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Bosonic quadratic Hamiltonians

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We discuss self-adjoint operators given formally by expressions quadratic in bosonic creation and annihilation operators. We give conditions when they can be defined as self-adjoint operators, possibly after an infinite renormalization. We also discuss explicit formulas for their infimum. Our main motivation comes from local quantum field theory, which furnishes interesting examples of bosonic quadratic Hamiltonians that require an infinite renormalization. *Published by AIP Publishing.*
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I. INTRODUCTION

Quantum bosonic quadratic Hamiltonians or *bosonic Bogoliubov Hamiltonians* are formally given by expressions of the form

$$\hat{H} = \sum h_{ij}\hat{a}_i^*\hat{a}_j + \frac{1}{2}\sum g_{ij}\hat{a}_i^*\hat{a}_j^* + \frac{1}{2}\sum \overline{g}_{ij}\hat{a}_i\hat{a}_j + c, \tag{1.1}$$

where $h = [h_{ij}]$ is a Hermitian matrix, $g = [g_{ij}]$ is a symmetric matrix, c is an arbitrary real number (possibly, infinite), and $\hat{a}_i^*, \hat{a}_j$ are the usual bosonic creation/annihilation operators. They are often used in quantum field theory (QFT) to describe free theories interacting with a given external classical field.[8,11] They are responsible for the *Casimir effect*.[11] Bogoliubov applied them to the theory of interacting Bose gas,[3] which justifies the name *Bogoliubov Hamiltonians*.

Bogoliubov Hamiltonians that are bounded from below are especially useful. Their infimum $E := \inf \hat{H}$ is often interesting physically.

Bogoliubov Hamiltonians have a surprisingly rich mathematical theory. In infinite dimensions, this theory sometimes involves interesting pathologies. For instance, $\hat{H}$ is often ill defined, but one can define its "infimum" E. In some situations, one needs to perform an infinite renormalization in order to define $\hat{H}$ or at least to compute E. This is typical for Bogoliubov Hamiltonians that are motivated by relativistic quantum field theory.[8] Another example of interesting mathematics related to Bogoliubov Hamiltonians can be found in a recent paper,[15] which contains a beautiful proof of diagonalizability of normally ordered Bogoliubov Hamiltonians under essentially optimal conditions.

Our paper is devoted to a systematic theory of bosonic Bogoliubov Hamiltonians in an abstract setting. We do not restrict ourselves to the normally ordered case [with $c = 0$ in (1.1)]. We start from a more general definition saying that a Bogoliubov Hamiltonian is the self-adjoint generator of a one-parameter unitary group on a bosonic Fock space that implements a symplectic group. There are interesting and physically important examples where the normally ordered Bogoliubov Hamiltonian is ill defined, whereas renormalized ones exist.[8]

The family of Bogoliubov Hamiltonians given by fixing h and g and varying $c \in \mathbb{R}$ in (1.1) can be understood as various quantizations of a single classical quadratic Hamiltonian

$$H = \sum h_{ij}a_i^*a_j + \frac{1}{2}\sum g_{ij}a_i^*a_j^* + \frac{1}{2}\sum \overline{g}_{ij}a_ia_j, \tag{1.2}$$

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where a_i, a_j^* are classical (commuting) variables. c, which appears in (1.1), can be understood as the ambiguity of quantization due to noncommutativity of $\hat{a}_i$ and $\hat{a}_j$. The most popular choice is probably $c = 0$, corresponding to the *normally (Wick) ordered Hamiltonian*. It will be denoted by $\hat{H}^{\mathrm{n}}$. The choice $c = \frac{1}{2}\sum_i h_{ii}$, which we call the *Weyl Bogoliubov Hamiltonian* and denote $\hat{H}^{\mathrm{w}}$, has its advantages as well. In some situations, however, one needs to consider other quantizations, where the constant c may turn out to be infinite and can be viewed as a renormalization counterterm. One particular possibility, which we call the *second order renormalized quantization* and denote $\hat{H}^{\mathrm{2ren}}$, plays an important role in quantum field theory in 1 + 3 dimensions. In the language of Feynman diagrams, $\hat{H}^{\mathrm{2ren}}$ corresponds to discarding loops of order 2 or less.

We will use the following notation for the infimum of the three main Bogoliubov Hamiltonians that we discuss:

$$E^{\mathrm{w}} := \inf \hat{H}^{\mathrm{w}}, \quad E^{\mathrm{n}} := \inf \hat{H}^{\mathrm{n}}, \quad E^{\mathrm{2ren}} := \inf \hat{H}^{\mathrm{2ren}}. \tag{1.3}$$

In physics, the infimum of the Hamiltonian appears under various names, e.g., vacuum energy, Casimir energy, vacuum polarization, and effective potential. Physicists often compute the vacuum energy without worrying whether the corresponding quantum Hamiltonian is well defined as a self-adjoint operator. Following this philosophy, we may consider E^{n} or E^{2ren} under conditions that are more general than the conditions for the existence of the corresponding Hamiltonians.

A. Comparison with literature

It is not always very easy to read the literature on Bogoliubov Hamiltonians and to compare statements in various papers. Their authors often use different conventions, terminology, and notations.

Most of these issues disappear when one fixes a basis in the 1-particle space, identifying it with $\mathbb{C}^m$. Then a Bogoliubov Hamiltonian is determined by two matrices, $h = [h_{ij}]$ and $g = [g_{ij}]$, and possibly a number c; see (1.1).

When we want to use a basis independent language, replacing $\mathbb{C}^m$ by an abstract Hilbert space $\mathcal{W}$, it is clear how to interpret h—it is a self-adjoint operator on $\mathcal{W}$. It is less obvious how to interpret g. One possibility is to view g as a symmetric tensor, that is, an element of $\otimes_{\mathrm{s}}^2 \mathcal{W}$. Often, however, it is preferable to view g as an operator from $\mathbb{C}^m$ to $\mathbb{C}^m$. These two $\mathbb{C}^m$ should be however viewed as two distinct spaces—one is the complex conjugate of the other; see, e.g., Ref. 8. The notion of a complex conjugate space is somewhat subtle and has a few equivalent, but superficially distinct, interpretations; see Subsection 1 of the Appendix. Various authors prefer distinct interpretations; see, e.g., the footnote Ref. 6 in Appendix A of Ref. 12. (Strictly speaking, this footnote refers to the fermionic case; however, the fermionic and bosonic cases are quite analogous.)

When we consider an infinite dimensional space, there are additional problems: various operators are often unbounded, are not trace class, or simply do not exist.[5]

Because of these two kinds of problems, our paper is divided into two parts. In the first part, we assume that the 1-particle space is finite dimensional and has a fixed orthonormal basis. All operators are represented by matrices. We do not worry about conceptual subtleties related to antilinear maps and the complex conjugate space. Infinite renormalization is not needed and all formulas are valid with no technical restrictions.

In the second part, the 1-particle space is an abstract space $\mathcal{W}$ of any dimension. We follow mostly the conceptual framework of Ref. 7. We distinguish between $\mathcal{W}$ and its complex conjugate $\overline{\mathcal{W}}$. We need to give technical conditions guaranteeing that various concepts and formulas survive into infinite dimensions.

Throughout this paper it is assumed that the reader is familiar with mathematical formalism of 2nd quantization. Properties of the metaplectic representation in the Fock space play an important role, such as the Shale theorem and formulas for the Bogoliubov implementers (2.17) and (2.18). These formulas were known to Friedrichs[10] and analysed later by Ruijsenaars[16,17] and Berezin.[2] We treat Ref. 7 as the basic reference on this subject, where, in particular, various questions related to the unboundedness of bosonic creation and annihilation operators are discussed in detail.

A major part of Sec. II is well known. Theorem 2.3 about diagonalizability of a quadratic Hamiltonian by a positive symplectic transformation is implicitly contained in Ref. 7 [see Theorem 11.20 (3) together with Theorem 18.5 (3)]. We come back to this issue in Sec. III, where an arbitrary dimension introduces additional technical issues. Note that a similar fact proven in Ref. 15 does not provide a construction of a distinguished diagonalizing operator.

The basic formula for the infimum of a quadratic Hamiltonian comes from Ref. 6. However, some of the formulas for the infimum of the normally ordered Hamiltonians, such as (2.73)–(2.75), seem to be new. In finite dimensions, they are not so interesting; however, they become quite useful in infinite dimensions.

It seems that the construction of the renormalized Hamiltonians described in Subsections II L and II M has never been presented in the literature in the abstract setting. Their importance is evident in concrete situations of quantum field theory described in Ref. 8. We give a brief discussion of the examples from QFT at the end of the Introduction.

Quadratic Hamiltonians in infinite dimensions is a rather technical topic of operator theory. Therefore, we prefer to give a self-contained treatment of this subject. Many results and definitions that we present are new; however, at some places we recall proofs contained in the literature.

Note that it would be awkward and restrictive to define Bogoliubov Hamiltonians in the infinite dimensional context by an expression of form (1.1). Instead, we define them as self-adjoint generators of one parameter unitary groups implementing Bogoliubov transformations. (In the bosonic context, the term "Bogoliubov transformations" is usually meant to denote "symplectic transformations.") The abstract approach makes it sometimes difficult to define some objects since we cannot refer to a formula of form (1.1). Fortunately, it is obvious how to define the Weyl Bogoliubov Hamiltonian—as the generator of a group inside the metaplectic group. It is less obvious how to define normally ordered Hamiltonians. The definition that we propose in Subsection III G seems to be new—in particular, it is more general than the definition of Ref. 6.

Subsections III G and III H give criteria for the existence of various quantizations. In these subsections, there is no assumption on the positivity of h. On the other hand, most results require the boundedness of g. Some results in this part of the paper come from Refs. 2 and 6. However, Theorem 3.18 (1), which gives a convenient criterion for the implementability of classical dynamics, seems to be new. It is useful in the context of examples from QFT discussed below.

In Subsections III I–III N, we adopt a different set of assumptions. In particular, we assume that h is positive and g is form bounded with respect to h with bound less than 1. This condition guarantees the positivity and diagonalizability of classical Hamiltonians.

Diagonalization of Bogoliubov Hamiltonians on the quantum level was considered already by Berezin[2] and then by Bach and Bru.[1] In a recent paper,[15] Napiórkowski, Nam, and Solovej gave a new beautiful proof of diagonalizability. In our paper, we repeat some of the arguments of Ref. 15, describing their result in Theorem 3.21, giving essentially optimal conditions for diagonalization. In distinction to Ref. 15, we show that there exists a distinguished positive symplectic operator diagonalizing a given Bogoliubov Hamiltonian.

In Theorem 3.23, we also describe a construction of normally ordered Bogoliubov Hamiltonians based on the form techniques (involving the so-called Kato, Lions, Lax, Milgram and Nelson (KLMN) theorem) presented in Ref. 15. This is an important improvement (even if it sounds technical) as compared to the results of Ref. 6, which were restricted to operator-type perturbations.

These theorems are complemented with new results. In Theorem 3.24, we show that the dynamics generated by the normally ordered Hamiltonian implements the corresponding classical dynamics. On a formal level, this theorem seems obvious; nevertheless, due to the unboundedness of various operators, it needs a careful proof. Another new result, easy in finite dimensions and rather technical in the general case, is the formula for the ground state energy described in Theorem 3.29. We also discuss a criterion for the existence of the Weyl Bogoliubov Hamiltonians in Theorem 3.31 and for the existence of the renormalized ground state energy in Theorem 3.32.

Let us mention some topics that are left out of our paper. We do not discuss time-dependent Bogoliubov Hamiltonians and the implementability and the phase of the corresponding scattering operator. This is interesting, especially in the context of charged relativistic fields in an external electromagnetic potential. An infinite renormalization is needed in order to define the vacuum energy. This topic on a partly heuristic level is discussed in Ref. 8. Its fermionic counterpart (a Dirac particle in an external electromagnetic potential) is better known in the literature; see, e.g., Ref. 9.

B. Applications to QFT

Let us first discuss the question of naturalness of the definition of various kinds of Bogoliubov Hamiltonians.

The Weyl Hamiltonian $\hat{H}^{\mathrm{w}}$ is the most natural. In fact, it is invariant with respect to symplectic transformations; see (2.31). Unfortunately, it is often ill defined.

The normally ordered Hamiltonian $\hat{H}^{\mathrm{n}}$ is naturally defined given a Fock representation. In particular, this is the case when we have a distinguished positive classical quadratic Hamiltonian which is treated as the "free" one. Then there exists a unique Fock representation where the free Hamiltonian can be quantized without any double creation/annihilation operators. It is usually quantized in the normally ordered form. We will denote it by $\hat{H}_0^{\mathrm{n}}$.

Suppose that we are interested in the "full" Hamiltonian, which is quadratic but more complicated than the free one and involves an interaction with external fields. We can then ask whether the corresponding classical Hamiltonian can be quantized. The most straightforward procedure seems to involve the normally ordered full quantum Hamiltonian $\hat{H}^{\mathrm{n}}$. The corresponding ground state energy then formally equals the difference of the "free Weyl ground state energy" and the "full Weyl ground state energy" (in typical situations both infinite).

It sometimes happens that $\hat{H}^{\mathrm{n}}$ is ill defined as well. Then we can try to subtract from $\hat{H}^{\mathrm{n}}$ another counterterm. In typical examples from QFT in 1 + 3 dimensions, it is enough to subtract the second order term in the perturbative expansion, obtaining finite $E^{2\mathrm{ren}}$. Sometimes, but not always, this makes $\hat{H}^{2\mathrm{ren}}$ well defined as well. This subtraction procedure in an abstract setting is explained in Subsections II L and II M. Below we briefly describe two examples from QFT where such a renormalization works. These examples are discussed in more detail in Ref. 8.

Consider the neutral massive scalar quantum field $\hat{\phi}(\vec{x})$. Its conjugate field is denoted $\hat{\pi}(\vec{x})$ with the usual equal time commutation relations

$$\begin{aligned}[\hat{\phi}(\vec{x}),\hat{\phi}(\vec{y})\} &= [\hat{\pi}(\vec{x}),\hat{\pi}(\vec{y})] = 0,\\ [\hat{\phi}(\vec{x}),\hat{\pi}(\vec{y})] &= \mathrm{i}\delta(\vec{x}-\vec{y}).\end{aligned}\tag{1.4}$$

The free Hamiltonian is defined in the standard way,

$$\hat{H}_0^{\mathrm{n}} := \int :\Big(\frac{1}{2}\hat{\pi}^2(\vec{x}) + \frac{1}{2}(\vec{\partial}\hat{\phi}(\vec{x}))^2 + \frac{1}{2}m^2\hat{\phi}^2(x)\Big):\mathrm{d}\vec{x},\tag{1.5}$$

where the double dots denote the normal ordering.

Suppose that the mass squared is perturbed by a Schwartz function $\kappa(\vec{x})$. One can check that the normally ordered full Hamiltonian does not exist. However, the 2nd order renormalized Hamiltonian is well defined. Formally, it can be written as

$$\hat{H}^{2\mathrm{ren}} := \int :\Big(\frac{1}{2}\hat{\pi}^2(\vec{x}) + \frac{1}{2}(\vec{\partial}\hat{\phi}(\vec{x}))^2 + \frac{1}{2}(m^2+\kappa(\vec{x}))\hat{\phi}^2(\vec{x})\Big):\mathrm{d}\vec{x} - E_2,\tag{1.6}$$

where the infinite counterterm E_2 is the contribution of a loop diagram with 2 vertices; see Subsection II M.

(1.6) is well defined, but physically somewhat artificial. To obtain a physically more satisfactory Hamiltonian, one needs to perform an additional finite subtraction, adding E_2^{ren} to (1.6), the renormalized value of E_2. The renormalization can be performed with the help of any method described in textbooks of QFT, e.g., by the Pauli-Villars method, by dispersion relations, or by dimensional regularization. All these methods are equivalent and one obtains a renormalized Hamiltonian with only *local* counterterms, which formally can be written as

$$\hat{H}^{\mathrm{ren}} := \int :\Big(\frac{1}{2}\hat{\pi}^2(\vec{x}) + \frac{1}{2}(\vec{\partial}\hat{\phi}(\vec{x}))^2 + \frac{1}{2}(m^2 + \kappa(\vec{x}))\hat{\phi}^2(\vec{x})\Big):\mathrm{d}\vec{x} - C\int |\kappa(\vec{x})|^2\mathrm{d}\vec{x}, \tag{1.7}$$

where C is infinite. This example is discussed in detail in Chap. III, Subsection C14 of Ref. 8.

The next example is more singular. Consider the charged massive scalar quantum field $\hat{\psi}(\vec{x})$, with $\hat{\psi}^*(\vec{x})$ denoting its Hermitian adjoint. The conjugate field will be denoted $\hat{\eta}(\vec{x})$ so that we have the commutation relations

$$\big[\hat{\psi}(\vec{x}), \hat{\psi}(\vec{y})\big] = [\hat{\psi}(\vec{x}), \hat{\eta}(\vec{y})] = [\hat{\eta}(\vec{x}), \hat{\eta}(\vec{y})] = 0, \tag{1.8}$$

$$\big[\hat{\psi}(\vec{x}), \hat{\eta}^*(\vec{y})\big] = [\hat{\psi}^*(\vec{x}), \hat{\eta}(\vec{y})] = \mathrm{i}\delta(\vec{x} - \vec{y}). \tag{1.9}$$

The free Hamiltonian is of course

$$\hat{H}_0^{\mathrm{n}} = \int :\Big(\hat{\eta}^*(\vec{x})\hat{\eta}(\vec{x}) + \vec{\partial}\hat{\psi}^*(\vec{x})\vec{\partial}\hat{\psi}(\vec{x}) + m^2\hat{\psi}^*(\vec{x})\hat{\psi}(\vec{x})\Big):\mathrm{d}\vec{x}.$$

Suppose now that we consider an external stationary electromagnetic potential, described by, say, Schwartz functions $(A^0, \vec{A})$. A candidate for the full Hamiltonian is

$$\begin{aligned}\hat{H}^{2\mathrm{ren}} = \int \mathrm{d}\vec{x}\, \Big(&\hat{\eta}^*(\vec{x})\hat{\eta}(\vec{x}) + \mathrm{i}eA_0(\vec{x})(\hat{\psi}^*(\vec{x})\hat{\eta}(\vec{x}) - \hat{\eta}^*(\vec{x})\hat{\psi}(\vec{x}))\\ &+ (\partial_i - \mathrm{i}eA_i(\vec{x}))\hat{\psi}^*(\vec{x})(\partial_i + \mathrm{i}eA_i(\vec{x}))\hat{\psi}(\vec{x}) + m^2\hat{\psi}^*(\vec{x})\hat{\psi}(\vec{x})\Big)\\ &- E_0 - E_2,\end{aligned} \tag{1.10}$$

where E_0 and E_2 are infinite counterterms that come from the expansion described in (2.101) ($E_1 = 0$ by the Furry theorem). Again, physically one prefers to add E_2^{ren} to (1.10), the renormalized value of E_2 so that all counterterms are local. One obtains the renormalized Hamiltonian formally written as

$$\begin{aligned}\hat{H}^{\mathrm{ren}} = \int \mathrm{d}\vec{x}\, \Big(&\hat{\eta}^*(\vec{x})\hat{\eta}(\vec{x}) + \mathrm{i}eA_0(\vec{x})(\hat{\psi}^*(\vec{x})\hat{\eta}(\vec{x}) - \hat{\eta}^*(\vec{x})\hat{\psi}(\vec{x}))\\ &+ (\partial_i - \mathrm{i}eA_i(\vec{x}))\hat{\psi}^*(\vec{x})(\partial_i + \mathrm{i}eA_i(\vec{x}))\hat{\psi}(\vec{x}) + m^2\hat{\psi}^*(\vec{x})\hat{\psi}(\vec{x})\Big)\\ &- E_0 - C\int (\partial_\mu A_\nu(\vec{x}) - \partial_\nu A_\mu(\vec{x}))(\partial^\mu A^\nu(\vec{x}) - \partial^\nu A^\mu(\vec{x}))\mathrm{d}\vec{x},\end{aligned} \tag{1.11}$$

where C is infinite. This example is worked out in detail in Chap. VI, Subsection B17 of Ref. 8.

Unfortunately, the classical dynamics is implementable only if the vector potential $\vec{A}$ vanishes everywhere. Therefore, both $\hat{H}^{2\mathrm{ren}}$ and $\hat{H}^{\mathrm{ren}}$ are well defined only in this case. However, the infimum of (1.11) is a well-defined gauge-invariant number also for nonzero $\vec{A}$.

Note that both Hamiltonians (1.7) and (1.11) can be derived from local Lagrangians. Therefore, even if the models based on these Hamiltonians do not satisfy Haag-Kastler axioms in the strict sense (because of the absence of translation invariance), they belong to local quantum field theory: they lead to nets satisfying the Einstein causality, and they have bounded from below Hamiltonians. At the same time, all of them require an infinite renormalization, typical for computations in perturbative quantum field theory.

Examples (1.7) and (1.11) are especially interesting in the context of more complicated interacting quantum field theories, where, typically, κ, respectively, A, are promoted to the role of quantum fields. Then E_2^{ren} can be interpreted as the value of certain renormalized diagrams involving the field $\hat{\phi}$, respectively, $\hat{\psi}$, in a loop and $\hat{\kappa}$, respectively, $\hat{A}$, in an external line. In particular, E_2^{ren} of the second example is usually called the *vacuum polarization (in scalar QED)*.

II. FINITE DIMENSIONS: BASIS DEPENDENT FORMALISM

Let us first describe the basic theory of bosonic quadratic Hamiltonians in finite dimensions, assuming that the one-particle space is $\mathbb{C}^m$. Seemingly, our formulas will depend on the choice of the canonical basis in $\mathbb{C}^m$. In reality, after an appropriate interpretation, they are basis independent. This interpretation will be given in Sec. III, when we discuss an arbitrary dimension.

Operators on $\mathbb{C}^m$ will be identified with matrices. If $h = [h_{ij}]$ is a matrix, then $\overline{h}$, h^*, and $h^\#$ will denote its complex conjugate, Hermitian conjugate, and transpose, respectively.

A. Creation/annihilation operators

We consider the bosonic Fock space $\Gamma_{\mathrm{s}}(\mathbb{C}^m)$. $\hat{a}_i, \hat{a}_j^*$ are the standard annihilation and creation operators on $\Gamma_{\mathrm{s}}(\mathbb{C}^m)$. $\hat{a}_i^*$ is the Hermitian conjugate of $\hat{a}_i$,

$$[\hat{a}_i, \hat{a}_j] = [\hat{a}_i, \hat{a}_j] = 0,$$
$$[\hat{a}_i, \hat{a}_j^*] = \delta_{ij}.$$

(We denote creation/annihilation operators with hats because we want to distinguish them from their classical analogs.)

We use the more or less standard notation for operators on Fock spaces. In particular, we use the standard notations $\Gamma(\cdot)$ and $\mathrm{d}\Gamma(\cdot)$, which will be recalled in Subsection III B. If $w = [w_i] \in \mathbb{C}^m$, then the corresponding creation/annihilation operators are

$$\hat{a}^*(w) := \sum_i w_i \hat{a}_i^*, \qquad \hat{a}(w) := \sum_i \overline{w}_i \hat{a}_i. \tag{2.1}$$

If $g = [g_{ij}]$ is a symmetric $m \times m$ matrix, then the corresponding double creation/annihilation operators are

$$\hat{a}^*(g) := \sum_{ij} g_{ij} \hat{a}_i^* \hat{a}_j^*, \qquad \hat{a}(g) := \sum_{ij} \overline{g}_{ij} \hat{a}_j \hat{a}_i.$$

B. Classical phase space

To specify a linear combination of operators $\hat{a}_i$ and $\hat{a}_j^*$, we need to choose a vector $(w, w') \in \mathbb{C}^m \oplus \mathbb{C}^m$,

$$\hat{\phi}(w, w') := \sum_i \hat{a}_i^* w_i + \sum_i \hat{a}_i w_i'. \tag{2.2}$$

(2.2) is self-adjoint if and only if $\overline{w} = w'$. Therefore, it is natural to introduce the doubled space $\mathbb{C}^m \oplus \mathbb{C}^m$ equipped with the *complex conjugation*

$$J \begin{bmatrix} w \\ w' \end{bmatrix} = \begin{bmatrix} \overline{w}' \\ \overline{w} \end{bmatrix}. \tag{2.3}$$

Vectors left invariant by J have the form

$$\begin{bmatrix} w \\ \overline{w} \end{bmatrix}, \; w \in \mathbb{C}^m. \tag{2.4}$$

They form a $2m$-dimensional real subspace of $\mathbb{C}^m \oplus \mathbb{C}^m$, which can be identified with $\mathbb{R}^{2m}$. [In what follows, when we speak of $\mathbb{R}^{2m}$ we usually mean the space of vectors of the form (2.4).]

