Spectral theory of Pauli-Fierz operators

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Abstract

We study spectral properties of Pauli-Fierz operators which are commonly used to describe the interaction of a small quantum system with a bosonic free field. We give precise estimates of the location and multiplicity of the singular spectrum of such operators. Applications of these estimates, which will be discussed elsewhere, concern spectral and ergodic theory of non-relativistic QED. Our proof has two ingredients: the Feshbach method, which is developed in an abstract framework, and Mourre theory applied to the operator restricted to the sector orthogonal to the vacuum.

Contents

1 Introduction

In this paper we study spectral properties of a certain class of self-adjoint operators which appear in non-relativistic physics. They are commonly used to describe the interaction of a small quantum system (an "atom") with a bosonic free field ("radiation" or a "heat bath"). We will refer to them as Pauli-Fierz operators (see [Bl, BFSS, DG, PF]).

In a few words, the main result of our paper can be described as follows: the predictions of the second-order perturbation theory for embedded eigenvalues of a large class of Pauli-Fierz operators are correct for a sufficiently small coupling constant. A large part of our argument is abstract and uses only certain structural properties of Pauli-Fierz operators which are common to many different problems of mathematical physics. Therefore we would like to devote the first part of the introduction to a description of the general structure of our results and arguments. Only afterwards will we explain them in the context of Pauli-Fierz operators.

1.1 The conjugate operator method

Let H be a self-adjoint operator and Θ a fixed open subset of the real line. First, we would like to describe two well-known methods used in the study of the spectrum of the operator H inside Θ: the analytic deformation method and Mourre theory. These two methods have a lot in common and can be viewed as two versions of one method that we will call the conjugate operator method. Although in this paper we will use only Mourre theory, it is helpful to keep in mind the intuition derived from the analytic deformation method.

In what follows $\sigma(B)$ will denote the spectrum of the operator B and $\sigma_{\rm pp}(B)$ will denote its pure point spectrum.

(1) The analytic deformation approach. One considers a family of operators

$$H(\xi) := \mathrm{e}^{\mathrm{i}\xi S} H \mathrm{e}^{-\mathrm{i}\xi S}, \tag{1.1}$$

where S is an appropriately chosen self-adjoint operator (sometimes called a conjugate operator). The basic assumptions that one imposes on H and S are the following:
(a) The family $H(\xi)$ is analytic in some strip $|\mathrm{Im}\xi| < a$.
(b) For $\mathrm{Im}\xi < 0$, the essential spectrum of $H(\xi)$ "moves down" below Θ, uncovering a region below the real axis, which belongs to the unphysical sheet of the complex plane.

In the uncovered region, $H(\xi)$ may have some discrete eigenvalues. One can show that these eigenvalues do not depend on ξ and that the eigenvalues of $H(\xi)$ contained in $\Theta \subset \mathbf{R}$ coincide with $\sigma_{\rm pp}(H) \cap \Theta$. The non-real eigenvalues of $H(\xi)$ are called resonances. All these eigenvalues can be studied by standard methods of perturbation theory developed for isolated eigenvalues.

The analytic deformation method gives the following practical criterion for the study of the spectral properties of H: If for $\mathrm{Im}\xi < 0$ the deformed operator $H(\xi)$ has no eigenvalues in Θ, then H has no pure point spectrum in Θ. Even if $H(\xi)$ has some real eigenvalues in Θ, the method allows one to exclude the singular continuous spectrum inside this set.

(2) Mourre Theory and Limiting Absorption Principle. This is an infinitesimal version of the analytic deformation approach. Probably the most advanced version of Mourre theory can be found in [BG, BGS]. Below we briefly describe the Mourre theory following essentially [BG].

One again considers a family of operators (1.1), where now ξ is restricted to the real line. Let $n = 0, 1, \ldots$, $0 < \theta \leq 1$ and $\nu = n + \theta$. The basic assumptions of the Mourre theory are:
(a_ν) The nth derivative of $\xi \mapsto (z - H(\xi))^{-1}$ is θ-Hölder continuous.
(b) (The Mourre estimate). For any $x \in \Theta$ there exists an open interval $I \ni x$, a positive number $C_0 > 0$ and a compact operator K such that

$$\mathbf{1}_I(H)\mathrm{i}[S, H]\mathbf{1}_I(H) \geq C_0\mathbf{1}_I(H) + K. \tag{1.2}$$

(Here $\mathbf{1}_I(H)$ denotes the spectral projection of H onto I.)

If (a_ν) holds with $\nu = 1$, (b) and some other technical assumptions hold, then one can show that $\sigma_{\mathrm{pp}}(H) \cap \Theta$ is a discrete set which consists of eigenvalues of finite multiplicity. If in addition $\nu > 1$ and $\mu > \frac{1}{2}$, then for $x \in \Theta \setminus \sigma_{\mathrm{pp}}(H)$ one can establish the existence of

$$\langle S\rangle^{-\mu}(x + \mathrm{i}0 - H)^{-1}\langle S\rangle^{-\mu} := \lim_{y\downarrow 0}\langle S\rangle^{-\mu}(x + \mathrm{i}y - H)^{-1}\langle S\rangle^{-\mu}. \tag{1.3}$$

($\langle S\rangle$ denotes $(1 + S^2)^{\frac{1}{2}}$.) Note that (1.3) implies the absence of singular continuous spectrum in Θ. Moreover, if $\nu \leq \mu + \frac{1}{2}$ then the function (1.3) is $C^{n-1}(\Theta \setminus \sigma_{\mathrm{pp}}(H))$ and its $(n-1)$st derivative is θ-Hölder continuous. Statements similar to the existence of (1.3) usually go under the name of the Limiting Absorption Principle.

The Mourre method in the form described above does not give much information about the location and the multiplicity of $\sigma_{\mathrm{pp}}(H)$. However, if for all $x \in \Theta$ there is no compact operator K in the Mourre estimate (1.2), then $\sigma_{\mathrm{pp}}(H) \cap \Theta$ is empty.

The two methods described above are complementary. The analytic deformation method typically yields stronger results and allows study of resonances, which are of considerable physical interest. This method, however, is usually applicable to a restricted class of operators that meet the analyticity condition. The Mourre theory approach is of much wider applicability but it yields weaker results. In particular, resonances cannot be studied with this approach.

The analytic deformation technique was started in [AC, BC]. For more information about the early literature on this subject see [Si, RS4].

The Mourre theory originated in [Mo] and was further developed in [ABG, AHS, BG, CFKS, FH, JMP, PSS].

Both these methods were first applied to Schrödinger operators where S was the generator of dilations. The generator of dilations is often applicable in situations where the spectrum of the operator covers the half-line. The subset of the lower half-plane uncovered by the analytic deformation in this case is the wedge $-a < \arg z \leq 0$.

Another choice of S that can be found in the literature is the generator of translations. With such S, the set uncovered by the analytic deformation is the strip $-a < \mathrm{Im} z \leq 0$. This choice was made in works devoted to the Stark operator. A similar choice (the generator of translations in energy) was made in [JP1, JP2] in the context of Pauli-Fierz operators, and we will keep the same S here.

Our treatment of the Mourre theory follows [BG]. One of the differences between our approach and [BG] is that we use weighted spaces instead of Besov spaces. This makes our treatment somewhat less general, but also more elementary than that of [BG]. There are some other differences, due to the special properties of Pauli-Fierz operators, which we will describe later.

1.2 The Feshbach Method

The structural properties of Pauli-Fierz operators which will play an important role in our paper can be described as follows. They are self-adjoint operators of the form

$$H = H_{\mathrm{fr}} + \lambda V,$$

on a Hilbert space $\mathcal{H}$. This Hilbert space has a distinguished decomposition

$$\mathcal{H} = \mathcal{H}^{\mathrm{v}} \oplus \mathcal{H}^{\overline{\mathrm{v}}}. \tag{1.4}$$

With respect to this decomposition, the full operator can be written as a 2×2 matrix

$$H = \begin{bmatrix} H^{\mathrm{vv}} & H^{\mathrm{v}\overline{\mathrm{v}}} \\ H^{\overline{\mathrm{v}}\mathrm{v}} & H^{\overline{\mathrm{v}}\overline{\mathrm{v}}} \end{bmatrix}. \tag{1.5}$$

We will use a similar notation for other operators, for instance,

$$\mathbf{1} = \begin{bmatrix} \mathbf{1}^{\mathrm{vv}} & 0 \\ 0 & \mathbf{1}^{\overline{\mathrm{v}}\overline{\mathrm{v}}} \end{bmatrix}. \tag{1.6}$$

We assume that the "free operator" has the form

$$H_{\mathrm{fr}} = \begin{bmatrix} H_{\mathrm{fr}}^{\mathrm{vv}} & 0 \\ 0 & H_{\mathrm{fr}}^{\overline{\mathrm{v}}\overline{\mathrm{v}}} \end{bmatrix}, \tag{1.7}$$

and that the perturbation has the form

$$V = \begin{bmatrix} 0 & V^{\mathrm{v}\overline{\mathrm{v}}} \\ V^{\overline{\mathrm{v}}\mathrm{v}} & V^{\overline{\mathrm{v}}\overline{\mathrm{v}}} \end{bmatrix}. \tag{1.8}$$

To explain the Feshbach formula in its simplest form, we will assume that $z \notin \sigma(H^{\overline{\mathrm{v}}\overline{\mathrm{v}}})$. We remark that this is not the most interesting case in the context of our paper, since in our case $\sigma(H^{\overline{\mathrm{v}}\overline{\mathrm{v}}}) = \mathbf{R}$ and we want to study embedded eigenvalues. However, the assumption $z \notin \sigma(H^{\overline{\mathrm{v}}\overline{\mathrm{v}}})$ allows us to explain the Feshbach method with the least amount of technical assumptions.

