mathrm{d}\sigma, \tag{8.50a}$$

$$F^b_{\theta,\alpha}(w) := \int\limits_{\gamma^b}\exp\Big(\frac{w}{\sigma}\Big)\sigma^{-\alpha-1}(\sigma-1)^{\frac{\alpha+\rho-1}{2}}\mathrm{d}\sigma. \tag{8.50b}$$

The above analysis shows that (for appropriate contours) the functions (8.50a) and (8.50b) satisfy the confluent equation.

8.14 Integral representations

Let us prove directly that integral (8.50a) and (8.50b) solve the confluent equation.

Theorem 8.2. *a) Let* $[0,1]\ni\tau\overset{\gamma}{\mapsto}t(\tau)$ *satisfy* $t^{a-c+1}\mathrm{e}^{t}(t-w)^{-a-1}\Big|_{t(0)}^{t(1)} = 0.$ *Then*

$$\mathcal{F}(a;c;w,\partial_w)\int\limits_{\gamma}t^{a-c}\mathrm{e}^{t}(t-w)^{-a}\mathrm{d}t = 0. \tag{8.51}$$

b) Let $[0,1]\ni\tau\overset{\gamma}{\mapsto}t(\tau)$ *satisfy* $\mathrm{e}^{\frac{w}{t}}t^{-c}(1-t)^{c-a}\Big|_{t(0)}^{t(1)} = 0.$ *Then*

$$\mathcal{F}(a;c;w,\partial_w)\int\limits_{\gamma}\mathrm{e}^{\frac{w}{t}}t^{-c}(1-t)^{c-a-1}\mathrm{d}t = 0. \tag{8.52}$$

Proof. We check that for any contour γ

$$\text{lhs of (8.51)} = -a\int_\gamma \mathrm{d}t\,\partial_t t^{a-c+1}\mathrm{e}^t(t-w)^{-a-1},$$

$$\text{lhs of (8.52)} = -\int_\gamma \mathrm{d}t\,\partial_t \mathrm{e}^{\frac{w}{t}}t^{-c}(1-t)^{c-a}.$$

□

8.15 Integral representations of standard solutions

Using the integral representations of type a) and attaching contours to $-\infty$, 0 and w we can obtain all standard solutions.

Similarly, using the integral representations of type b) and attaching contours to $0-0$, 1 and ∞ we can obtain all standard solutions.

Here is the list of contours:

	a)	**b)**
~ 1 **at** 0:	$]-\infty,(0,w)^+,-\infty[$,	$[1,+\infty[$;
$\sim w^{-\alpha}$ **at** 0:	$[0,w]$,	$(0-0)^+$;
$\sim w^{-a}$ **at** $+\infty$:	$]-\infty,0]$,	$]-\infty,0]$;
$\sim (-w)^{b-1}\mathrm{e}^w$ **at** $-\infty$:	$[w,-\infty[$,	$[0,1]$.

$(0,w)^+$ means that we bypass 0 and w counterclockwise. $(0-0)^+$ means that the contour departs from 0 on the negative side, encircles it and then comes back again from the negative side.

Here are the explicit formulas for a)-type integral representations:

$$\text{all } \theta,\alpha: \tag{8.53a}$$

$$\frac{1}{2\pi\mathrm{i}}\int\limits_{]-\infty,(0,w)^+-\infty[} t^{\frac{-1+\theta-\alpha}{2}}\mathrm{e}^t(t-w)^{\frac{-1-\theta-\alpha}{2}}\mathrm{d}t = \mathbf{F}_{\theta,\alpha}(w),$$

$$\mathrm{Re}(1-\alpha) > |\mathrm{Re}\theta| : \tag{8.53b}$$

$$\int_0^w t^{\frac{-1+\theta-\alpha}{2}} \mathrm{e}^t (w-t)^{\frac{-1-\theta-\alpha}{2}} \mathrm{d}t$$

$$= w^{-\alpha} \mathbf{F}^{\mathrm{I}}_{\theta,-\alpha}(w), \quad w \notin]-\infty, 0];$$

$$\int_w^0 (-t)^{\frac{-1+\theta-\alpha}{2}} \mathrm{e}^t (t-w)^{\frac{-1-\theta-\alpha}{2}} \mathrm{d}t$$

$$= (-w)^{-\alpha} \mathbf{F}^{\mathrm{I}}_{\theta,-\alpha}(w), \quad w \notin [0, \infty[;$$

$$\mathrm{Re}(1+\theta-\alpha) > 0 : \tag{8.53c}$$

$$\int_{-\infty}^0 (-t)^{\frac{-1+\theta-\alpha}{2}} \mathrm{e}^t (w-t)^{\frac{-1-\theta-\alpha}{2}} \mathrm{d}t$$

$$= w^{\frac{-1-\theta-\alpha}{2}} \tilde{F}^{\mathrm{I}}_{\theta,\alpha}(-w^{-1}), \quad w \notin]-\infty, 0];$$

$$xs\mathrm{Re}(1-\theta-\alpha) > 0 : \tag{8.53d}$$

$$\int_{-\infty}^w (-t)^{\frac{-1+\theta-\alpha}{2}} \mathrm{e}^t (w-t)^{\frac{-1-\theta-\alpha}{2}} \mathrm{d}t$$

$$= \mathrm{e}^w (-w)^{\frac{-1+\theta-\alpha}{2}} \tilde{F}^{\mathrm{I}}_{-\theta,\alpha}(w^{-1}), \quad w \notin [0, \infty[.$$

We also present explicit formulas for b)-type integral representations:

$$\mathrm{Re}(1+\alpha) > |\mathrm{Re}\theta| : \tag{8.54a}$$

$$\int_{[1,+\infty[} \mathrm{e}^{\frac{w}{t}} t^{-1-\alpha} (t-1)^{\frac{-1-\theta+\alpha}{2}} \mathrm{d}t$$

$$= \mathbf{F}^{\mathrm{I}}_{\theta,\alpha}(w);$$

$$\text{all } \theta, \alpha: \tag{8.54b}$$

$$\frac{1}{2\pi \mathrm{i}} \int_{(0-0)^+} \mathrm{e}^{\frac{w}{t}} t^{-1-\alpha} (1-t)^{\frac{-1-\theta+\alpha}{2}} \mathrm{d}t$$

$$= w^{-\alpha} \mathbf{F}_{\theta,-\alpha}(w), \quad \mathrm{Re}w > 0;$$

$$\operatorname{Re}(1+\theta+\alpha)>0: \tag{8.54c}$$
$$\int_{-\infty}^{0} \mathrm{e}^{\frac{w}{t}}(-t)^{-1-\alpha}(1-t)^{\frac{-1-\theta+\alpha}{2}}\mathrm{d}t$$
$$= w^{\frac{-1-\theta-\alpha}{2}}\tilde{F}^{\mathrm{I}}_{\theta,-\alpha}(-w^{-1}),\quad \operatorname{Re}w>0;$$

$$\operatorname{Re}(1-\theta+\alpha)>0: \tag{8.54d}$$
$$\int_{0}^{1} \mathrm{e}^{\frac{w}{t}}t^{-1-\alpha}(1-t)^{\frac{-1-\theta+\alpha}{2}}\mathrm{d}t$$
$$= \mathrm{e}^{w}(-w)^{\frac{-1+\theta-\alpha}{2}}\tilde{F}^{\mathrm{I}}_{-\theta,-\alpha}(w^{-1}),\quad \operatorname{Re}w<0.$$

8.16 Connection formulas

The two solutions with a simple behavior at infinity can be expressed as linear combination of the solutions with a simple behavior at zero:

$$w^{\frac{-1-\theta-\alpha}{2}}\tilde{F}_{\theta,\pm\alpha}(-w^{-1}) = \frac{\pi\mathbf{F}_{\theta,\alpha}(w)}{\sin\pi(-\alpha)\Gamma\left(\frac{1+\theta-\alpha}{2}\right)} + \frac{\pi w^{-\alpha}\mathbf{F}_{\theta,-\alpha}(w)}{\sin\pi\alpha\Gamma\left(\frac{1+\theta+\alpha}{2}\right)},\quad w\notin]-\infty,0]; \tag{8.55a}$$

$$\mathrm{e}^{w}(-w)^{\frac{-1+\theta-\alpha}{2}}\tilde{F}_{-\theta,\pm\alpha}(w^{-1}) = \frac{\pi\mathbf{F}_{\theta,\alpha}(w)}{\sin\pi(-\alpha)\Gamma\left(\frac{1-\theta-\alpha}{2}\right)} + \frac{\pi(-w)^{-\alpha}\mathbf{F}_{\theta,-\alpha}(w)}{\sin\pi\alpha\Gamma\left(\frac{1-\theta+\alpha}{2}\right)},\quad w\notin[0,+\infty[. \tag{8.55b}$$

Note that (8.55a) uses a different domain from (8.55b). This is natural, however it is inconvenient when we want to rewrite (8.55) in the matrix form, because on the right hand side of (8.55a) and (8.55b) the second standard solutions differ by a phase factor.