Operators on $\mathbb{C}^m \oplus \mathbb{C}^m$ that commute with J, or equivalently preserve $\mathbb{R}^{2m}$, have the form

$$R = \begin{bmatrix} p & q \\ \overline{q} & \overline{p} \end{bmatrix} \tag{2.5}$$

and will be called *J-real*. Note that if we know the restriction of R to (2.4), then we can uniquely extend it to a (complex linear) operator on $\mathbb{C}^m \oplus \mathbb{C}^m$.

The operator

$$S = \begin{bmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{bmatrix} \tag{2.6}$$

determines the commutation relations

$$\left[\hat{\phi}(w_1, w_1')^*, \hat{\phi}(w_2, w_2')\right] = (w_1|w_2) - (w_1'|w_2') = ((w_1, w_1')|S(w_2, w_2')). \tag{2.7}$$

Instead of quantum operators $\hat{a}_i^*$ and $\hat{a}_j$, one can also consider classical (commuting) variables a_i, a_j^*, $i = 1, \dots, m$, such that a_i^* is the complex conjugate of a_i and the following Poisson bracket relations hold:

$$\begin{aligned} \{a_i, a_j\} = \{a_i, a_j\} = 0, \\ \{a_i, a_j^*\} = -\mathrm{i}\delta_{ij}. \end{aligned} \tag{2.8}$$

Setting

$$\phi(w, w') := \sum_i a_i^* w_i + \sum_i a_i w_i', \tag{2.9}$$

we can rewrite (2.8) as

$$\left\{\phi(w_1, w_1')^*, \phi(w_2, w_2')\right\} = -\mathrm{i}(w_1|w_2) + \mathrm{i}(w_1'|w_2') = -\mathrm{i}((w_1, w_1')|S(w_2, w_2')). \tag{2.10}$$

In particular, $\phi(w, \overline{w})$ are real, and (2.10) can be rewritten as

$$\{\phi(w_1, \overline{w}_1)^*, \phi(w_2, \overline{w}_2)\} = 2\mathrm{Im}(w_1|w_2) = \mathrm{Im}((w_1, \overline{w}_1)|S(w_2, \overline{w}_2)). \tag{2.11}$$

Thus S determines a symplectic structure on $\mathbb{R}^{2m}$ (and sometimes S itself is called, incorrectly, a symplectic form).

C. Symplectic transformations

In this subsection, we recall some basic facts concerning the symplectic and metaplectic groups. We follow mostly Ref. 7.

We say that an operator R on $\mathbb{C}^m \oplus \mathbb{C}^m$ is *symplectic* if it is J-real and preserves S,

$$R^* S R = S. \tag{2.12}$$

We denote by $Sp(\mathbb{R}^{2m})$ the group of all symplectic transformations.

Note that if R is symplectic, then so is R^*. In fact, iS is symplectic, and

$$R^* = \mathrm{i}SR^{-1}(\mathrm{i}S)^{-1}. \tag{2.13}$$

The operator

$$R = \begin{bmatrix} p & q \\ \overline{q} & \overline{p} \end{bmatrix} \tag{2.14}$$

satisfies (2.12) if and only if

$$p^* p - q^{\#}\overline{q} = \mathbb{1}, \quad p^* q - q^{\#}\overline{p} = 0,$$

$$pp^* - qq^* = \mathbb{1}, \quad pq^{\#} - qp^{\#} = 0.$$

Note that

$$pp^* \geq \mathbb{1}, \quad p^* p \geq \mathbb{1}.$$

Hence p^{-1} is well defined, and we can set

$$d_1 := q^{\#}(p^{\#})^{-1}, \tag{2.15}$$

$$d_2 := q\overline{p}^{-1}. \tag{2.16}$$

We have $d_1^{\#} = d_1$ and $d_2 = d_2^{\#}$.

D. Metaplectic transformations

Let U be a unitary operator on $\Gamma_{\rm s}(\mathbb{C}^m)$. Let R be a symplectic transformation written as (2.14). We say that U *implements* R if

$$U\hat{a}_i^* U^* = \hat{a}_j^* p_{ji} + \hat{a}_j \overline{q}_{ji},$$
$$U\hat{a}_i U^* = \hat{a}_j^* q_{ji} + \hat{a}_j \overline{p}_{ji}.$$

U will be called a *(Bogoliubov) implementer of* R. Every symplectic transformation has an implementer, unique up to a phase factor. One can distinguish some canonical choices: the *natural implementer* $U_R^{\rm nat}$ and a pair of *metaplectic implementers* $\pm U_R^{\rm met}$,

$$U_R^{\rm nat} := |\det pp^*|^{-\frac{1}{4}} {\rm e}^{-\frac{1}{2}\hat{a}^*(d_2)}\Gamma((p^*)^{-1}){\rm e}^{\frac{1}{2}\hat{a}(d_1)}, \tag{2.17}$$
$$\pm U_R^{\rm met} := \pm(\det p^*)^{-\frac{1}{2}} {\rm e}^{-\frac{1}{2}\hat{a}^*(d_2)}\Gamma((p^*)^{-1}){\rm e}^{\frac{1}{2}\hat{a}(d_1)}. \tag{2.18}$$

See, e.g., Theorem 11.33 and Definition 11.36 of Ref. 7.

It is easy to see that the set of Bogoliubov implementers is a group. It is sometimes called the *c-metaplectic group* $Mp^c(\mathbb{R}^{2m})$.

It is a little less obvious, but also true, that the set of metaplectic Bogoliubov implementers is a subgroup of $Mp^c(\mathbb{R}^{2m})$. It is called the *metaplectic group* $Mp(\mathbb{R}^{2m})$.

We have a homomorphism $Mp^c(\mathbb{R}^{2m}) \ni U \mapsto R \in Sp(\mathbb{R}^{2m})$, where U implements R.

Various homomorphisms related to the metaplectic group can be described by the following diagram:

$$\begin{array}{ccccccc}
 & & 1 & & 1 & & 1 \\
 & & \downarrow & & \downarrow & & \downarrow \\
1 & \rightarrow & \mathbb{Z}_2 & \rightarrow & U(1) & \rightarrow & U(1) \rightarrow 1 \\
 & & \downarrow & & \downarrow & & \downarrow \\
1 & \rightarrow & Mp(\mathbb{R}^{2m}) & \rightarrow & Mp^c(\mathbb{R}^{2m}) & \rightarrow & U(1) \rightarrow 1\,. \\
 & & \downarrow & & \downarrow & & \downarrow \\
1 & \rightarrow & Sp(\mathbb{R}^{2m}) & \rightarrow & Sp(\mathbb{R}^{2m}) & \rightarrow & 1 \\
 & & \downarrow & & \downarrow & & \\
 & & 1 & & 1 & &
\end{array} \tag{2.19}$$

E. Positive symplectic transformations

Positive symplectic transformations are especially important. They satisfy

$$p = p^*, \quad p > 0, \quad q = q^\#. \tag{2.20}$$

For positive transformations, d_1 equals d_2, and it will be simply denoted by d. We have

$$d := q(p^\#)^{-1}.$$

The natural implementer coincides in this case with one of the metaplectic implementers

$$U_R^{\rm nat} := (\det p)^{-\frac{1}{2}} {\rm e}^{-\frac{1}{2}a^*(d)}\Gamma(p^{-1}){\rm e}^{\frac{1}{2}a(d)}.$$

Positive symplectic transformations have special properties. In particular, one can diagonalize them in an explicit way. We will need this later on.

Proposition 2.1. Assume that R is positive symplectic and $\mathrm{Ker}(p - \mathbb{1}) = \{0\}$.*Then q is invertible so that we can define $u := q|q|^{-1}$ with $|q| := \sqrt{q^*q}$. Besides,*

$$M := \frac{1}{\sqrt{2}}\begin{bmatrix} \mathbb{1} & -u \\ u^* & \mathbb{1} \end{bmatrix} \tag{2.21}$$

is unitary and diagonalizes R,

$$R = M \begin{bmatrix} p + \sqrt{p^2 - \mathbb{1}} & 0 \\ 0 & \overline{p} - \sqrt{\overline{p}^2 - \mathbb{1}} \end{bmatrix} M^*. \tag{2.22}$$

Proof. We have the polar decomposition $q = u|q|$. u is a unitary operator and we have $|\overline{q}| = u|q|u^*$ Now (2.22) follows using $u\overline{p}u^* = p$, $|q| = \sqrt{\overline{p}^2 - \mathbb{1}}$, and $\sqrt{\mathbb{1} + |q|^2} = \overline{p}$. □

F. Classical quadratic Hamiltonians

It is easy to analyze generators of 1-parameter symplectic groups. In fact, $\mathrm{e}^{\mathrm{i}tB} \in Sp(\mathbb{R}^{2m})$ for any $t \in \mathbb{R}$ if and only if BS is J-real and self-adjoint. All such operators can be written as

$$B = \begin{bmatrix} h & -g \\ \overline{g} & -\overline{h} \end{bmatrix}, \tag{2.23}$$

where h and g are $m \times m$ matrices satisfying $h = h^*$ and $g = g^{\#}$. Note that iB is J-real, and

$$SB = B^* S. \tag{2.24}$$

With every such operator B, we associate another operator A_B by

$$A_B := BS = \begin{bmatrix} h & g \\ \overline{g} & \overline{h} \end{bmatrix}. \tag{2.25}$$

As we noted above, A_B is self-adjoint and J-real. The corresponding *classical quadratic Hamiltonian* is the expression

$$H_B = \sum h_{ij} a_i^* a_j + \frac{1}{2} \sum g_{ij} a_i^* a_j^* + \frac{1}{2} \sum \overline{g}_{ij} a_i a_j, \tag{2.26}$$

which can be viewed as a quadratic function on the classical phase space. Moreover,

$$\{H_B, \phi(w, w')\} = -\mathrm{i}\phi(w_1, w_1'), \qquad \begin{bmatrix} w_1 \\ w_1' \end{bmatrix} = B \begin{bmatrix} w \\ w' \end{bmatrix}. \tag{2.27}$$

Clearly, for any symplectic R,

$$A_{RBR^{-1}} = R A_B R^*. \tag{2.28}$$

In what follows, we will often abuse the terminology: A_B will also be called a *classical Hamiltonian* just as H_B. B will be called a *symplectic generator*. Besides, we will often drop the subscript B from H_B and A_B.

G. Quantum quadratic Hamiltonians

Let B be a symplectic generator of form (2.23).

By a *quantization of* H_B (2.26), we will mean an operator on $\Gamma_{\mathrm{s}}(\mathbb{C}^m)$ of the form

$$\hat{H}_B^c := \sum h_{ij} \hat{a}_i^* \hat{a}_j + \frac{1}{2} \sum g_{ij} \hat{a}_i^* \hat{a}_j^* + \frac{1}{2} \sum \overline{g}_{ij} \hat{a}_i \hat{a}_j + c, \tag{2.29}$$

where c is an arbitrary real constant. By an abuse of terminology, we will usually say that (2.29) is a *quantization of* B (2.25). We will often drop the subscript B from $\hat{H}_B^c$, and c will be replaced by other superscripts corresponding to some special choices.

Two quantizations of B are especially useful: the *Weyl* (or *symmetric*) *quantization* $\hat{H}_B^{\mathrm{w}}$ and the *normally ordered* (or *Wick*) *quantization* $\hat{H}_B^{\mathrm{n}}$,

$$\hat{H}_B^{\mathrm{w}} := \frac{1}{2}\sum h_{ij}\hat{a}_i^*\hat{a}_j + \frac{1}{2}\sum h_{ij}\hat{a}_j\hat{a}_i^* + \frac{1}{2}\sum g_{ij}\hat{a}_i^*\hat{a}_j^* + \frac{1}{2}\sum \overline{g}_{ij}\hat{a}_i\hat{a}_j,$$
$$\hat{H}_B^{\mathrm{n}} := \sum h_{ij}\hat{a}_i^*\hat{a}_j + \frac{1}{2}\sum g_{ij}\hat{a}_i^*\hat{a}_j^* + \frac{1}{2}\sum \overline{g}_{ij}\hat{a}_i\hat{a}_j.$$

The following is the relation between these two quantizations:

$$\hat{H}_B^{\mathrm{w}} = \hat{H}_B^{\mathrm{n}} + \frac{1}{2}\mathrm{Tr}h. \tag{2.30}$$

Note a special relationship of the Weyl quantization to the metaplectic group (defined in Subsection II C): for any B, $\mathrm{e}^{\mathrm{i}t\hat{H}_B^{\mathrm{w}}}$ belongs to $Mp(\mathbb{R}^{2m})$; see, e.g., Theorem 11.34 of Ref. 7. Besides, if R is symplectic and U_R is its implementer, then

$$U_R\hat{H}_B^{\mathrm{w}}U_R^* = \hat{H}_{RBR^{-1}}^{\mathrm{w}}. \tag{2.31}$$

H. Diagonalization of quadratic Hamiltonians

In this subsection, we show that if $A_B > 0$, then A_B can be *diagonalized*. By this, we mean that we can find a symplectic transformation R that kills off-diagonal terms of A_B,

$$A_B = R\begin{bmatrix} h_{\mathrm{dg}} & 0 \\ 0 & \overline{h}_{\mathrm{dg}} \end{bmatrix} R^*, \tag{2.32}$$

for some h_{dg}. Of course, h_{dg} has to be positive.

Clearly, this is equivalent to *diagonalizing* B, that is, to killing its off-diagonal terms,

$$B = R\begin{bmatrix} h_{\mathrm{dg}} & 0 \\ 0 & -\overline{h}_{\mathrm{dg}} \end{bmatrix} R^{-1}. \tag{2.33}$$

On the quantum level, this is equivalent to finding a unitary operator U that removes double annihilators and double creators. The free constant then equals the infimum of the quantum Hamiltonian,

$$U^*\hat{H}^{\mathrm{w}}U = \mathrm{d}\Gamma(h_{\mathrm{dg}}) + E^{\mathrm{w}},$$
$$U^*\hat{H}^{\mathrm{n}}U = \mathrm{d}\Gamma(h_{\mathrm{dg}}) + E^{\mathrm{n}}.$$

As a preparation for a construction of a diagonalizing operator, let us prove the following proposition. In this proposition, we will use the function $\mathrm{sgn}t := \begin{cases} 1, & t > 0, \\ 0, & t = 0, \\ -1, & t < 0. \end{cases}$

Proposition 2.2. Suppose that $A_B > 0$.

(1) *The operator B has only real nonzero eigenvalues. Therefore, sgn can be interpreted as a holomorphic function on a neighborhood of spB, and we can define* sgn(B) *by the standard holomorphic functional calculus.*
(2) *A symplectic transformation R diagonalizes B if* and only if

$$\mathrm{sgn}(B) = RSR^{-1}. \tag{2.34}$$

Proof. It is useful to endow the space $\mathbb{C}^m \oplus \mathbb{C}^m$ with the scalar product given by the positive operator SAS. More precisely, if $v = (v_1, v_2)$, $w = (w_1, w_2) \in \mathbb{C}^m \oplus \mathbb{C}^m$, we set

$$(v|w)_{\rm en} = (v|SASw) = (v_1|hw_1) - (v_1|gw_2) - (v_2|\overline{g}w_1) + (v_2|\overline{h}w_2). \tag{2.35}$$

(2.35) is sometimes called the *energy scalar product*.

Note that we also have the original scalar product

$$(v|w) = (v_1|w_1) + (v_2|w_2),$$

which is used for a basic notation such as the Hermitian adjoints.

First note that B is self-adjoint in the energy scalar product and has a zero null space. Indeed

$$\begin{aligned}(v|Bw)_{\rm en} &= (v|SASASw)\\ &= (ASv|SASw) = (Bv|w)_{\rm en},\\ (Bv|Bv)_{\rm en} &= (v|SASASASv) > 0, \quad v \neq 0.\end{aligned}$$

This shows (1).

Now let R be symplectic. Set

$$B_{\rm dg} := R^{-1}BR, \quad A_{\rm dg} := B_{\rm dg}S = R^{-1}AR^{*-1}.$$

Then, by functional calculus,

$$\operatorname{sgn}(B) = R\operatorname{sgn}(B_{\rm dg})R^{-1}. \tag{2.36}$$

R diagonalizes A if and only if

$$A_{\rm dg} = \begin{bmatrix} h_{\rm dg} & 0 \\ 0 & \overline{h}_{\rm dg} \end{bmatrix}, \quad B_{\rm dg} = \begin{bmatrix} h_{\rm dg} & 0 \\ 0 & -\overline{h}_{\rm dg} \end{bmatrix}. \tag{2.37}$$

A is strictly positive, hence are $A_{\rm dg}$ and $h_{\rm dg}$. Therefore,

$$\operatorname{sgn}(B_{\rm dg}) = S. \tag{2.38}$$

Together with (2.36), this implies (2.34).

Conversely, suppose that (2.34) holds. Together with (2.36), this implies (2.38). Hence $B_{\rm dg}$ is diagonal. □

It is possible to find a distinguished positive symplectic transformation R diagonalizing B.

Theorem 2.3. *Suppose that $A_B > 0$.*

(1) i $\operatorname{sgn}(B)$ *is symplectic.*
(2) $\mathrm{R}_0 := \operatorname{sgn}(B)S$ *is symplectic and has positive eigenvalues.*
(3) *Using holomorphic calculus and the principal square root (which for positive arguments has positive values), define*

$$R := R_0^{\frac{1}{2}}. \tag{2.39}$$

Then R is positive symplectic and diagonalizes B.

(4) *The following is an alternative formula for R_0, where the square root can be interpreted in terms of functional calculus for self-adjoint operators:*

$$R_0 = SA_B^{-\frac{1}{2}}(A_B^{\frac{1}{2}}SA_BSA_B^{\frac{1}{2}})^{\frac{1}{2}}A_B^{-\frac{1}{2}}S. \tag{2.40}$$

Proof. B satisfies (2.24). Hence for any function f holomorphic on the spectrum of B,

$$Sf(B)S^{-1} = f(B^*). \tag{2.41}$$

In particular,

$$S\mathrm{sgn}(B)S^{-1} = \mathrm{sgn}(B^*). \tag{2.42}$$

But sgn is real; hence $\mathrm{sgn}(B^*) = \mathrm{sgn}(B)^*$. Besides, away from 0, we have $\mathrm{sgn}(t) = \mathrm{sgn}(t)^{-1}$. Hence, $\mathrm{sgn}(B) = \mathrm{sgn}(B)^{-1}$. Therefore, (2.42) can be rewritten as

$$S\mathrm{sgn}(B)S^{-1} = \mathrm{sgn}(B)^{*-1}. \tag{2.43}$$

Hence,

$$(\mathrm{i}\,\mathrm{sgn}(B))^* S \mathrm{i}\,\mathrm{sgn}(B) = S. \tag{2.44}$$

This means that $\mathrm{i}\,\mathrm{sgn}(B)$ preserves S. Besides,

$$\mathrm{i}\,\mathrm{sgn}B = \big(-(\mathrm{i}B)^2\big)^{\frac{1}{2}}(\mathrm{i}B)^{-1} \tag{2.45}$$

is also J-real. Thus we have shown that $\mathrm{i}\,\mathrm{sgn}(B)$ is symplectic.

$-\mathrm{i}S$ is also symplectic. Therefore, so is $R_0 = (\mathrm{i}\,\mathrm{sgn}(B))(-\mathrm{i}S)$.

Now,

$$\begin{aligned} R_0 &= (B^2)^{\frac{1}{2}}B^{-1}S & (2.46)\\ &= (ASAS)^{\frac{1}{2}}SA^{-1}S & (2.47)\\ &= SA^{-\frac{1}{2}}(A^{\frac{1}{2}}SASA^{\frac{1}{2}})^{\frac{1}{2}}A^{-\frac{1}{2}}S. & (2.48)\end{aligned}$$

Therefore, (2.40) is true and R_0 is a positive self-adjoint operator for the original scalar product. Hence it has positive eigenvalues.

$R_0 = R_0^*$ and R_0 is symplectic. Hence,

$$SR_0S^{-1} = R_0^{-1}.$$

Hence for any Borel function f,

$$Sf(R_0)S^{-1} = f(R_0^{-1}).$$

Choosing f to be the (positive) square root, we obtain

$$SR_0^{\frac{1}{2}}S^{-1} = R_0^{-\frac{1}{2}}.$$

Thus $R := R_0^{\frac{1}{2}}$ is symplectic, positive, and self-adjoint for the original scalar product.

Now

$$\mathrm{sgn}(B) = R^2S = RSR^{-1}.$$

Hence (2.34) is true. □

I. Positive Weyl Bogoliubov Hamiltonians

Theorem 2.4. (1) *If $A_B \geq 0$, then the Weyl quantization of B is positive. Hence all quantizations of B are bounded from below.*
(2) *If B possesses a quantization that is bounded from below, then $A_B \geq 0$.*

Proof. (1) Let $A \geq 0$. Then there exists a symplectic transformation R and a decomposition $\mathbb{C}^m = \mathbb{C}^{m_1} \oplus \mathbb{C}^{m-m_1}$ such that RAR^* decomposes into the direct sum of the following two terms:

$$\begin{bmatrix} h_{\mathrm{dg}} & 0 \\ 0 & \overline{h}_{\mathrm{dg}} \end{bmatrix} \quad \text{on } \mathbb{C}^{m_1} \oplus \mathbb{C}^{m_1},$$

$$\frac{1}{2}\begin{bmatrix} \mathbb{1} & \mathbb{1} \\ \mathbb{1} & \mathbb{1} \end{bmatrix} \quad \text{on } \mathbb{C}^{m-m_1} \oplus \mathbb{C}^{m-m_1},$$

where $h_{\rm dg} \geq 0$ and can be assumed to be diagonal. This is a well-known fact proven, e.g., in Refs. 13 and 7. It is a very special case of a more general and more complicated classification of quadratic forms on a symplectic space called Williamson's theorem, proven, e.g., in Refs. 19 and 14. If we strengthen the assumption and demand that $h > 0$, it follows also from the diagonalizability of A (Theorem 2.3). Thus, after an application of the transformation R, and a diagonalization of $h_{\rm dg}$, the classical Hamiltonian becomes

$$H_{RBR^{-1}} = \sum_{i=1}^{m_1} h_{{\rm dg},ii} a_i^* a_i + \sum_{m_1+1}^{m} \frac{1}{2}(a_i^* + a_i)^2. \tag{2.49}$$

After application of an implementer of R, the quantum Weyl Hamiltonian becomes

$$U_R \hat{H}_B^{\rm w} U_R^* = \hat{H}_{RBR^{-1}}^{\rm w} = \sum_{i=1}^{m_1} \frac{1}{2} h_{{\rm dg},ii}(\hat{a}_i^* \hat{a}_i + \hat{a}_i \hat{a}_i^*) + \sum_{m_1+1}^{m} \frac{1}{2}(\hat{a}_i^* + \hat{a}_i)^2. \tag{2.50}$$

Thus $\hat{H}_B^{\rm w}$ is positive.