For $z \notin \sigma(H^{\overline{\mathrm{v}}\overline{\mathrm{v}}})$ we introduce the following objects:

$$\begin{aligned} W_{\mathrm{v}}(z) &:= H^{\mathrm{v}\overline{\mathrm{v}}}(z\mathbf{1}^{\overline{\mathrm{v}}\overline{\mathrm{v}}} - H^{\overline{\mathrm{v}}\overline{\mathrm{v}}})^{-1}H^{\overline{\mathrm{v}}\mathrm{v}}, \\ G_{\mathrm{v}}(z) &:= z\mathbf{1}^{\mathrm{vv}} - H^{\mathrm{vv}} - W_{\mathrm{v}}(z). \end{aligned} \tag{1.9}$$

In the physics literature, the operator $W_{\mathrm{v}}(z)$ is sometimes called *the self-energy.* We propose to call $G_{\mathrm{v}}(z)$ *the resonance function.*

One can show that $z \notin \sigma(H)$ iff $0 \notin \sigma(G_{\mathrm{v}}(z))$. Moreover, if $0 \notin \sigma(G_{\mathrm{v}}(z))$, then one can express the resolvent of H in terms of the resolvent of $H^{\overline{\mathrm{v}}\overline{\mathrm{v}}}$ with the help of the following identity:

$$\begin{aligned} (z-H)^{-1} &= \left(\mathbf{1}^{\mathrm{vv}} + (z\mathbf{1}^{\overline{\mathrm{v}}\overline{\mathrm{v}}} - H^{\overline{\mathrm{v}}\overline{\mathrm{v}}})^{-1}H^{\overline{\mathrm{v}}\mathrm{v}}\right) G_{\mathrm{v}}^{-1}(z) \left(\mathbf{1}^{\mathrm{vv}} + H^{\mathrm{v}\overline{\mathrm{v}}}(z\mathbf{1}^{\overline{\mathrm{v}}\overline{\mathrm{v}}} - H^{\overline{\mathrm{v}}\overline{\mathrm{v}}})^{-1}\right) \\ &\quad + (z\mathbf{1}^{\overline{\mathrm{v}}\overline{\mathrm{v}}} - H^{\overline{\mathrm{v}}\overline{\mathrm{v}}})^{-1}. \end{aligned} \tag{1.10}$$

We call (1.10) *the Feshbach formula.* This formula was discovered independently by many physicists and mathematicians and it is known under a variety of names – the Grushin, Krein, Livshic formula. In the physics literature, where it is especially widely used and known (see for instance [CT]), it is usually called the Feshbach formula, and we keep this name. It was used recently in a context similar to ours in [BFS1, BFS2]. We refer the reader to [BFS1, How, MeMo] for more information on the literature about this formula.

1.3 Combining the Feshbach Method with the Mourre Theory

Next we want to describe how we study $\sigma_{\mathrm{pp}}(H)$ embedded in $\sigma(H)$ with the help of the Feshbach formula. To that end we study the boundary values of $(z-H)^{-1}$ at the real axis using the expression (1.10). We choose an operator S on $\mathcal{H}$ of the form

$$S = \begin{bmatrix} 0 & 0 \\ 0 & S^{\overline{\mathrm{v}}\overline{\mathrm{v}}} \end{bmatrix}. \tag{1.11}$$

Let us list the most important additional properties of H and S that we use in our analysis.
(a'_ν) The family $H(\xi)^{\overline{\mathrm{v}}\overline{\mathrm{v}}}$ satisfies an assumption analogous to (a_ν).
(b') The following global Mourre estimate holds:

$$\mathrm{i}[S^{\overline{\mathrm{v}}\overline{\mathrm{v}}}, H^{\overline{\mathrm{v}}\overline{\mathrm{v}}}] \geq C_0 > 0. \tag{1.12}$$

(c$'$) $V^{\rm v\overline{v}}(1+|S|)^{\nu-\frac{1}{2}}$ is bounded.

Using (a$'_\nu$) with $\nu>1$, (b$'$) and some additional technical assumptions, we develop the Mourre theory for $H^{\overline{\rm vv}}$, which implies that $H^{\overline{\rm vv}}$ satisfies the Limiting Absorption Principle uniformly on the whole real line. More precisely, for $\mu>\frac{1}{2}$ we prove that the limit

$$\langle S^{\overline{\rm vv}}\rangle^{-\mu}((x+{\rm i}0)\mathbf{1}^{\overline{\rm vv}}-H^{\overline{\rm vv}})^{-1}\langle S^{\overline{\rm vv}}\rangle^{-\mu}:=\lim_{y\downarrow 0}\langle S^{\overline{\rm vv}}\rangle^{-\mu}((x+{\rm i}y)\mathbf{1}^{\overline{\rm vv}}-H^{\overline{\rm vv}})^{-1}\langle S^{\overline{\rm vv}}\rangle^{-\mu} \tag{1.13}$$

exists and is uniformly bounded in x and λ. Moreover, if $\nu\geq\mu+\frac{1}{2}=n+1+\theta$ for some $0<\theta\leq 1$ then the function (1.13) is in $C^{n-1}(\mathbf{R})$ and its $(n-1)$st derivative is θ-Hölder continuous.

As we have mentioned before, our treatment of the Mourre theory follows [BG]. Nevertheless, there are some important differences. First, the spectrum of $H^{\overline{\rm vv}}$ covers the whole real line while [BG] make the assumption that operator has a spectral gap. More importantly, in our case the commutator ${\rm i}[H^{\overline{\rm vv}},S^{\overline{\rm vv}}]$ is not bounded relatively to $H^{\overline{\rm vv}}$. This leads to some difficulties related to the infrared problem of QED which require delicate arguments.

If (a$'_\nu$), (b$'$) and (c$'$) with $\nu>1$ hold, one shows, using (1.13), that the limit

$$W_{\rm v}(x+{\rm i}0):=\lim_{y\downarrow 0}W_{\rm v}(x+{\rm i}y)$$

exists and satisfies appropriate regularity properties. Now, it follows from the Feshbach formula that if $x\in\mathbf{R}$ and $0\notin\sigma(G_{\rm v}(x+{\rm i}0))$, then the Limiting Absorption Principle for H holds. Moreover, we can show that $0\in\sigma_{\rm disc}(G_{\rm v}(x+{\rm i}0))$ implies $x\in\sigma_{\rm pp}(H)$ and that the multiplicity of 0 as the eigenvalue of $G_{\rm v}(x+{\rm i}0)$ is equal to the multiplicity of x as the eigenvalue of H. Thus, the study of $\sigma_{\rm pp}(H)$ can be reduced to the study of $G_{\rm v}(x+{\rm i}0)$.

An additional property useful in our analysis is the bound

$$\langle S\rangle^{-\mu}(x+{\rm i}0-H^{\overline{\rm vv}})^{-1}\langle S\rangle^{-\mu}-\langle S\rangle^{-\mu}(x+{\rm i}0-H^{\overline{\rm vv}}_{\rm fr})^{-1}\langle S\rangle^{-\mu}=O(\lambda^\kappa), \tag{1.14}$$

where $\kappa=\frac{\nu-1}{\nu}$. These results together with the Feshbach formula give us a lot of control over the resolvent of the full operator H. In particular, we are able to describe the approximate location of the pure point spectrum, to give sharp estimates on its multiplicity and to rule out the singular continuous spectrum.

1.4 Main results

As we have said before, our results concern a certain class of Pauli-Fierz operators. While we still postpone the description of these operators, let us mention that for our purposes their most important properties are the following: They are self-adjoint operators of the form described in (1.5), (1.7), (1.8). In addition, they satisfy a certain hypothesis, called $S(\nu)$, which resembles the assumption (a$_\nu$) of the Mourre theory.

We introduce the following auxiliary object:

$$w(z) := V^{\mathrm{v}\overline{\mathrm{v}}}(z\mathbf{1}^{\overline{\mathrm{v}}\overline{\mathrm{v}}} - H_{\mathrm{fr}}^{\overline{\mathrm{v}}\overline{\mathrm{v}}})^{-1}V^{\overline{\mathrm{v}}\mathrm{v}}.$$

Note that $\lambda^2 w(z)$ is the second-order approximation to the self-energy $W_{\mathrm{v}}(z)$.

If $\nu > \frac{1}{2}$, it follows from the Mourre theory for $H_{\mathrm{fr}}^{\overline{\mathrm{v}}\overline{\mathrm{v}}}$ that $w(z)$ has the boundary values on the real line which we denote by $w(x+\mathrm{i}0)$. It is easy to see that $w(x+\mathrm{i}0)$ is a dissipative operator, that is, $\mathrm{Im}w(x+\mathrm{i}0) \leq 0$.

Let us describe the main results of our paper. Let H be a Pauli-Fierz operator satisfying appropriate conditions. We assume $S(\nu)$ with $\nu > 1$ and set $\kappa = \frac{\nu-1}{\nu}$.

(a) Our first result is Theorem 6.2. In this theorem we show that outside of an $O(\lambda^2)$ neighborhood of $\sigma(H^{\mathrm{vv}})$, the spectrum of H is purely absolutely continuous and that the Limiting Absorption Principle holds.