Let us introduce the matrix

$$A_{\theta,\alpha} := \frac{\pi}{\sin(\pi\alpha)}\begin{bmatrix} \frac{-1}{\Gamma\left(\frac{1+\theta-\alpha}{2}\right)} & \frac{\mathrm{e}^{-\frac{\mathrm{i}\pi}{2}\alpha}}{\Gamma\left(\frac{1+\theta+\alpha}{2}\right)} \\ \frac{-1}{\Gamma\left(\frac{1-\theta-\alpha}{2}\right)} & \frac{\mathrm{e}^{\frac{\mathrm{i}\pi}{2}\alpha}}{\Gamma\left(\frac{1-\theta+\alpha}{2}\right)} \end{bmatrix},$$

satisfying

$$A_{\theta,\alpha}^{-1} = \frac{\mathrm{i}\mathrm{e}^{\frac{\mathrm{i}\pi}{2}\theta}}{2}\begin{bmatrix} \frac{\mathrm{e}^{\frac{\mathrm{i}\pi\alpha}{2}}}{\Gamma(\frac{1-\theta+\alpha}{2})} & \frac{-\mathrm{e}^{-\frac{\mathrm{i}\pi\alpha}{2}}}{\Gamma(\frac{1+\theta+\alpha}{2})} \\ \frac{1}{\Gamma(\frac{1-\theta-\alpha}{2})} & \frac{-1}{\Gamma(\frac{1+\theta-\alpha}{2})} \end{bmatrix}, \tag{8.56}$$

$$\det A_{\theta,\alpha} = -\frac{\mathrm{i}\pi\mathrm{e}^{-\frac{\mathrm{i}\pi}{2}\theta}}{2\sin(\pi\alpha)}. \tag{8.57}$$

Then we have for $\mathrm{Im} w > 0$

$$\begin{bmatrix} w^{\frac{-1-\theta-\alpha}{2}}\tilde{F}_{\theta,\pm\alpha}(-w^{-1}) \\ \mathrm{e}^{w}(-w)^{\frac{-1+\theta-\alpha}{2}}\tilde{F}_{-\theta,\pm\alpha}(w^{-1}) \end{bmatrix} = A_{\theta,\alpha}\begin{bmatrix} \mathbf{F}_{\theta,\alpha}(w) \\ (-\mathrm{i}w)^{-\alpha}\mathbf{F}_{\theta,-\alpha}(w) \end{bmatrix}. \tag{8.58}$$

Let us show how to derive connection formulas from integral representations of type a). We have

$$\begin{aligned} &\Big(\int\limits_{]-\infty,0-\mathrm{i}0]} + \int\limits_{[0-\mathrm{i}0,w]} - \int\limits_{]-\infty,0+\mathrm{i}0]} - \int\limits_{[0+\mathrm{i}0,w]}\Big) t^{\frac{-1+\theta-\alpha}{2}}\mathrm{e}^{t}(t-w)^{\frac{-1-\theta-\alpha}{2}}\mathrm{d}t \\ = &\int\limits_{]-\infty,(0,w)^+,-\infty[} t^{\frac{-1+\theta-\alpha}{2}}\mathrm{e}^{t}(t-w)^{\frac{-1-\theta-\alpha}{2}}\mathrm{d}t, \qquad w \notin [-\infty,0[; \end{aligned} \tag{8.59a}$$

$$\begin{aligned} &\Big(\int\limits_{]-\infty,w-\mathrm{i}0]} + \int\limits_{[w-\mathrm{i}0,0]} - \int\limits_{]-\infty,w+\mathrm{i}0]} - \int\limits_{[w+\mathrm{i}0,0]}\Big) t^{\frac{-1+\theta-\alpha}{2}}\mathrm{e}^{t}(t-w)^{\frac{-1-\theta-\alpha}{2}}\mathrm{d}t \\ = &\int\limits_{]-\infty,(0,w)^+,-\infty[} t^{\frac{-1+\theta-\alpha}{2}}\mathrm{e}^{t}(t-w)^{\frac{-1-\theta-\alpha}{2}}\mathrm{d}t, \qquad w \notin]0,+\infty]. \end{aligned} \tag{8.59b}$$

We obtain

$$\begin{aligned} &-\sin(\pi\alpha)w^{\frac{-1-\theta-\alpha}{2}}\tilde{F}^{\mathrm{I}}_{\theta,\alpha}(-w^{-1}) + \cos\frac{\pi(\theta+\alpha)}{2}w^{-\alpha}\mathbf{F}^{\mathrm{I}}_{\theta,-\alpha}(w) \\ =&\pi\mathbf{F}_{\theta,\alpha}(w), \qquad w \notin [-\infty,0[; \end{aligned} \tag{8.60a}$$

$$\begin{aligned} &-\sin(\pi\alpha)\mathrm{e}^{w}(-w)^{\frac{-1+\theta-\alpha}{2}}\tilde{F}^{\mathrm{I}}_{-\theta,\alpha}(w^{-1}) + \cos\frac{\pi(\theta-\alpha)}{2}(-w)^{-\alpha}\mathbf{F}^{\mathrm{I}}_{\theta,-\alpha}(w) \\ =&\pi\mathbf{F}_{\theta,\alpha}(w), \qquad w \notin]0,+\infty]. \end{aligned} \tag{8.60b}$$

This implies (8.55).

9 Heat equation in 1 dimension and the Hermite equation

The goal of this section is to derive the Hermite equation together with its symmetries from the heat equation in 1 dimension, which in turn comes from the Laplace equation in 5 and 3 dimensions.

The first part of this section describes main steps of the derivation of the Hermite equation. They are parallel to those of the derivation of the ${}_1\mathcal{F}_1$ equation:

(1) We start from the Schrödinger Lie algebra sch(1) and group Sch(1) considered as a subalgebra of so(5), resp. a subgroup of O(5), acting in 5 dimensions. The main initial operator is the Laplacian Δ_5.

(2) We descend onto 3 dimensions. The 5-dimensional Laplacian Δ_5 becomes the 3-dimensional Laplacian Δ_3.

(3) We descend on 1 + 1 dimensions. The Laplacian Δ_3 becomes the heat operator $\mathcal{L}_1$. The representations $B^{\mathrm{sch},\eta}$ and $\alpha^{\mathrm{sch},\eta}$ with $\eta = -\frac{1}{2}$ and $\eta = -\frac{5}{2}$ are generalized symmetries of $\mathcal{L}_1$.

(4) We choose coordinates w, s, so that the Cartan operator is expressed in terms of s. We compute $\mathcal{L}_1$, $B^{\mathrm{sch},\eta}$ and $\alpha^{\mathrm{sch},\eta}$ in the new coordinates.

(5) We make an ansatz that diagonalizes the Cartan operator, whose eigenvalue becomes a parameter, denoted by λ. $\mathcal{L}_1$, $B^{\mathrm{sch},\eta}$ and $\alpha^{\mathrm{sch},\eta}$ involve now only the single variable w. $2s^2\mathcal{L}_1$ turns out to be the Hermite operator. The generalized symmetries of $\mathcal{L}_1$ yield transmutation relations and discrete symmetries of the Hermite operator.

(As in the previous section, in our presentation we omit the step 2).

In the remaining part of this section we develop the theory of the Hermite equation and its solutions. Its organization is parallel to that of all other sections on individual equations, and especially of Sect. 6 on the Gegenbauer equation. In particular, the Gegenbauer equation can be derived by a quadratic relation from the ${}_2\mathcal{F}_1$ equation in essentially the same way as the Hermite equation can be derived from the ${}_1\mathcal{F}_1$ equation.

9.1 sch(1) in 5 dimensions

We again consider $\mathbb{R}^5$ with the coordinates

$$z_0, z_{-2}, z_2, z_{-3}, z_3 \tag{9.1}$$

and the scalar product given by

$$\langle z|z\rangle = z_0^2 + 2z_{-2}z_2 + 2z_{-3}z_3. \tag{9.2}$$

We keep the notation from so(5)—remember that sch(1) is a subalgebra of so(5).