(2) Consider the family of coherent vectors

$$\Omega_w := {\rm e}^{\hat{a}^*(w) - \hat{a}(w)} \Omega, \quad w \in \mathbb{C}^m. \tag{2.51}$$

Note that

$${\rm e}^{-\hat{a}^*(w)+\hat{a}(w)} \hat{a}_i^* {\rm e}^{\hat{a}^*(w)-\hat{a}(w)} = \hat{a}_i^* + \overline{w}_i, \quad {\rm e}^{-\hat{a}^*(w)+\hat{a}(w)} \hat{a}_i {\rm e}^{\hat{a}^*(w)-\hat{a}(w)} = \hat{a}_i + w_i. \tag{2.52}$$

Obviously, if one of the quantizations of B is bounded from below, then so are all of them. Let $\hat{H}_B^{\rm n}$ be bounded from below by $-c$. Then, using (2.52), we obtain

$$\begin{aligned} -c &\leq (\Omega_w | \hat{H}_B^{\rm n} \Omega_w) && (2.53)\\ &= \left(\Omega | {\rm e}^{-\hat{a}^*(w)+\hat{a}(w)} \hat{H}_B^{\rm n} {\rm e}^{\hat{a}^*(w)-\hat{a}(w)} \Omega\right) && (2.54)\\ &= \sum h_{ij} \overline{w}_i w_j + \frac{1}{2} \sum g_{ij} \overline{w}_i \overline{w}_j + \frac{1}{2} \sum \overline{g}_{ij} w_i w_j. && (2.55) \end{aligned}$$

Thus the classical Hamiltonian is a quadratic polynomial and is bounded from below. But if a quadratic polynomial is bounded from below, then it is non-negative. □

Note that by the above theorem, every B satisfying $A_B \geq 0$, besides $\hat{H}_B^{\rm w}$ and $\hat{H}_B^{\rm n}$, possesses another natural quantization: the *zero infimum quantization* $\hat{H}_B^{\rm z}$ fixed by the condition

$$\inf \hat{H}_B^{\rm z} = 0. \tag{2.56}$$

The infimum of the Weyl Bogoliubov Hamiltonians can be computed from several formulas described in the following theorem borrowed from Refs. 6 and 7:

Theorem 2.5. *Assume that $A_B \geq 0$. Then*

$$\begin{aligned} E_B^{\rm w} := \inf \hat{H}_B^{\rm w} &= \frac{1}{4} {\rm Tr} \sqrt{B^2} && (2.57)\\ &= \frac{1}{4} {\rm Tr} \begin{bmatrix} h^2 - g\overline{g} & -hg + g\overline{h} \\ \overline{g}h - \overline{h}\overline{g} & \overline{h}^2 - \overline{g}g \end{bmatrix}^{\frac{1}{2}} && (2.58)\\ &= \frac{1}{4} {\rm Tr} \sqrt{A^{\frac{1}{2}} SASA^{\frac{1}{2}}} && (2.59)\\ &= \frac{1}{4} {\rm Tr} \int \frac{B^2}{(B^2 + \tau^2)} \frac{{\rm d}\tau}{\pi}. && (2.60) \end{aligned}$$

Proof. Let R be as in the proof of Theorem 2.4. Clearly,

$$\inf(\hat{a}_i^*\hat{a}_i + \hat{a}_i\hat{a}_i^*) = 1, \tag{2.61}$$

$$\inf(\hat{a}_i^* + \hat{a}_i)^2 = 0. \tag{2.62}$$

Hence, by (2.50),

$$\inf \hat{H}_B^{\mathrm{w}} = \inf U_R\hat{H}_B^{\mathrm{w}}U_R^* = \inf \hat{H}_{RBR^{-1}}^{\mathrm{w}} = \frac{1}{2}\sum h_{\mathrm{dg},ii} = \frac{1}{2}\mathrm{Tr}h_{\mathrm{dg}} \tag{2.63}$$

$$= \frac{1}{4}\mathrm{Tr}\sqrt{B_{\mathrm{dg}}^2} = \frac{1}{4}\mathrm{Tr}R\sqrt{B^2}R^{-1} = \frac{1}{4}\mathrm{Tr}\sqrt{B^2}. \tag{2.64}$$

This gives (2.57), which implies (2.58) and (2.59).

(2.60) follows by an application of identity (A5). □

J. Infimum of normally ordered Hamiltonians

As usual, we have

$$B = \begin{bmatrix} h & -g \\ \overline{g} & -\overline{h} \end{bmatrix}, \qquad B_0 = \begin{bmatrix} h & 0 \\ \overline{g} & 0 \end{bmatrix}, \tag{2.65}$$

$$A_B = A = \begin{bmatrix} h & g \\ \overline{g} & \overline{h} \end{bmatrix}, \qquad A_0 = \begin{bmatrix} h & 0 \\ 0 & \overline{h} \end{bmatrix}. \tag{2.66}$$

It is convenient to set

$$G := B - B_0 = \begin{bmatrix} 0 & -g \\ \overline{g} & 0 \end{bmatrix}, \tag{2.67}$$

$$A_\sigma = A_0 + \sigma GS = \begin{bmatrix} h & \sigma g \\ \sigma\overline{g} & \overline{h} \end{bmatrix}, \tag{2.68}$$

$$B_\sigma := B_0 + \sigma G = \begin{bmatrix} h & -\sigma g \\ \sigma\overline{g} & -\overline{h} \end{bmatrix}, \quad \sigma \in \mathbb{R}. \tag{2.69}$$

The following are a few formulas for the infimum of the normally ordered Hamiltonian.

Theorem 2.6. *Assume that $A_B \geq 0$. Then*

$$E_B^{\mathrm{n}} := \inf \hat{H}_B^{\mathrm{n}} = E_B^{\mathrm{w}} - \frac{1}{2}\mathrm{Tr}h \tag{2.70}$$

$$= \frac{1}{4}\mathrm{Tr}\left(\sqrt{B^2} - \sqrt{B_0^2}\right) \tag{2.71}$$

$$= \frac{1}{4}\mathrm{Tr}\left(\begin{bmatrix} h^2 - g\overline{g} & -hg + g\overline{h} \\ \overline{g}h - \overline{h}\overline{g} & \overline{h}^2 - \overline{g}g \end{bmatrix}^{\frac{1}{2}} - \begin{bmatrix} h & 0 \\ 0 & \overline{h} \end{bmatrix}\right) \tag{2.72}$$

$$= \frac{1}{4}\int_0^1 \mathrm{d}\sigma \mathrm{Tr}\frac{B_\sigma}{\sqrt{B_\sigma^2}}G \tag{2.73}$$

$$= \frac{1}{4}\int_0^1 \mathrm{d}\sigma \mathrm{Tr}A_\sigma^{\frac{1}{2}}(A_\sigma^{\frac{1}{2}}SA_\sigma SA_\sigma^{\frac{1}{2}})^{-\frac{1}{2}}A_\sigma^{\frac{1}{2}}GS \tag{2.74}$$

$$= \frac{1}{4}\int_0^1 \mathrm{d}\sigma \int \frac{\mathrm{d}\tau}{\pi}(1-\sigma)\mathrm{Tr}\frac{1}{(A_\sigma + \mathrm{i}\tau S)}SG\frac{1}{(A_\sigma + \mathrm{i}\tau S)}SG. \tag{2.75}$$

Proof. (2.70)–(2.72) follow immediately from Theorem 2.5 and (2.30).

Starting from (2.71), let us prove (2.73),

$$\frac{1}{4}\mathrm{Tr}(\sqrt{B^2}-\sqrt{B_0^2}) \tag{2.76}$$

$$=\frac{1}{4}\int \mathrm{Tr}\Big(\frac{B^2}{B^2+\tau^2}-\frac{B_0^2}{B_0^2+\tau^2}\Big)\frac{\mathrm{d}\tau}{\pi} \tag{2.77}$$

$$=-\frac{1}{4}\int \mathrm{Tr}\Big(\frac{1}{B^2+\tau^2}-\frac{1}{B_0^2+\tau^2}\Big)\frac{\tau^2\mathrm{d}\tau}{\pi} \tag{2.78}$$

$$=-\frac{1}{4}\int_0^1 \mathrm{d}\sigma\int\frac{\mathrm{d}}{\mathrm{d}\sigma}\mathrm{Tr}\frac{1}{B_\sigma^2+\tau^2}\frac{\tau^2\mathrm{d}\tau}{\pi} \tag{2.79}$$

$$=\frac{1}{4}\int_0^1 \mathrm{d}\sigma\int\mathrm{Tr}\frac{1}{B_\sigma^2+\tau^2}(B_\sigma G+GB_\sigma)\frac{1}{B_\sigma^2+\tau^2}\frac{\tau^2\mathrm{d}\tau}{\pi} \tag{2.80}$$

$$=\frac{1}{2}\int_0^1 \mathrm{d}\sigma\int\mathrm{Tr}\frac{B_\sigma}{(B_\sigma^2+\tau^2)^2}G\frac{\tau^2\mathrm{d}\tau}{\pi} \tag{2.81}$$

$$=\frac{1}{4}\int_0^1 \mathrm{d}\sigma\mathrm{Tr}\frac{B_\sigma}{\sqrt{B_\sigma^2}}G, \tag{2.82}$$

where at the end we used identity (A6).

(2.73) together with identity (2.46) = (2.48) implies (2.74).

Now, starting from (2.73), we prove (2.75),

$$\frac{1}{4}\int_0^1 \mathrm{d}\sigma\int\frac{\mathrm{d}\tau}{\pi}\mathrm{Tr}\frac{B_\sigma}{B_\sigma^2+\tau^2}G \tag{2.83}$$

$$=\frac{1}{8}\int_0^1 \mathrm{d}\sigma\int\frac{\mathrm{d}\tau}{\pi}\mathrm{Tr}\Big(\frac{1}{(B_\sigma+\mathrm{i}\tau)}+\frac{1}{(B_\sigma-\mathrm{i}\tau)}\Big)G \tag{2.84}$$

$$=\frac{1}{4}\int_0^1 \mathrm{d}\sigma\int\frac{\mathrm{d}\tau}{\pi}\mathrm{Tr}\frac{1}{(B_\sigma+\mathrm{i}\tau)}G \tag{2.85}$$

$$=\frac{1}{4}\int_0^1 \mathrm{d}\sigma\int_0^\sigma \mathrm{d}\sigma_1\int\frac{\mathrm{d}\tau}{\pi}\frac{\mathrm{d}}{\mathrm{d}\sigma_1}\mathrm{Tr}\frac{1}{(B_{\sigma_1}+\mathrm{i}\tau)}G \tag{2.86}$$

$$=-\frac{1}{4}\int_0^1 (1-\sigma)\mathrm{d}\sigma\int\frac{\mathrm{d}\tau}{\pi}\mathrm{Tr}\frac{1}{(B_\sigma+\mathrm{i}\tau)}G\frac{1}{(B_\sigma+\mathrm{i}\tau)}G \tag{2.87}$$

$$=-\frac{1}{4}\int_0^1 (1-\sigma)\mathrm{d}\sigma\int\frac{\mathrm{d}\tau}{\pi}\mathrm{Tr}\frac{1}{(A_\sigma+\mathrm{i}\tau S)}SG\frac{1}{(A_\sigma+\mathrm{i}\tau S)}SG. \tag{2.88}$$

In (2.86) ⇒ (2.87), we used

$$\frac{\mathrm{d}}{\mathrm{d}\sigma_1}\frac{1}{(B_{\sigma_1}+\mathrm{i}\tau)}=-\frac{1}{(B_{\sigma_1}+\mathrm{i}\tau)}G\frac{1}{(B_{\sigma_1}+\mathrm{i}\tau)}. \tag{2.89}$$

□

K. Loop expansion

Suppose now that

$$B_0=\begin{bmatrix}h_0 & 0\\ 0 & -\overline{h}_0\end{bmatrix} \tag{2.90}$$

is a "free" symplectic generator. We assume that $h_0 > 0$. Note that we allow h_0 to be different from h.

We set

$$A_0 := B_0 S = \begin{bmatrix} h_0 & 0 \\ 0 & \overline{h}_0 \end{bmatrix}, \tag{2.91}$$

$$V := B^2 - B_0^2 = \begin{bmatrix} h^2 - h_0^2 - g\overline{g} & -hg + g\overline{h} \\ \overline{g}h - \overline{h}\overline{g} & \overline{h}^2 - \overline{h}_0^2 - \overline{g}g \end{bmatrix}. \tag{2.92}$$

(2.60) can be rewritten as

$$\begin{aligned} E^{\mathrm{w}} &= \frac{1}{4}\mathrm{Tr}\int \frac{B_0^2}{B_0^2+\tau^2}\frac{\mathrm{d}\tau}{\pi} + \frac{1}{4}\mathrm{Tr}\int \frac{1}{B^2+\tau^2}V\frac{1}{B_0^2+\tau^2}\tau^2\frac{\mathrm{d}\tau}{\pi} \\ &= \sum_{j=0}^{k} L_j + \frac{1}{4}\mathrm{Tr}\int \frac{(-1)^k}{B_0^2+\tau^2}V\frac{1}{B^2+\tau^2}\Big(V\frac{1}{B_0^2+\tau^2}\Big)^k\tau^2\frac{\mathrm{d}\tau}{\pi} \\ &= \sum_{j=0}^{\infty} L_j, \end{aligned}$$

where

$$L_0 = \frac{1}{4}\mathrm{Tr}\int \frac{B_0^2}{B_0^2+\tau^2}\frac{\mathrm{d}\tau}{\pi} = \frac{1}{4}\mathrm{Tr}|B_0| = \frac{1}{2}\mathrm{Tr}h_0, \tag{2.93}$$

$$L_j = \frac{1}{4}\mathrm{Tr}\int \frac{(-1)^{j+1}}{B_0^2+\tau^2}\Big(V\frac{1}{B_0^2+\tau^2}\Big)^j\tau^2\frac{\mathrm{d}\tau}{\pi} \tag{2.94}$$

$$= \frac{1}{4}\mathrm{Tr}\int \frac{(-1)^{j+1}}{2j}\Big(V\frac{1}{B_0^2+\tau^2}\Big)^j\frac{\mathrm{d}\tau}{\pi}, \quad j = 1, 2, \ldots. \tag{2.95}$$

The last identity for L_j follows by a cyclic relocation of operators under the trace and by an application of integration by parts.

We can further simplify the formula for L_1,

$$L_1 = \mathrm{Tr}\int \frac{1}{8}V\frac{1}{B_0^2+\tau^2}\frac{\mathrm{d}\tau}{\pi} = \frac{1}{8}\mathrm{Tr}V\frac{1}{|B_0|} = \frac{1}{4}\mathrm{Tr}(h^2 - h_0^2 - g\overline{g})h_0^{-1}. \tag{2.96}$$

The constant L_j arises in the diagrammatic expansion as the evaluation of the loop with j vertices. To obtain this, introduce the "time variable" t and the "Feynman propagator"

$$G(t) := \frac{\mathrm{e}^{-|B_0|t}}{2|B_0|}.$$

Clearly, τ can be interpreted as the "energy variable" and

$$\frac{1}{B_0^2+\tau^2} = \int G(t)\mathrm{e}^{\mathrm{i}t\tau}\mathrm{d}t.$$

Therefore,

$$L_j = \frac{1}{4}\int \mathrm{d}t_{j-1}\cdots\int \mathrm{d}t_1 \mathrm{Tr}VG(t_j - t_1)VG(t_1 - t_2)\cdots VG(t_{j-1} - t_j) \tag{2.97}$$

$$= \lim_{T\to\infty}\frac{1}{2T}\frac{1}{4}\int_{-T}^{T}\mathrm{d}t_j\int_{-T}^{T}\mathrm{d}t_{j-1}\cdots\int_{-T}^{T}\mathrm{d}t_1 \mathrm{Tr}VG(t_j - t_1)VG(t_1 - t_2)\cdots VG(t_{j-1} - t_j). \tag{2.98}$$

L. Renormalization I

Note that in general V (2.92) contains terms of the 1st and 2nd order. Explicitly, let λ be a "coupling constant." Let $h = h_0 + \lambda h_1$ and replace g with λg (to keep track of the order of perturbation). Then $V = \lambda V_1 + \lambda^2 V_2$, where

$$V_1 := \begin{bmatrix} h_0h_1 + h_1h_0 & -h_0g + g\overline{h}_0 \\ \overline{g}h_0 - \overline{h}_0\overline{g} & \overline{h}_0\overline{h}_1 + \overline{h}_0\overline{h}_1 \end{bmatrix}, \tag{2.99}$$

$$V_2 := \begin{bmatrix} h_1^2 - gg^* & -h_1g + g\overline{h}_1 \\ \overline{g}h_1 - \overline{h}_1\overline{g} & \overline{h}_1^2 - g^*g \end{bmatrix}. \tag{2.100}$$

We can expand E^{w} with respect to the coupling constant λ,

$$E^{\mathrm{w}} = \sum_{n=0}^{\infty} \lambda^n E_n. \tag{2.101}$$

We have

$$L_0 = E_0 = \frac{1}{2}\mathrm{Tr}h_0. \tag{2.102}$$

However, in general, L_n of higher orders differ from $\lambda^n E_n$.

There are situations when it is useful to introduce the *nth order renormalized vacuum energy*

$$E^{n\mathrm{ren}} := E^{\mathrm{w}} - \sum_{j=0}^{n} \lambda^j E_j = \sum_{j=n+1}^{\infty} \lambda^j E_j \tag{2.103}$$

and the *nth order renormalized Hamiltonian*

$$\hat{H}^{n\mathrm{ren}} := \hat{H}^{\mathrm{w}} - \sum_{j=0}^{n} \lambda^j E_j \tag{2.104}$$

so that $E^{n\mathrm{ren}} = \mathit{inf}\, \hat{H}^{n\mathrm{ren}}$. The numbers $\lambda^0 E_0, \dots, \lambda^n E_n$ can be called *counterterms*.

The above construction is relevant, e.g., in the theory of charged scalar fields in external electromagnetic potentials in 1 + 3 dimensions. In this case, E_0, E_1, and E_2 are infinite so that one is forced to perform the 2nd order renormalization. $\hat{H}^{2\mathrm{ren}}$ is usually also ill defined. However $E^{2\mathrm{ren}}$ is typically finite. Note that we have a somewhat paradoxical situation: the Hamiltonian does not exist; however, the "infimum of the Hamiltonian" is well defined.

Actually, physically, $E^{2\mathrm{ren}}$ is still somewhat artificial. It is natural to make an additional finite subtraction so that all counterterms are formally *local*; see Ref. 8. The resulting finite quantity E^{ren} is sometimes called the (renormalized) *vacuum energy* or the *vacuum polarization*.

M. Renormalization II

Suppose now that

$$h_1^2 = g\overline{g}, \quad h_1 g = g\overline{h}_1. \tag{2.105}$$

(2.105) implies $V_2 = 0$. Therefore, the loop expansion coincides with the expansion into powers of λ. Putting $\lambda = 1$, we thus have

$$E_n = L_n, \quad n = 0, 1, \dots . \tag{2.106}$$

We can compute the loop with one vertex,

$$L_1 = \frac{1}{4}\mathrm{Tr}(h_0h_1 + h_1h_0)h_0^{-1} = \frac{1}{2}\mathrm{Tr}h_1. \tag{2.107}$$

Thus

$$L_0 + L_1 = \frac{1}{2}\mathrm{Tr}(h_0 + h_1) = \frac{1}{2}\mathrm{Tr}h. \tag{2.108}$$

Therefore, the loop expansion for the infimum of the normally ordered Hamiltonian amounts to omitting L_0 and L_1,

$$\inf H^{\mathrm{n}} = E^{\mathrm{w}} - \frac{1}{2}\mathrm{Tr}h = \sum_{n=2}^{\infty} L_n = E^{1\mathrm{ren}}. \tag{2.109}$$

Note that L_1 and especially L_0 are often infinite. Sometimes, L_2 is infinite as well. Then we can renormalize the vacuum energy even further obtaining

$$E^{2\text{ren}} := E^{\text{w}} - L_0 - L_1 - L_2 = \sum_{n=3}^{\infty} L_n \tag{2.110}$$

$$= -\frac{1}{4}\int \operatorname{Tr}\frac{1}{B_0^2+\tau^2}V\frac{1}{B^2+\tau^2}\Big(V\frac{1}{B_0^2+\tau^2}\Big)^2\tau^2\frac{\mathrm{d}\tau}{\pi}. \tag{2.111}$$

We also have the 2nd order renormalized Hamiltonian

$$\hat{H}^{2\text{ren}} := \hat{H}^{\text{w}} - L_0 - L_1 - L_2 \tag{2.112}$$

so that

$$E^{2\text{ren}} = \inf \hat{H}^{2\text{ren}}. \tag{2.113}$$

The situation described in this subsection is typical for a charged particle in an external electrostatic potential (without a vector potential), as well as for a neutral scalar particle with a mass-like perturbation.[8] Under rather broad assumptions, the 2nd order renormalized Hamiltonian $\hat{H}^{2\text{ren}}$ is then a well-defined self-adjoint operator.

Actually, as in Subsection II L, subtracting only $L_0 + L_1 + L_2$ is somewhat artificial from the physical point of view. To obtain physically relevant objects, one performs an additional finite renormalization so that all counterterms are formally local, obtaining a finite *renormalized vacuum energy* E^{ren} and a well-defined *renormalized Hamiltonian* $\hat{H}^{\text{ren}}$ so that $E^{\text{ren}} = \inf \hat{H}^{\text{ren}}$. See Ref. 8 for details.

III. ARBITRARY DIMENSIONS: BASIS INDEPENDENT FORMALISM

In this section, we consider Bogoliubov Hamiltonians in any dimension. Unlike in Sec. II, we will use a basis independent notation.

We will use the standard notation for the Hilbert-Schmidt and trace class norms,

$$\|g\|_2 := \sqrt{\operatorname{Tr} g^* g}, \qquad \|g\|_1 := \operatorname{Tr}\sqrt{g^* g}. \tag{3.1}$$

A. Doubled space in abstract setting

Let $\mathcal{W}$ be a Hilbert space. $\mathcal{W}$ will serve as the *1-particle space*.

Let $\overline{\mathcal{W}}$ be another Hilbert space with a fixed antiunitary map $\chi : \mathcal{W} \to \overline{\mathcal{W}}$. $\overline{\mathcal{W}}$ will be called the *complex conjugate of* $\mathcal{W}$.

We will often use the doubled space $\mathcal{W} \oplus \overline{\mathcal{W}}$ equipped with the conjugation

$$J = \begin{bmatrix} 0 & \chi^{-1} \\ \chi & 0 \end{bmatrix}. \tag{3.2}$$

A *J-real* operator is an operator on $\mathcal{W} \oplus \overline{\mathcal{W}}$ commuting with J. Bounded J-real operators have the form

$$R = \begin{bmatrix} p & q \\ \chi q \chi & \chi p \chi^{-1} \end{bmatrix}, \tag{3.3}$$

for some $p \in B(\mathcal{W})$ and $q \in B(\overline{\mathcal{W}}, \mathcal{W})$.

J-real operators leave invariant the real subspace of vectors

$$\begin{bmatrix} w \\ \chi w \end{bmatrix}, \; w \in \mathcal{W},$$

which we will denote by $\mathcal{Y}$. Note also that every J-real operator in $B(\mathcal{W} \oplus \overline{\mathcal{W}})$ restricts to an operator in $B(\mathcal{Y})$, and conversely, each operator in $B(\mathcal{Y})$ extends uniquely to an operator in $B(\mathcal{W} \oplus \overline{\mathcal{W}})$.

In what follows, we will usually write $\overline{w}$ for χw. We will write $\overline{p}$ and $\overline{q}$ for $\chi p \chi^{-1}$ and $\chi q \chi$. We will write $p^{\#}$ and $q^{\#}$ for $\chi p^* \chi^{-1}$ and $\chi^{-1} q^* \chi^{-1}$. In Subsection 1 of the Appendix, we explain why it is natural to use this simplified notation.