(b) Let k be an isolated eigenvalue of H^{vv}. Theorem 6.3 describes the structure of the spectrum of H in an $O(\lambda^2)$ neighborhood of k. Let p_k be the projection of $H_{\mathrm{fr}}^{\mathrm{vv}}$ onto k. Set

$$w_k := p_k w(k+\mathrm{i}0)p_k. \tag{1.15}$$

It is easy to see that this operator is dissipative. If

$$\sigma(w_k) \cap \mathbf{R} = \emptyset, \tag{1.16}$$

then we will say that the Fermi Golden Rule assumption for k holds. Under this assumption we can show that the spectrum of H is purely absolutely continuous in a neighborhood of k and that the Limiting Absorption Principle holds.

If the Fermi Golden Rule assumption fails, we show that outside an $O(\lambda^{2+\kappa})$ neighborhood of $k + \lambda^2\sigma(w_k)$, the spectrum of H is purely absolutely continuous and that the Limiting Absorption Principle holds.

(c) If the Fermi Golden Rule assumption fails and $m \in \sigma_{\mathrm{disc}}(w_k) \cap \mathbf{R}$, Theorem 6.4 describes the spectrum of H in a neighborhood of $k + \lambda^2 m$ where, by second-order perturbation theory, we can expect some eigenvalues of H. Let $p_{k,m}$ be the projection of w_k onto m (we will prove that this projection is orthogonal). We know from (b) that $\sigma_{\mathrm{pp}}(H)$ around $k + \lambda^2 m$ is located in an $O(\lambda^{2+\kappa})$ neighborhood of $k + \lambda^2 m$. In Theorem 6.4 we show that if $S(\nu)$ holds with $\nu > 2$, then the dimension of this point spectrum is not bigger than $\dim p_{k,m}$. Moreover, the Limiting Absorption Principle holds away from $\sigma_{\mathrm{pp}}(H)$.

To summarize, we show that isolated eigenvalues of H^{vv}, which may give rise to a cluster of eigenvalues of the size $O(\lambda^2)$, split into subclusters of size $O(\lambda^{2+\kappa})$ with the multiplicities estimated from above by the predictions of second-order perturbation theory. Outside these eigenvalues, the Limiting Absorption Principle holds. Note that as $\nu \to \infty$, $\kappa \to 1$, as expected from the analytic case.

Our results in (b) and (c) hold even if k has infinite multiplicity (this situation is typical for Pauli-Fierz Liouvilleans which arise in quantum statistical mechanics).

Our approach is reminiscent of what can be found in the early literature on stationary scattering theory, eq. in [Fr]. It has a lot in common with typical presentations of the perturbation of embedded eigenvalues found in physics textbooks [He]. The Fermi Golden Rule idea goes back to Dirac [Di] (for the history of the name see [Ha], Section I.1.5).

1.5 Pauli-Fierz operators

In this section we describe the operators that we study in our paper. They belong to the class of the so-called Pauli-Fierz operators, which are often used in quantum physics as generators of approximate dynamics of a (usually small) quantum system interacting with a free Bose gas. The class of operators that we study is quite abstract, with few specific assumptions; physical examples of Hamiltonians and Liouvilleans belonging to this class will be given in Sections 1.6-1.9.

Suppose that this small system is described by a Hilbert space $\mathcal{K}$ and a self-adjoint Hamiltonian K. The one-particle bosonic space is denoted by $\mathfrak{h}$ and the one-particle energy operator by ω. After the second quantization, the bosons are described by the symmetric Fock space $\Gamma(\mathfrak{h})$ and their Hamiltonian is $\mathrm{d}\Gamma(\omega)$. The Hilbert space of the composite system is $\mathcal{H} := \mathcal{K} \otimes \Gamma(\mathfrak{h})$. The free Pauli-Fierz operator has the form

$$H_{\mathrm{fr}} = K \otimes \mathbf{1} + \mathbf{1} \otimes \mathrm{d}\Gamma(\omega). \tag{1.17}$$

The interacting Pauli-Fierz operator is given by

$$H = H_{\mathrm{fr}} + \lambda V, \tag{1.18}$$

where $V = \varphi(\alpha)$, λ is a real constant and $\varphi(\alpha)$ is the field operator corresponding to $\alpha \in \mathcal{B}(\mathcal{K}, \mathcal{K} \otimes \mathfrak{h})$.

We remark that from the physical point of view one might wish to consider a more general class of operators which also have a quadratic term in the field. In fact, it is for reasons of space that we have decided to discuss the linear case only – our techniques easily extend to couplings which are quadratic in the field.

In the next section we will say a few words on the physical origin of Pauli-Fierz operators. Let us stress that they are interesting also from the purely mathematical point of view and they have been studied (under various names) by rigorous methods by many authors, eg [AH, Ar, BFS1, BFS2, BFSS, BS, HuSp1, HuSp2, JP1, JP2, Sk, DG].

Let us now state the most important additional assumption that we impose on the Pauli-Fierz operators in our paper. We suppose that, for some auxiliary Hilbert space $\mathfrak{g}$,

$$\mathfrak{h} = L^2(\mathbf{R}) \otimes \mathfrak{g}, \text{ and } \omega \text{ is the multiplication operator by } \omega \in \mathbf{R}. \tag{1.19}$$

The reader may find it surprising that the bosonic energy is unbounded both from below and above. Further on we will explain how usual physical systems with the energy bounded from below fit in our framework.

We split the Hilbert space into $\mathcal{H} = \mathcal{H}^{\mathrm{v}} \oplus \mathcal{H}^{\overline{\mathrm{v}}}$, where $\mathcal{H}^{\mathrm{v}} := \mathcal{K} \otimes \Gamma_0(\mathfrak{h})$ is the vacuum sector. We will call $\mathcal{H}^{\overline{\mathrm{v}}} := (\mathcal{H}^{\mathrm{v}})^{\perp}$ the radiation sector. It is easy to see that the operators H, H_{fr} and V are of the form (1.5), (1.7), (1.8). Note in particular that $H^{\mathrm{vv}} = K$.

The conjugate operator S, which plays the crucial role in the Mourre theory, is chosen to be $S := \mathbf{1} \otimes \mathrm{d}\Gamma(s)$, where $s = -\mathrm{i}\partial_{\omega}$ acts on $\mathfrak{h}$. Note that in the absence of interaction we have

$$\mathrm{i}[S, H_{\mathrm{fr}}] = N,$$

where N is the number operator. Therefore, on $\mathcal{H}^{\overline{\mathrm{v}}}$ we have a global Mourre estimate:

$$\mathrm{i}[S^{\overline{\mathrm{vv}}}, H_{\mathrm{fr}}^{\overline{\mathrm{vv}}}] = N^{\overline{\mathrm{vv}}} \geq 1.$$

A similar estimate holds in the interacting case: for sufficiently small λ, we can find $C_0 > 0$ such that

$$\mathrm{i}[S^{\overline{\mathrm{vv}}}, H^{\overline{\mathrm{vv}}}] \geq C_0 N^{\overline{\mathrm{vv}}}. \tag{1.20}$$

This relation will be the initial building block in our development of the Mourre theory.

1.6 From nonrelativistic QED to Pauli-Fierz Hamiltonians

In this section we briefly describe how Pauli-Fierz operators arise in physics. We follow [PF, CT, RZ, BFS1].

We start from the Hamiltonian of nonrelativistic QED. Suppose that we are given a system of N nonrelativistic particles. Assume that the ith particle has mass m_i and charge distribution $\rho_i(x)$. (If we suppose that the particles are pointlike we would obtain ultraviolet divergences. A smeared out charge distribution serves as an ultraviolet cut-off.) Suppose that the particles are in an external electrostatic potential Φ and interact with photons.

The full system is described by the Hilbert space $L^2(\mathbf{R}^{3N}) \otimes \Gamma(\mathbf{R}^3 \otimes \mathbf{C}^2)$ with the Hamiltonian equal to

$$\begin{aligned} H \;&= \sum_{i=1}^{N} \Big((2m_i)^{-1}(D_i - A_{\rho_i}(x_i))^2 + Q_i(x_i) \Big) + \sum_{1 \leq i < j \leq N} Q_{ij}(x_i - x_j) \\ &+ \sum_{s=1,2} \int a_s^*(k) a_s(k) |k| \mathrm{d}k, \end{aligned} \tag{1.21}$$

where

$$\begin{aligned} Q_i(x) \;&= \int \rho_i(x-y)\Phi(y)\mathrm{d}y, \\ Q_{ij}(x_i - x_j) \;&= \int\int \frac{\rho_i(x_i - y)\rho_j(x_j - y')}{|y - y'|} \mathrm{d}y\mathrm{d}y', \\ A(x) \;&= \sum_{s=1,2} (2\pi)^{-\frac{3}{2}} \int (2|k|)^{-\frac{1}{2}} \epsilon_s(k)(\mathrm{e}^{\mathrm{i}kx} a_s(k) + \mathrm{e}^{-\mathrm{i}kx} a_s^*(k))\mathrm{d}k, \\ A_{\rho_i}(x) \;&= \int A(x-y)\rho_i(y)\mathrm{d}y, \end{aligned}$$

and $\epsilon_s(k)$ are polarization vectors (an orthogonal basis of the orthogonal complement to k inside $\mathbf{R}^3$).