Lie algebra sch(1)**.** The Cartan algebra

$$\begin{aligned} M &= z_{-2}\partial_{z_{-2}} - z_2\partial_{z_2} - z_{-3}\partial_{z_{-3}} + z_3\partial_{z_3}, && (9.3a)\\ B_{3,2} &= z_{-3}\partial_{z_2} - z_{-2}\partial_{z_3}. && (9.3b)\end{aligned}$$

Root operators

$$\begin{aligned} B_{3,0} &= z_{-3}\partial_{z_0} - z_0\partial_{z_3}, && (9.4a)\\ B_{2,0} &= z_{-2}\partial_{z_0} - z_0\partial_{z_2}, && (9.4b)\\ B_{3,-2} &= z_{-3}\partial_{z_{-2}} - z_2\partial_{z_3}, && (9.4c)\\ B_{-3,2} &= z_3\partial_{z_2} - z_{-2}\partial_{z_{-3}}. && (9.4d)\end{aligned}$$

Weyl symmetries

$$\begin{aligned} \iota K(z_0, z_{-2}, z_2, z_{-3}, z_3) &= K(z_0, z_{-2}, z_2, z_{-3}, z_3), && (9.5a)\\ \kappa K(z_0, z_{-2}, z_2, z_{-3}, z_3) &= K(z_0, -z_{-3}, -z_3, z_{-2}, z_2), && (9.5b)\\ \kappa^2 K(z_0, z_{-2}, z_2, z_{-3}, z_3) &= K(z_0, -z_{-2}, -z_2, -z_{-3}, -z_3), && (9.5c)\\ \kappa^3 K(z_0, z_{-2}, z_2, z_{-3}, z_3) &= K(z_0, z_{-3}, z_3, -z_{-2}, -z_2). && (9.5d)\end{aligned}$$

Laplacian

$$\Delta_5 = \partial_{z_0}^2 + 2\partial_{z_{-2}}\partial_{z_2} + 2\partial_{z_{-3}}\partial_{z_3}. \tag{9.6}$$

9.2 sch(1) in 3 dimensions

We descend on the level of $\mathbb{R}^3$, with the variables y_0, y_{-2}, y_2 and the scalar product given by

$$\langle y|y\rangle = y_0^2 + 2y_{-2}y_2.$$

Lie algebra sch(1)**.** Cartan algebra

$$\begin{aligned} M^{\mathrm{fl},\eta} &= y_0\partial_{y_0} + 2y_{-2}\partial_{y_{-2}} - \eta,\\ B_{3,2}^{\mathrm{fl}} &= \partial_{y_2}.\end{aligned}$$

Root operators

$$\begin{aligned} B_{3,0}^{\mathrm{fl}} &= \partial_{y_0},\\ B_{2,0}^{\mathrm{fl}} &= y_{-2}\partial_{y_0} - y_0\partial_{y_2},\\ B_{3,-2}^{\mathrm{fl}} &= \partial_{y_{-2}},\\ B_{-3,2}^{\mathrm{fl},\eta} &= y_{-2}\big(y_0\partial_{y_0} + y_{-2}\partial_{y_{-2}} - \eta\big) - \frac{1}{2}y_0^2\partial_{y_2}.\end{aligned}$$

Weyl symmetries

$$\begin{aligned}
\iota^{\mathrm{fl},\eta} f(y_0,y_{-2},y_2) &= f(y_0,y_{-2},y_2),\\
\kappa^{\mathrm{fl},\eta} f(y_0,y_{-2},y_2) &= y_{-2}^{\eta} f\Big(\frac{y_0}{y_{-2}},-\frac{1}{y_{-2}},\frac{y_0^2+2y_{-2}y_2}{2y_{-2}}\Big),\\
\big(\kappa^{\mathrm{fl},\eta}\big)^2 f(y_0,y_{-2},y_2) &= (-1)^{\eta} f(-y_0,y_{-2},y_2),\\
\big(\kappa^{\mathrm{fl},\eta}\big)^3 f(y_0,y_{-2},y_2) &= (-y_{-2})^{\eta} f\Big(-\frac{y_0}{y_{-2}},-\frac{1}{y_{-2}},\frac{y_0^2+2y_{-2}y_2}{2y_{-2}}\Big).
\end{aligned}$$

Laplacian

$$\Delta_5^{\mathrm{fl}} = \partial_{y_0}^2 + 2\partial_{y_{-2}}\partial_{y_2}.$$

9.3 $\mathrm{sch}(1)$ in $1+1$ dimensions

We descend onto the level of $\mathbb{R}\oplus\mathbb{R}$, as described in Subsect. 7.4. We rename y_{-2} to t.

Lie algebra $\mathrm{sch}(1)$**.** Cartan algebra:

$$M^{\mathrm{sch},\eta} = y_0\partial_{y_0} + 2t\partial_t - \eta, \tag{9.7a}$$

$$B_{3,2} = 1. \tag{9.7b}$$

Root operators

$$B_{3,0}^{\mathrm{sch}} = \partial_{y_0}, \tag{9.8a}$$

$$B_{2,0}^{\mathrm{sch}} = t\partial_{y_0} - y_0, \tag{9.8b}$$

$$B_{3,-2}^{\mathrm{sch}} = \partial_t, \tag{9.8c}$$

$$B_{-3,2}^{\mathrm{sch},\eta} = t(y_0\partial_{y_0} + t\partial_t - \eta) - \frac{1}{2}y_0^2. \tag{9.8d}$$

Weyl symmetry

$$\iota^{\mathrm{sch},\eta} h(y_0,t) = h(y_0,t), \tag{9.9a}$$

$$\kappa^{\mathrm{sch},\eta} h(y_0,t) = t^{\eta}\exp(\tfrac{y_0^2}{2t})h(\tfrac{y_0}{t},-\tfrac{1}{t}), \tag{9.9b}$$

$$\big(\kappa^{\mathrm{sch},\eta}\big)^2 h(y_0,t) = (-1)^{\eta} h(-y_0,t), \tag{9.9c}$$

$$\big(\kappa^{\mathrm{sch},\eta}\big)^3 h(y_0,t) = (-t)^{\eta}\exp(\tfrac{y_0^2}{2t})h(-\tfrac{y_0}{t},-\tfrac{1}{t}). \tag{9.9d}$$

Heat operator

$$\Delta_5^{\mathrm{sch}} = \mathcal{L}_1 = \partial_{y_0}^2 + 2\partial_t. \tag{9.10}$$

9.4 $\mathrm{sch}(1)$ in the coordinates w, s

Let us define new coordinates

$$w = \frac{y_0}{\sqrt{2\,t}}\,, \qquad s = \sqrt{t}\,, \tag{9.11}$$

with the reverse transformation

$$y_0 = \sqrt{2}\,s\,w\,, \qquad t = s^2\,. \tag{9.12}$$

Lie algebra $\mathrm{sch}(1)$**.** Cartan algebra

$$\begin{aligned} M^{\mathrm{sch},\eta} &= s\,\partial_s - \eta, \\ B_{3\,2} &= 1. \end{aligned}$$

Root operators

$$\begin{aligned} B_{3,0}^{\mathrm{sch}} &= \frac{1}{\sqrt{2}\,s}\ \partial_w, \\ B_{2,0}^{\mathrm{sch}} &= \frac{s}{\sqrt{2}}\ (\,\partial_w - 2w)\,, \\ B_{3,-2}^{\mathrm{sch}} &= \frac{1}{2\,s^2}\ (-w\,\partial_w + s\partial_s)\,, \\ B_{-3,2}^{\mathrm{sch},\eta} &= \frac{s^2}{2}\ \left(w\,\partial_w + s\partial_s - 2\eta - 2w^2\right). \end{aligned}$$

Weyl symmetries

$$\begin{aligned} \iota^{\mathrm{sch},\eta}h(w,s) &= h(w,s), \\ \kappa^{\mathrm{sch},\eta}h(w,s) &= s^{2\eta}\mathrm{e}^{w^2}h(\mathrm{i}w, -\tfrac{\mathrm{i}}{s}), \\ (\kappa^{\mathrm{sch},\eta})^2h(w,s) &= (-1)^{\eta}h(-w,s), \\ (\kappa^{\mathrm{sch},\eta})^3h(w,s) &= (-s^2)^{\eta}\mathrm{e}^{w^2}h(-\mathrm{i}w, -\tfrac{\mathrm{i}}{s}). \end{aligned}$$

Heat operator

$$\mathcal{L}_1 = \frac{1}{2s^2}\left(\partial_w^2 - 2w\,\partial_w + 2s\,\partial_s\right)\,. \tag{9.13}$$

9.5 Hermite operator

Let us set $\eta = -\frac{1}{2}$ and use the ansatz

$$h(w,s) = s^{-\lambda-\frac{1}{2}}S(w). \tag{9.14}$$

Clearly,

$$\begin{aligned} M^{\mathrm{sch},-\frac{1}{2}}h &= -\lambda h, && (9.15)\\ s^{\lambda+\frac{1}{2}}2s^2\mathcal{L}_1 h &= \mathcal{S}_\lambda(w,\partial_w)S(w), && (9.16) \end{aligned}$$

where we have introduced the *Hermite operator*

$$S_\lambda(w,\partial_w) := \partial_w^2 - 2w\partial_w - 2\lambda - 1. \tag{9.17}$$

We will also use an alternative notation

$$S(a;w,\partial_w) := \partial_w^2 - 2w\partial_w - 2a, \tag{9.18}$$

so that

$$\lambda = a - \frac{1}{2}, \quad a = \lambda + \frac{1}{2}. \tag{9.19}$$

9.6 Quadratic transformation

Let us go back to 2+1 dimensions and the heat operator

$$\mathcal{L}_2 = 2\partial_{y_{-1}}\partial_{y_1} + 2\partial_t. \tag{9.20}$$

Let us use the reduction described in Subsect. 3.14, and then applied in Subsect. 6.4:

$$y_0 := \sqrt{2y_{-1}y_1}, \quad u := \sqrt{\frac{y_1}{y_{-1}}}. \tag{9.21}$$

In the new variables,

$$\begin{aligned} N_1 &= u\partial_u, && (9.22)\\ \mathcal{L}_2 &= \left(\partial_{y_0} + \frac{1}{2y_0}\right)^2 - \frac{1}{y_0^2}\left(u\partial_u - \frac{1}{2}\right)\left(u\partial_u + \frac{1}{2}\right) + 2\partial_t. && (9.23) \end{aligned}$$