To reduce the formalism of this section to that of Sec. II, it suffices to set $\mathcal{W} = \mathbb{C}^m$ and replace χ with the complex conjugation.

B. Fock spaces

If $\mathcal{D}$ is a vector space of any dimension (with or without a Hilbert space structure), then we can introduce its *algebraic n-th symmetric power*, denoted by $\overset{\mathrm{al}n}{\otimes_{\mathrm{s}}}\mathcal{D}$, and the *algebraic bosonic Fock space*

$$\overset{\mathrm{al}}{\Gamma}_{\mathrm{s}}(\mathcal{W}) := \overset{\mathrm{al}}{\underset{n=0}{\overset{\infty}{\oplus}}} \overset{\mathrm{al}n}{\otimes_{\mathrm{s}}}\mathcal{D},$$

which is the space of finite symmetric tensor products of vectors of $\mathcal{D}$.[7] If $\mathcal{W}$ is a Hilbert space, then we prefer to use the Hilbert space versions of the above constructions. Thus $\otimes_{\mathrm{s}}^n\mathcal{W}$ will denote the *n-th symmetric tensor power of $\mathcal{W}$ in the sense of Hilbert spaces* and, as usual, the *bosonic Fock space over the one-particle space $\mathcal{W}$* is defined as

$$\Gamma_{\mathrm{s}}(\mathcal{W}) := \underset{n=0}{\overset{\infty}{\oplus}} \otimes_{\mathrm{s}}^n \mathcal{W}.$$

$\Omega := (1, 0, \cdots)$ denotes the vacuum vector and

$$\begin{aligned}\Gamma_{\mathrm{s}}^{\mathrm{fin}}(\mathcal{W}) &:= \overset{\mathrm{al}}{\underset{n=0}{\overset{\infty}{\oplus}}} \otimes_{\mathrm{s}}^n\mathcal{W} \\ &= \{(\Psi^0, \dots, \Psi^n, \cdots) \in \Gamma_{\mathrm{s}}(\mathcal{W}) \mid \Psi^n = 0 \text{ for all but a finite number of } n\}\end{aligned}$$

is the *finite particle bosonic Fock space*.

Note that if $\mathcal{D}$ is dense in $\mathcal{W}$, then $\overset{\mathrm{al}}{\Gamma}_{\mathrm{s}}(\mathcal{D})$ is dense in $\Gamma_{\mathrm{s}}(\mathcal{W})$.

If h is an operator on $\mathcal{W}$, $\mathrm{d}\Gamma(h)$ will denote

$$\mathrm{d}\Gamma(h)\big\lceil_{\otimes_{\mathrm{s}}^n\mathcal{W}} := \sum_{j=1}^n \underbrace{1 \otimes \cdots \otimes 1}_{j-1} \otimes h \otimes \underbrace{1 \otimes \cdots \otimes 1}_{n-j}\big\lceil_{\otimes_{\mathrm{s}}^n\mathcal{W}}.$$

If q is an operator on $\mathcal{W}$ of norm less than 1, we define $\Gamma(q) : \Gamma_{\mathrm{s}}(\mathcal{W}) \to \Gamma_{\mathrm{s}}(\mathcal{W})$ by

$$\Gamma(q)\big\lceil_{\otimes_{\mathrm{s}}^n\mathcal{H}} := q \otimes \cdots \otimes q\big\lceil_{\otimes_{\mathrm{s}}^n\mathcal{H}}.$$

C. Quadratic forms on Fock spaces

For any operator h on $\mathcal{W}$ such that $h \geq c$, its *form domain* is defined as

$$\mathrm{Dom}(|h|^{\frac{1}{2}}) = (\mathbb{1} + |h|)^{-\frac{1}{2}}\mathcal{W}. \tag{3.4}$$

For $w_1, w_2 \in \mathrm{Dom}(|h|^{\frac{1}{2}})$, we can define $(w_i|hw_2)$. $\mathrm{Dom}(|h|^{\frac{1}{2}})$ is a Hilbert space for the scalar product $(w_1|(h+c+\mathbb{1})w_2)$. We say that $\mathcal{D}$ is a *form core of h* if it is a dense subspace of the form domain of h.

Lemma 3.1. Suppose that $h \geq 0$ and $\mathcal{D}$ is a form core of h. Then $\overset{\mathrm{al}}{\Gamma}_{\mathrm{s}}(\mathcal{D})$ is a form core of $\mathrm{d}\Gamma(h)$.

Proof. It is easy to see that

$$\mathbb{1} + \mathrm{d}\Gamma(h) \leq \Gamma(\mathbb{1} + h). \tag{3.5}$$

Hence,

$$\Gamma(\mathbb{1} + h)^{-\frac{1}{2}}\Gamma_{\mathrm{s}}(\mathcal{W}) \subset (\mathbb{1} + \mathrm{d}\Gamma(h))^{-\frac{1}{2}}\Gamma_{\mathrm{s}}(\mathcal{W}). \tag{3.6}$$

Let $\Psi \in (\mathbb{1} + \mathrm{d}\Gamma(h))^{-\frac{1}{2}}\Gamma_{\mathrm{s}}(\mathcal{W})$. Set

$$\Psi_n := \mathbb{1}_{[0,n]}(\Gamma(\mathbb{1} + h))\Psi. \tag{3.7}$$

By the spectral theorem and the fact that $\mathbb{1} + \mathrm{d}\Gamma(h)$ and $\Gamma(\mathbb{1} + h)$ commute with one another, $\Psi_n \in \mathrm{Dom}\Gamma(\mathbb{1} + h)^{\frac{1}{2}}$ and $\Psi_n \to \Psi$ in $(\mathbb{1} + \mathrm{d}\Gamma(h))^{-\frac{1}{2}}\Gamma_{\mathrm{s}}(\mathcal{W})$. Hence

$$\Gamma(\mathbb{1}+h)^{-\frac{1}{2}}\Gamma_{\mathrm{s}}(\mathcal{W}) \text{ is dense in } (\mathbb{1}+\mathrm{d}\Gamma(h))^{-\frac{1}{2}}\Gamma_{\mathrm{s}}(\mathcal{W}). \tag{3.8}$$

Now $\mathcal{D}$ is dense in $(\mathbb{1}+h)^{-\frac{1}{2}}\mathcal{W}$. Hence

$$\overset{\mathrm{al}}{\Gamma}_{\mathrm{s}}(\mathcal{D}) \text{ is dense in } \Gamma_{\mathrm{s}}((\mathbb{1}+h)^{-\frac{1}{2}}\mathcal{W}) = \Gamma(\mathbb{1}+h)^{-\frac{1}{2}}\Gamma_{\mathrm{s}}(\mathcal{W}). \tag{3.9}$$

Putting together (3.8) and (3.9), we obtain

$$\overset{\mathrm{al}}{\Gamma}_{\mathrm{s}}(\mathcal{D}) \text{ is dense in } (\mathbb{1}+\mathrm{d}\Gamma(h))^{-\frac{1}{2}}\Gamma_{\mathrm{s}}(\mathcal{W}). \tag{3.10}$$

But the RHS of (3.10) is the form domain of $\mathrm{d}\Gamma(\mathrm{h})$. □

D. Creation/annihilation operators

For any $w \in \mathcal{W}$, $\hat{a}(w)$ and $\hat{a}^*(w)$ denote the usual *annihilation/creation operators*,

$$\hat{a}^*(w)\Psi := \sqrt{n+1}\, w \otimes_{\mathrm{s}} \Psi, \qquad \Psi \in \otimes_{\mathrm{s}}^n \mathcal{W}, \tag{3.11}$$

$$\hat{a}(w)\Psi := \sqrt{n+1}\,(w| \otimes 1^{\otimes n}\,\Psi, \qquad \Psi \in \otimes_{\mathrm{s}}^{n+1}\mathcal{W}. \tag{3.12}$$

These operators, originally well defined on $\Gamma_{\mathrm{s}}^{\mathrm{fin}}(\mathcal{W})$, extend to closed operators on $\Gamma_{\mathrm{s}}(\mathcal{W})$. We set

$$\hat{\phi}(w,\overline{w}') := \hat{a}^*(w) + \hat{a}(w'). \tag{3.13}$$

Note that $\hat{\phi}(w,\overline{w})$ are self-adjoint. One can also introduce the so-called *Weyl operators* $\mathrm{e}^{\mathrm{i}\hat{\phi}(w,\overline{w})}$.

Remark 3.2. Sometimes we may want to define creation/annihilation operators for w that do not belong to $\mathcal{W}$, but are functionals, possibly unbounded, with domain $\mathcal{D} \subset \mathcal{W}$. Then we can still define the annihilation operator $\hat{a}(w)$ by formula (3.12), at least for $\Psi \in \overset{\mathrm{al}}{\Gamma}_{\mathrm{s}}(\mathcal{D})$. If w is unbounded, then $\hat{a}(w)$ is not closable. Besides, (3.11), the definition of $\hat{a}^*(w)$ as an operator, is incorrect. However, we can interpret both $\hat{a}(w)$ and $\hat{a}^*(w)$ as quadratic forms on $\overset{\mathrm{al}}{\Gamma}_{\mathrm{s}}(\mathcal{D})$.

The following inequality is sometimes called the *N_τ-estimate*:

Proposition 3.3. Let $h > 0$ and $w \in \mathcal{W}$. Then

$$\|\hat{a}(w)\Phi\|^2 \le (w|h^{-1}w)(\Phi|\mathrm{d}\Gamma(h)\Phi). \tag{3.14}$$

Therefore,

$$\|\mathrm{d}\Gamma(h)^{-\frac{1}{2}}\hat{a}^*(w)\| \le \|h^{-\frac{1}{2}}w\|. \tag{3.15}$$

Proof. Clearly,

$$|w)(w| \le \left(w|h^{-1}w\right)h. \tag{3.16}$$

Applying $\mathrm{d}\Gamma$, we obtain

$$\hat{a}^*(w)\hat{a}(w) = \mathrm{d}\Gamma(|w)(w|) \le (w|h^{-1}w)\mathrm{d}\Gamma(h). \tag{3.17}$$

□

Let $g \in \otimes_{\mathrm{s}}^2\mathcal{W}$. We define the annihilation and creation operators associated with g as follows:

$$\hat{a}^*(g)\Psi := \sqrt{n+2}\sqrt{n+1}\, g \otimes_{\mathrm{s}} \Psi, \qquad \Psi \in \otimes_{\mathrm{s}}^n\mathcal{W}, \tag{3.18}$$

$$\hat{a}(g)\Psi := \sqrt{n+2}\sqrt{n+1}\,(g| \otimes \mathbb{1}^{\otimes n}\,\Psi, \qquad \Psi \in \otimes_{\mathrm{s}}^{n+2}\mathcal{W}. \tag{3.19}$$

Again, these operators, originally defined on $\Gamma_{\mathrm{s}}^{\mathrm{fin}}(\mathcal{W})$, extend to closed operators on $\Gamma_{\mathrm{s}}(\mathcal{W})$.

Remark 3.4. Again, if g does not belong to $\oplus_{\rm s}^2\mathcal{W}$, but is a functional with the domain $\overset{\rm al}{\otimes}{}_{\rm s}^2\mathcal{D} \subset \otimes_{\rm s}^2\mathcal{W}$, then we can define $\hat{a}(g)$ and $\hat{a}^*(g)$ as quadratic forms on $\overset{\rm al}{\Gamma}_{\rm s}(\mathcal{D})$.

It is important to note that each $g \in \otimes^2\mathcal{W}$ defines a linear Hilbert-Schmidt operator from $\overline{\mathcal{W}}$ to $\mathcal{W}$, denoted by the same symbol g, by the identity

$$(w_1 \otimes w_2|g) = (w_2|g\chi w_1). \tag{3.20}$$

This provides an isometric isomorphism of $\otimes^2\mathcal{W}$ with $B^2(\overline{\mathcal{W}}, \mathcal{W})$—the space of Hilbert-Schmidt operators from $\overline{\mathcal{W}}$ to $\mathcal{W}$. Symmetric tensors (elements of $\otimes_{\rm s}^2\mathcal{W}$) are mapped onto symmetric operators (where the *symmetry* of g means $g = g^{\#}$).

Let us state the following fact about this identification:

Proposition 3.5. Let $p_1, p_2 \in B(\mathcal{W})$. Then the tensor $p_1 \otimes p_2\, g$ corresponds to the operator $p_1 g p_2^{\#}$.

Proposition 3.6. Let $w \in \mathcal{W}$, $h \in B(\mathcal{W})$, and *$g \in \mathcal{W} \otimes_{\rm s} \mathcal{W}$. Then the following identities are true:*

$$[{\rm d}\Gamma(h), \hat{a}^*(w)] = \hat{a}^*(hw), \qquad [{\rm d}\Gamma(h), \hat{a}(w)] = -\hat{a}(hw), \tag{3.21}$$

$$[\hat{a}(g), \hat{a}^*(w)] = 2\hat{a}^*(g\overline{w}), \qquad [\hat{a}^*(g), \hat{a}(w)] = -2\hat{a}(g\overline{w}). \tag{3.22}$$

E. Symplectic and metaplectic transformations in infinite dimensions

As in (2.6), we introduce the operator

$$S = \begin{bmatrix} 1\!\!1 & 0 \\ 0 & -1\!\!1 \end{bmatrix}. \tag{3.23}$$

Let $R \in B(\mathcal{W} \oplus \overline{\mathcal{W}})$. As in Subsection II C, R is called symplectic if $R^*SR = S$. Bounded symplectic transformations form a group, which we denote by $Sp(\mathcal{Y})$.

Various properties of symplectic operators described in Subsection II C are valid in the present setting.

Theorem 3.7. *Let*

$$R = \begin{bmatrix} p & q \\ \overline{q} & \overline{p} \end{bmatrix} \in Sp(\mathcal{Y}). \tag{3.24}$$

Then the following conditions are equivalent:

(1) *There exists a unitary U such that*

$$\begin{aligned} U\hat{a}^*(w)U^* &= \hat{a}^*(pw) + \hat{a}(q\overline{w}), \\ U\hat{a}(w)U^* &= \hat{a}^*(q\overline{w}) + \hat{a}(pw), \quad w \in \mathcal{W}. \end{aligned} \tag{3.25}$$

(2) *There exists a unitary U such that*

$$U{\rm e}^{{\rm i}\hat{\phi}(w,\overline{w})}U^* = {\rm e}^{{\rm i}\hat{\phi}(w',\overline{w}')}, \quad R\begin{bmatrix} w \\ \overline{w} \end{bmatrix} = \begin{bmatrix} w' \\ \overline{w}' \end{bmatrix}, \quad w \in \mathcal{W}. \tag{3.26}$$

(3) *There exists a $*$-automorphism α_R of $B(\Gamma_{\rm s}(\mathcal{W}))$ such that*

$$\alpha_R({\rm e}^{{\rm i}\hat{\phi}(w,\overline{w})}) = {\rm e}^{{\rm i}\hat{\phi}(w',\overline{w}')}, \quad R\begin{bmatrix} w \\ \overline{w} \end{bmatrix} = \begin{bmatrix} w' \\ \overline{w}' \end{bmatrix}, \quad w \in \mathcal{W}. \tag{3.27}$$

Let (1), (2), and (3) be true. Then U [common for (1) and (2)] is uniquely determined up to a phase factor. Besides, α_R is uniquely defined.

If R satisfies the conditions of the above theorem, then we say that R is *implementable*. The unitary U is called a *(Bogoliubov) implementer of* R. α_R is called the *Bogoliubov automorphism associated with* R.

We leave the proof of this theorem to the reader. Let us only mention that to show (3) ⇒ (2), we need to use Proposition A.1. To obtain the uniqueness of α_R, we use the weak density of linear combinations of Weyl operators in $B(\Gamma_s(\mathcal{W}))$.

$Sp_{\rm res}(\mathcal{Y})$ will denote the *restricted symplectic group*, which consists of $R \in Sp(\mathcal{Y})$ such that q is Hilbert-Schmidt. The importance of $Sp_{\rm res}(\mathcal{Y})$ is due to the Shale theorem,[18] which we quote below in the form given in Ref. 7.

Theorem 3.8. *Let $R \in Sp(\mathcal{Y})$. Then R is implementable if* and only if $R \in Sp_{\rm res}(\mathcal{Y})$. *For such R, we can define the natural implementer of R,*

$$U_R^{\rm nat} := |\det pp^*|^{-\frac14} \mathrm{e}^{-\frac12 \hat{a}^*(d_2)} \Gamma((p^*)^{-1}) \mathrm{e}^{\frac12 \hat{a}(d_1)}, \tag{3.28}$$

where d_2 and *d_1 are defined as in (2.15)* and *(2.16). All implementers of $R \in Sp_{\rm res}(\mathcal{Y})$ coincide with $U_R^{\rm nat}$ up to a phase factor.*

Bogoliubov implementers form a group, which is denoted by $Mp^c(\mathcal{Y})$. We have a short exact sequence

$$\mathbb{1} \to U(1) \to Mp^c(\mathcal{Y}) \to Sp_{\rm res}(\mathcal{Y}) \to \mathbb{1}.$$

Let us mention the following criterion, which was used in Ref. 15:

Proposition 3.9. If $R^*R - \mathbb{1}$ is Hilbert-Schmidt, then $R \in Sp_{\rm res}(\mathcal{Y})$.

Proof.

$$R^*R = \begin{bmatrix} p^*p + q^\#\overline{q} & p^*q + q^\#\overline{p} \\ p^\#\overline{q} + q^*p & p^\#\overline{p} + q^*q \end{bmatrix} = \begin{bmatrix} \mathbb{1} + 2q^\#\overline{q} & 2p^*q \\ 2p^\#\overline{q} & \mathbb{1} + 2q^*q \end{bmatrix}.$$

Now

$$\begin{aligned} \|\mathbb{1} - R^*R\|_2^2 &= 8\mathrm{Tr}q^*q + 8\mathrm{Tr}q^*qq^*q + 8\mathrm{Tr}q^*pp^*q && (3.29)\\ &= 16\mathrm{Tr}q^*qq^*q + 16\mathrm{Tr}q^*q \ \geq\ 16\mathrm{Tr}q^*q. && (3.30) \end{aligned}$$

□

$Sp_{\rm af}(\mathcal{Y})$ will denote the *anomaly-free symplectic group*, which consists of $R \in Sp(\mathcal{Y})$ such that $\mathbb{1} - p$ is a trace class.[7]

Proposition 3.10. $Sp_{\rm af}(\mathcal{Y})$ is a subgroup of $Sp_{\rm res}(\mathcal{Y})$.

Proof. We have

$$q^*q = p^*p - \mathbb{1} = (p^* - \mathbb{1})p + p - \mathbb{1}. \tag{3.31}$$

Therefore, $\|p - \mathbb{1}\|_1 < \infty$ implies $\|q\|_2 < \infty$. □

For $R \in Sp_{\rm af}(\mathcal{Y})$, we can define a pair of *metaplectic Bogoliubov implementers*

$$\pm U_R^{\rm met} := \pm(\det p^*)^{-\frac12} \mathrm{e}^{-\frac12 \hat{a}^*(d_2)} \Gamma((p^*)^{-1}) \mathrm{e}^{\frac12 \hat{a}(d_1)}. \tag{3.32}$$

They form a group, which we denote by $Mp_{\rm af}(\mathcal{Y})$.[7] We have a short exact sequence

$$\mathbb{1} \to \mathbb{Z}_2 \to Mp_{\rm af}(\mathcal{Y}) \to Sp_{\rm af}(\mathcal{Y}) \to \mathbb{1}.$$

F. Classical quadratic Hamiltonians

In this subsection, we consider strongly continuous 1-parameter groups of symplectic transformations. The following proposition describes their generators:

Proposition 3.11. Let $\mathrm{i}B$ be a generator of a 1-parameter group on $\mathcal{W} \oplus \overline{\mathcal{W}}$. The following statements are equivalent:

(1) e^{iBt}, where $t \in \mathbb{R}$, *is a strongly continuous 1-parameter group of symplectic transformations.*
(2) iB *is* J-*real*, $SB^* \supset BS$.
(3) $A_B := BS$ *is J-real and* $A_B^* \supset A_B$ *(in other words,* A_B *is Hermitian).*

Proof. We have for $w_1, w_2 \in \mathrm{Dom}(B)$,

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{e}^{\mathrm{i}tB}w_1|S\mathrm{e}^{\mathrm{i}tB}w_2)\Big|_{t=0} = -\mathrm{i}(Bw_1|Sw_2) + \mathrm{i}(w_1|SBw_2). \tag{3.33}$$

Hence preservation of S by $\mathrm{e}^{\mathrm{i}tB}$ is equivalent to $(SAS)^* = B^*S \supset SB = SAS$, which means that SAS is Hermitian. This is equivalent to A being Hermitian. □

For brevity, we will say that B is a *symplectic generator* if iB generates a one-parameter group of symplectic transformations. Similarly as in Sec. III E, $A_B := BS$ will be sometimes called the *classical Hamiltonian of* B, and we will often write A instead of A_B.

Note that in finite dimensions, the converse of Proposition 3.11 (3) is true: If A is Hermitian and J-real, then $B := AS$ is a symplectic generator. This is probably not the case in infinite dimensions.

G. Bogoliubov Hamiltonians

Let B be, as usual, a symplectic generator, and $A = BS$. We will write

$$\mathrm{e}^{\mathrm{i}tB} = \begin{bmatrix} p_t & q_t \\ \overline{q}_t & \overline{p}_t \end{bmatrix}. \tag{3.34}$$

Theorem 3.12. *The following conditions are equivalent:*

(1) *There exists a self-adjoint operator* $\hat{H}$ *on* $\Gamma_{\mathrm{s}}(\mathcal{W})$ *such that* $\mathrm{e}^{\mathrm{i}t\hat{H}}$ *implements* $\mathrm{e}^{\mathrm{i}tB}$ *for any* $t \in \mathbb{R}$.
(2) *There exists* α_t*, a 1-parameter* C_0^**-group of* $*$*-automorphisms of* $B(\Gamma_{\mathrm{s}}(\mathcal{W}))$*, such that*

$$\alpha_t(\mathrm{e}^{\mathrm{i}\hat{\phi}(w,\overline{w})}) = \mathrm{e}^{\mathrm{i}\hat{\phi}(w_t,\overline{w}_t)}, \quad \begin{bmatrix} w_t \\ \overline{w}_t \end{bmatrix} = \mathrm{e}^{\mathrm{i}tB}\begin{bmatrix} w \\ \overline{w} \end{bmatrix}, \quad w \in \mathcal{W}. \tag{3.35}$$

(3) $\lim_{t\to 0} \|q_t\|_2 = 0$.

Let (1), (2), and (3) be true. Then α_t *is determined uniquely.* $\hat{H}$ *is uniquely defined up to an additive constant.*

$\hat{H}$ will be called a *quantization of* B. We will also say that $\hat{H}$ is a *quantum quadratic Hamiltonian*, or shorter, a *Bogoliubov Hamiltonian*. If the equivalent conditions of the above theorem are satisfied, then we will say that B *possesses quantizations.*

Proof. (1) ⇔ (2) is a consequence of Proposition A.2. We need to show that (1), (2) ⇔ (3).

If $\mathrm{e}^{\mathrm{i}tB}$, $t \in \mathbb{R}$, is implementable, then $\|q_t\|_2 < \infty$, for all $t \in \mathbb{R}$.

If $\lim\limits_{t\to 0} \|q_t\|_2 = 0$, then $\|q_t\|_2 < \infty$, for small enough t. But since $Sp_{\mathrm{res}}(\mathcal{Y})$ is a group, $\|q_t\|_2 < \infty$, for all $t \in \mathbb{R}$.