Let

$$\begin{array}{ll} H_{\rm matter} & := \sum_{i=1}^{N} \Big((2m_i)^{-1}D_i^2 + Q_i(x_i)\Big) + \sum_{1\leq i<j\leq N} Q_{ij}(x_i - x_j) \\ H_{\rm photon} & := \sum_{s=1,2} \int a_s^*(k)a_s(k)|k|\mathrm{d}k. \end{array} \tag{1.22}$$

Suppose that $H_{\rm matter}$ has discrete eigenvalues at the lower part of its spectrum. Let $\mathcal{K}_{\rm I} \subset L^2(\mathbf{R}^{3N})$ be the subspace spanned by the eigenvectors of the n lowest lying eigenvalues of $H_{\rm matter}$, and let $P_{\rm I}$ be the orthogonal projection onto $\mathcal{K}_{\rm I}$. Set $K_{\rm I} := P_{\rm I}H_{\rm matter}$. If one is interested in the physical processes that involve only bound states of the matter, then it is natural to restrict states of the system to the subspace

$$\mathcal{K}_{\rm I} \otimes \Gamma(L^2(\mathbf{R}^3)\otimes \mathbf{C}^2). \tag{1.23}$$

In this approximation the dynamics is generated by the following effective Hamiltonian:

$$H_{\rm I} := P_{\rm I}HP_{\rm I} = K_I + H_{\rm photon} + V_1 + V_2, \tag{1.24}$$

where

$$\begin{array}{rl} V_1 & = \sum_s \int \Big(g_s(k)a_s^*(k) + g_s(-k)a_s(k)\Big)\mathrm{d}k \\ V_2 & = \sum_s \int\int \Big(g(k_1,k_2)a_s^*(k_1)a_s^*(k_2) + g(-k_1,-k_2)a_s(k_1)a_s(k_2) \\ & \qquad\qquad +2g(k_1,-k_2)a_s^*(k_1)a_s(k_2)\Big)\mathrm{d}k_1\mathrm{d}k_2, \\ g_s(k) & = \sum_i (2m_i)^{-1}\frac{\hat\rho_i(k)}{\sqrt{2|k|}}P_{\rm I}\epsilon_s D_i \mathrm{e}^{\mathrm{i}kx_i}P_{\rm I}, \\ g(k_1,k_2) & = \sum_i (2m_i)^{-1}\frac{\hat\rho_i(k_1)}{\sqrt{2|k_1|}}\frac{\hat\rho_i(k_2)}{\sqrt{2|k_2|}}P_{\rm I}\mathrm{e}^{\mathrm{i}(k_1+k_2)x}P_{\rm I}. \end{array}$$

One often drops the higher order term V_2 and keeps just V_1. Without V_2, the Hamiltonian $H_{\rm I}$ has the form

$$H_{\rm I} = K_{\rm I}\otimes \mathbf{1} + 1\otimes \mathrm{d}\Gamma(|k|) + \int (\alpha_{\rm I}(k)a^*(k) + \alpha_{\rm I}^*(k)a(k)\mathrm{d}k.$$

and this operator belongs to the class of Pauli-Fierz operators.

An alternative strategy, due to Pauli-Fierz, is also commonly used to approximate the Hamiltonian (1.21) [PF, BFS1]. First one uses the unitary transformation

$$U = \exp\Big(\mathrm{i}\textstyle\sum_i x_i A_{\rho_i}(0)\Big),$$

obtaining

$$\begin{array}{rl} UHU^* & = H_{\rm matter} + \int \Big|\sum_i \hat\rho_i(k)x_i\Big|^2\mathrm{d}k + H_{\rm photon} \\ & \qquad + \sum_i x_i \int \epsilon_s(k)\sqrt{\frac{|k|}{2}}\Big(\hat\rho_i(k)a^*(k) + \hat\rho_i(-k)a(k)\Big)\mathrm{d}k + \cdots, \end{array}$$

where $\cdots$ denotes the terms that depend on $A(x_i) - A(0)$. Let $\mathcal{K}_{\mathrm{II}} \subset L^2(\mathbf{R}^{3N})$ denote the subspace spanned by the eigenvectors of the n lowest lying eigenvalues of

$$H_{\mathrm{matter,II}} := H_{\mathrm{matter}} + \int \Big|\sum_i \hat{\rho}_i(k) x_i\Big|^2 \mathrm{d}k$$

and P_{II} the projection onto $\mathcal{K}_{\mathrm{II}}$. Set $K_{\mathrm{II}} = P_{\mathrm{II}} H_{\mathrm{matter,II}}$. Then one can argue that one should use the Hilbert space

$$\mathcal{K}_{\mathrm{II}} \otimes \Gamma(L^2(\mathbf{R}^3) \otimes \mathbf{C}^2) \tag{1.25}$$

and the following effective Hamiltonian:

$$\begin{aligned} H_{\mathrm{II}} := P_{\mathrm{II}} U H U^* P_{\mathrm{II}} &= K_{\mathrm{II}} + H_{\mathrm{photon}} \\ &+ B \textstyle\int \epsilon_s(k) \sqrt{\frac{|k|}{2}} \Big(\hat{\rho}_i(k) a^*(k) + \hat{\rho}(-k) a(k)\Big) \mathrm{d}k, \end{aligned} \tag{1.26}$$

where

$$B = \sum_i P_{\mathrm{II}} x_i P_{\mathrm{II}}.$$

Again, (1.26) has the form

$$H_{\mathrm{II}} = K_{\mathrm{II}} \otimes \mathbf{1} + \mathbf{1} \otimes \mathrm{d}\Gamma(|k|) + \int (\alpha_{\mathrm{II}}(k) a^*(k) + \alpha_{\mathrm{II}}^*(k) a(k) \mathrm{d}k$$

and belongs to the class of Pauli-Fierz operators.

The advantage of H_{II} over H_{I} is a milder infrared behavior of the interaction:

$$\alpha_{\mathrm{I}}(k) \sim C|k|^{-\frac{1}{2}}, \quad \alpha_{\mathrm{II}}(k) \sim C|k|^{\frac{1}{2}}, \quad |k| \to 0.$$

For further use let us note that the one photon space $\tilde{\mathfrak{h}} = L^2(\mathbf{R}^3) \otimes \mathbf{C}^2$, by using polar coordinates, can be identified with $L^2(\mathbf{R}_+) \otimes \mathfrak{g}$, where $\mathfrak{g} = L^2(S^2) \otimes \mathbf{C}^2$ and S^2 denotes the unit sphere. After this change of coordinates the photon energy becomes the operator of multiplication by $\tilde{\omega} \in \mathbf{R}_+$. The infrared behavior of the interaction is then

$$\alpha_{\mathrm{I}}(\tilde{\omega}) \sim C\tilde{\omega}^{\frac{1}{2}}, \qquad \alpha_{\mathrm{II}}(\tilde{\omega}) \sim C\tilde{\omega}^{\frac{3}{2}}, \quad \tilde{\omega} \to 0. \tag{1.27}$$

Clearly, the operators (1.24) and (1.26) have a positive one-photon energy and do not satisfy the assumption (1.19). We will explain below several different means by which such operators can be naturally fit into the class of operators which satisfy the assumption (1.19).

Finally, we mention that the Pauli-Fierz operators also arise in solid state physics, where they are used to describe the interaction of phonons with a quantum system with finitely many degrees of freedom. In this case, the form of the function α is dictated by the particular physical situation one wishes to describe and to a large extent α can be an arbitrary function [LCD].

1.7 Pauli-Fierz Liouvilleans

The most common description of a quantum system is based on the so-called Schrödinger picture. Pure states are described by rays in a Hilbert space $\mathcal{H}$ and the generator of dynamics is a self-adjoint operator H called a Hamiltonian. Hamiltonians describing realistic quantum systems are usually bounded from below.

It is often advantageous to use the so-called Heisenberg picture. In this picture, the generator of the dynamics is sometimes called Liouvillean and the states (including mixed ones) are described by positive Hilbert-Schmidt operators, see [Ha, HHW] for details. The space of Hilbert-Schmidt operators is unitarily equivalent to $\mathcal{H} \otimes \overline{\mathcal{H}}$ and the Liouvillean is unitarily equivalent to

$$L_0 = H \otimes \mathbf{1} - \mathbf{1} \otimes \overline{H} \tag{1.28}$$

(the bar denotes complex conjugation). In the sequel we will distinguish between the Hamiltonian H, which is bounded from below, and the operator L_0, which we will call the "zero temperature Liouvillean".

The above description of the Liouvillean formalism is actually valid only at the zero temperature. If we deal with an infinitely extended system at a positive temperature, it is appropriate to use a more sophisticated formalism, which involves von Neumann algebras in standard forms and the Araki-Woods representation of CCR. This approach is explained in [AW, BR, JP1, JP2] and it will also be the subject of our forthcoming paper [DJP].