Therefore,

$$\begin{aligned} (uy_0)^{\frac{1}{2}}\mathcal{L}_2(uy_0)^{-\frac{1}{2}} &= -\frac{1}{y_0^2}N_1\left(N_1 - 1\right) + \mathcal{L}_1, && (9.24a)\\ (u^{-1}y_0)^{\frac{1}{2}}\mathcal{L}_2(u^{-1}y_0)^{-\frac{1}{2}} &= -\frac{1}{y_0^2}N_1\left(N_1 + 1\right) + \mathcal{L}_1. && (9.24b) \end{aligned}$$

Compare the coordinates (8.12) for 2+1 dimensions and the coordinates (9.11) for 1+1 dimensions. The coordinate s are the same. This is not the case of w, so let us rename w from (9.11) as v. We then have $w = v^2$. We also have

$$uy_0 = \sqrt{2}su_1, \quad u^{-1}y_0 = \sqrt{2}wu_1^{-1}s.$$

Hence on functions that do not depend on u we obtain

$$s^{\frac12}u_1^{\frac12}\mathcal{L}_2 s^{-\frac12}u_1^{-\frac12} = \mathcal{L}_1, \tag{9.25a}$$

$$s^{\frac12}u_1^{-\frac12}v\mathcal{L}_2 s^{-\frac12}u_1^{\frac12}v^{-1} = \mathcal{L}_1. \tag{9.25b}$$

Thus by a quadratic transformation we can transform the Hermite equation into a special case of the confluent equation:

$$\mathcal{S}_\lambda(v,\partial_v) \quad = \quad 4\mathcal{F}_{\lambda,-\frac12}(w,\partial_w), \tag{9.26a}$$

$$v^{-1}\mathcal{S}_\lambda(v,\partial_v)v \quad = \quad 4\mathcal{F}_{\lambda,\frac12}(w,\partial_w), \tag{9.26b}$$

where

$$w = v^2, \qquad v = \sqrt{w}.$$

9.7 Transmutation relations and discrete symmetries

The heat operator satisfies the generalized symmetries

$$B^{\mathrm{sch},-\frac52}\mathcal{L}_1 \quad = \quad \mathcal{L}_1 B^{\mathrm{sch},-\frac12}, \quad B \in \mathrm{sch}(1); \tag{9.27a}$$

$$\alpha^{\mathrm{sch},-\frac52}\mathcal{L}_1 \quad = \quad \mathcal{L}_1 \alpha^{\mathrm{sch},-\frac12}, \quad \alpha \in \mathrm{Sch}(1). \tag{9.27b}$$

Equation (9.27a) applied to the roots of sch(1) implies the transmutation relations of the Hermite operator:

$$\begin{array}{rrcll}
\partial_w & \mathcal{S}_\lambda & = & \mathcal{S}_{\lambda+1} & \partial_w, \\
(\partial_w - 2w) & \mathcal{S}_\lambda & = & \mathcal{S}_{\lambda-1} & (\partial_w - 2w), \\
(w\partial_w + \lambda + \frac12) & w^2\mathcal{S}_\lambda & = & w^2\mathcal{S}_{\lambda+2} & (w\partial_w + \lambda + \frac12), \\
(w\partial_w - \lambda + \frac12 - 2w^2) & w^2\mathcal{S}_\lambda & = & w^2\mathcal{S}_{\lambda-2} & (w\partial_w - \lambda + \frac12 - 2w^2).
\end{array}$$

Relation (9.27a) applied to the Weyl symmetries of sch(1) implies the discrete symmetries of the Hermite operator, described below.

The following operators equal $\mathcal{S}_\lambda(w,\partial_w)$ for an appropriate w:

$$w = \pm v: \qquad \mathcal{S}_\lambda(v,\partial_v), \tag{9.28a}$$

$$w = \pm \mathrm{i}v: \qquad -\exp(-v^2)\mathcal{S}_{-\lambda}(v,\partial_v)\exp(v^2). \tag{9.28b}$$

9.8 Factorizations of the heat operator

Special role is played by two distinguished subalgebras of sch(2).

First note the commutation relations

$$[B_{2,0}, B_{3,0}] = B_{3,2}. \tag{9.29}$$

Therefore, we have the following distinguished subalgebra in sch(1) isomorphic to heis(2):

$$\mathrm{heis}_0(2) \quad \text{spanned by} \quad B_{2,0},\ B_{3,0},\ B_{3,2}. \tag{9.30}$$

Let us define

$$\mathcal{C}_0 = 2\,B_{2,0}B_{3,0} + 2M - B_{3,2} \tag{9.31a}$$

$$= 2\,B_{3,0}B_{2,0} + 2M + B_{3,2}. \tag{9.31b}$$

We have the commutation relations

$$[\mathcal{C}_0, B_{2,0}] = -2B_{2,0}(B_{3,2} - 1),$$
$$[\mathcal{C}_0, B_{3,0}] = 2B_{3,0}(B_{3,2} - 1),$$
$$[\mathcal{C}_0, B_{3,2}] = 0.$$

But $B_{3,2}^{\mathrm{sch},\eta} = 1$. Therefore, on the level of $\mathbb{R} \oplus \mathbb{R}$ the operator $\mathcal{C}_0^{\mathrm{sch},\eta}$ can be treated as a kind of a Casimir operator of $\mathrm{heis}_0(2)$: it commutes with all elements of $\mathrm{heis}_0(2)$. Note the identity

$$2t\mathcal{L}_1 = \mathcal{C}_0^{\mathrm{sch},-\frac{1}{2}}. \tag{9.32}$$

Second, consider $B_{-3,2}$, $B_{3,-2}$, M. They are contained both in sch(6) and in sch(5). Therefore, the subalgebra $\mathrm{so}_{23}(3)$, described in Sect. 8.7 in the context of sch(6), is also contained in sch(5). Recall that its Casimir operator is

$$\mathcal{C}_{23} = 4\,B_{3,-2}B_{-3,2} - (M+1)^2 + 1 \tag{9.33a}$$

$$= 4\,B_{-3,2}B_{3,-2} - (M-1)^2 + 1. \tag{9.33b}$$

By (4.21) we have

$$(2z_{-2}z_2 + 2z_{-3}z_3)\Delta_5^{\diamond} = \mathcal{C}_{23}^{\diamond,-\frac{1}{2}} - \frac{3}{4}. \tag{9.34}$$

Inserting (9.33) into (9.34) we obtain

$$(2z_{-2}z_2 + 2z_{-3}z_3)\Delta_5^{\diamond}$$

$$=4B_{2,-3}B_{-2,3} - \left(M + \frac{3}{2}\right)\left(M + \frac{1}{2}\right), \tag{9.35a}$$

$$=4B_{-2,3}B_{2,-3} - \left(M - \frac{3}{2}\right)\left(M - \frac{1}{2}\right), \tag{9.35b}$$

where the B, N_1 and M operators should be decorated with the superscript $^{\diamond,-\frac12}$.

Let us sum up the factorizations in the variables y_0, t obtained with the help of the two subalgebras:

$$\begin{aligned} 2t\mathcal{L}_1 &= 2\,B_{2,0}B_{3,0} - (-2M+1) && (9.36a)\\ &= 2\,B_{3,0}B_{2,0} - (-2M-1), && (9.36b)\\ -y_0^2\mathcal{L}_1 &= 4B_{2,-3}B_{-2,3} - \Big(M+\frac32\Big)\Big(M+\frac12\Big) && (9.36c)\\ &= 4B_{-2,3}B_{2,-3} - \Big(M-\frac32\Big)\Big(M-\frac12\Big), && (9.36d) \end{aligned}$$

where the B, N_1 and M operators should be equipped with the superscript $^{\mathrm{sch},-\frac12}$.