Thus, in all cases (1)–(3), we can define

$$U_t^{\mathrm{nat}} := U_{\mathrm{e}^{\mathrm{i}tB}}^{\mathrm{nat}} \tag{3.36}$$

[see (3.28)]. Set

$$\alpha_t(C) := U_t^{\mathrm{nat}} C U_{-t}^{\mathrm{nat}}. \tag{3.37}$$

Clearly, $t \mapsto \alpha_t$ is a 1-parameter group of $*$-automorphisms satisfying (3.35). The proof will be completed if we show the equivalence of the following statements:

(i) $t \mapsto U_t^{\mathrm{nat}}$ is strongly continuous at zero;
(ii) $t \mapsto \alpha_t$ is a C_0^*-group of $*$-automorphisms;

(iii) $d_{2,t} := q_t \overline{p}_t^{-1}$ satisfies $\lim_{t\to 0} \|d_{2,t}\|_2 = 0$;
(iv) $\lim_{t\to 0} \|q_t\|_2 = 0$.

(i) ⇒ (ii): We easily see that if $t \mapsto U_t^{\rm nat}$ is strongly continuous at zero and if C is a bounded operator, then $t \mapsto U_t^{\rm nat} C U_{-t}^{\rm nat}$ is weakly continuous at zero. This implies that $t \mapsto \alpha_t$ is a C_0^*-group.
(ii) ⇒ (iii): Let $|\Omega)(\Omega|$ denote the orthogonal projection onto Ω. We have

$$\begin{aligned}\big(\Omega|\alpha(|\Omega)(\Omega|)\Omega\big) &= |(\Omega|U_t^{\rm nat}\Omega)|^2\\ &= |\det p_t p_t^*|^{-\frac12} = \det\left(\mathbb{1} - d_{2,t}^* d_{2,t}\right)^{\frac12} &(3.38)\\ &= \exp\Big(\frac12 \operatorname{Tr}\log\left(\mathbb{1} - d_{2,t}^* d_{2,t}\right)\Big). &(3.39)\end{aligned}$$

In (3.38), we used the identity

$$p_t^{\#-1}\overline{p}_t^{-1} = \mathbb{1} - d_{2,t}^* d_{2,t}. \tag{3.40}$$

(ii) implies that (3.39) goes to 1 for $t \to 0$. This is equivalent to $\lim_{t\to 0} \operatorname{Tr}\log(\mathbb{1} - d_{2,t}^* d_{2,t}) = 0$, which is equivalent to $\lim_{t\to 0} \operatorname{Tr} d_{2,t}^* d_{2,t} = 0$.
(iii) ⇒ (i): We have

$$U_t^{\rm nat} e^{i\hat\phi(w,\overline{w})}\Omega = e^{i\hat\phi(w_t,\overline{w}_t)} |\det p_t p_t^*|^{-\frac14} e^{-\frac12 a^*(d_{2,t})}\Omega. \tag{3.41}$$

But $t \mapsto e^{i\hat\phi(w_t,\overline{w}_t)}$ is strongly continuous. By (3.40), $\lim_{t\to 0} |\det p_t p_t^*|^{-\frac14} = 1$. Besides, (iii) implies that $\lim_{t\to 0} e^{-\frac12 a^*(d_{2,t})}\Omega = \Omega$. Therefore, (3.41) is continuous at $t = 0$. But the span of $e^{i\hat\phi(w,\overline{w})}\Omega$ is dense and $U_t^{\rm nat}$ is unitary. Hence $U_t^{\rm nat}$ is strongly continuous at $t = 0$.
(iii) ⇔ (iv) follows from the identities

$$\begin{aligned} q_t q_t^* &= d_{2,t}^* d_{2,t}\left(\mathbb{1} - d_{2,t}^* d_{2,t}\right)^{-1}, &(3.42)\\ d_{2,t}^* d_{2,t} &= q_t q_t^*\left(\mathbb{1} + q_t q_t^*\right)^{-1}. &(3.43)\end{aligned}$$

□

Below we describe three distinguished quantizations.

(1) If the group $e^{it\hat H}$ implementing e^{itB} is contained in $Mp_{\rm af}(\mathcal{Y})$, then $\hat H$ will be called *Weyl*. It is easy to see that for a given symplectic generator B, its *Weyl quantization*, if it exists, is unique. We will denote it by $\hat H_B^{\rm w}$. An alternative name for $\hat H_B^{\rm w}$ is the *symmetric quantization of B.*
(2) We say that a quantization $\hat H$ of B is *normally ordered* if

$$\frac{\rm d}{{\rm d}t}(\Omega|e^{it\hat H}\Omega)\Big|_{t=0} = 0. \tag{3.44}$$

Again, a given symplectic generator B possesses at most one *normally ordered quantization*. We will denote it by $\hat H_B^{\rm n}$. An alternative name for $\hat H_B^{\rm n}$ is the *Wick quantization of B*.
(3) If B possesses a quantization, which is bounded from below, then all of its quantizations are bounded from below. Then one can introduce the *zero-infimum quantization* $\hat H_B^{\rm z}$ fixed by the condition

$$\inf \hat H_B^{\rm z} = 0.$$

Let us stress that there exist B that possess quantizations, but they do not possess $\hat H_B^{\rm w}$, $\hat H_B^{\rm n}$, or $\hat H_B^{\rm z}$.

We will usually drop the subscript B in the above symbols.

Note that whereas the definitions of $\hat H^{\rm w}$ and $\hat H^{\rm z}$ are quite obvious, it is less clear how to generalize the concept of normally ordered Bogoliubov Hamiltonian to infinite dimensions. In the following proposition, we formulate another condition, which could be considered as another candidate for a definition of $\hat H^{\rm n}$.

Proposition 3.13. Suppose that B possesses a quantization $\hat{H}$ such that $\Omega \in \mathrm{Dom}(|\hat{H}|^{\frac{1}{2}})$ (the vacuum belongs to the form domain of $\hat{H}$). Then B possesses the normally ordered quantization.

Proof. We easily check that

$$\hat{H}^{\mathrm{n}} := \hat{H} - (\Omega|\hat{H}\Omega) \tag{3.45}$$

satisfies (3.44). □

Theorem 3.14. *Consider (3.34).*

(1) *The condition*

$$\lim_{t\to 0} \|p_t - \mathbb{1}\|_1 = 0 \tag{3.46}$$

is equivalent to B possessing the Weyl quantization $\hat{H}^{\mathrm{w}}$. If this is the case, then

$$\mathrm{e}^{\mathrm{i}t\hat{H}^{\mathrm{w}}} = (\det p_t^*)^{-\frac{1}{2}} \mathrm{e}^{-\frac{1}{2}\hat{a}^*(d_{2,t})} \Gamma((p_t^*)^{-1}) \mathrm{e}^{\frac{1}{2}\hat{a}(d_{1,t})}, \tag{3.47}$$

where the sign of the square root is determined by continuity.

(2) *Suppose that there exists a self-adjoint operator h on $\mathcal{W}$ such that*

$$\lim_{t\to 0} \frac{\|\mathrm{e}^{-\mathrm{i}th} p_t - \mathbb{1}\|_1}{t} = 0. \tag{3.48}$$

Then B possesses the normally ordered quantization $\hat{H}^{\mathrm{n}}$ and

$$\mathrm{e}^{\mathrm{i}t\hat{H}^{\mathrm{n}}} = (\det p_t^* \mathrm{e}^{\mathrm{i}th})^{-\frac{1}{2}} \mathrm{e}^{-\frac{1}{2}\hat{a}^*(d_{2,t})} \Gamma((p_t^*)^{-1}) \mathrm{e}^{\frac{1}{2}\hat{a}(d_{1,t})}, \tag{3.49}$$

where the sign of the square root is determined by continuity. The operator h that appears in (3.48) is uniquely defined.

(3) *Suppose that the assumptions of (2) hold. In addition, assume that h in (3.48) is a trace class. Then B possesses both normally ordered and Weyl quantizations, and*

$$\hat{H}^{\mathrm{n}} + \frac{1}{2}\mathrm{Tr}h = \hat{H}^{\mathrm{w}}. \tag{3.50}$$

Proof. (1): $\lim_{t\to 0} \|\mathbb{1} - p_t\|_1 = 0$ implies that $\mathrm{e}^{\mathrm{i}tB} \in Sp_{\mathrm{af}}(\mathcal{Y})$ at least for small t. But $Sp_{\mathrm{af}}(\mathcal{Y})$ is a group. Therefore, $\mathrm{e}^{\mathrm{i}tB} \in Sp_{\mathrm{af}}(\mathcal{Y})$ for all $t \in \mathbb{R}$.

Besides, $t \mapsto \mathrm{e}^{\mathrm{i}tB}$ is continuous in the topology of $Sp_{\mathrm{af}}(\mathcal{Y})$ at zero. By the group property of $Sp_{\mathrm{af}}(\mathcal{Y})$, it is continuous for all $t \in \mathbb{R}$.

Hence, U_t^{met} given by (3.47) is well defined. U_t^{met} obviously is one of the metaplectic implementers of $\mathrm{e}^{\mathrm{i}tB}$. We have

$$(\Omega|U_t^{\mathrm{met}}\Omega) = (\det p_t^*)^{-\frac{1}{2}}, \tag{3.51}$$

which depends continuously on t.

Using that $Mp(\mathcal{Y})$ is a group and the continuity of (3.51), we see that U_t^{met} satisfies the group property. Next, repeating the argument of the proof of Theorem 3.12, we see that U_t^{met} is continuous on coherent vectors.

Thus U_t^{met} is a strongly continuous group of metaplectic implementers of $\mathrm{e}^{\mathrm{i}tB}$. Hence B possesses the Weyl quantization.

Conversely, if B possesses the Weyl quantization $\hat{H}^{\mathrm{w}}$, then $U_t^{\mathrm{met}} = \mathrm{e}^{\mathrm{i}t\hat{H}^{\mathrm{w}}}$. Then (3.51) is true. But $\lim_{t\to 0} \|\mathbb{1} - p_t\|_1 = 0$ is equivalent to the continuity of the rhs of (3.51).

(2): (3.48) implies

$$\lim_{t\to 0} \|\mathrm{e}^{-\mathrm{i}th} p_t - \mathbb{1}\|_1 = 0. \tag{3.52}$$

Therefore, the identity

$$q_t^* q_t = p_t^* p_t - \mathbb{1} = (p_t^* \mathrm{e}^{\mathrm{i}th} - \mathbb{1})\mathrm{e}^{-\mathrm{i}th} p_t + \mathrm{e}^{-\mathrm{i}th} p_t - \mathbb{1} \tag{3.53}$$

shows that $\lim_{t\to 0}\|q_t\|_2 = 0$. Therefore, (3.49) is well defined and depends continuously on t. Clearly,

$$|\det p_t^* \mathrm{e}^{\mathrm{i}th}|^2 = \det p_t^* p_t. \tag{3.54}$$

Hence, (3.49) differs from $U^{\mathrm{nat}}_{\mathrm{e}^{\mathrm{i}tB}}$ by a phase factor. We check by direct calculation that it satisfies the group property.[6]

Using (3.52) and the differentiability of the determinant in the trace norm, for small enough t we have

$$|\det p_t^* \mathrm{e}^{\mathrm{i}th} - 1| \le c\|p_t^* \mathrm{e}^{\mathrm{i}th} - \mathbb{1}\|_1. \tag{3.55}$$

Hence (3.48) implies

$$\lim_{t\to 0} \frac{\det p_t^* \mathrm{e}^{\mathrm{i}th} - 1}{t} = 0. \tag{3.56}$$

Therefore, using also (3.52),

$$\lim_{t\to 0} \frac{(\det p_t^* \mathrm{e}^{\mathrm{i}th})^{-\frac{1}{2}} - 1}{t} = 0. \tag{3.57}$$

By (3.49),

$$(\Omega|\mathrm{e}^{\mathrm{i}t\hat{H}^{\mathrm{n}}}\Omega) = (\det p_t^* \mathrm{e}^{\mathrm{i}th})^{-\frac{1}{2}}. \tag{3.58}$$

Hence, (3.44) is true.

Suppose that for h_1 and h_2 we have (3.48). Let $w, w' \in \mathcal{W}$ be normalized. Then

$$\frac{1}{t}|(w|\mathrm{e}^{\mathrm{i}th_1}w') - (w|\mathrm{e}^{\mathrm{i}th_2}w')| \le \frac{1}{t}\|\mathrm{e}^{\mathrm{i}th_1} - \mathrm{e}^{\mathrm{i}th_2}\| \tag{3.59}$$

$$\le \frac{1}{t}\|\mathrm{e}^{\mathrm{i}th_1} - \mathrm{e}^{\mathrm{i}th_2}\|_1 \tag{3.60}$$

$$\le \frac{1}{t}\|\mathrm{e}^{\mathrm{i}th_1} - p_t\|_1 + \frac{1}{t}\|p_t - \mathrm{e}^{\mathrm{i}th_2}\|_1 \to 0. \tag{3.61}$$

Hence $h_1 = h_2$ by Lemma A.4.

(3): Using $\|h\|_1 < \infty$, we can write

$$\det p_t = \det \mathrm{e}^{\mathrm{i}th} \det \mathrm{e}^{-\mathrm{i}th} p_t = \mathrm{e}^{\mathrm{i}t\mathrm{Tr}h} \det \mathrm{e}^{-\mathrm{i}th} p_t. \tag{3.62}$$

Thus we see that both (3.47) and (3.49) are well defined and

$$\mathrm{e}^{\mathrm{i}t\hat{H}^{\mathrm{w}}} = \mathrm{e}^{\mathrm{i}t\frac{1}{2}\mathrm{Tr}h}\mathrm{e}^{\mathrm{i}t\hat{H}^{\mathrm{n}}}. \tag{3.63}$$

□

H. Criteria for existence of quantizations of classical Hamiltonians

In this subsection, we restrict our study to symplectic generators that are bounded perturbations of diagonal symplectic generators.

We will always assume that h is a self-adjoint operator on $\mathcal{W}$ and $g = g^\#$. Besides,

$$B := \begin{bmatrix} h & -g \\ \overline{g} & -\overline{h} \end{bmatrix}, \qquad B_0 := \begin{bmatrix} h & 0 \\ 0 & -\overline{h} \end{bmatrix}, \tag{3.64}$$

$$A = BS = \begin{bmatrix} h & g \\ \overline{g} & \overline{h} \end{bmatrix}, \qquad A_0 = B_0 S = \begin{bmatrix} h & 0 \\ 0 & \overline{h} \end{bmatrix}, \quad G = \begin{bmatrix} 0 & -g \\ \overline{g} & 0 \end{bmatrix}. \tag{3.65}$$

The following proposition is immediate:

Proposition 3.15. If g is bounded, then B is a symplectic generator. Besides, A is self-adjoint.

Proof. Clearly, B_0 is a symplectic generator and A_0 is self-adjoint. We can add a bounded perturbation without destroying these properties. □

The following theorem is a slightly strengthened version of a criterion due to Berezin;[2] see also Ref. 6. Throughout this subsection we set

$$f(t) := \int_0^t \mathrm{e}^{\mathrm{i}sh} g \mathrm{e}^{\mathrm{i}s\bar{h}} \mathrm{d}s. \tag{3.66}$$

Theorem 3.16. (1) *Suppose that g is bounded and $\lim_{t\to 0} \|f(t)\|_2 = 0$. Then B possesses quantizations.*
(2) *In addition to assumptions of (1) suppose that $\lim_{t\to 0} \|\bar{g}f(t)\|_1 = 0$. Then B possesses the normally ordered quantization.*
(3) *In addition to assumptions of (2) suppose that $\|h\|_1 < \infty$. Then B possesses both the Weyl and the normally ordered quantizations, and*

$$\hat{H}^{\mathrm{n}} + \frac{1}{2}\mathrm{Tr}h = \hat{H}^{\mathrm{w}}. \tag{3.67}$$

Proof. (1): Using repeatedly the identity

$$f(2t) = f(t) + \mathrm{e}^{\mathrm{i}th} f(t) \mathrm{e}^{\mathrm{i}t\bar{h}}, \tag{3.68}$$

we see that $\|f(t)\|_2$ is finite for all t.

Set

$$V(t) := \mathrm{e}^{\mathrm{i}tB}\mathrm{e}^{-\mathrm{i}tB_0}, \tag{3.69}$$

$$G(t) := \mathrm{e}^{\mathrm{i}tB_0} G \mathrm{e}^{-\mathrm{i}tB_0} = \begin{bmatrix} 0 & -\mathrm{e}^{\mathrm{i}sh} g \mathrm{e}^{\mathrm{i}s\bar{h}} \\ \mathrm{e}^{-\mathrm{i}s\bar{h}} \bar{g} \mathrm{e}^{-\mathrm{i}sh} & 0 \end{bmatrix}, \tag{3.70}$$

$$F(t) := \int_0^t G(s)\mathrm{d}s = \begin{bmatrix} 0 & -f(t) \\ \overline{f(t)} & 0 \end{bmatrix}. \tag{3.71}$$

From

$$V(t) = \mathbb{1} + \mathrm{i}\int_0^t V(s)G(s)\mathrm{d}s \tag{3.72}$$

and $\|G(t)\| = \|G\|$, we obtain

$$\|V(t)\| \le \mathrm{e}^{|t|\,\|G\|}. \tag{3.73}$$

Iterating (3.72) gives

$$V(t) = \mathbb{1} + \mathrm{i}F(t) - \int_0^t V(s)G(s)\mathrm{e}^{\mathrm{i}sB_0}F(t-s)\mathrm{e}^{-\mathrm{i}sB_0}\mathrm{d}s. \tag{3.74}$$

Therefore,

$$\|V(t) - \mathbb{1}\|_2 \le \|F(t)\|_2 + \int_0^t \|V(s)\|\|G\|\|F(t-s)\|_2 \mathrm{d}s. \tag{3.75}$$

But $\|F(t)\|_2 = \sqrt{2}\|f(t)\|_2$ and $\|G\| = \|g\|$. Hence $\|V(t) - \mathbb{1}\|_2$ is finite and goes to zero as $t \to 0$. Arguing as in Proposition 3.9, we obtain

$$16\|q(t)\|_2^2 \le \|V(t) - \mathbb{1}\|_2. \tag{3.76}$$

Therefore, $\|q(t)\|_2$ is finite and goes to zero as $t \to 0$. This means that the assumption of Theorem 3.12 (3) is satisfied. Hence B possesses quantizations.

(2): We rewrite (3.74) as

$$\begin{bmatrix} p_t - \mathrm{e}^{\mathrm{i}th} & q_t + \mathrm{i}f(t)\mathrm{e}^{-\mathrm{i}t\bar{h}} \\ \bar{q}_t - \mathrm{i}\overline{f(t)}\mathrm{e}^{\mathrm{i}th} & \bar{p}_t - \mathrm{e}^{-\mathrm{i}th} \end{bmatrix} = \mathrm{e}^{\mathrm{i}tB} - \mathrm{e}^{\mathrm{i}tB_0} - \mathrm{i}F(t)\mathrm{e}^{\mathrm{i}tB_0} \tag{3.77}$$

$$= -\int_0^t V(s)G(s)\mathrm{e}^{\mathrm{i}sB_0}F(t-s)\mathrm{e}^{\mathrm{i}(t-s)B_0}\mathrm{d}s. \tag{3.78}$$

Therefore, by (A9),

$$2\|p_t - \mathrm{e}^{\mathrm{i}th}\|_1 \le \|\mathrm{e}^{\mathrm{i}tB} - \mathrm{e}^{\mathrm{i}tB_0} - \mathrm{i}F(t)\mathrm{e}^{\mathrm{i}tB_0}\|_1 \tag{3.79}$$

$$\le \int_0^t \|V(s)\|\|GF(t-s)\mathrm{e}^{\mathrm{i}(t-s)B_0}\|_1 \mathrm{d}s. \tag{3.80}$$

Using $\|GF(t)\|_1 = 2\|\overline{g}f(t)\|_1$ and $\lim\limits_{t\to 0}\|\overline{g}f(t)\|_1 = 0$, we see that (3.80) is $o(t)$. Thus we obtain $\|p_t - \mathrm{e}^{\mathrm{i}th}\|_1 = o(t)$. This means that the assumption of Theorem 3.14 (2) is satisfied. Hence, B possesses the normally ordered quantization.

(3): We apply Theorem 3.14 (3). □

The assumptions of Theorem 3.16 are not very convenient to verify. Our next aim is to formulate criteria for the existence of quantizations, which are more convenient to check.

Define

$$\gamma(g) := (h\otimes \mathbb{1} + \mathbb{1}\otimes h)^{-1} g, \tag{3.81}$$

where we use the tensor interpretation of g and assume that $g \in \mathrm{Dom}(h\otimes \mathbb{1} + \mathbb{1}\otimes h)^{-1}$.

Proposition 3.17. In the operator interpretation, $\gamma(g)$ corresponds to

$$\gamma(g) = \mathrm{i}\lim_{\epsilon\searrow 0}\int_0^\infty \mathrm{e}^{-\epsilon t}\mathrm{e}^{-\mathrm{i}th} g \mathrm{e}^{-\mathrm{i}t\overline{h}}\mathrm{d}t \tag{3.82}$$

and satisfies

$$h\gamma(g) + \gamma(g)\overline{h} = g. \tag{3.83}$$

For $h > 0$ we can "Wick rotate" the formula (3.82) and write

$$\gamma(g) = \int_0^\infty \mathrm{e}^{-th} g \mathrm{e}^{-t\overline{h}}\mathrm{d}t. \tag{3.84}$$

Proof. By Proposition 3.5 and $\overline{h} = h^{\#}$, we can identify the operator $\mathrm{e}^{-\mathrm{i}th} g \mathrm{e}^{-\mathrm{i}t\overline{h}}$ with the tensor

$$\mathrm{e}^{-\mathrm{i}th}\otimes \mathrm{e}^{-\mathrm{i}th} g = \mathrm{e}^{-\mathrm{i}t(h\otimes\mathbb{1}+\mathbb{1}\otimes h)} g. \tag{3.85}$$

Clearly,

$$\mathrm{i}\int_0^\infty \mathrm{e}^{-\epsilon t}\mathrm{e}^{-\mathrm{i}t(h\otimes\mathbb{1}+\mathbb{1}\otimes h)} g = (h\otimes \mathbb{1} + \mathbb{1}\otimes h - \mathrm{i}\epsilon)^{-1} g \underset{\epsilon\searrow 0}{\to} (h\otimes \mathbb{1} + \mathbb{1}\otimes h)^{-1} g, \tag{3.86}$$

where we use the usual Hilbert space convergence, which proves (3.82).

Set

$$\gamma_\epsilon(g) := \mathrm{i}\int_0^\infty \mathrm{e}^{-\epsilon s}\mathrm{e}^{-\mathrm{i}sh} g \mathrm{e}^{-\mathrm{i}s\overline{h}}\mathrm{d}s. \tag{3.87}$$

We compute

$$\mathrm{e}^{-\mathrm{i}th}\gamma_\epsilon(g)\mathrm{e}^{-\mathrm{i}t\overline{h}} = \mathrm{i}\int_t^\infty \mathrm{e}^{-\epsilon(s-t)}\mathrm{e}^{-\mathrm{i}sh} g \mathrm{e}^{-\mathrm{i}s\overline{h}}\mathrm{d}s \tag{3.88}$$

$$= -\mathrm{i}\int_0^t \mathrm{e}^{-\mathrm{i}sh} g \mathrm{e}^{-\mathrm{i}s\overline{h}}\mathrm{e}^{-\epsilon s}\mathrm{d}s + \mathrm{e}^{\epsilon t}\gamma_\epsilon(g). \tag{3.89}$$

We differentiate with respect to t at $t = 0$, obtaining

$$-\mathrm{i}(h\gamma_\epsilon(g) + \gamma_\epsilon(g)\overline{h}) = -\mathrm{i}g + \epsilon\gamma_\epsilon(g). \tag{3.90}$$

Taking the limit as $\epsilon \searrow 0$, we obtain (3.83).

The proof of (3.84) is almost the same as that of (3.82). □

We will also write

$$\gamma(G) := \begin{bmatrix} 0 & -\gamma(g) \\ \overline{\gamma(g)} & 0 \end{bmatrix}. \tag{3.91}$$

Note that in the operator interpretation, we have

$$[B_0, \gamma(G)] = G. \tag{3.92}$$

The following criterion is a consequence of Theorem 3.16.