Let us briefly explain the basic ideas related to a Pauli-Fierz system at temperature $T \geq 0$. (For simplicity, in the sequel we will drop the dependence on the spin of the photon). Instead of the Hilbert space (1.23) or (1.25) one should use a larger Hilbert space:

$$\mathcal{K} \otimes \overline{\mathcal{K}} \otimes \Gamma(L^2(\mathbf{R}^3) \oplus \overline{L^2(\mathbf{R}^3)}) \tag{1.29}$$

($\overline{\mathcal{K}}$ and $\overline{L^2(\mathbf{R}^3)}$ denote the complex conjugate of the spaces $\mathcal{K}$ and $L^2(\mathbf{R}^3)$). The $L^2(\mathbf{R}^3)$ part of the Fock space corresponds to the "excitations" over the Gibbs state, and the $\overline{L^2(\mathbf{R}^3)}$ part corresponds to the "holes". The former will be described by creation/annihilation operators denoted by $a_{\mathrm{l}}^*(k)$ and $a_{\mathrm{l}}(k)$ and the latter will be described by $a_{\mathrm{r}}^*(k)$ and $a_{\mathrm{r}}(k)$. Let $\rho(k) := (\mathrm{e}^{|k|/T} - 1)^{-1}$ for $T > 0$ and $\rho = 0$ for $T = 0$. The function $\rho(k)$ (the Planck law) describes the momentum distribution of the Bose gas in thermal equilibrium at temperature T. The Liouvillean at temperature T corresponding to the Pauli-Fierz Hamiltonian (1.24) or (1.26) is equal to

$$L_T = L_{\mathrm{fr}} + \lambda V_T \tag{1.30}$$

where

$$
\begin{aligned}
L_{\mathrm{fr}} \;&:= K\otimes\mathbf{1}\otimes\mathbf{1}-\mathbf{1}\otimes\overline{K}\otimes\mathbf{1}\\
&+\mathbf{1}\otimes\mathbf{1}\otimes\int a_{\mathrm{l}}^*(k)a_{\mathrm{l}}(k)|k|\mathrm{d}k-\mathbf{1}\otimes\mathbf{1}\otimes\int a_{\mathrm{r}}^*(k)a_{\mathrm{r}}(k)|k|\mathrm{d}k\\
V_T \;&:= \int\alpha(k)\otimes\mathbf{1}\otimes\Big(\sqrt{1+\rho(k)}a_{\mathrm{l}}^*(k)+\sqrt{\rho(k)}a_{\mathrm{r}}(k)\Big)\mathrm{d}k\\
&+\int\alpha^*(k)\otimes\mathbf{1}\otimes\Big(\sqrt{1+\rho(k)}a_{\mathrm{l}}(k)+\sqrt{\rho(k)}a_{\mathrm{r}}^*(k)\Big)\mathrm{d}k\\
&-\int\mathbf{1}\otimes\overline{\alpha}(k)\otimes\Big(\sqrt{\rho(k)}a_{\mathrm{l}}(k)+\sqrt{1+\rho(k)}a_{\mathrm{r}}^*(k)\Big)\mathrm{d}k\\
&-\int\mathbf{1}\otimes\overline{\alpha}^*(k)\otimes\Big(\sqrt{\rho(k)}a_{\mathrm{l}}^*(k)+\sqrt{1+\rho(k)}a_{\mathrm{r}}(k)\Big)\mathrm{d}k.
\end{aligned}
$$

For $T>0$, the form of the Liouvillean is dictated by Tomita-Takesaki modular theory [BR, JP2, DJP], while for $T=0$, L_T reduces to the zero-temperature Liouvillean L_0. One can use polar coordinates in both copies of $\mathbf{R}^3$ appearing in (1.29), as described at the end of the last subsection, and then "glue" them together using the exponential law for bosonic systems, see [JP1, JP2, DJP] for details (for an example of how is this "gluing" done see Section 5.2 below). Then (1.29) becomes $\mathcal{K}\otimes\overline{\mathcal{K}}\otimes\Gamma(\mathbf{R}\otimes\mathfrak{g})$ and the Liouvillean (1.30) becomes a Pauli-Fierz operator satisfying the condition (1.19). Thus the main results of our paper can be applied to Pauli-Fierz Liouvilleans at temperature $T\geq 0$ (including the case of $T=0$). We will indicate below the main ideas of this application.

1.8 Return to equilibrium

One says that a quantum dynamical system has the property of return to equilibrium if all normal states converge to a unique stationary state as $t\to\pm\infty$ (see eg [Ro1, Ro2, JP2, JP3]). If the stationary state is faithful then this property holds if the Liouvillean has no singular spectrum except for a simple eigenvalue zero [JP2]. The results of our paper can be used to prove the return to equilibrium for a large class of Pauli-Fierz systems in the whole range of temperatures $T>0$, uniformly in the coupling constant. The details of this application require use of techniques of algebraic quantum statistical mechanics and will be given in our forthcoming paper [DJP], so below we just briefly indicate some of the main ideas involved in this application.

Suppose that the operator K has pure point spectrum. Then the pure point spectrum of L_{fr} consists of differences of eigenvalues of K. In particular, 0 is an eigenvalue of degeneracy at least $\dim\mathcal{K}$. All these eigenvalues are embedded in the continuous spectrum of L_{fr} and one may ask how many of them survive perturbation V_T.

If the temperature T is positive, then the general theory of perturbations of KMS states due to Araki guarantees that the perturbed Liouvillean L_T has at least one eigenstate with eigenvalue zero – the vector representative of the KMS state of the perturbed system (see eg [BR]).

It has been proven in [AH, BFS1] (see also [DuSp, Sp1, Sp2], and [BFS3] for the physically realistic model (1.21)) that a Pauli-Fierz Hamiltonian H has a ground state. This implies that the zero temperature Liouvillean L_0 has an eigenvalue 0.

Thus for any $T \geq 0$ we know that 0 is an eigenvalue of the perturbed Liouvillean L_T. This is reflected by the fact that the operator w_0, defined for L_T by (1.15), always has a zero eigenvalue.

Suppose that we assume that 0 is a simple eigenvalue of w_0 and for any $k \neq 0$, the operator w_k has spectrum away from the real line. (These assumptions can be shown to hold generically [Fri, Sp3]). Suppose that the assumptions of Theorem 6.4 are satisfied. Then this theorem implies that, for a small coupling constant, L_T has no singular spectrum, except possibly for a simple eigenvalue at zero. But since we know that the KMS/ground state survives as a bound state of L_T, we conclude that the singular spectrum of L_T consists exactly of a simple eigenvalue zero.

Under appropriate conditions on the interaction, the assumptions of Theorem 6.4 can be checked uniformly in the temperature. In this case, we obtain a proof of the return to equilibrium property for Pauli-Fierz systems with a small coupling constant uniformly in the temperature $T > 0$.

If α has an analytic continuation to a strip along the real axis then there exists an alternative proof of the return to equilibrium based on the analytic deformation method [JP1, JP2]. In many respects, this method yields stronger results than the Mourre theory. However, the analyticity condition this method requires is never satisfied in the zero-temperature case. Moreover, the analytic deformation method of [JP1, JP2] works for $|\lambda| \leq \Lambda(T)$, where $\Lambda(T) \downarrow 0$ as $T \downarrow 0$, and so this method does not yield the return to equilibrium property uniformly in the temperature $T > 0$.

1.9 "Gluing non-physical free bosons"

Recall that Hamiltonians H are related to zero-temperature Liouvilleans L_0 by the formula (1.28). Therefore, if we study properties of L_0, as described in the previous section, we can learn about some of the properties of H. For instance, L_0 satisfies $\dim \mathbf{1}_{\{0\}}(L_0) = 1$ iff H satisfies $\dim \mathbf{1}^{\mathrm{pp}}(H) = 1$.

There exists, however, a more direct method of studying spectral properties of Pauli-Fierz Hamiltonians by applying the results of our paper. This method is described in detail in Section 5.2. Its main idea is to add a non-physical copy of the free bosonic field. In this way, the one-particle space becomes isomorphic to $L^2(\mathbf{R}) \otimes \mathfrak{g}$. After this modification we obtain an extended Hilbert space and an extended Pauli-Fierz operator, which satisfies the condition (1.19).

The pure point and singular continuous spectrum of the Pauli-Fierz operator do not change after gluing the non-physical bosons. Therefore, the spectral results we prove for the extended Pauli-Fierz operator remain valid for the Pauli-Fierz Hamiltonian.

Consider a Pauli-Fierz Hamiltonian with positive boson energy, where $\alpha(\tilde{\omega})$ behaves as $\tilde{\omega}^\delta$ around zero. It is easy to see that, after gluing the nonphysical bosons, Hypothesis

$S(\nu)$ is satisfied with $\nu > \delta + \frac{1}{2}$. Therefore, Theorems 6.2 and 6.3 hold for $\delta > \frac{1}{2}$ and Theorem 6.4 holds for $\delta > \frac{3}{2}$.

By (1.27), Theorems 6.2 and 6.3 do not cover the effective Hamiltonian H_{I} of (1.24) but apply to H_{II} of (1.26). Unfortunately, Theorem 6.4 does not cover either H_{I} or H_{II}.

1.10 Comparison with the literature

In the literature one can find other applications of the Mourre theory to zero-temperature Pauli-Fierz operators. Probably the earliest is contained in [HuSp2], where the massive radiation field is considered. The conjugate operator that is used there is the second quantization of the generator of translations in the energy variable. However, unlike in our paper, in [HuSp2] the energy variable is restricted to the positive half-line, hence this operator is not self-adjoint.

The massless case, which is physically more important and technically more demanding due to infrared difficulties, was first considered in [BFS1]. This paper contains several interesting results. One of them is based on the Mourre theory using the second quantization of the generator of dilations, which is applied to study the spectrum of H in regions away from $\sigma(K)$.

Concerning the Mourre theory in the massless case, the first relatively complete results are presented in [Sk]. This work uses the same non-self-adjoint S as [HuSp2], which is however approximated with a sequence of self-adjoint operators.