In the coordinates w, s we need to make the replacements

$$\begin{aligned} t &\to s^2, && (9.37a)\\ y_0^2 &\to 2w^2s^2. && (9.37b) \end{aligned}$$

9.9 Factorizations of the Hermite operator

The factorizations of $\mathcal{L}_1$ described in Subsect. 9.8 yield the following factorizations of the Hermite operator:

$$\begin{aligned} \mathcal{S}_\lambda &= \big(\partial_w - 2w\big)\partial_w - 2\lambda - 1\\ &= \partial_w\big(\partial_w - 2w\big) - 2\lambda + 1,\\ w^2\mathcal{S}_\lambda &= \Big(w\partial_w + \lambda - \frac32\Big)\Big(w\partial_w - \lambda + \frac12 - 2w^2\Big) + \Big(\lambda - \frac32\Big)\Big(\lambda - \frac12\Big)\\ &= \Big(w\partial_w - \lambda - \frac32 - 2w^2\Big)\Big(w\partial_w + \lambda + \frac12\Big) + \Big(\lambda + \frac32\Big)\Big(\lambda + \frac12\Big). \end{aligned}$$

9.10 Standard solutions

The Hermite equation has only one singular point, ∞. One can define two kinds of solutions with a simple asymptotics at ∞. They can be derived from the expressions of Subsect. 8.11, using (9.26) and (9.28b)

$$\sim w^{-a} \text{ for } w \to +\infty: \qquad S_\lambda(w) := w^{-\lambda-\frac{1}{2}}\tilde{F}_{\lambda,\frac{1}{2}}(-w^{-2}) = w^{-a}F\Big(\frac{a}{2}, \frac{a+1}{2}; -; -w^{-2}\Big),$$

$$\sim(-\mathrm{i}w)^{a-1}\mathrm{e}^{w^2} \text{ for } w \to +\mathrm{i}\infty: \qquad \mathrm{e}^{w^2}S_{-\lambda}(-\mathrm{i}w) = (-\mathrm{i}w)^{\lambda-\frac{1}{2}}\mathrm{e}^{w^2}\tilde{F}_{-\lambda,\frac{1}{2}}(w^{-2}) = (-\mathrm{i}w)^{a-1}\mathrm{e}^{w^2}F\Big(\frac{1-a}{2}, \frac{2-a}{2}; -; -w^{-2}\Big).$$

9.11 Recurrence relations

Each of the following recurrence relations corresponds to a root of sch(1):

$$\begin{aligned}
\partial_w S_\lambda(w) &= -\Big(\frac{1}{2}+\lambda\Big)S_{\lambda+1}(w),\\
(\partial_w - 2w)S_\lambda(w) &= -2S_{\lambda-1}(w),\\
\Big(w\partial_w + \frac{1}{2}+\lambda\Big)S_\lambda(w) &= \frac{1}{2}\Big(\frac{1}{2}+\lambda\Big)\Big(\frac{3}{2}+\lambda\Big)S_{\lambda+2}(w),\\
\Big(w\partial_w + \frac{1}{2}-\lambda-2w^2\Big)S_\lambda(w) &= -2S_{\lambda-2}(w).
\end{aligned}$$

The first pair corresponds correspond to the celebrated annihilation and creation operators in the theory of quantum harmonic oscillator. The second pair are the double annihilation and creation operators.

9.12 Wave packets for the heat equation in 1 dimensions

Consider the space $\mathbb{R}\oplus\mathbb{R}$ and the heat equation given by the operator $\mathcal{L}_1 = \partial_y^2 + 2\partial_t$. Recall that

$$M^{\mathrm{sch},-\frac{1}{2}} = y\partial_y + 2t\partial_t + \frac{1}{2}. \tag{9.38}$$

Set

$$G_\lambda^a(y,t) := \int\limits_{\gamma^a} t^{-\frac{1}{2}}\exp\Big(\frac{(y-\tau^{-1})^2}{2t}\Big)\tau^{-\frac{3}{2}+\lambda}\mathrm{d}\tau, \tag{9.39a}$$

$$G_\lambda^b(y,t) := \int\limits_{\gamma^b} \mathrm{e}^{-\sqrt{2}y\tau-t\tau^2}\tau^{-\frac{1}{2}+\lambda}\mathrm{d}\tau. \tag{9.39b}$$

Proposition 9.1. *We have*

$$\mathcal{L}_1 G^a_\lambda = 0, \qquad \mathcal{L}_1 G^b_\lambda = 0; \tag{9.40a}$$

$$M^{\mathrm{sch},-\frac12} G^a_\lambda = -\lambda G^a_\lambda, \qquad M^{\mathrm{sch},-\frac12} G^b_\lambda = -\lambda G^b_\lambda. \tag{9.40b}$$

Proof. Set

$$g^a(y,t) := t^{-\frac12} \exp\frac{(y-1)^2}{2t}, \tag{9.41a}$$

$$g^b(y,t) := \mathrm{e}^{-\sqrt{2}y-t}. \tag{9.41b}$$

We have

$$G^a_\lambda = \int\limits_{\gamma^a} \tau^{-1+\frac12+\lambda} g^a(\tau y, \tau^2 t)\mathrm{d}\tau, \tag{9.42a}$$

$$G^b_\lambda = \int\limits_{\gamma^b} \tau^{-1+\frac12+\lambda} g^b(\tau y, \tau^2 t)\mathrm{d}\tau. \tag{9.42b}$$

Clearly, g^a and g^b solve the heat equation. By (9.42b), G^a_λ, resp. G^b_λ are wave packets made out of rotated g^a, resp. g^b. Therefore, they also solve the heat equation.

If the contours satisfy the requirements of Prop. 7.1, then (9.42b) implies (9.40b). □

Let us express these wave packets in the coordinates w, s:

$$G^a_\lambda(w,s) = \int s^{-1} \exp\left(\left(w - \frac{1}{\sqrt{2}\tau s}\right)^2\right) \tau^{-2+\frac12+\lambda}\mathrm{d}\tau, \tag{9.43a}$$

$$G^b_\lambda(w,s) = \int \mathrm{e}^{-2sw\tau - s^2\tau^2} \tau^{-1+\frac12+\lambda}\mathrm{d}\tau. \tag{9.43b}$$

In (9.43a) we set $\sigma := w - \frac{1}{\sqrt{2}\tau s}$, so that $\tau = \frac{1}{(w-\sigma)\sqrt{2}s}$. In (9.43b) we set $\sigma := s\tau$, so that $\tau = \frac{\sigma}{s}$. We obtain

$$G^a_\lambda(w,s) = (\sqrt{2})^{\frac12-\lambda} s^{-\frac12-\lambda} F^a_\lambda(w), \tag{9.44a}$$

$$G^b_\lambda(w,s) = s^{-\frac12-\lambda} F^b_\lambda(w), \tag{9.44b}$$

where

$$F^a_\lambda(w) := \int\limits_{\gamma^a} \mathrm{e}^{\sigma^2} (w-\sigma)^{-\frac12-\lambda}\mathrm{d}\sigma, \tag{9.45a}$$

$$F^b_\lambda(w) := \int\limits_{\gamma^b} \mathrm{e}^{-2\sigma w - \sigma^2} \sigma^{-1+\frac12+\lambda}\mathrm{d}\sigma. \tag{9.45b}$$

The above analysis shows that for appropriate contours (9.45a) and (9.45b) are solutions of the Hermite equation.

9.13 Integral representations

Below we directly describe the two kinds of integral representations of solutions, without passing through additional variables.

Theorem 9.2. *a) Let* $[0,1] \ni \tau \overset{\gamma}{\mapsto} t(\tau)$ *satisfy* $\mathrm{e}^{t^2}(t-w)^{-a-1}\Big|_{t(0)}^{t(1)} = 0$. *Then*

$$\mathcal{S}(a; w, \partial_w) \int\limits_\gamma \mathrm{e}^{t^2}(t-w)^{-a}\mathrm{d}t = 0. \tag{9.46}$$

b) Let $[0,1] \ni \tau \mapsto t(\tau)$ *satisfy* $\mathrm{e}^{-t^2-2wt}t^a\Big|_{t(0)}^{t(1)} = 0$. *Then*

$$\mathcal{S}(a; w, \partial_w) \int\limits_\gamma \mathrm{e}^{-t^2-2wt}t^{a-1}\mathrm{d}t = 0. \tag{9.47}$$

Proof. We check that for any contour γ

$$\begin{aligned} \text{lhs of (9.46)} &= -a\int\limits_\gamma \mathrm{d}t\, \partial_t \mathrm{e}^{t^2}(t-w)^{-a-1}, \\ \text{lhs of (9.47)} &= -2\int\limits_\gamma \mathrm{d}t\, \partial_t \mathrm{e}^{-t^2-2wt}t^a. \end{aligned}$$

We can also deduce the second representation from the first by the discrete symmetry (9.28b). □

9.14 Integral representations of standard solutions

In type a) representations the integrand has a singular point at 0 and goes to zero as $t \to \pm\infty$. We can thus use contours with such endpoints. We will see that they give all standard solutions.

In type b) representations the integrand has a singular point at w and goes to zero as $t \to \pm\mathrm{i}\infty$. Using contours with such endpoints, we will also obtain all standard solutions.

	a)	**b)**
$\sim w^{-a}$ **for** $w \to +\infty$:	$[0,\infty[$,	$]-\mathrm{i}\infty, w^-, -\mathrm{i}\infty[$;
$\sim (-\mathrm{i}w)^{a-1}\mathrm{e}^{w^2}$ **for** $w \to +\mathrm{i}\infty$:	$]-\infty, 0^+, -\infty[$,	$[w, \mathrm{i}\infty[$.