Theorem 3.18. (1) *Suppose that g is bounded and $g = g_1 + g_2$, where $\|g_1\|_2 < \infty$ and $\|\gamma(g_2)\|_2 < \infty$. Then the assumptions of Theorem 3.16 are satisfied, and hence B possesses quantizations.*
(2) *Suppose that $\|g\|_2 < \infty$. Then B possesses the normally ordered quantization.*
(3) *Suppose that $\|h\|_1 < \infty$ and $\|g\|_2 < \infty$. Then B possesses both the Weyl and the normally ordered quantizations. Besides,*

$$\hat{H}^{\mathrm{w}} = \hat{H}^{\mathrm{n}} + \mathrm{Tr}h. \tag{3.93}$$

Proof. (1): Set

$$f_i(t) := \int_0^t \mathrm{e}^{\mathrm{i}sh} g_i \mathrm{e}^{\mathrm{i}s\bar{h}} \mathrm{d}s. \tag{3.94}$$

It is clear that $\lim_{t\to 0} \|f_1(t)\|_2 = 0$. The fact that $\lim_{t\to 0} \|f_2(t)\|_2 = 0$ follows from

$$\int_0^t \mathrm{e}^{\mathrm{i}sh} g_2 \mathrm{e}^{\mathrm{i}s\bar{h}} \mathrm{d}s = -\mathrm{i}\int_0^t \frac{\mathrm{d}}{\mathrm{d}s}\mathrm{e}^{\mathrm{i}sh}\gamma(g_2)\mathrm{e}^{\mathrm{i}s\bar{h}}\mathrm{d}s = -\mathrm{i}\mathrm{e}^{\mathrm{i}th}\gamma(g_2)\mathrm{e}^{\mathrm{i}t\bar{h}} + \mathrm{i}\gamma(g_2), \tag{3.95}$$

where we used (3.83). Hence assumptions of Theorem 3.16 (1) are satisfied.

(2): Clearly, $\|\bar{g}f(t)\|_1 \le t\|g\|_2^2$. Hence assumptions of Theorem 3.16 (2) are satisfied.

Now (3) follows immediately from Theorem 3.16 (3). □

The following is another condition that implies the assumptions of (1) of Theorem 3.18 and hence the existence of quantizations of B:

Proposition 3.19. $h > 0$ and $\|h^{-\frac{1}{2}} g \bar{h}^{-\frac{1}{2}}\|_2 < \infty$ implies $\|\gamma(g)\|_2 < \infty$.

Proof. $h^{-\frac{1}{2}} g \bar{h}^{-\frac{1}{2}}$ corresponds to $h^{-\frac{1}{2}} \otimes h^{-\frac{1}{2}} g$ in the tensor interpretation. Clearly,

$$2h \otimes h \le (h \otimes \mathbb{1} + \mathbb{1} \otimes h)^2. \tag{3.96}$$

Hence,

$$h^{-1} \otimes h^{-1} \ge 2(h \otimes \mathbb{1} + \mathbb{1} \otimes h)^{-2}. \tag{3.97}$$

Therefore,

$$\|h^{-\frac{1}{2}} g \bar{h}^{-\frac{1}{2}}\|_2 = \|h^{-\frac{1}{2}} \otimes h^{-\frac{1}{2}} g\| \ge \sqrt{2}\|(h \otimes \mathbb{1} + \mathbb{1} \otimes h)^{-1} g\| = \sqrt{2}\|\gamma(g)\|_2. \tag{3.98}$$

□

I. Positive classical Hamiltonians and their diagonalization

The following theorem is an extension of Theorem 2.3 to arbitrary dimensions. It says that a large class of classical Hamiltonians can be diagonalized by a positive symplectic transformation. This theorem is implicitly contained in Ref. 7 [see Theorem 11.20 (3) together with Theorem 18.5 (3)]. Reference 15 contains also a related result about the diagonalizability of classical Hamiltonians. It does not provide, however, a construction of a distinguished diagonalizing operator.

We will use the notation introduced in (3.64) and (3.65). We will assume that $h > 0$. It will not be necessary to assume that g is bounded—we will assume that $g = g^{\#}$ is a bilinear form with the right domain $\mathrm{Dom}|\bar{h}|^{\frac{1}{2}}$ and the left domain $\mathrm{Dom}|h|^{\frac{1}{2}}$.

Theorem 3.20. *Let h be positive and*

$$\|h^{-\frac{1}{2}} g \overline{h}^{-\frac{1}{2}}\| =: a < 1. \tag{3.99}$$

Then A is a positive self-adjoint operator with the form domain $\mathrm{Dom} A_0^{\frac{1}{2}}$. *The corresponding B is a symplectic generator.*

Besides,

$$R_0 = SA^{-\frac{1}{2}}(A^{\frac{1}{2}} SASA^{\frac{1}{2}})^{\frac{1}{2}} A^{-\frac{1}{2}} S \tag{3.100}$$

is a bounded invertible positive symplectic operator, so is

$$R = R_0^{\frac{1}{2}}. \tag{3.101}$$

R diagonalizes B and A, that is, for some positive self-adjoint h_{dg}

$$B = R \begin{bmatrix} h_{\mathrm{dg}} & 0 \\ 0 & -\overline{h_{\mathrm{dg}}} \end{bmatrix} R^{-1}, \tag{3.102}$$

$$A = R \begin{bmatrix} h_{\mathrm{dg}} & 0 \\ 0 & \overline{h_{\mathrm{dg}}} \end{bmatrix} R^{*}. \tag{3.103}$$

Moreover,

$$\frac{(1-a)^{\frac{1}{4}}}{(1+a)^{\frac{1}{4}}} \le \|R\| \le \frac{(1+a)^{\frac{1}{4}}}{(1-a)^{\frac{1}{4}}}. \tag{3.104}$$

Proof. GS is a form bounded perturbation of A_0,

$$|(v|GSv)| \le a(v|A_0 v), \quad v \in \mathrm{Dom}(A_0^{\frac{1}{2}}).$$

Therefore, A extends to a positive self-adjoint operator by the KLMN theorem.

$A = A_0 + GS$ satisfies

$$(1-a)A_0 \le A \le (1+a)A_0. \tag{3.105}$$

Similarly, $SAS = A_0 + SG$ extends to a positive operator satisfying

$$(1-a)A_0 \le SAS \le (1+a)A_0. \tag{3.106}$$

Therefore,

$$A^{\frac{1}{2}} SASA^{\frac{1}{2}} \ge (1-a)A^{\frac{1}{2}} A_0 A^{\frac{1}{2}} \ge \frac{(1-a)}{(1+a)} A^2, \tag{3.107}$$

$$A^{\frac{1}{2}} SASA^{\frac{1}{2}} \le (1+a)A^{\frac{1}{2}} A_0 A^{\frac{1}{2}} \le \frac{(1+a)}{(1-a)} A^2. \tag{3.108}$$

Hence,

$$\frac{\sqrt{1-a}}{\sqrt{1+a}} A \le (A^{\frac{1}{2}} SASA^{\frac{1}{2}})^{\frac{1}{2}} \le \frac{\sqrt{1+a}}{\sqrt{1-a}} A. \tag{3.109}$$

Thus R_0, defined by (3.100), is a well-defined bounded invertible positive operator, so is R.

Repeating the arguments of the proof of Theorem 2.3, we obtain (3.102) and (3.103). By (3.102), we have

$$\mathrm{e}^{\mathrm{i}tB} = R \begin{bmatrix} \mathrm{e}^{\mathrm{i}t h_{\mathrm{dg}}} & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}t\overline{h_{\mathrm{dg}}}} \end{bmatrix} R^{-1}. \tag{3.110}$$

(3.110) is clearly symplectic. Hence B is a symplectic generator. □

For further use, we note that we can rewrite (3.100) as follows:

$$R_0 = SA^{-\frac12}\Big(\int \frac{\tau^2}{(\tau^2 + A^{\frac12}SASA^{\frac12})}\frac{\mathrm{d}\tau}{2\pi}\Big)A^{-\frac12}S.$$

As a side remark, note that $h > 0$ and (3.99) do not only imply $A > 0$, but the converse implication is "almost true." More precisely, set $\mathcal{W}_0 := (\mathrm{Ker}h)^{\perp}$. Then $A \geq 0$ is equivalent to the following conditions:

(1) $h \geq 0$,
(2) $\mathrm{Ker}g \supset \mathcal{W}_0^{\perp}$ (and hence, since $g = g^{\#}$, we have $\mathrm{Ran}g \subset \mathcal{W}_0$),
(3) $\|h^{-\frac12} g \overline{h}^{-\frac12}\| \leq 1$, in the sense of operators from $\mathcal{W}_0$.

J. Implementable diagonalizability of positive Hamiltonians

The following theorem is due to Ref. 15. The proof that we present below follows closely that of Ref. 15, with only minor modifications.

Theorem 3.21. *In addition to the assumptions of Theorem 3.20, suppose that*

$$\|h^{-\frac12} g \overline{h}^{-\frac12}\|_2 < \infty. \tag{3.111}$$

Let R be the symplectic operator given by Theorem 3.20 and q be given by (3.24). Then

$$\|q\|_2 \leq 2\frac{1}{(1-a)}\|h^{-\frac12} g \overline{h}^{-\frac12}\|_2. \tag{3.112}$$

In particular, $R \in Sp_{\mathrm{res}}(\mathcal{Y})$ and hence R is implementable.

Let us note that, by Proposition 3.19, $h > 0$ and (3.111) imply the assumptions of (1) of Theorem 3.18 about the existence of quantizations. Therefore, we already know that the assumptions of Theorem 3.21 imply the existence of quantizations of B. However, Theorem 3.21 implies that these quantizations have some important additional properties: e.g., they are bounded from below and possess a ground state.

In fact, R possesses a Bogoliubov implementer U. If h_{dg} is given by (3.110), then

$$U\mathrm{d}\Gamma(h_{\mathrm{dg}})U^* \tag{3.113}$$

is the zero-infimum quantization of B, where obviously $\mathrm{d}\Gamma(h_{\mathrm{dg}})$ possesses a ground state.

Proof of Theorem 3.21. We start from estimating $R^*R - \mathbb{1} = R^2 - \mathbb{1} = R_0 - \mathbb{1}$. We have

$$\begin{aligned}
&S(R_0 - \mathbb{1})S \\
&= A^{-\frac12}(A^{\frac12}SASA^{\frac12})^{\frac12}A^{-\frac12} - A^{-\frac12}(A^2)^{\frac12}A^{-\frac12} \\
&= \int \frac{\mathrm{d}\tau}{\pi}A^{-\frac12}\Big(\frac{A^{\frac12}SASA^{\frac12}}{\tau^2 + A^{\frac12}SASA^{\frac12}} - \frac{A^2}{\tau^2 + A^2}\Big)A^{-\frac12} \\
&= -\int \frac{\tau^2\mathrm{d}\tau}{\pi}A^{-\frac12}\Big(\frac{1}{\tau^2 + A^{\frac12}SASA^{\frac12}} - \frac{1}{\tau^2 + A^2}\Big)A^{-\frac12} \\
&= \int \frac{\tau^2\mathrm{d}\tau}{\pi}A^{\frac12}\frac{1}{\tau^2 + A^{\frac12}SASA^{\frac12}}A^{-\frac12}(SAS - A)\frac{1}{\tau^2 + A^2} \\
&=: \int \frac{\tau^2\mathrm{d}\tau}{\pi}T(\tau).
\end{aligned}$$

Now, for any $\epsilon > 0$,

$$
\begin{aligned}
&\pm 2T(\tau)\\
&= \pm 2A^{-\frac{1}{2}}\frac{1}{\tau^2+A^{\frac{1}{2}}SASA^{\frac{1}{2}}}A^{\frac{1}{2}}(SAS-A)\frac{1}{\tau^2+A^2}\\
&\leq \epsilon^{-1}\frac{1}{\tau^2+A^2}(SAS-A)A^{\frac{1}{2}}\frac{1}{A^{\frac{1}{2}}SASA^{\frac{1}{2}}}A^{\frac{1}{2}}(SAS-A)\frac{1}{\tau^2+A^2}\\
&\quad + \epsilon A^{-\frac{1}{2}}\frac{A^{\frac{1}{2}}SASA^{\frac{1}{2}}}{\tau^2+A^{\frac{1}{2}}SASA^{\frac{1}{2}}}A^{-\frac{1}{2}}\\
&=: \epsilon^{-1}T_1(\tau)+\epsilon T_2(\tau).
\end{aligned}
$$

We deal with the second term

$$
\begin{aligned}
&\int \frac{\tau^2 \mathrm{d}\tau}{\pi}T_2(\tau)\\
&= \int \frac{\tau^2 \mathrm{d}\tau}{\pi}A^{-\frac{1}{2}}\frac{A^{\frac{1}{2}}SASA^{\frac{1}{2}}}{\tau^2+A^{\frac{1}{2}}SASA^{\frac{1}{2}}}A^{-\frac{1}{2}}\\
&= A^{-\frac{1}{2}}(A^{\frac{1}{2}}SASA^{\frac{1}{2}})^{\frac{1}{2}}A^{-\frac{1}{2}}\\
&= R_0.
\end{aligned}
$$

Next we treat the first term

$$
\begin{aligned}
K &:= \int \frac{\tau^2 \mathrm{d}\tau}{\pi}T_1(\tau)\\
&= \int \frac{\tau^2 \mathrm{d}\tau}{\pi}\frac{1}{\tau^2+A^2}(SAS-A)S\frac{1}{A}S(SAS-A)\frac{1}{\tau^2+A^2}.
\end{aligned}
$$

We have

$$
\begin{aligned}
\mathrm{Tr}K &= \int \frac{\tau^2 \mathrm{d}\tau}{\pi}\frac{1}{(\tau^2+A^2)^2}(SAS-A)S\frac{1}{A}S(SAS-A) && (3.114)\\
&= \frac{1}{2}\mathrm{Tr}\frac{1}{A}(SAS-A)S\frac{1}{A}S(SAS-A) && (3.115)\\
&\leq \frac{1}{2(1-a)^2}\mathrm{Tr}\frac{1}{A_0}(SAS-A)\frac{1}{A_0}(SAS-A) && (3.116)\\
&= 4\frac{1}{(1-a)^2}\mathrm{Tr}\overline{h}^{-1}\overline{g}h^{-1}g. && (3.117)
\end{aligned}
$$

Thus we have proved that

$$\pm 2S(R_0-\mathbb{1})S \leq \epsilon^{-1}K+\epsilon SR_0S, \tag{3.118}$$

where K is positive operator with a trace bounded by (3.117). We rewrite (3.118) with sign + as

$$(2-\epsilon)S(R_0-\mathbb{1})S \leq \epsilon^{-1}K+\epsilon. \tag{3.119}$$

Let $s_n(C)$ denote the nth singular value of an operator C, that means, the nth eigenvalue of $|C| := \sqrt{C^*C}$ in the descending order. We will write for brevity $\lambda_n := s_n(|q|^2)$.

Using U defined in (2.21), we have

$$R_0-\mathbb{1} = R^2-\mathbb{1} = 2U\begin{bmatrix} u(|q|^2+|q|\sqrt{\mathbb{1}+|q|^2})u^* & 0\\ 0 & |q|^2-|q|\sqrt{\mathbb{1}+|q|^2}\end{bmatrix}U^*. \tag{3.120}$$

Therefore,

$$s_n(R_0-\mathbb{1}) = 2(\lambda_n+\sqrt{\lambda_n+\lambda_n^2}). \tag{3.121}$$

Thus,

$$2(2-\epsilon)(\lambda_n+\sqrt{\lambda_n+\lambda_n^2}) \leq \epsilon^{-1}s_n(K)+\epsilon. \tag{3.122}$$

Let c be an arbitrary positive number. Let

$$\lambda_n \le c. \tag{3.123}$$

Clearly, (3.122) implies

$$2(2-\epsilon)\sqrt{\lambda_n} \le \epsilon^{-1} s_n(K) + \epsilon. \tag{3.124}$$

Taking into account (3.123), we obtain

$$4\sqrt{\lambda_n} \le \epsilon^{-1} s_n(K) + \epsilon(1 + 2\sqrt{c}). \tag{3.125}$$

Optimizing with respect to ϵ, we obtain

$$4\sqrt{\lambda_n} \le 2\sqrt{s_n(K)}\sqrt{1 + 2\sqrt{c}}. \tag{3.126}$$

Hence,

$$\lambda_n \le s_n(K)\frac{1 + 2\sqrt{c}}{4}. \tag{3.127}$$

Let

$$c \le \lambda_n. \tag{3.128}$$

Clearly, (3.122) implies

$$4(2-\epsilon)\lambda_n \le \epsilon^{-1} s_n(K) + \epsilon. \tag{3.129}$$

Taking into account (3.128), we obtain

$$\lambda_n \le \frac{1}{\epsilon(8 - \epsilon(4 + c^{-1}))} s_n(K). \tag{3.130}$$

Optimizing with respect to ϵ, we obtain

$$\lambda_n \le \frac{4 + c^{-1}}{16} s_n(K). \tag{3.131}$$

Setting $c = \frac{1}{4}$ in (3.127) and (3.131), we obtain

$$\lambda_n \le \frac{1}{2} s_n(K). \tag{3.132}$$

Hence,

$$\|q\|_2^2 = \sum_{n=1}^{\infty} \lambda_n \le \frac{1}{2} \sum_{n=1}^{\infty} s_n(K) = \frac{1}{2}\mathrm{Tr}K. \tag{3.133}$$

This together with (3.117) yields (3.112). □

K. Normally ordered Hamiltonian

In this subsection, we give conditions on B that guarantee the existence of a bounded from below normally ordered quantization. We follow Ref. 15, whose approach is based on quadratic forms. Similar results were contained in Ref. 6. They were however weaker since only operator bounded perturbations were used in Ref. 6.

Suppose that $\Phi, \Psi \in \Gamma_s(\mathcal{W})$. Define the *reduced 1-body density operator* $\gamma_{\Psi,\Phi}$ and the *pairing operator* $\alpha_{\Psi,\Phi}$ as follows:

$$\begin{aligned}(w_1|\gamma_{\Psi,\Phi} w_2) &:= (\Phi|\hat{a}^*(w_2)\hat{a}(w_1)\Psi),\\ (\alpha_{\Phi,\Psi}\overline{w}_2|w_1) = (\alpha_{\Phi,\Psi}|w_1 \otimes w_2) &:= (\Phi|\hat{a}^*(w_2)\hat{a}^*(w_1)\Psi), \qquad w_1, w_2 \in \mathcal{W}.\end{aligned}$$

(Note that, as usual for similar objects, $\alpha_{\Psi,\Phi}$ has two interpretations: as a symmetric operator from $\overline{\mathcal{W}}$ to $\mathcal{W}$ or as an element of the Hilbert space $\otimes_s^2\mathcal{W}$. We will treat the former interpretation as the standard one.)

We will write

$$\gamma_\Phi := \gamma_{\Phi,\Phi}, \quad \alpha_\Phi := \alpha_{\Phi,\Phi}.$$

Note that

$$\alpha^{\#}_{\Phi,\Psi} = \alpha_{\Phi,\Psi}, \tag{3.134}$$

$$\begin{bmatrix} \gamma_\Phi & \alpha_\Phi \\ \overline{\alpha_\Phi} & \mathbb{1} + \overline{\gamma_\Phi} \end{bmatrix} \geq 0. \tag{3.135}$$

For further use, note that (3.135) is equivalent to

$$\gamma_\Phi \geq 0, \quad \gamma_\Phi \geq \alpha_\Phi(\mathbb{1} + \overline{\gamma_\Phi})^{-1}\overline{\alpha_\Phi}. \tag{3.136}$$

Clearly, if h is an operator on $\mathcal{W}$ and $g \in \otimes_{\mathrm{s}}^2 \mathcal{W}$, then

$$(\Phi|\mathrm{d}\Gamma(h)\Psi) = \mathrm{Tr}\gamma_{\Phi,\Psi}h, \tag{3.137}$$

$$(\Phi|\hat{a}^*(g)\Psi) = \mathrm{Tr}\alpha^*_{\Phi,\Psi}g, \tag{3.138}$$

$$(\Phi|\hat{a}(g)\Psi) = \mathrm{Tr}\alpha_{\Psi,\Phi}g^*. \tag{3.139}$$

Note that (3.138) and (3.139) are still true if g is an unbounded functional on $\otimes_{\mathrm{s}}^2 \mathcal{W}$ with domain $\overset{\mathrm{al}}{\otimes}{}_{\mathrm{s}}^2 \mathcal{D}$, provided that $\Psi, \Phi \in \overset{\mathrm{al}}{\Gamma}_{\mathrm{s}}(\mathcal{D})$, where $\mathcal{D} = \mathrm{Dom} h^{-\frac{1}{2}}$, as discussed in Remark 3.4.

The following proposition provides a key estimate for the construction of normally ordered Bogoliubov Hamiltonians:

Proposition 3.22. Assume that $\|h^{-\frac{1}{2}} g \overline{h}^{-\frac{1}{2}}\| \leq 1$ *and* $\mathrm{Tr} g^* h^{-1} g < \infty$. *Let* $\|h^{-\frac{1}{2}} g \overline{h}^{-\frac{1}{2}}\| \leq c$. *Then for* $\Phi \in \Gamma_{\mathrm{s}}(\mathcal{W})$ *with* $\|\Phi\| = 1$,

$$(\Phi|\hat{a}^*(g)\Phi) \leq c(\Phi|\mathrm{d}\Gamma(h)\Phi) + \frac{1}{2c}\mathrm{Tr}(g^* h^{-1} g). \tag{3.140}$$

Proof.

$$\begin{aligned}
(\Phi|\hat{a}^*(g)\Phi) &= |\mathrm{Tr}\overline{\alpha_\Phi} g| \\
&= \left|\mathrm{Tr}(\mathbb{1} + \overline{\gamma_\Phi})^{-\frac{1}{2}}\overline{\alpha}_\Phi h^{\frac{1}{2}} h^{-\frac{1}{2}} g (\mathbb{1} + \overline{\gamma_\Phi})^{\frac{1}{2}}\right| \\
&\leq \left(\mathrm{Tr} h^{\frac{1}{2}} \alpha_\Phi (\mathbb{1} + \overline{\gamma_\Phi})^{-1} \alpha^*_\Phi h^{\frac{1}{2}}\right)^{\frac{1}{2}} \\
&\quad \times \left(\mathrm{Tr} h^{-\frac{1}{2}} g (\mathbb{1} + \overline{\gamma_\Phi}) g^* h^{-\frac{1}{2}}\right)^{\frac{1}{2}} \\
&\leq \left(\mathrm{Tr} h^{\frac{1}{2}} \gamma_\Phi h^{\frac{1}{2}}\right)^{\frac{1}{2}} \left(\mathrm{Tr} h^{-\frac{1}{2}} g g^* h^{-\frac{1}{2}} + \|h^{-\frac{1}{2}} g h^{-\frac{1}{2}}\|^2 \mathrm{Tr} h^{\frac{1}{2}} \overline{\gamma_\Phi} h^{\frac{1}{2}}\right)^{\frac{1}{2}} \\
&= \left((\Phi|\mathrm{d}\Gamma(h)\Phi)\right)^{\frac{1}{2}} \left(\mathrm{Tr} g^* h^{-1} g + \|h^{-\frac{1}{2}} g h^{-\frac{1}{2}}\|^2 (\Phi|\mathrm{d}\Gamma(h)\Phi)\right)^{\frac{1}{2}}.
\end{aligned}$$

Then we use the inequality

$$\sqrt{x(y + c_0^2 x)} \leq cx + \frac{y}{2c} \tag{3.141}$$

valid for $x, y \geq 0$, $c > c_0$. □

Theorem 3.23. *Assume that* $\|h^{-\frac{1}{2}} g \overline{h}^{-\frac{1}{2}}\| < 1$ *and* $\mathrm{Tr} g^* h^{-1} g < \infty$.*Then the quadratic form*

$$\mathrm{d}\Gamma(h) + \frac{1}{2}\hat{a}^*(g) + \frac{1}{2}\hat{a}(g) \tag{3.142}$$

defined on the form domain of $\mathrm{d}\Gamma(h)$ *is closed and bounded from below by* $-\frac{1}{2}\mathrm{Tr}(g^* h^{-1} g)$. *Hence it defines a self-adjoint operator, which we temporarily denote by C. It satisfies that*

$$(1 + \mathrm{d}\Gamma(h))^{\frac{1}{2}} (\mathrm{i} + C)^{-1} (1 + \mathrm{d}\Gamma(h))^{\frac{1}{2}} \ \textit{is bounded}. \tag{3.143}$$

Proof. By Proposition 3.22,

$$\frac{1}{2}|(\Phi|(\hat{a}^*(g)+\hat{a}(g))\Phi)| \le c(\Phi|\mathrm{d}\Gamma(h)\Phi) + \frac{1}{2c}\mathrm{Tr} g^* h^{-1} g\|\Phi\|^2. \tag{3.144}$$

Setting $c := \|h^{-\frac{1}{2}} g \overline{h}^{-\frac{1}{2}}\| < 1$ and using the KLMN theorem, we see that form (3.142) is closed and bounded from below and hence defines a bounded from below self-adjoint operator C. Setting $c = 1$, we see that

$$-\frac{1}{2}\mathrm{Tr}(g^* h^{-1} g) < C. \tag{3.145}$$

(3.143) is also a consequence of the KLMN theorem. □

Theorem 3.24. *The operator defined in Theorem 3.23 is the normally ordered quantization of B. In other words, following the notation introduced in Subsection III G, $C = \hat{H}_B^{\mathrm{n}}$.*

On a formal level, the above theorem is essentially obvious. However, there are technical difficulties for which we will need a few technical lemmas. In these lemmas, we use $h \ge 0$ and $\|h^{-\frac{1}{2}} g \overline{h}^{-\frac{1}{2}}\| < 1$. Note that under this assumption, $\mathrm{i}\tau$ belongs to the resolvent set of B for $\tau \neq 0$.