Another choice of S, a suitably modified second quantization of the generator of dilations, is used in [BFSS]. Note that the generator of dilations should in principle allow one to treat perturbations with a more singular infrared behavior than the generator of translations.

All of the above papers, including ours, give results which are valid for a small coupling constant. All values of the coupling constant are covered in [DG], where the Mourre theory for a massive radiation field is developed. Unfortunately, the techniques of [DG] do not seem applicable to the massless case considered here.

Let us remark that in all of the above works the Mourre theory is studied on the whole Hilbert space. The distinct feature of our method is that we apply first Mourre theory to $H^{\overline{\mathrm{vv}}}$ and then use the Feshbach method. We believe that this approach is natural and that it gives more precise information on the location and multiplicity of embedded eigenvalues.

Among other works related to our paper we mention those in which the complex deformation technique is applied to Pauli-Fierz operators. In the case of bosons with a positive mass the complex scaling was first used in [OY].

In the case of massless bosons, the complex scaling method was studied by [BFS1, BFS2]. These papers contain a technique, which the authors call the renormalization group, that is used to study the properties of the spectrum of the deformed operator. Among the results of the paper one can find the proof that eigenvalues satisfying the

Fermi Golden Rule turn into resonances, under the assumption of the dilation analyticity. In [BFS3] the complex scaling has been applied to the full Hamiltonian (1.21).

The papers [JP1, JP2] are the main predecessors of our work – these works treated positive temperature Liouvilleans, introduced the method of gluing negative and positive energy bosons and the analytic deformation generated by the second quantization of the translation operator.

Some of the results of our paper are quite general. These results concern spectral analysis of a relatively arbitrary linear operator and are related to the Feshbach method. Similar ideas can be found throughout the literature, notably in [BFS1, BFS2, GGK]. We believe that these results are of interest outside of the context of Pauli-Fierz operators. In particular, the following of our general results appear to be new: Proposition 3.2 about real eigenvalues of a dissipative operator, Proposition 3.7 and Theorem 3.8 about the Feshbach method for embedded eigenvalues, and Theorem 3.12 and Corollary 3.13 about estimating the number of embedded eigenvalues.

Almost one year after the original version of our paper was circulated, a very interesting, closely related paper appeared [BFS4]. This paper describes a proof of the return to equilibrium property for a certain class of Pauli-Fierz systems uniformly in temperature for a small coupling constant. As we mentioned above, a similar result (for a somewhat different class of interactions) is a relatively easy consequence of the main result of our paper and will be the subject of our forthcoming paper. We remark that the methods of [BFS4] are completely different from ours. One of the main features of [BFS4] is the use of the generator of dilations, whereas in our approach the major role is played by the generator of translations. We will compare these two methods in our forthcoming paper [DJP].

1.11 Organization of the paper

The paper is organized as follows.

In Chapter 2 we introduce notation and, for reference purposes, state some general facts about operators in Hilbert spaces.

In Chapter 3 we describe some properties of self-adjoint operators in a Hilbert space decomposed as a direct sum of two Hilbert spaces. They are centered around the Feshbach formula. This formula leads to certain identities for the projections onto eigenvectors of the operator H, which we found appealing and useful in our analysis. It also leads to certain precise estimates on the number of eigenvalues of the operator H. In spite of the fact that they are general and simple, some of the results of Chapter 3 appear to be new. Problems involving embedded eigenvalues arise naturally in spectral geometry, number theory and mathematical physics, and we hope that some of the results of this chapter will find applications outside of our work.

We present in a parallel way results concerning the spectrum of H outside of $\sigma(H^{\overline{\mathrm{vv}}})$ and the embedded point spectrum of H inside $\sigma(H^{\overline{\mathrm{vv}}})$. The results about the spectrum outside $\sigma(H^{\overline{\rm vv}})$ are less technical and can be partly found in the literature, eg. in [BFS1]. They are not used in the remaining part of our paper. On the other hand, the more difficult results concerning the embedded spectrum inside $\sigma(H^{\overline{\rm vv}})$ are among the most important tools of our paper. We believe that it is helpful for the reader to compare these two types of results.

In Chapter 4 we review the basic notions of quantum field theory. This chapter makes the paper essentially self-contained.

In Chapter 5 we introduce Pauli-Fierz Hamiltonians and discuss some of their basic properties.

The main results of the paper are stated in Chapter 6.

Chapter 7 describes the proof of the Limiting Absorption Principle for the operator $H^{\overline{\rm vv}}$. As we have stressed before, our proof follows the arguments of [BG]. In Section 7.1 we derive a bound on the boundary value of the resolvent of $H^{\overline{\rm vv}}$, which is the basic ingredient of the Limiting Absorption Principle, following essentially the original arguments of [Mo] and [PSS], with modifications due to [BG]. The main additional difficulty is the infrared problem, which we handle following [JP1]. In Section 7.2 we study the regularity of the boundary value of the resolvent of $H^{\overline{\rm vv}}$, following [BG]. In Section 7.4 we estimate the difference of the full and the free resolvent.

Chapter 8 completes the proof of our main results. The main tool is the Feshbach formula, which is applied several times to various decompositions of our Hilbert space.

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2 Preliminaries

In this section we set the notation and, for reference purposes, recall some definitions and facts which will be used in the paper.

$\mathbf{N} := \{0, 1, 2, \ldots\}$ denotes the set of natural numbers (including 0). We set $\langle t\rangle := \sqrt{1+t^2}$. We will also use the shorthand

$$\mathbf{C}_\pm := \{z \in \mathbf{C}\ :\ \pm{\rm Im} z > 0\},$$

$$\mathbf{R}_\pm := \{x \in \mathbf{R}\ :\ \pm x > 0\}.$$

The closure of a set $\Omega \subset \mathbf{C}$ we denote by $\overline{\Omega}$.

If $\Omega \subset \mathbf{C}$ and $r > 0$, we set

$$B(\Omega, r) := \{z \in \mathbf{C} \;:\; \mathrm{dist}(\Omega, z) < r\},$$
$$\overline{B}(\Omega, r) := \{z \in \mathbf{C} \;:\; \mathrm{dist}(\Omega, z) \leq r\}.$$

In particular, for $k \in \mathbf{C}$,

$$B(k, r) := B(\{k\}, r), \quad \overline{B}(k, r) := \overline{B}(\{k\}, r)$$

denotes the open/closed ball of center k and radius r.

If $\Theta \subset \mathbf{R}$, we set

$$I(\Theta, r) := \{x \in \mathbf{R} \;:\; \mathrm{dist}(\Theta, x) < r\},$$
$$\overline{I}(\Theta, r) := \{x \in \mathbf{R} \;:\; \mathrm{dist}(\Theta, x) \leq r\}.$$

In particular, for $k \in \mathbf{R}$,

$$I(k, r) := I(\{k\}, r), \quad \overline{I}(k, r) := \overline{I}(\{k\}, r)$$

denotes the open/closed interval of center k and radius r.

Let $\mathcal{H}$ be a Hilbert space. The inner product on $\mathcal{H}$ we denote by $(\cdot|\cdot)$.

Let $\mathcal{H}_1, \mathcal{H}_2$ be Hilbert spaces. We denote by $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ the Banach space of all bounded operators from $\mathcal{H}_1$ to $\mathcal{H}_2$. If these two spaces are the same and equal to $\mathcal{H}$, we will write simply $\mathcal{B}(\mathcal{H})$.

Let $\Omega \subset \mathbf{C}$. In this paper we will often deal with operator-valued functions

$$\Omega \ni z \mapsto A(z) \in \mathcal{B}(\mathcal{H}). \tag{2.31}$$

Unless otherwise specified, the various limits of such functions are always defined with respect to the norm of the Banach space $\mathcal{B}(\mathcal{H})$.

Definition 2.1 *Assume that $z_0 \in \Omega$ is not an isolated point of Ω. We say that the function (2.31) is differentiable at z_0 with derivative $A'(z_0) \in \mathcal{B}(\mathcal{H})$ if*

$$\lim_{\substack{z \to z_0 \\ z \in \Omega, z \neq z_0}} \|(z - z_0)^{-1}(A(z) - A(z_0)) - A'(z_0)\| = 0.$$

We say that the function $A(z)$ is differentiable on Ω if it is differentiable at every non-isolated point of Ω.

As usual, we denote the n-th derivative by $\partial_z^n A(z)$.

Definition 2.2 *Let $n \in \mathbf{N}$. We say that a function $\Omega \ni z \mapsto A(z)$ is in the class $C_{\mathrm{u}}^n(\Omega)$ if it has a continuous n-th derivative on Ω and satisfies the bound*

$$\|\partial_z^m A(z)\| \leq C_m, \quad z \in \Omega, \quad m = 0, \ldots, n. \tag{2.32}$$

Let $\ell : \overline{\mathbf{R}}_+ \mapsto \overline{\mathbf{R}}_+$ be a positive continuous function with $\ell(0) = 0$. We say that the function $\Omega \ni z \mapsto A(z)$ is in the class $C_{\mathrm{u}}^{n,\ell}(\Omega)$ if $A \in C_{\mathrm{u}}^{n}(\Omega)$ and there exists C such that

$$\|\partial_z^n A(z_1) - \partial_z^n A(z_2)\| \leq C\ell(|z_1 - z_2|), \qquad z_1, z_2 \in \Omega. \tag{2.33}$$

If we have a family of functions A_λ defined on sets Ω_λ, $\lambda \in \mathcal{I}$, we say that A_λ is of the class $C_{\mathrm{u}}^{n}(\Omega_\lambda)$ or $C_{\mathrm{u}}^{n,\ell}(\Omega_\lambda)$ uniformly in λ if the constants C in (2.33) and C_m in (2.32) can be chosen independently of $\lambda \in \mathcal{I}$.