It is convenient to introduce alternatively normalized solutions:

$$S_\lambda^{\mathrm{I}}(w) \quad := \quad 2^{-\lambda-\frac{1}{2}}\Gamma\Big(\lambda+\frac{1}{2}\Big)S_\lambda(w).$$

Here are integral representations of type a):

$$\text{all } \lambda: \tag{9.48}$$

$$-\mathrm{i}\int\limits_{]-\mathrm{i}\infty, w^-, \mathrm{i}\infty[} \mathrm{e}^{t^2}(w-t)^{-\lambda-\frac{1}{2}}\mathrm{d}t = \sqrt{\pi}S_\lambda(w), \qquad w \notin]-\infty, 0];$$

$$\mathrm{Re}\lambda < \frac{1}{2}: \tag{9.49}$$

$$-\mathrm{i}\int\limits_{[w,\mathrm{i}\infty[} \mathrm{e}^{t^2}(-\mathrm{i}(t-w))^{-\lambda-\frac{1}{2}}\mathrm{d}t = \mathrm{e}^{w^2}S_{-\lambda}^{\mathrm{I}}(-\mathrm{i}w), \qquad w \notin [0,\infty[.$$

And here are integral representations of type b):

$$-\frac{1}{2} < \mathrm{Re}\lambda: \tag{9.50}$$

$$\int\limits_0^\infty \mathrm{e}^{-t^2-2tw}t^{\lambda-\frac{1}{2}}\mathrm{d}t = S_\lambda^{\mathrm{I}}(w), \qquad w \notin]-\infty, 0];$$

$$\text{all } \lambda: \tag{9.51}$$

$$\int\limits_{]-\infty, 0^+, \infty[} \mathrm{e}^{-t^2-2tw}(\mathrm{i}t)^{\lambda-\frac{1}{2}}\mathrm{d}t = \sqrt{\pi}\mathrm{e}^{w^2}S_{-\lambda}(-\mathrm{i}w), \qquad w \notin [0,\infty[.$$

10 The Helmholtz equation in 2 dimensions and the ${}_0\mathcal{F}_1$ equation

The goal of this section is to derive the ${}_0\mathcal{F}_1$ equation together with its symmetries from the Helmoltz equation in 2 dimensions. The symmetries of these equations, together with its derivation, are the simplest and the best known. In particular, we do not need to consider *generalized symmetries*.

Here are the main steps from the derivation:

(1) We start from the Helmholtz operator $\Delta_2 - 1$. The Lie algebra aso(2) and group ASO(2) acting in 2 dimensions, are the obvious symmetries of this operator.
(2) We choose coordinates w, u, so that the Cartan element is expressed in terms of u. We compute $\Delta_2 - 1$ and the representations of aso(2) and ASO(2) in the new coordinates.
(3) We make an ansatz diagonalizing the Cartan element, whose eigenvalue α becomes a parameter. The only variable left is w. The Helmholtz operator $\Delta_2 - 1$ becomes the ${}_0\mathcal{F}_1$ operator. The symmetries of $\Delta_2 - 1$ yield transmutation relations and discrete symmetries of the ${}_0\mathcal{F}_1$ operator.

The remaining part of this section is to a large extent parallel to their analogs in Sects 5, 6, 8 and 9. Essentially all subsections have their counterparts there. The only exception is Subsect. 10.4 on the equivalence of the ${}_0\mathcal{F}_1$ equation with a subclass of the ${}_1\mathcal{F}_1$ equation, and its many-dimensional unravelling. This equivalence is obtained by a quadratic transformation, which is quite different from the quadratic transformations for the Gegenbauer and Hermite equation considered in Subsects 6.4, resp. 9.6.

10.1 aso(2)

We consider $\mathbb{R}^2$ with split coordinates x_-, x_+ and the scalar product

$$\langle x|x\rangle = 2x_- x_+. \tag{10.1}$$

Lie algebra $\mathrm{aso}(\mathbb{C}^2)$**.** Cartan operator

$$N = -x_-\partial_{x_-} + x_+\partial_{x_+}. \tag{10.2}$$

Root operators

$$B_- = \partial_{x_-}, \tag{10.3a}$$

$$B_+ = \partial_{x_+}. \tag{10.3b}$$

Weyl symmetry

$$\tau f(x_-, x_+) = f(x_+, x_-). \tag{10.4}$$

Helmholtz operator

$$\Delta_2 - 1 = 2\partial_{x_-}\partial_{x_+} - 1. \tag{10.5}$$

10.2 Variables w, u

We introduce the coordinates

$$w = \frac{x_- x_+}{2}, \qquad u = x_+. \tag{10.6}$$

Lie algebra aso(2)**.** Cartan operator

$$N = u\partial_u.$$

Root operators

$$\begin{aligned} B_+ &= \frac{u}{2}\partial_w, \\ B_- &= \frac{1}{u}(w\partial_w + u\partial_u). \end{aligned}$$

Weyl symmetry

$$\tau f(w,u) = f\Big(w, \frac{w}{u}\Big).$$

Helmholtz operator

$$\Delta_2 - 1 = w\partial_w^2 + (1 + u\partial_u)\partial_w - 1. \tag{10.7}$$

10.3 The ${}_0\mathcal{F}_1$ operator

Let us make the ansatz

$$f(w,u) = u^\alpha F(w). \tag{10.8}$$

Clearly,

$$Nf = \alpha f, \tag{10.9}$$

$$u^{-\alpha}(\Delta_2 - 1)f = \mathcal{F}_\alpha(w, \partial_w)F, \tag{10.10}$$

where we have introduced the ${}_0\mathcal{F}_1$ *operator*

$$\mathcal{F}_\alpha(w, \partial_w) := w\partial_w^2 + (1+\alpha)\partial_w - 1. \tag{10.11}$$

Instead of the *Lie-algebraic parameter* α one could also use the *classical parameter* c

$$\alpha := c - 1, \quad c = \alpha + 1, \tag{10.12}$$

so that the ${}_0\mathcal{F}_1$ operator becomes

$$\mathcal{F}(c; w, \partial_w) := w\partial_w^2 + c\partial_w - 1. \tag{10.13}$$

10.4 Equivalence with a subclass of the confluent equation

The ${}_0\mathcal{F}_1$ equation is equivalent to a subclass of the ${}_1\mathcal{F}_1$ equation by a quadratic transformation. This quadratic transformation is however quite different from transformations described in Subsect. 3.14, and then applied to derive the Gegenbauer quation and the Hermite equation. In this subsection we derive this equivalence starting from the heat equation in 2 dimensions.

First let us recall some elements of our derivation of the ${}_1\mathcal{F}_1$ operator. As described in Sect. 8, it was obtained from the heat operator (8.11) together with Cartan operators (8.8a), (8.8c):

$$\frac{t}{2}\mathcal{L}_2 = \frac{t}{2}\big(2\partial_t + 2\partial_{y_{-1}}\partial_{y_1}\big), \tag{10.14a}$$

$$M = y_{-1}\partial_{y_{-1}} + y_1\partial_{y_1} + 2t\partial_t + 1, \tag{10.14b}$$

$$N_1 = -y_{-1}\partial_{y_{-1}} + y_1\partial_{y_1}. \tag{10.14c}$$

(We set $\eta = -1$ and dropped the superscript ${}^{\mathrm{sch},-1}$). Recall that substituting the coordinates (8.12)

$$w = \frac{y_{-1}y_1}{t}\ , \qquad u_1 = \frac{y_1}{\sqrt{t}}\ , \qquad s = \sqrt{t} \tag{10.15}$$

we obtain

$$\frac{t}{2}\mathcal{L}_2 = w\partial_w^2 + (u\partial_u + 1 - w)\partial_w + \frac{1}{2}(-u\partial_u + s\partial_s), \tag{10.16a}$$

$$M = s\partial_s + 1, \tag{10.16b}$$

$$N_1 = u_1\partial_{u_1}. \tag{10.16c}$$

After we set $M = -\theta$, $N_1 = \alpha$, (10.16a) becomes $\mathcal{F}_{\theta,\alpha}(w, \partial_w)$.

Consider now

$$\begin{aligned}
&\frac{2t^2}{y_{-1}y_1}\mathrm{e}^{-\frac{y_{-1}y_1}{2t}}\mathcal{L}_2\mathrm{e}^{\frac{y_{-1}y_1}{2t}} \\
=&\frac{2t}{y_{-1}y_1}\big(y_{-1}\partial_{y_{-1}} + y_1\partial_{y_1} + 2t\partial_t + 1\big) + \frac{4t^2}{y_{-1}y_1}\partial_{y_{-1}}\partial_{y_1} - 1 \\
=&\frac{2t}{y_{-1}y_1}M + 2\partial_{x_-}\partial_{x_+} - 1, \\
&\mathrm{e}^{-\frac{y_{-1}y_1}{2t}}N_1\mathrm{e}^{\frac{y_{-1}y_1}{2t}} = N_1 = -2x_-\partial_{x_-} + 2x_+\partial_{x_+},
\end{aligned} \tag{10.17}$$

where we introduced new variables

$$x_- = \frac{y_{-1}^2}{2\sqrt{2}t}, \quad x_+ = \frac{y_1^2}{2\sqrt{2}t}. \tag{10.18}$$

Therefore, on the subspace $M=0$ we have

$$\begin{aligned} \frac{2t^2}{y_{-1}y_1}\mathrm{e}^{-\frac{y_{-1}y_1}{2t}}\mathcal{L}_2\mathrm{e}^{\frac{y_{-1}y_1}{2t}} &=\Delta_2-1,\\ \mathrm{e}^{-\frac{y_{-1}y_1}{2t}}N_1\mathrm{e}^{\frac{y_{-1}y_1}{2t}} &= 2N, \end{aligned} \tag{10.19}$$

where Δ_2-1 is the Helmholtz operator (10.5) and N the Cartan operator (10.2). Remember, that in Subsect. 10.2 we express these operators in the coordinates (10.6). To avoid a clash of symbols, we rename w from (10.6) into v:

$$v=\frac{y_-y_+}{2},\quad u=y_+. \tag{10.20}$$

Recall that in the v,u coordinates we have

$$\Delta_2-1=v\partial_v^2+(1+u\partial_u)\partial_v-1, \tag{10.21a}$$

$$N=u\partial_u, \tag{10.21b}$$

so that (10.21a) on $N=\alpha$ becomes $\mathcal{F}_\alpha(v,\partial_v)$.