Lemma 3.25. For $\tau \neq 0$, $B(\tau^2 + B^2)^{-1}$ has a dense range.

Proof. We write

$$B(\tau^2 + B^2)^{-1} = (\mathrm{i}\tau + B)^{-1}(-\mathrm{i}\tau + B)^{-1} B. \tag{3.146}$$

We will show that (3.146) has a dense range when restricted to Dom$B$.

First note that $B = AS$, where A is self-adjoint and S is unitary. Hence Dom$B$ = SDom$A$ and BDom$B$ = ADom$A$. This shows that BDom$B$ is dense.

Then we apply Lemma A.3 twice to the bounded operators with dense range $(\mathrm{i}\tau + B)^{-1}$ and $(-\mathrm{i}\tau + B)^{-1}$. □

Lemma 3.26. For $\tau \neq 0$, the operator $A_0^{-\frac{1}{2}} B(\tau^2 + B^2)^{-1}$ is bounded.

Proof. First note that

$$\|(\mathrm{i}\tau S + A_0)^{-\frac{1}{2}} GS(\mathrm{i}\tau S + A_0)^{-\frac{1}{2}}\| = \|h^{\frac{1}{2}} g \overline{h}^{\frac{1}{2}}\| < 1. \tag{3.147}$$

Next we check that all the terms on the right of the following identity are bounded:

$$A_0^{-\frac{1}{2}} B(\mathrm{i}\tau + B)^{-1} = (1 + A_0^{-\frac{1}{2}} GSA_0^{-\frac{1}{2}}) \tag{3.148}$$

$$\times A_0^{\frac{1}{2}}(\mathrm{i}\tau S + A_0)^{-\frac{1}{2}} \tag{3.149}$$

$$\times (1 + (\mathrm{i}\tau S + A_0)^{-\frac{1}{2}} GS(\mathrm{i}\tau S + A_0)^{-\frac{1}{2}})^{-1} \tag{3.150}$$

$$\times (\mathrm{i}\tau S + A_0)^{-\frac{1}{2}}. \tag{3.151}$$

[To see that (3.150) is well defined, we use (3.147).] Therefore, $A_0^{-\frac{1}{2}} B(\mathrm{i}\tau + B)^{-1}$ is bounded, which obviously implies the boundedness of $A_0^{-\frac{1}{2}} B(\tau^2 + B^2)^{-1}$. □

Proof of Theorem 3.24. Consider $w \in \mathrm{Ran} B(B^2 + 1)^{-1}$. By Lemma 3.25, such w are dense in $\mathcal{W}$. Set

$$\begin{bmatrix} w_t \\ \overline{w}_t \end{bmatrix} := \mathrm{e}^{\mathrm{i}tB} \begin{bmatrix} w \\ \overline{w} \end{bmatrix}. \tag{3.152}$$

By Lemma 3.26, $\|h^{-\frac{1}{2}}w_t\|$ is uniformly bounded. Therefore, by Propostion 3.3,

$$(1+\mathrm{d}\Gamma(h))^{-\frac{1}{2}}\hat{\phi}(w_t,\overline{w}_t)(1+\mathrm{d}\Gamma(h))^{-\frac{1}{2}} \tag{3.153}$$

is uniformly bounded. Hence, by (3.143), so is

$$k(t) := (C+\mathrm{i})^{-1}\mathrm{e}^{-\mathrm{i}tC}\hat{\phi}(w_t,\overline{w}_t)\mathrm{e}^{\mathrm{i}tC}(C+\mathrm{i})^{-1}. \tag{3.154}$$

We know that $(w_t,\overline{w}_t)\in\mathrm{Dom}(B)$. But this does not necessarily imply that $w_t\in\mathrm{Dom}h$. It only implies $w_t\in\mathrm{Dom}h^{\frac{1}{2}}$. Therefore, strictly speaking, we cannot write

$$\frac{\mathrm{d}}{\mathrm{d}t}w_t = \mathrm{i}hw_t - \mathrm{i}g\overline{w}_t, \tag{3.155}$$

but only

$$\frac{\mathrm{d}}{\mathrm{d}t}h^{-\frac{1}{2}}w_t = \mathrm{i}h^{-\frac{1}{2}}hw_t - \mathrm{i}h^{-\frac{1}{2}}g\overline{w}_t. \tag{3.156}$$

However, using the boundedness of $(\mathrm{i}+C)^{-1}(\mathrm{i}+\mathrm{d}\Gamma(h))^{\frac{1}{2}}$ and Proposition 3.3, it is sufficient to compute

$$\begin{aligned}
&(\mathrm{i}+C)^{-1}\mathrm{i}[C,\hat{\phi}(w_t,\overline{w}_t)](\mathrm{i}+C)^{-1} && (3.157)\\
&= (\mathrm{i}+C)^{-1}\Big(\hat{a}^*(hw_t)+\hat{a}(g\overline{w}_t)-\hat{a}(hw_t)-\hat{a}^*(g\overline{w}_t)\Big)(\mathrm{i}+C)^{-1} && (3.158)\\
&= (\mathrm{i}+C)^{-1}\hat{\phi}(\tfrac{\mathrm{d}}{\mathrm{d}t}w_t,\tfrac{\mathrm{d}}{\mathrm{d}t}\overline{w}_t)(\mathrm{i}+C)^{-1}. && (3.159)
\end{aligned}$$

Therefore,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}k(t) &= (C+\mathrm{i})^{-1}\mathrm{e}^{-\mathrm{i}tC}\Big(-\mathrm{i}[C,(\hat{\phi}(w_t,\overline{w}_t))] && (3.160)\\
&\quad + \hat{\phi}\Big(\tfrac{\mathrm{d}}{\mathrm{d}t}w_t,\tfrac{\mathrm{d}}{\mathrm{d}t}\overline{w}_t\Big)\Big)\,\mathrm{e}^{\mathrm{i}tC}(C+\mathrm{i})^{-1} = 0. && (3.161)
\end{aligned}$$

This shows that $k(t)$ does not depend on t. Therefore,

$$(C+\mathrm{i})^{-1}\mathrm{e}^{\mathrm{i}tC}\hat{\phi}(w,\overline{w})\mathrm{e}^{-\mathrm{i}tC}(C+\mathrm{i})^{-1} = (C+\mathrm{i})^{-1}\hat{\phi}(w_t,\overline{w}_t)(C+\mathrm{i})^{-1}. \tag{3.162}$$

This proves that $\mathrm{e}^{\mathrm{i}tC}$ implements $\mathrm{e}^{\mathrm{i}tB}$.

Clearly, $\gamma_\Omega = 0$, $\alpha_\Omega = 0$, and $\Omega\in\mathrm{Dom}(\mathrm{d}\Gamma(h)^{\frac{1}{2}}) = \mathrm{Dom}(|C|^{\frac{1}{2}})$. Therefore,

$$(\Omega|C\Omega) = 0. \tag{3.163}$$

Thus, by Proposition 3.13, the operator temporarily denoted as C is the normally ordered quantization of B. □

L. Infimum of normally ordered Hamiltonians

In Subsection II J, in the finite dimensional context, we defined E_B^{n} as the infimum of the normally ordered Hamiltonian $\hat{H}_B^{\mathrm{n}}$. In infinite dimensions, it is useful to define E_B^{n} independently of whether $\hat{H}_B^{\mathrm{n}}$ exists or not.

As a basic condition on the symplectic generator B, we assume that $h>0$, $\|h^{-\frac{1}{2}}g\overline{h}^{-\frac{1}{2}}\|<1$. As in (2.68), for $\sigma\in\mathbb{R}$, we set

$$A_\sigma := A_0+\sigma GS = \begin{bmatrix} h & \sigma g\\ \sigma\overline{g} & \overline{h}\end{bmatrix} \tag{3.164}$$

so that $A = A_1$.

Out of the formulas for E^{n} listed in (2.70)–(2.75) valid in finite dimensions, the most suitable one for infinite dimensions seems to be (2.74), which we choose as the definition of E^{n},

$$E_B^{\mathrm{n}} := \frac{1}{8}\int_0^1 \mathrm{d}\sigma\,\mathrm{Tr}A_\sigma^{\frac{1}{2}}(A_\sigma^{\frac{1}{2}}SA_\sigma SA_\sigma^{\frac{1}{2}})^{-\frac{1}{2}}A_\sigma^{\frac{1}{2}}GS, \tag{3.165}$$

provided that the above integral is well defined.

(2.75) is another formula for E^{n} useful in infinite dimensions:

Proposition 3.27. We have

$$E_B^{\mathrm{n}} := \frac{1}{8}\int_0^1 \mathrm{d}\sigma \int \frac{\mathrm{d}\tau}{\pi}(1-\sigma)\mathrm{Tr}\frac{1}{(A_\sigma + \mathrm{i}\tau S)}SG\frac{1}{(A_\sigma + \mathrm{i}\tau S)}SG. \tag{3.166}$$

More precisely, if (3.166) is well defined as a convergent integral, then it coincides with (3.165).

Below we list a few criteria for the existence of E_B^{n}.

Theorem 3.28. (1) *Let* $\|g\|_1 < \infty$. *Then* E_B^{n} *is well defined by (3.165).*
(2) *Let* $s_- < \frac{1}{2} < s_+$. *Suppose that* $\mathrm{Tr}g\overline{h}^{-s_-}g^*h^{-s_-} < \infty$ *and* $\mathrm{Tr}g\overline{h}^{-s_+}g^*h^{-s_+} < \infty$. *Then* E_B^{n} *is well defined by (3.165) or (3.166).*
(3) *Suppose that* $\mathrm{Tr}gh^{-1}g^* < \infty$. *Then* E_B^{n} *is well defined by (3.165) or (3.166).*

Proof. (1): Repeating the arguments that lead to inequality (3.109), we obtain for $\sigma \in [0, 1]$

$$\frac{\sqrt{1-a\sigma}}{\sqrt{1+a\sigma}}A_\sigma \le (A_\sigma^{\frac{1}{2}}SA_\sigma SA_\sigma^{\frac{1}{2}})^{\frac{1}{2}} \le \frac{\sqrt{1+a\sigma}}{\sqrt{1-a\sigma}}A_\sigma. \tag{3.167}$$

Therefore, $Y := A_\sigma^{\frac{1}{2}}(A_\sigma^{\frac{1}{2}}SA_\sigma SA_\sigma^{\frac{1}{2}})^{-\frac{1}{2}}A_\sigma^{\frac{1}{2}}$ is uniformly bounded.

We apply inequality (A7) to the operator Y and $X := GS$. We obtain

$$|E^{\mathrm{n}}| \le \frac{1}{8}\int_0^1 \mathrm{d}\sigma|\mathrm{Tr}A_\sigma^{\frac{1}{2}}(A_\sigma^{\frac{1}{2}}SA_\sigma SA_\sigma^{\frac{1}{2}})^{-\frac{1}{2}}A_\sigma^{\frac{1}{2}}GS| \tag{3.168}$$

$$\le \frac{1}{8}\int_0^1 \mathrm{d}\sigma\|A_\sigma^{\frac{1}{2}}(A_\sigma^{\frac{1}{2}}SA_\sigma SA_\sigma^{\frac{1}{2}})^{-\frac{1}{2}}A_\sigma^{\frac{1}{2}}\|\mathrm{Tr}\sqrt{G^2} \le c\mathrm{Tr}\sqrt{G^2}. \tag{3.169}$$

But $\mathrm{Tr}\sqrt{G^2} = 2\mathrm{Tr}\sqrt{\overline{g}g}$. This proves (1).

(2): First note that for $0 \le s \le 1$,

$$\|A_0^{\frac{s}{2}}(A_0 + \mathrm{i}\tau S)^{-\frac{1}{2}}\| \le \tau^{-\frac{1}{2}+\frac{s}{2}}. \tag{3.170}$$

Indeed,

$$A_0^{\frac{s}{2}}(A_0 + \mathrm{i}\tau S)^{-\frac{1}{2}} = A_0^{\frac{s}{2}}(A_0 + \mathrm{i}\tau S)^{-\frac{s}{2}} \times (A_0 + \mathrm{i}\tau S)^{-\frac{1}{2}+\frac{s}{2}}, \tag{3.171}$$

where the first term is bounded by 1 and the second term is bounded by $\tau^{-\frac{1}{2}+\frac{s}{2}}$.

Moreover,

$$\|(A_0 + \mathrm{i}\tau S)^{-\frac{1}{2}}GS(A_0 + \mathrm{i}\tau S)^{-\frac{1}{2}}\| \le \|A_0^{-\frac{1}{2}}GSA_0^{-\frac{1}{2}}\| = a < 1. \tag{3.172}$$

Therefore, we can write

$$(A_\sigma + \mathrm{i}\tau S)^{-1} = (A_0 + \mathrm{i}\tau S)^{-\frac{1}{2}}\left(\mathbb{1} - \sigma(A_0 + \mathrm{i}\tau S)^{-\frac{1}{2}}GS(A_0 + \mathrm{i}\tau S)^{-\frac{1}{2}}\right)^{-1}(A_0 + \mathrm{i}\tau S)^{-\frac{1}{2}}. \tag{3.173}$$

(3.173) together with (3.170) and (3.172) yields

$$\|A_0^{\frac{s}{2}}(A_\sigma + \mathrm{i}\tau S)^{-1}A_0^{\frac{s}{2}}\| \le (1-a)^{-1}\tau^{-1+s}. \tag{3.174}$$

Now,

$$\Big|\mathrm{Tr}\frac{1}{(A_\sigma+\mathrm{i}\tau S)}GS\frac{1}{(A_\sigma+\mathrm{i}\tau S)}G\Big| \tag{3.175}$$

$$\leq \Big\|A_0^{\frac{s}{2}}(A_\sigma+\mathrm{i}\tau S)^{-1}A_0^{\frac{s}{2}}\Big\|^2 \mathrm{Tr}GA_0^{-s}G^*A_0^{-s} \tag{3.176}$$

$$\leq (1-a)^{-2}\tau^{2s-2}\mathrm{Tr}GA_0^{-s}G^*A_0^{-s}, \tag{3.177}$$

where we first used inequality (A8) with $Y=Z:=A_0^{\frac{s}{2}}(A_\sigma+\mathrm{i}\tau S)^{-1}A_0^{\frac{s}{2}}$ and $X:=A_0^{-\frac{s}{2}}GA_0^{-\frac{s}{2}}$, and then we applied (3.174). Thus

$$|E^{\mathrm{n}}| \leq \frac{1}{8}\int_0^1 \mathrm{d}\sigma \int \frac{\mathrm{d}\tau}{\pi}(1-\sigma)\Big|\mathrm{Tr}\frac{1}{(A_\sigma+\mathrm{i}\tau S)}GS\frac{1}{(A_\sigma+\mathrm{i}\tau S)}G\Big| \tag{3.178}$$

$$\leq c\mathrm{Tr}GA_0^{-s_+}G^*A_0^{-s_+}\int_0^1 \tau^{2s_+-2}\mathrm{d}\tau + c\mathrm{Tr}GA_0^{-s_-}G^*A_0^{-s_-}\int_1^\infty \tau^{2s_--2}\mathrm{d}\tau. \tag{3.179}$$

But $\mathrm{Tr}GA_0^{-s_\pm}GA_0^{-s_\pm}=2\mathrm{Tr}\overline{g}h^{-s_\pm}g\overline{h}^{-s_\pm}$. This proves (2).

(3): Applying (A8) to

$$Y=Z:=(\mathbb{1}-\sigma(A_0+\mathrm{i}\tau S)^{-\frac{1}{2}}GS(A_0+\mathrm{i}\tau S)^{-\frac{1}{2}})^{-1},\quad X:=(A_0+\mathrm{i}\tau S)^{-\frac{1}{2}}GS(A_0+\mathrm{i}\tau S)^{-\frac{1}{2}},$$

and using (3.173) and (3.172), we obtain

$$\Big|\mathrm{Tr}\frac{1}{(A_\sigma+\mathrm{i}\tau S)}GS\frac{1}{(A_\sigma+\mathrm{i}\tau S)}GS\Big| \tag{3.180}$$

$$= \mathrm{Tr}XYXY \leq \|Y\|^2\mathrm{Tr}XX^* \tag{3.181}$$

$$\leq \frac{1}{(1-\sigma a)^2}\mathrm{Tr}\frac{1}{(A_0^2+\tau^2)^{\frac{1}{2}}}G\frac{1}{(A_0^2+\tau^2)^{\frac{1}{2}}}G^* \tag{3.182}$$

$$\leq \frac{1}{(1-\sigma a)^2}\mathrm{Tr}G\frac{1}{(A_0^2+\tau^2)}G^*. \tag{3.183}$$

Therefore,

$$|E^{\mathrm{n}}| \leq \frac{1}{8}\int_0^1 \mathrm{d}\sigma \int \frac{\mathrm{d}\tau}{\pi}(1-\sigma)\Big|\mathrm{Tr}\frac{1}{(A_\sigma+\mathrm{i}\tau S)}GS\frac{1}{(A_\sigma+\mathrm{i}\tau S)}GS\Big| \tag{3.184}$$

$$\leq \frac{1}{8}\int_0^1 \mathrm{d}\sigma\frac{(1-\sigma)}{(1-a\sigma)^2}\int\frac{\mathrm{d}\tau}{\pi}\mathrm{Tr}G\frac{1}{(A_0^2+\tau^2)}G^* \tag{3.185}$$

$$= \frac{(-\log(1-a)-a)}{8a^2}\mathrm{Tr}G\frac{1}{A_0}G^*. \tag{3.186}$$

But $\mathrm{Tr}G\frac{1}{A_0}G=2\mathrm{Tr}gh^{-1}g^*$. This proves (3). □

Theorem 3.29. *Suppose that* $\mathrm{Tr}gh^{-1}g^*<\infty$*, as in Theorem 3.28 (3). Let* $\hat{H}_B^{\mathrm{n}}$ *be defined as in Subsection III K. Let* E_B^{n} *be defined as in (3.165). Then*

$$E_B^{\mathrm{n}}=\inf \hat{H}_B^{\mathrm{n}}. \tag{3.187}$$

If $\mathcal{W}$ is finite dimensional, then (3.187) was proven in Theorem 2.6. In our proof, we will reduce the full problem to this case. The proof will be divided into several steps.

Step 1. *Suppose that there exists a finite dimensional* $\mathcal{W}_0$ *such that* $\mathrm{Ran}g\subset\mathcal{W}_0$ *and* h *preserves* $\mathcal{W}_0$*. Then (3.187) is true.*

Proof. Set $\mathcal{W}_1:=\mathcal{W}_0^\perp$. Note that $g=g^\#$ implies that $\overline{\mathcal{W}_1}\subset\mathrm{Ker}g$. Let h_0 and g_0 denote the restrictions of g and h to $\mathcal{W}_0$. Let h_1 denote the restriction of h to $\mathcal{W}_1$. Consider the symplectic generator on $\mathcal{W}_0$,

$$B_0 := \begin{bmatrix} h_0 & -g_0 \\ \overline{g}_0 & -\overline{h}_0 \end{bmatrix}, \tag{3.188}$$

and the corresponding normally ordered Bogoliubov Hamiltonian

$$\hat{H}_0^{\mathrm{n}} := \hat{H}_{B_0}^{\mathrm{n}} = \mathrm{d}\Gamma(h_0) + \frac{1}{2}\left(\hat{a}^*(g_0) + \hat{a}(g_0)\right). \tag{3.189}$$

We will write E_0^{n}, respectively, E^{n}, for $E_{B_0}^{\mathrm{n}}$, respectively, E_B^{n}.

We have the decomposition

$$\Gamma_{\mathrm{s}}(\mathcal{W}) \simeq \Gamma_{\mathrm{s}}(\mathcal{W}_0) \otimes \Gamma_{\mathrm{s}}(\mathcal{W}_1). \tag{3.190}$$

The operator $\hat{H}^{\mathrm{n}}$ can be decomposed as

$$\hat{H}^{\mathrm{n}} \simeq \hat{H}_0^{\mathrm{n}} \otimes \mathbb{1} + \mathbb{1} \otimes \mathrm{d}\Gamma(h_1). \tag{3.191}$$

We have

$$\inf \hat{H}^{\mathrm{n}} = \inf \hat{H}_0^{\mathrm{n}} = E_0^{\mathrm{n}} = E^{\mathrm{n}}, \tag{3.192}$$

where in the middle step we used the finite dimension of $\mathcal{W}_0$. □

Step 2. *Suppose that g is finite dimensional and $\mathbb{1}_{[\delta,\delta^{-1}]}(h)g = g$. Then (3.187) is true.*

Proof. Let $\epsilon > 0$. Let us set

$$\pi_{\epsilon,n} := \mathbb{1}_{[(1+\epsilon)^n,(1+\epsilon)^{n+1}[}(h), \tag{3.193}$$

$$h_\epsilon := \sum_{n=-\infty}^{\infty} (1+\epsilon)^{n+1}\pi_{\epsilon,n}. \tag{3.194}$$

Note that

$$(1+\epsilon)^{-1}h_\epsilon \le h \le h_\epsilon. \tag{3.195}$$

Hence,

$$\mathrm{d}\Gamma\left((1+\epsilon)^{-1}h_\epsilon\right) \le \mathrm{d}\Gamma(h) \le \mathrm{d}\Gamma(h_\epsilon). \tag{3.196}$$

Now

$$\hat{H}_{\epsilon,-}^{\mathrm{n}} := \mathrm{d}\Gamma\left((1+\epsilon)^{-1}h_\epsilon\right) + \frac{1}{2}\left(\hat{a}^*(g) + \hat{a}(g)\right) \le \hat{H}^{\mathrm{n}} \le \mathrm{d}\Gamma(h_\epsilon) + \frac{1}{2}\left(\hat{a}^*(g) + \hat{a}(g)\right) =: H_{\epsilon,+}^{\mathrm{n}}.$$

Let $\mathcal{W}_{\epsilon,0}$ be the smallest subspace of $\mathcal{W}$ containing Ran$g$ and left invariant by h_ϵ. In other words,

$$\mathcal{W}_{\epsilon,0} := \mathrm{Span}\{\pi_{\epsilon,n}w \ : \ w \in \mathrm{Ran}g\}. \tag{3.197}$$

Note that $\pi_{\epsilon,n}\mathrm{Ran}g = 0$ for $|n|$ large enough. Therefore, $\mathcal{W}_{\epsilon,0}$ is finite dimensional.