For $0 < \theta \leq 1$ we define functions ℓ_θ on $\overline{\mathbf{R}}_+$ by the formula

$$\ell_\theta(\tau) = \begin{cases} \tau^\theta & \text{if } 0 < \theta < 1 \\ \tau\,(1 + \ln(1 + \tau^{-1})) & \text{if } \theta = 1. \end{cases} \tag{2.34}$$

The classes $C_{\mathrm{u}}^{n,\ell_\theta}$ will figure in the Limiting Absorption Principle which we will establish in this paper. In the sequel we use the shorthands $C_{\mathrm{u}}^{n,\theta} = C_{\mathrm{u}}^{n,\ell_\theta}$.

Let $\Omega \subset \mathbf{C}$ and $\Gamma \subset \partial\Omega$. We say that a continuous function $\Omega \ni z \mapsto A(z)$ extends by continuity to $\Omega \cup \Gamma$ if for every $z_0 \in \Gamma$ the limit

$$A(z_0) := \lim_{z \to z_0, z \in \Omega} A(z)$$

exists. We will denote the functions extended by continuity with the same letter. If $\Omega \subset \mathbf{C}_\pm$ and $A(z)$ extends by continuity to a part of the real axis, we denote by $A(x \pm \mathrm{i}0)$ its values along $\mathbf{R}$.

In the development of Mourre theory, we will make use of the following two simple facts.

Proposition 2.3 *Let Ω be an open convex set, and let*

$$\overline{\mathbf{R}}_+ \times \Omega \ni (\epsilon, z) \mapsto A(\epsilon, z) \in \mathcal{B}(\mathcal{H})$$

be a bounded function which is continuously differentiable in each variable separately. Assume further that for some constants C and $0 < \theta \leq 1$,

$$\sup_{z \in \Omega} \|\partial_\epsilon^k \partial_z^l A(\epsilon, z)\| < C\epsilon^{-1}\ell_\theta(\epsilon), \qquad k + l = 1.$$

Then, the function $\Omega \ni z \mapsto A(0, z)$ is in the class $C_{\mathrm{u}}^{0,\theta}(\Omega)$.

Proof. For $z_1, z_2 \in \Omega$ and $\epsilon > 0$ we have

$$\begin{aligned} \|A(0, z_1) - A(0, z_2)\| &\leq \textstyle\int_0^\epsilon \|\partial_\tau A(\tau, z_1)\| \mathrm{d}\tau + |z_1 - z_2| \int_0^1 \|\partial_z A(\epsilon, z_1 + t(z_2 - z_1))\| \mathrm{d}t \\ &\quad + \textstyle\int_0^\epsilon \|\partial_\tau A(\tau, z_2)\| \mathrm{d}\tau \\ &\leq 2C \textstyle\int_0^\epsilon \tau^{-1}\ell_\theta(\tau)\mathrm{d}\tau + C|z_1 - z_2|\epsilon^{-1}\ell_\theta(\epsilon). \end{aligned} \tag{2.35}$$

Using the form of functions ℓ_θ one easily shows that for some constants C_θ and all $\epsilon \geq 0$,

$$\int_0^\epsilon \tau^{-1}\ell_\theta(\tau)\mathrm{d}\tau \leq C_\theta \ell_\theta(\epsilon).$$

Combining this estimate with (2.35) and setting $\epsilon = |z_1 - z_2|$ we derive that $A(0, z) \in C_{\mathrm{u}}^{0,\theta}(\Omega)$. $\square$

Proposition 2.4 *Let A_λ, $\lambda \in \mathcal{I}$, be a family of functions defined on open convex sets $\Omega_\lambda \subset \mathbf{C}$. If the family A_λ is of the class $C_{\mathrm{u}}^{n,\ell}(\Omega_\lambda)$ uniformly in λ, then the functions A_λ extend by continuity to $\overline{\Omega}_\lambda$ and the family A_λ is of the class $C_{\mathrm{u}}^{n,\ell}(\overline{\Omega}_\lambda)$ uniformly in λ.*

The proof of this proposition is elementary and we will skip it.

Let H be a closed operator on $\mathcal{H}$. We denote the domain of H by $\mathcal{D}(H)$ and the spectrum of H by $\sigma(H)$. The numerical range of H is defined by

$$\mathfrak{N}(H) := \{(\psi|H\psi) \ : \ \psi \in \mathcal{D}(H), \ \|\psi\| = 1\}.$$

The vector space $\mathcal{D}(H)$ equipped with the graph norm

$$\|\psi\|_H = \|\psi\| + \|H\psi\|,$$

is a Banach space. A vector space $\mathcal{C} \subset \mathcal{D}(H)$ is called a core of H if $\mathcal{C}$ is dense in $\mathcal{D}(H)$ in the graph norm. The following useful fact is well known:

Lemma 2.5 *Let A and B be closed operators and let $\mathcal{D}(B)$ be a core of A. Then any core of B is a core of A.*

For any closed operator H, we say that Ω is an isolated subset of $\sigma(H)$ if it is relatively closed and open subset of $\sigma(H)$. If in addition Ω is bounded, then there exists a simple closed path γ which separates Ω and $\sigma(H) \setminus \Omega$, and we can define the spectral projection of H onto Ω by the formula

$$\mathbf{1}_\Omega(H) = \frac{1}{2\pi\mathrm{i}} \oint_\gamma (z - H)^{-1}\mathrm{d}z.$$

It is easy to show the following fact:

Lemma 2.6 *Let Ω be a bounded isolated subset of $\sigma(H)$ and p is a projection such that* $\mathrm{Ran}\mathbf{1}_\Omega(H) = \mathrm{Ran}p$ *and $[p, (z - H)^{-1}] = 0$ for $z \notin \sigma(H)$. Then $p = \mathbf{1}_\Omega(H)$.*

An isolated point of $\sigma(H)$ will be called an isolated eigenvalue of H. An isolated eigenvalue z_0 of H is called semisimple if $(z-H)^{-1}$ has a simple pole at z_0, or equivalently, if

$$\mathrm{Ran}\mathbf{1}_{\{z_0\}}(H) = \mathrm{Ker}(H - z_0).$$

We denote by $\sigma_{\mathrm{disc}}(H)$ the discrete spectrum of H, that is, the set of all isolated eigenvalues e such that $\dim \mathbf{1}_{\{e\}}(H) < \infty$. The essential spectrum of H is defined by $\sigma_{\mathrm{ess}}(H) := \sigma(H) \setminus \sigma_{\mathrm{disc}}(H)$.

If H is a self-adjoint operator, we denote by $\sigma_{\mathrm{pp}}(H)$, $\sigma_{\mathrm{sc}}(H)$ and $\sigma_{\mathrm{ac}}(H)$ the pure point, singular continuous and absolutely continuous spectrum of H. The singular spectrum is defined by $\sigma_{\mathrm{sing}}(H) = \overline{\sigma_{\mathrm{pp}}(H)} \cup \sigma_{\mathrm{sc}}(H)$. If Θ a Borel subset of $\mathbf{R}$, then $\mathbf{1}_{\Theta}(H)$ will denote the spectral projection of H onto Θ. We denote by $\mathbf{1}_{\Theta}^{\mathrm{pp}}(H)$, $\mathbf{1}_{\Theta}^{\mathrm{sc}}(H)$, $\mathbf{1}_{\Theta}^{\mathrm{ac}}(H)$ the spectral projections of H onto Θ associated to the pure point, singular continuous and absolutely continuous spectrum.

We now recall some standard results about linear operators that will be used throughout the paper.

Proposition 2.7 *Let H be a self-adjoint operator. If $z \in \mathbf{C} \setminus \sigma(H)$ then*

$$\|(z-H)^{-1}\| = \frac{1}{\mathrm{dist}(z, \sigma(H))}.$$

An immediate consequence of this proposition is

Proposition 2.8 *Let H be a self-adjoint and V a bounded operator. Then, $\sigma(H+V) \subset \overline{B}(\sigma(H), \|V\|)$ and for $z \in \mathbf{C} \setminus \overline{B}(\sigma(H), \|V\|)$ one has the bound*

$$\|(z-(H+V))^{-1}\| \leq \frac{1}{\mathrm{dist}(z, \sigma(H)) - \|V\|}.$$

The concept of the numerical range allows to formulate related results for closed operators.

Proposition 2.9 *Let H be a closed operator such that $\mathcal{D}(H) = \mathcal{D}(H^*)$. Then, $\sigma(H) \subset \overline{\mathfrak{N}(H)}$ and, for $z \in \mathbf{C} \setminus \overline{\mathfrak{N}(H)}$, one has the bound*

$$\|(z-H)^{-1}\| \leq \frac{1}{\mathrm{dist}(z, \mathfrak{N}(H))}.$$

Proposition 2.10 *Let H be a closed operator such that $\mathcal{D}(H) = \mathcal{D}(H^*)$, and let V be a bounded operator. Then, $\sigma(H+V) \subset \overline{B}(\mathfrak{N}(H), \|V\|)$, and for $z \in \mathbf{C} \setminus \overline{B}(\mathfrak{N}(H), \|V\|)$ one has the bound*

$$\|(z-(H+V))^{-1}\| \leq \frac{1}{\mathrm{dist}(z, \mathfrak{N}(H)) - \|V\|}.$$

We say that the operator B is A-bounded if $\mathcal{D}(B) \supset \mathcal{D}(A)$ and

$$\|B\phi\| \leq a\|A\phi\| + b\|\phi\|, \quad \phi \in \mathcal{D}(A). \tag{2.36}$$

The infimum of possible values of a in (2.36) is called the A-bound of B. Recall that if A is closed and the A-bound of B is less than 1, then $A+B$ is closed on $\mathcal{D}(A)$. Clearly, if for some $z_0 \notin \sigma(A)$, $\|B_1(z_0-A)^{-1}\| = a$, where $B = B_1 + B_2$ and B_2 is bounded, then the A-bound of B is less than or equal to a.