Now we can compare the coordinates w,u_1 and v,u

$$v=\frac{y_{-1}^2y_1^2}{16t^2}=\Big(\frac{w}{4}\Big)^2,\quad u=\frac{y_1^2}{2\sqrt{2}t}=\frac{u_1^2}{2\sqrt{2}}. \tag{10.22}$$

This leads to the so-called *Kummer's 2nd transformation*, which reduces the ${}_0\mathcal{F}_1$ equation to a special class of the confluent equation by a quadratic transformation:

$$\mathcal{F}_\alpha(v,\partial_v)=\frac{4}{w}\mathrm{e}^{-w/2}\mathcal{F}_{0,2\alpha}(w,\partial_w)\mathrm{e}^{w/2}, \tag{10.23}$$

or, in classical parameters

$$\mathcal{F}(c;v,\partial_v)=\frac{4}{w}\mathrm{e}^{-w/2}\mathcal{F}\Big(c-\frac{1}{2};2c-1;w,\partial_w\Big)\mathrm{e}^{w/2}, \tag{10.24}$$

where $w=\pm4\sqrt{v}$, $v=\left(\frac{w}{4}\right)^2$.

10.5 Transmutation relations and symmetries

The following symmetries of the Helmholtz operator are obvious:

$$B(\Delta_2-1)=(\Delta_2-1)B;\quad B\in\mathrm{aso}(2); \tag{10.25a}$$

$$\alpha(\Delta_2-1)=(\Delta_2-1)\alpha;\quad \alpha\in\mathrm{ASO}(2). \tag{10.25b}$$

Applying (10.25a) to the roots of aso(2) we obtain the trasmutation relations

$$\begin{aligned}\partial_w \quad \mathcal{F}_\alpha &= \mathcal{F}_{\alpha+1} \quad \partial_w,\\ (w\partial_w+\alpha) \quad \mathcal{F}_\alpha &= \mathcal{F}_{\alpha-1} \quad (w\partial_w+\alpha).\end{aligned}$$

Applying (10.25b) to the Weyl symmetry of aso(2) we obtain the symmetry

$$w^{-\alpha}\,\mathcal{F}_{-\alpha}\,w^{\alpha} = \mathcal{F}_\alpha.$$

10.6 Factorizations

The factorizations

$$\Delta_2 - 1 = 2B_-B_+ - 1 \tag{10.26a}$$

$$= 2B_+B_- - 1, \tag{10.26b}$$

are completely obvious. They yield the factorizations of the ${}_0\mathcal{F}_1$ operator:

$$\begin{aligned}\mathcal{F}_\alpha &= \big(w\partial_w+\alpha+1\big)\partial_w - 1\\ &= \partial_w\big(w\partial_w+\alpha\big) - 1.\end{aligned}$$

10.7 The ${}_0F_1$ function

The ${}_0\mathcal{F}_1$ equation has a regular singular point at 0. Its indices at 0 are equal to 0, $\alpha = 1-c$.

If $c \neq 0,-1,-2,\dots$, then the only solution of the ${}_0F_1$ equation ~ 1 at 0 is called the ${}_0F_1$ *function*. It is

$$F(c;w) := \sum_{j=0}^{\infty} \frac{1}{(c)_j}\frac{w^j}{j!}. \tag{10.27}$$

It is defined for $c \neq 0,-1,-2,\dots$. Sometimes it is more convenient to consider the function

$$\mathbf{F}(c;w) := \frac{F(c;w)}{\Gamma(c)} = \sum_{j=0}^{\infty}\frac{1}{\Gamma(c+j)}\frac{w^j}{j!} \tag{10.28}$$

defined for all c.

Using (10.24), we can express the ${}_0F_1$ function in terms of the confluent function

$$F(c;w) = \mathrm{e}^{-2\sqrt{w}}F\Big(\frac{2c-1}{2};2c-1;4\sqrt{w}\Big) \tag{10.29a}$$

$$= \mathrm{e}^{2\sqrt{w}}F\Big(\frac{2c-1}{2};2c-1;-4\sqrt{w}\Big). \tag{10.29b}$$

We will usually prefer to use the Lie-algebraic parameters:

$$F_\alpha(w) := F(\alpha+1;w), \tag{10.30a}$$
$$\mathbf{F}_\alpha(w) := \mathbf{F}(\alpha+1;w). \tag{10.30b}$$

10.8 Standard solutions

We have two standard solutions corresponding to two indices of the regular singular point $w=0$. Besides, using Tricomi's function described in Subsect. 8.11, we have an additional solution with a special behavior at ∞:

$$\begin{aligned}
&\sim 1 \text{ at } 0: && F_\alpha(w) = \mathrm{e}^{-2\sqrt{w}} F_{0,2\alpha}\big(4\sqrt{w}\big)\\
& && \quad = \mathrm{e}^{2\sqrt{w}} F_{0,2\alpha}\big(-4\sqrt{w}\big);\\
&\sim w^{-\alpha} \text{ at } 0: && w^{-\alpha}F_{-\alpha}(w) = w^{-\alpha}\mathrm{e}^{-2\sqrt{w}} F_{0,-2\alpha}\big(4\sqrt{w}\big)\\
& && \quad = w^{-\alpha}\mathrm{e}^{2\sqrt{w}} F_{0,-2\alpha}\big(-4\sqrt{w}\big);\\
&\sim \mathrm{e}^{-2\sqrt{w}} w^{-\frac{\alpha}{2}-\frac14},\ w\to+\infty: && \tilde F_\alpha(w) := \mathrm{e}^{-2\sqrt{w}} w^{-\frac{\alpha}{2}-\frac14} \tilde F_{0,2\alpha}\Big(-\frac{1}{4\sqrt{w}}\Big)\\
& && \quad = \mathrm{e}^{-2\sqrt{w}} w^{-\frac{\alpha}{2}-\frac14} \tilde F_{0,-2\alpha}\Big(-\frac{1}{4\sqrt{w}}\Big).
\end{aligned}$$

Note that the third standard solution is a new function closely related to the MacDonald function. It satisfies the identity

$$\tilde F_\alpha(w) = w^{-\alpha}\tilde F_{-\alpha}(w). \tag{10.31}$$

Its asymptotics

$$\tilde F_\alpha(w) \sim \exp(-2w^{\frac12})w^{-\frac{\alpha}{2}-\frac14} \tag{10.32}$$

is valid in the sector $|\arg w| < \pi/2-\epsilon$ for $|w|\to\infty$.

10.9 Recurrence relations

The following recurrence relations follow from the transmutation relations

$$\partial_w \mathbf{F}_\alpha(w) = \mathbf{F}_{\alpha+1}(w),$$

$$(w\partial_w+\alpha)\,\mathbf{F}_\alpha(w) = \mathbf{F}_{\alpha-1}(w).$$

10.10 Wave packets

Obviously, for any τ the function $\exp\left(\frac{x_-}{\sqrt{2}\tau}+\frac{\tau x_+}{\sqrt{2}}\right)$ solves the Helmholtz equation. Therefore, for appropriate contours γ,

$$f(x_-,x_+) := \frac{1}{2\pi \mathrm{i}}\int\limits_\gamma \exp\left(\frac{x_-}{\sqrt{2}\tau}+\frac{\tau x_+}{\sqrt{2}}\right)\tau^{-\alpha-1}\mathrm{d}\tau \tag{10.33}$$

solves

$$(\Delta_2 - 1)f = 0, \tag{10.34}$$

$$Nf = \alpha f. \tag{10.35}$$

Substituting the coordinates w, u we obtain

$$\begin{aligned} f(w,u) &= \int\limits_\gamma \exp\left(\frac{w}{\tau u\sqrt{2}}+\frac{\tau u}{\sqrt{2}}\right)\tau^{-\alpha-1}\mathrm{d}\tau \\ &= u^\alpha 2^{-\frac{\alpha}{2}}\int\limits_\gamma \exp\left(\frac{w}{s}+s\right)s^{-\alpha-1}\mathrm{d}s, \end{aligned} \tag{10.36}$$

where we made the substitution $s=\frac{\tau u}{\sqrt{2}}$. Therefore,

$$F(w) = \int\limits_\gamma \exp\left(\frac{w}{s}+s\right)s^{-\alpha-1}\mathrm{d}s. \tag{10.37}$$

solves the ${}_0F_1$ equation.