Thus $\hat{H}_{\epsilon,\pm}^{\mathrm{n}}$ satisfy the conditions of Step 1, and so

$$\inf \hat{H}_{\epsilon,\pm}^{\mathrm{n}} = E_{\epsilon,\pm}^{\mathrm{n}} \tag{3.198}$$

in the obvious notation. Using Lemma 3.30, we show that

$$\lim_{\epsilon\to 0} E_{\epsilon,\pm}^{\mathrm{n}} = E^{\mathrm{n}}. \tag{3.199}$$

Besides,

$$\inf \hat{H}_{\epsilon,-}^{\mathrm{n}} \le \inf H^{\mathrm{n}} \le \inf \hat{H}_{\epsilon,+}^{\mathrm{n}}. \tag{3.200}$$

□

Step 3. *Suppose that for some $\delta > 0$ we have $\mathbb{1}_{[\delta,\delta^{-1}]}(h)g = g$. Then (3.187) is true.*

Proof. We know that $h^{-\frac{1}{2}}g$ is Hilbert-Schmidt. Finite dimensional operators are dense in Hilbert-Schmidt operators. Therefore, given $\epsilon > 0$, we can find a finite dimensional g_ϵ such that $g_\epsilon = g_\epsilon^{\#}$, $\mathbb{1}_{[\delta,\delta^{-1}]}(h)g = g$, and

$$\|h^{-\frac{1}{2}}g - h^{-\frac{1}{2}}g_\epsilon\|_2 = \sqrt{\mathrm{Tr}(\overline{g} - \overline{g}_\delta')h^{-1}(g - g_\delta')} < \epsilon^2. \tag{3.201}$$

Now, the Hilbert-Schmidt norm dominates the operator norm. Hence, (3.201) implies

$$\|h^{-\frac{1}{2}}(g - g_\epsilon)\| \le \epsilon. \tag{3.202}$$

As a consequence,

$$\|h^{-\frac{1}{2}}(g - g_\epsilon)\overline{h}^{-\frac{1}{2}}\| \le \epsilon\delta^{-\frac{1}{2}}. \tag{3.203}$$

Therefore, by (3.203),

$$\|h^{-\frac{1}{2}}g_\epsilon\overline{h}^{-\frac{1}{2}}\| \le \|h^{-\frac{1}{2}}g\overline{h}^{-\frac{1}{2}}\| + \|h^{-\frac{1}{2}}(g - g_\epsilon)\overline{h}^{-\frac{1}{2}}\| \tag{3.204}$$

$$\le a + \epsilon\delta^{-\frac{1}{2}} =: a_1. \tag{3.205}$$

By choosing ϵ small enough, we can guarantee that $a_1 < 1$.

Now

$$\hat{H}^{\mathrm{n}} = (1-\nu)\mathrm{d}\Gamma(h) + \frac{1}{2}(\hat{a}^*(g_\epsilon) + \hat{a}(g_\epsilon)) \tag{3.206}$$

$$+\nu\mathrm{d}\Gamma(h) + \frac{1}{2}(\hat{a}^*(g - g_\epsilon) + \hat{a}(g - g_\epsilon)) \tag{3.207}$$

$$\ge (1-\nu)\mathrm{d}\Gamma(h) + \frac{1}{2}(\hat{a}^*(g_\epsilon) + \hat{a}(g_\epsilon)) - \frac{\epsilon^2}{\nu}, \tag{3.208}$$

$$\hat{H}^{\mathrm{n}} = (1+\nu)\mathrm{d}\Gamma(h) + \frac{1}{2}(\hat{a}^*(g_\epsilon) + \hat{a}(g_\epsilon)) \tag{3.209}$$

$$-\nu\mathrm{d}\Gamma(h) + \frac{1}{2}(\hat{a}^*(g - g_\epsilon) + \hat{a}(g - g_\epsilon)) \tag{3.210}$$

$$\le (1+\nu)\mathrm{d}\Gamma(h) + \frac{1}{2}(\hat{a}^*(g_\epsilon) + \hat{a}(g_\epsilon)) + \frac{\epsilon^2}{\nu}. \tag{3.211}$$

The Hamiltonians $(1 \pm \nu)\mathrm{d}\Gamma(h) + \frac{1}{2}(\hat{a}^*(g_\epsilon) + \hat{a}(g_\epsilon))$ satisfy the assumptions of Step 2. □

Step 4. *(3.187) is true without additional assumptions.*

Proof. $h^{-\frac{1}{2}}g$ is Hilbert-Schmidt and $\text{s-}\lim_{\delta\to 0}\mathbb{1}_{[\delta,\delta^{-1}]}(h) = \mathbb{1}$. Hence, for any $\epsilon > 0$, we can find $1 \ge \delta > 0$ such that if we set

$$g_\delta := \mathbb{1}_{[\delta,\delta^{-1}]}(h)g\mathbb{1}_{[\delta,\delta^{-1}]}(\overline{h}), \tag{3.212}$$

then

$$\|h^{-\frac{1}{2}}g - h^{-\frac{1}{2}}g_\delta\|_2 = \sqrt{\mathrm{Tr}(\overline{g} - \overline{g}_\delta)h^{-1}(g - g_\delta)} < \epsilon. \tag{3.213}$$

Note that

$$\|h^{-\frac{1}{2}}g_\delta\overline{h}^{-\frac{1}{2}}\| \le \|h^{-\frac{1}{2}}g\overline{h}^{-\frac{1}{2}}\| = a. \tag{3.214}$$

Then we estimate similarly as at the end of Step 3. We argue that we need only estimates about the Hamiltonians $(1 \pm \nu)\mathrm{d}\Gamma(h) + \frac{1}{2}(\hat{a}^*(g_\delta) + \hat{a}(g_\delta))$, which satisfy the assumptions of Step 3. □

A result about the continuity of E_B^{n} with respect to h, which we needed in the above proof, is described below.

Lemma 3.30. Let

$$B = \begin{bmatrix} h & -g \\ \overline{g} & -\overline{h} \end{bmatrix}, \quad B' = \begin{bmatrix} h' & -g \\ \overline{g} & -\overline{h}' \end{bmatrix} \tag{3.215}$$

with

$$\|h^{-\frac{1}{2}} g \overline{h}^{-\frac{1}{2}}\|, \ \|h'^{-\frac{1}{2}} g \overline{h}'^{-\frac{1}{2}}\| \leq a, \qquad \mathrm{Tr} g h^{-1} g^*, \ \mathrm{Tr} g h'^{-1} g^* \leq a_1.$$

Then

$$\|E_B^{\mathrm{n}} - E_{B'}^{\mathrm{n}}\| \leq c \|h^{-\frac{1}{2}}(h - h') h'^{-\frac{1}{2}}\|, \tag{3.216}$$

where c depends only on a and a_1.

Proof.

$$\begin{aligned} & E_B^{\mathrm{n}} - E_{B'}^{\mathrm{n}} \\ = & \frac{1}{8}\int_0^1 \mathrm{d}\sigma \int \frac{\mathrm{d}\tau}{\pi}(1-\sigma)\mathrm{Tr}\frac{1}{(A_\sigma + \mathrm{i}\tau S)}(A_0' - A_0)\frac{1}{(A_\sigma' + \mathrm{i}\tau S)} SG \frac{1}{(A_\sigma + \mathrm{i}\tau S)} SG \\ & + \frac{1}{8}\int_0^1 \mathrm{d}\sigma \int \frac{\mathrm{d}\tau}{\pi}(1-\sigma)\mathrm{Tr}\frac{1}{(A_\sigma' + \mathrm{i}\tau S)} SG \frac{1}{(A_\sigma + \mathrm{i}\tau S)}(A_0' - A_0)\frac{1}{(A_\sigma' + \mathrm{i}\tau S)} SG. \end{aligned}$$

Then we argue similarly as in the proof of Theorem 3.28 (3). □

M. Weyl Bogoliubov Hamiltonian

Weyl Bogoliubov Hamiltonians play a central role in the theory of Bogoliubov Hamiltonians, providing the simplest algebraic formulas. Unfortunately, in infinite dimensions, they are usually ill defined.

If $A_B \geq 0$, then we can define

$$E_B^{\mathrm{w}} := \frac{1}{4}\mathrm{Tr}\sqrt{A_B^{\frac{1}{2}} S A_B S A_B^{\frac{1}{2}}}, \tag{3.217}$$

which is a nonnegative number, often infinite. Recall that in finite dimensions, it coincides with the infimum of $\hat{H}_B^{\mathrm{w}}$.

The following theorem gives (rather restrictive) conditions when we can define the Weyl quantization in any dimension.

Theorem 3.31. *Assume that $h > 0$, $\|h^{-\frac{1}{2}} g \overline{h}^{-\frac{1}{2}}\| =: a < 1$, and $\mathrm{Tr} h < \infty$. Then the following hold:*

(1) $\mathrm{Tr} g^* h^{-1} g < \infty$.

(2) $\|g\|_1 < \infty$.

(3) *By (1), we can define $\hat{H}_B^{\mathrm{n}}$ as in Subsection III K and E_B^{n} is well defined as in Subsection III L, and by Theorem 3.29,*

$$\inf H_B^{\mathrm{n}} = E_B^{\mathrm{n}}. \tag{3.218}$$

(4) *By Theorem 3.18 (3), H_B^{w} is well defined. We have*

$$\hat{H}_B^{\mathrm{w}} := \hat{H}_B^{\mathrm{n}} + \frac{1}{2}\mathrm{Tr} h. \tag{3.219}$$

(5) $\sqrt{A_B^{\frac{1}{2}} S A_B S A_B^{\frac{1}{2}}}$ *is a trace class so that E_B^{w} is finite. We have*

$$E_B^{\mathrm{w}} = E_B^{\mathrm{n}} + \frac{1}{2}\mathrm{Tr} h, \tag{3.220}$$

$$E_B^{\mathrm{w}} = \inf \hat{H}_B^{\mathrm{w}}. \tag{3.221}$$

Proof. We have

$$h^{-\frac{1}{2}}gg^*h^{-\frac{1}{2}} = h^{-\frac{1}{2}}g\overline{h}^{-\frac{1}{2}}\overline{h}\overline{h}^{-\frac{1}{2}}g^*h^{-\frac{1}{2}}. \tag{3.222}$$

But $h^{-\frac{1}{2}}g\overline{h}^{-\frac{1}{2}}$ and $\overline{h}^{-\frac{1}{2}}g^*h^{-\frac{1}{2}}$ are bounded and $\overline{h}$ is a trace class. Therefore, (3.222) is a trace class. Hence (1) is true and

$$\|g\|_1 = \|h^{\frac{1}{2}}h^{-\frac{1}{2}}g\overline{h}^{-\frac{1}{2}}\overline{h}^{\frac{1}{2}}\|_1 \le \|h^{\frac{1}{2}}\|_2\|h^{-\frac{1}{2}}g\overline{h}^{-\frac{1}{2}}\|\|\overline{h}^{\frac{1}{2}}\|_2 \le a\|h\|_1 \tag{3.223}$$

proves (2).

We know that h is a trace class. Besides, $\|g\|_1 < \infty$ implies $\|g\|_2 < \infty$. Therefore, the assumptions of Theorem 3.18 (3) are satisfied. Therefore, $\hat{H}_B^{\mathrm{w}}$ is well defined.

By (3.107) and (3.108),

$$\sqrt{A^{\frac{1}{2}}SASA^{\frac{1}{2}}} \le \frac{(1+a)^{\frac{1}{2}}}{(1-a)^{\frac{1}{2}}}A \le \frac{(1+a)^{\frac{3}{2}}}{(1-a)^{\frac{1}{2}}}A_0. \tag{3.224}$$

But A_0 is a trace class, so is the left hand side of (3.224).

By repeating the arguments of Theorem 2.6, we see that (3.220) is true. By Theorem 3.18 (3), we have (3.219). Combining (3.218), (3.220), and (3.219), we obtain (3.221). □

N. Infimum of the 2nd order renormalized Hamiltonian

Recall that in Subsections II L and II M, we discussed the 2nd order renormalized Hamiltonians $\hat{H}_B^{\mathrm{2ren}}$ and its infimum E_B^{2ren} in the context of finite dimensions. These objects are of course especially interesting in infinite dimensions.

Note that it may happen that E_B^{2ren} is well defined and $\hat{H}_B^{\mathrm{2ren}}$ is not. In this subsection, we discuss only E_B^{2ren}, without asking whether $\hat{H}_B^{\mathrm{2ren}}$ exists.

We will use the framework of Subsection II M with $\lambda = 1$. That means we assume that $h_0 > 0$, $h = h_0 + h_1$, $h_1^2 = g\overline{g}$, and $h_1 g = g\overline{h}_1$. Recall that we have

$$B_0 = \begin{bmatrix} h_0 & 0 \\ 0 & -\overline{h}_0 \end{bmatrix}, \tag{3.225}$$

$$A_0 = B_0 S = \begin{bmatrix} h_0 & 0 \\ 0 & \overline{h}_0 \end{bmatrix}, \tag{3.226}$$

$$B_0^2 = A_0^2 = \begin{bmatrix} h_0^2 & 0 \\ 0 & \overline{h}_0^2 \end{bmatrix}, \tag{3.227}$$

$$V = B^2 - B_0^2 = \begin{bmatrix} h_0h_1 + h_1h_0 & -h_0g + g\overline{h}_0 \\ \overline{g}h_0 - \overline{h}_0\overline{g} & \overline{h}_0\overline{h}_1 + \overline{h}_0\overline{h}_1 \end{bmatrix}. \tag{3.228}$$

We use (2.111) to define E_B^{2ren},

$$E^{\mathrm{2ren}} := -\frac{1}{4}\int \mathrm{Tr}\frac{1}{B_0^2+\tau^2}V\frac{1}{B^2+\tau^2}\Big(V\frac{1}{B_0^2+\tau^2}\Big)^2\tau^2\frac{\mathrm{d}\tau}{\pi}, \tag{3.229}$$

provided that the above integral is well defined. Below we give a simple criterion for the well definedness of E^{2ren}.

Theorem 3.32. *Suppose that*

$$\|B_0^{-1}VB_0^{-1}\| =: a_1 < 1, \tag{3.230}$$

and

$$L_3 := \frac{1}{4\cdot 6}\int \mathrm{Tr}\Big(V\frac{1}{B_0^2+\tau^2}\Big)^3\frac{\mathrm{d}\tau}{\pi}, \tag{3.231}$$

$$L_4 := -\frac{1}{4\cdot 8}\int \mathrm{Tr}\Big(V\frac{1}{B_0^2+\tau^2}\Big)^4\frac{\mathrm{d}\tau}{\pi} \tag{3.232}$$

are finite. (In the case of L_4, the meaning of the assumption is clear since the integrand is always positive. This does not need to be the case of L_3—here we assume that the integrand is integrable.) Then $E^{2\mathrm{ren}}$ is well defined.

Proof. Assumption (3.230) is equivalent to

$$a_1 B_0^2 \leq V \leq a_1 B_0^2. \tag{3.233}$$

Therefore,

$$B^2 = B_0^2 + V \geq (1-a_1)B_0^2. \tag{3.234}$$

Hence, with $c := \frac{1}{1-a_1} > 0$,

$$\frac{1}{(B^2+\tau^2)} \leq c\frac{1}{(B_0^2+\tau^2)}. \tag{3.235}$$

Besides,

$$E^{2\mathrm{ren}} := \frac{1}{4\cdot 6}\int \mathrm{Tr}\Big(V\frac{1}{B_0^2+\tau^2}\Big)^3\frac{\mathrm{d}\tau}{\pi} \tag{3.236}$$

$$-\frac{1}{4}\int \mathrm{Tr}\Big(\frac{1}{B_0^2+\tau^2}V\Big)^2\frac{1}{B^2+\tau^2}\Big(V\frac{1}{B_0^2+\tau^2}\Big)^2\tau^2\frac{\mathrm{d}\tau}{\pi}. \tag{3.237}$$

The first term is precisely L_3. The second term is controlled by L_4 because

$$0 \leq \frac{1}{4}\int \mathrm{Tr}\Big(\frac{1}{B_0^2+\tau^2}V\Big)^2\frac{1}{B^2+\tau^2}\Big(V\frac{1}{B_0^2+\tau^2}\Big)^2\tau^2\frac{\mathrm{d}\tau}{\pi} \tag{3.238}$$

$$\leq \frac{c}{4}\int \mathrm{Tr}\Big(\frac{1}{B_0^2+\tau^2}V\Big)^2\frac{1}{B_0^2+\tau^2}\Big(V\frac{1}{B_0^2+\tau^2}\Big)^2\tau^2\frac{\mathrm{d}\tau}{\pi} \tag{3.239}$$

$$= \frac{c}{4\cdot 8}\int \mathrm{Tr}\Big(V\frac{1}{B_0^2+\tau^2}\Big)^4\frac{\mathrm{d}\tau}{\pi} = -cL_4. \tag{3.240}$$

To pass from (3.239) to (3.240), we use identity (2.95), which involves integration by parts. □

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APPENDIX: USEFUL STRUCTURES AND RESULTS

1. Complex conjugate space

This appendix is a side remark about complex conjugate spaces. This well known but abstract and somewhat confusing concept appears naturally in the context of Bogoliubov Hamiltonians. We follow Ref. 7.

Let $\mathcal{W}$ be a Hilbert space. By definition, a space *complex conjugate to* $\mathcal{W}$ is another Hilbert space $\overline{\mathcal{W}}$ equipped with a fixed anti-unitary map $\chi : \mathcal{W} \to \overline{\mathcal{W}}$.

In the literature, various authors use several concrete realizations of χ and $\overline{\mathcal{W}}$.

(1) We can assume that $\mathcal{W} = \overline{\mathcal{W}}$ and χ is antiunitary on $\mathcal{W}$ satisfying $\chi^2 = \mathbb{1}$. Suppose that we choose a basis fixed by χ (which is always possible). Then χ amounts to conjugating the components of vectors in this basis.

(2) We can also identify $\overline{\mathcal{W}}$ with the space of continuous linear functionals on $\mathcal{W}$. We then define χ to be the *Riesz isomorphism*, that is,

$$\langle \chi z | w \rangle := (z|w), \tag{A1}$$

where $(\cdot|\cdot)$ denotes the scalar product and $\langle\cdot|\cdot\rangle$ denotes the action of a linear functional. If we choose an orthonormal basis in $\mathcal{W}$ and the dual basis in $\overline{\mathcal{W}}$, then again χ amounts to conjugating the components of vectors in this basis.

(3) Finally, one can set $\overline{\mathcal{W}} = \mathcal{W}$ as the real vector space, changing only the complex structure to the opposite one and the scalar product to the complex conjugate of the original scalar product. χ is defined to be the identity operator. If we fix any basis, then similarly as before, χ is conjugating the components of vectors.

Interpretation (1) is the most naive one. It is often natural—especially if $\mathcal{W}$ is L^2 of some measure space. It is used, e.g., in Sec. 1 or in Ref. 12. Interpretation (2) is used in Ref. 15. Interpretation (3) is probably the most "orthodox" option—it does not invoke anything besides the vector space structure. In particular, it does not involve the Hilbert space structure of $\mathcal{W}$.

Note that for all three interpretations, in typical bases, the action of χ is equivalent to conjugating components of vectors. Similarly $\chi p \chi^{-1}$ and $\chi q \chi$ amount to conjugating matrix elements of p and q.

2. $*$-automorphisms of the algebra of bounded operators

Let $\mathcal{H}$ be a Hilbert space. A bijective linear map α on $B(\mathcal{H})$ is a $*$-*automorphism* if

$$\alpha(BC) = \alpha(B)\alpha(C), \quad \alpha(C^*) = \alpha(C)^*, \quad B, C \in B(\mathcal{H}). \tag{A2}$$

Proposition A.1 (Example 3.2.14, Ref. 4). The following statements are equivalent:

(1) *α is a $*$-automorphism of $B(\mathcal{H})$.*
(2) *There exists a unitary $U \in B(\mathcal{H})$ such that*

$$\alpha(C) = UCU^*, \quad C \in B(\mathcal{H}). \tag{A3}$$

If (1) and (2) hold, then U is determined uniquely up to a phase factor.

Let $\mathbb{R} \ni t \mapsto \alpha_t$ be a 1-parameter group of $*$-automorphisms of $B(\mathcal{H})$. We say that it is a C_0^*-*group* if $t \mapsto \alpha_t(C)$ for any $C \in B(\mathcal{H})$ is weakly continuous.

Proposition A.2 (Example 3.2.35, Ref. 4). The following statements are equivalent:

(1) *$t \mapsto \alpha_t$ is a C_0^*-group of $*$-automorphisms of $B(\mathcal{H})$.*
(2) *There exists a self-adjoint H on $\mathcal{H}$ such that*

$$\alpha_t(C) = \mathrm{e}^{\mathrm{i}tH} C \mathrm{e}^{-\mathrm{i}tH}, \quad C \in B(\mathcal{H}). \tag{A4}$$

If (1) and (2) hold, then H is defined uniquely up to an additive real constant.

3. Useful identities and inequalities

$$\int \frac{\mathrm{d}\tau}{\pi} \frac{1}{(C^2 + \tau^2)} = \frac{1}{\sqrt{C^2}}. \tag{A5}$$

$$\int \frac{\tau^2 \mathrm{d}\tau}{\pi} \frac{1}{(C^2 + \tau^2)^2} = \frac{1}{2\sqrt{C^2}}. \tag{A6}$$

$$|\mathrm{Tr} XY| \leq \|Y\| \mathrm{Tr}\sqrt{XX^*}. \tag{A7}$$

$$|\mathrm{Tr}XYXZ| \leq \sqrt{\mathrm{Tr}XYY^*X^*}\sqrt{XZZ^*X^*} \leq \|Y\|\|Z\|\mathrm{Tr}XX^*. \tag{A8}$$

$$\|a_{11}\|_1 + \|a_{22}\|_1 \leq \left\|\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\right\|_1. \tag{A9}$$

4. Useful lemmas

Lemma A.3. Let C be a bounded operator on $\mathcal{W}$ with a dense range. Let $\mathcal{D}$ be a dense subspace of $\mathcal{W}$. Then $C\mathcal{D}$ is dense.

Proof. Let $w \in \mathcal{W}$. We can find $v_1 \in \mathcal{W}$ such that $\|w - Cv_1\| < \frac{\epsilon}{2}$. We can find $v_2 \in \mathcal{D}$ such that $\|v_1 - v_2\| < \frac{\epsilon}{2\|C\|}$. Now

$$\|w - Cv_2\| < \|w - w_1\| + \|Cv_1 - Cv_2\| < \epsilon. \tag{A10}$$

□

Lemma A.4. Suppose that h_1 and h_2 are self-adjoint operators on $\mathcal{W}$ such that for any $w, w' \in \mathcal{W}$,

$$\lim_{t\to 0} \frac{1}{t}\Big((w|\mathrm{e}^{\mathrm{i}th_1}w') - (w|\mathrm{e}^{\mathrm{i}th_2}w')\Big) = 0. \tag{A11}$$

Then $h_1 = h_2$.

Proof. Let $w \in \mathrm{Dom}h_1$ and $w \in \mathrm{Dom}h_2$. Then (A11) implies

$$(h_1 w|w') = (w|h_2 w'). \tag{A12}$$

This means that $h_1 \subset h_2^* = h_2$ and $h_2 \subset h_1 = h_1^*$. Hence $h_1 = h_2$. □

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