10.11 Integral representations

There are three kinds of integral representations of solutions to the ${}_0F_1$ equation. The first is suggested by the previous subsection. Representations of the first kind will be called *Bessel-Schläfli type representations*. The next two are inherited from the confluent equation by 2nd Kummer's identity. We will call them *Poisson-type representations*.

Theorem 10.1. *i)* **Bessel-Schläfli type representations.** *Suppose that* $[0,1]\ni t\mapsto \gamma(t)$ *satisfies*

$$\mathrm{e}^t\mathrm{e}^{\frac{w}{t}}t^{-c}\Big|_{\gamma(0)}^{\gamma(1)} = 0.$$

Then

$$\mathcal{F}(c;w,\partial_w)\int\limits_\gamma \mathrm{e}^t\mathrm{e}^{\frac{w}{t}}t^{-c}\mathrm{d}t=0. \tag{10.38}$$

ii) **Poisson type a) representations.** *Let the contour γ satisfy*

$$(t^2-w)^{-c+3/2}\mathrm{e}^{2t}\Big|_{\gamma(0)}^{\gamma(1)}=0.$$

Then

$$\mathcal{F}(c;w,\partial_w)\int\limits_\gamma (t^2-w)^{-c+1/2}\mathrm{e}^{2t}\mathrm{d}t=0. \tag{10.39}$$

iii) **Poisson type b) representations.** *Let the contour γ satisfy*

$$(t^2-1)^{c-1/2}\mathrm{e}^{2t\sqrt{w}}\Big|_{\gamma(0)}^{\gamma(1)}=0.$$

Then

$$\mathcal{F}(c;w,\partial_w)\int\limits_\gamma (t^2-1)^{c-3/2}\mathrm{e}^{2t\sqrt{w}}\mathrm{d}t=0. \tag{10.40}$$

Proof. We check that for any contour γ

$$\text{lhs of (10.38)}=-\int\limits_\gamma \mathrm{d}t\partial_t\mathrm{e}^t\mathrm{e}^{\frac{w}{t}}t^{-c}.$$

This proves i).
To prove both Poisson type representations we use the quadratic relation (10.24). Using the type a) representation for solutions of ${}_1\mathcal{F}_1$ (8.51), for appropriate contours γ and γ', we see that

$$\begin{aligned}&\mathrm{e}^{-2\sqrt{w}}\int\limits_\gamma \mathrm{e}^s s^{-c+\frac12}(s-4\sqrt{w})^{-c+\frac12}\mathrm{d}s\\ =\;&2^{-2c+2}\int\limits_{\gamma'} \mathrm{e}^{2t}(t^2-w)^{-c+\frac12}\mathrm{d}t\end{aligned}$$

is annihilated by $\mathcal{F}(c)$, where we set $t=\frac{s}{2}-\sqrt{w}$. This proves ii).

Similarly, by the type b) representation for solutions of ${}_1\mathcal{F}_1$ (8.52),

$$\begin{aligned}&\mathrm{e}^{-2\sqrt{w}}\int\limits_\gamma \mathrm{e}^{\frac{4\sqrt{w}}{s}}s^{-2c+1}(1-s)^{c-\frac32}\mathrm{d}s\\ =\;&-2^{-2c+2}\int\limits_{\gamma'} \mathrm{e}^{2t\sqrt{w}}(1-t^2)^{c-\frac32}\mathrm{d}t\end{aligned}$$

is annihilated by $\mathcal{F}(c)$, where we set $t=\frac{2}{s}-1$. This proves iii). □

10.12 Integral representations of standard solutions

In Bessel-Schläfli type representations the integrand goes to zero as $t \to -\infty$ and $t \to 0-0$ (the latter for $\mathrm{Re} w > 0$). Therefore, contours ending at these points yield solutions. We will see that in this way we can obtain all 3 standard solutions.

We can also obtain all solutions using Poisson type representations (which are actually special cases of representations for solutions of the confluent equation).

	Bessel-Schläfli	**Poisson type a)**	**Poisson type b)**
~ 1 **at** 0:	$]-\infty, 0^+, \infty[$	$[-1, 1]$	
$\sim w^{-\alpha}$ **at** 0:	$(0-0)^+$		$[-\sqrt{w}, \sqrt{w}]$
$\sim \mathrm{e}^{-2\sqrt{w}} w^{-\frac{\alpha}{2}-\frac{1}{4}}$ **for** $w \to +\infty$:	$]-\infty, 0]$	$]-\infty, -1]$	$]-\infty, -\sqrt{w}]$

Here are Bessel-Schläfli type representations. They are valid for all values of α and $\mathrm{Re} w > 0$:

$$\frac{1}{2\pi \mathrm{i}} \int\limits_{]-\infty, 0^+, -\infty[} \mathrm{e}^t \mathrm{e}^{\frac{w}{t}} t^{-\alpha-1} \mathrm{d}t = \mathbf{F}_\alpha(w), \tag{10.41}$$

$$\frac{1}{2\pi \mathrm{i}} \int\limits_{[(0-0)^+]} \mathrm{e}^t \mathrm{e}^{\frac{w}{t}} t^{-\alpha-1} \mathrm{d}t = w^{-\alpha} \mathbf{F}_{-\alpha}(w), \tag{10.42}$$

$$\int\limits_{-\infty}^{0} \mathrm{e}^t \mathrm{e}^{\frac{w}{t}} (-t)^{-\alpha-1} \mathrm{d}t = \pi^{\frac{1}{2}} \tilde{F}_\alpha(w). \tag{10.43}$$

Next we give Poisson type representations, valid for $w \notin]-\infty, 0]$:

$$\mathrm{Re}\alpha > -\frac{1}{2} : \tag{10.44}$$

$$\int\limits_{-1}^{1} (1-t^2)^{\alpha-\frac{1}{2}} \mathrm{e}^{2t\sqrt{w}} \mathrm{d}t = \Gamma(\alpha + \frac{1}{2}) \sqrt{\pi} \mathbf{F}_\alpha(w),$$

$$\frac{1}{2} > \mathrm{Re}\alpha : \tag{10.45}$$

$$\int\limits_{-\sqrt{w}}^{\sqrt{w}} (w-t^2)^{-\alpha-\frac{1}{2}} \mathrm{e}^{2t} \mathrm{d}t = \Gamma\Big(-\alpha + \frac{1}{2}\Big) \sqrt{\pi} w^{-\alpha} \mathbf{F}_{-\alpha}(w);$$

$$\mathrm{Re}\alpha > -\frac{1}{2}: \tag{10.46}$$

$$\int_{-\infty}^{-1}(t^2-1)^{\alpha-\frac{1}{2}}\mathrm{e}^{2t\sqrt{w}}\mathrm{d}t = \frac{1}{2}\Gamma\Big(\alpha+\frac{1}{2}\Big)\tilde{F}_\alpha(w),$$

$$\mathrm{Re}\alpha < \frac{1}{2}: \tag{10.47}$$

$$\int_{-\infty}^{-\sqrt{w}}(t^2-w)^{-\alpha-\frac{1}{2}}\mathrm{e}^{2t}\mathrm{d}t = \frac{1}{2}\Gamma\Big(-\alpha+\frac{1}{2}\Big)\tilde{F}_\alpha(w).$$

10.13 Connection formulas

From integral representations we easily obtain connection formulas. As the basis we can use the solutions with a simple behavior at zero:

$$\tilde{F}_\alpha(w) = \frac{\sqrt{\pi}}{\sin\pi(-\alpha)}\mathbf{F}_\alpha(w) + \frac{\sqrt{\pi}}{\sin\pi\alpha}w^{-\alpha}\mathbf{F}_{-\alpha}(w).$$

Alternatively, we can use the basis conisting of the $\tilde{F}$ function and its clockwise or anti-clockwise analytic continuation around 0:

$$\begin{aligned}\mathbf{F}_\alpha(w) &= \frac{1}{2\sqrt{\pi}}\Big(\mathrm{e}^{\pm\mathrm{i}\pi(\alpha+\frac{1}{2})}\tilde{F}_\alpha(w) + \mathrm{e}^{\mp\mathrm{i}\pi(\alpha+\frac{1}{2})}\tilde{F}_\alpha(\mathrm{e}^{\mp\mathrm{i}2\pi}w)\Big),\\ w^{-\alpha}\mathbf{F}_{-\alpha}(w) &= \frac{1}{2\sqrt{\pi}}\Big(\mathrm{e}^{\mp\mathrm{i}\pi(\alpha-\frac{1}{2})}\tilde{F}_\alpha(w) - \mathrm{e}^{\mp\mathrm{i}\pi(\alpha-\frac{1}{2})}\tilde{F}_\alpha(\mathrm{e}^{\mp\mathrm{i}2\pi}w)\Big).\end{aligned}$$

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