Proposition 2.11 *Suppose that A, B are operators such that A is closed, $\mathcal{D}(B) \supset \mathcal{D}(A)$ and, for some $z_0 \in \mathbf{C}$, we have*

$$\|B_1(z_0 - A)^{-1}\| < 1, \qquad \|(z_0 - A)^{-1}B_1\| < 1,$$

where $B = B_1 + B_2$ and B_2 is bounded. Then

$$(A + B)^* = A^* + B^*. \tag{2.37}$$

Proof. Replacing A with $A - z_0$, where $z_0 \notin \sigma(A)$, we can assume that $z_0 = 0$. We can also subtract the bounded operator B_2 from B without affecting (2.37).

Clearly, $\mathcal{D}(A^* + B^*) \supset \mathcal{D}(A^*)$ and

$$(A + B)^*\Big|_{\mathcal{D}(A^*)} = A^* + B^*.$$

We want to show that

$$\mathcal{D}(A^*) \supset \mathcal{D}((A + B)^*). \tag{2.38}$$

Let $\psi \in \mathcal{D}(A)$. Since $\|A^{-1}B\| < 1$, we know that $\mathbf{1} + A^{-1}B$ has a bounded inverse. Hence $\psi_1 := (1 + A^{-1}B)^{-1}\psi$ satisfies

$$\|\psi_1\| \leq C_1\|\psi\|. \tag{2.39}$$

Since $\|BA^{-1}\| < 1$, the operator $\mathbf{1} + BA^{-1}$ has also a bounded inverse. But, on $\mathcal{D}(A)$, $(\mathbf{1} + A^{-1}B)^{-1} = A^{-1}(\mathbf{1} + BA^{-1})^{-1}A$. Therefore, $\mathbf{1} + BA^{-1}$ is bounded as an operator on $\mathcal{D}(A)$. Thus $\psi_1 \in \mathcal{D}(A)$.

Now let $\phi \in \mathcal{D}((A + B)^*)$. Using that $\psi_1 \in \mathcal{D}(A) = \mathcal{D}(A + B)$, we have

$$|(\phi|(A + B)\psi_1)| \leq C\|\psi_1\|. \tag{2.40}$$

Clearly,

$$\begin{aligned} |(\phi|A\psi)| &= |(\phi|A(1 + A^{-1}B)\psi_1) \\ &= (\phi|(A + B)\psi_1) \\ &\leq C\|\psi_1\| \leq C_1\|\psi\|, \end{aligned}$$

where in the last steps we used (2.40) and (2.39). This shows that $\phi \in \mathcal{D}(A^*)$, and ends the proof of the inclusion (2.38). □

3 General theory

3.1 Dissipative operators

We begin with

Definition 3.1 *A closed operator B is called dissipative if $\mathfrak{N}(B) \subset \overline{\mathbf{C}}_-$.*

On $\mathcal{D}(B) \cap \mathcal{D}(B^*)$ we can define

$$\mathrm{Re}B := \frac{1}{2}(B + B^*), \qquad \mathrm{Im}B := \frac{1}{2\mathrm{i}}(B - B^*).$$

Clearly, $\mathrm{Re}B$ and $\mathrm{Im}B$ are symmetric operators.

In all our applications the following condition will be satisfied:

$$\mathcal{D}(B) = \mathcal{D}(B^*), \ \ \mathrm{Im}B \text{ is bounded and } \mathrm{Re}B \text{ is self-adjoint on } \mathcal{D}(B). \tag{3.41}$$

Clearly, under this condition B is dissipative iff $\mathrm{Im}B \leq 0$.

Proposition 3.2 *Let B be a dissipative operator satisfying (3.41) and $e \in \mathbf{R}$. Then,*
(i) $\mathrm{Ker}(B - e) = \mathrm{Ran}\mathbf{1}_{\{\mathrm{e}\}}(\mathrm{Re}B) \cap \mathrm{Ran}\mathbf{1}_{\{0\}}(\mathrm{Im}B)$.
(ii) *Let p be the orthogonal projection onto* $\mathrm{Ker}(B - e)$. *Then* $0 = [p, B]$.
(iii) *If in addition $e \in \sigma_{\mathrm{disc}}(B)$, then the eigenvalue e is semisimple and $p = \mathbf{1}_{\{\mathrm{e}\}}(B)$.*

Proof. Let $\psi \in \mathrm{Ker}(B - e)$. Then $\psi \in \mathcal{D}(B) = \mathcal{D}(B^*)$ and

$$0 = (\psi|(B - e)\psi) = (\psi|(B^* - e)\psi).$$

Hence $0 = (\psi|\mathrm{Im}B\psi)$. Since $\mathrm{Im}B \leq 0$, we derive that $0 = \mathrm{Im}B\psi$. This and $0 = (B - e)\psi$ yield that $e\psi = \mathrm{Re}B\psi$. Hence

$$\mathrm{Ran}\mathbf{1}_{\{e\}}(\mathrm{Re}B) \cap \mathrm{Ran}\mathbf{1}_{\{0\}}(\mathrm{Im}B) \supset \mathrm{Ran}\mathbf{1}_{\{e\}}(B).$$

The inclusion $\subset$ is obvious, and Part (i) follows.

By Part (i) we have

$$p \leq \mathbf{1}_{\{e\}}(\mathrm{Re}(B)), \quad p \leq \mathbf{1}_{\{0\}}(\mathrm{Im}(B)).$$

Hence

$$0 = [p, \mathrm{Re}B] = [p, \mathrm{Im}B].$$

This implies (ii).

To establish Part (iii), we note that if $e \in \sigma_{\mathrm{disc}}(B)$ then e is a pole of $(z - B)^{-1}$. By Proposition 2.9 this pole is simple. Hence e is a semisimple eigenvalue of B. An application of Lemma 2.6 completes the proof of (iii). □

If e is an isolated real eigenvalue of B with $\dim \mathbf{1}_{\{e\}}(B) = \infty$, then $\mathrm{Ker}(B - e)$ may be strictly smaller than $\mathrm{Ran}\mathbf{1}_{\{e\}}(B)$. (We thank E. Skibsted for pointing this out to us).

Proposition 3.3 *Let B be a bounded dissipative operator on a Hilbert space $\mathcal{H}$ such that $\sigma(B) \cap \mathbf{R} \subset \sigma_{\rm disc}(B)$. Then*

$$c := \sup_{z \in \mathbf{C}_+} \left\| (z - B)^{-1} \mathbf{1}_{\sigma(B) \setminus \mathbf{R}}(B) \right\| < \infty, \tag{3.42}$$

and, for $z \in \overline{\mathbf{C}}_+ \setminus \sigma(B)$,

$$\|(z - B)^{-1}\| \le \max\Big(\frac{1}{\mathrm{dist}(z, \sigma(B) \cap \mathbf{R})}, c\Big). \tag{3.43}$$

Proof. By Proposition 3.2

$$\mathbf{1}_{\sigma(B) \cap \mathbf{R}} = \sum_{e \in \sigma(B) \cap \mathbf{R}} \mathbf{1}_{\{e\}}(B),$$

is an orthogonal projection. Therefore

$$\|(z - B)^{-1}\| = \max\Big(\|(z - B)^{-1}\mathbf{1}_{\sigma(B) \cap \mathbf{R}}(B)\|, \|(z - B)^{-1}\mathbf{1}_{\sigma(B) \setminus \mathbf{R}}(B)\|\Big). \tag{3.44}$$

Also by Proposition 3.2, $B\mathbf{1}_{\sigma(B) \cap \mathbf{R}}(B)$ is self-adjoint. Hence

$$\|(z - B)^{-1}\mathbf{1}_{\sigma(B) \cap \mathbf{R}}(B)\| = \frac{1}{\mathrm{dist}(z, \sigma(B) \cap \mathbf{R})}. \tag{3.45}$$

Since $B\mathbf{1}_{\sigma(B) \setminus \mathbf{R}}(B)$ is a bounded operator with spectrum contained in $\mathbf{C}_-$, we clearly have (3.42). Now (3.44), (3.42) and (3.45) imply (3.43). □

The following result is an immediate consequence of the previous proposition.

Proposition 3.4 *Let B be a bounded dissipative operator such that $\sigma(B) \cap \mathbf{R} \subset \sigma_{\rm disc}(B)$, and let c be the constant defined in (3.42). If V is a bounded operator such that $c\|V\| < 1$, then*

$$\sigma(B + V) \cap \overline{\mathbf{C}}_+ \subset \overline{B}(\sigma(B) \cap \mathbf{R}, \|V\|).$$

Furthermore, for $z \in \overline{\mathbf{C}}_+ \setminus \overline{B}(\sigma(B) \cap \mathbf{R}, \|V\|)$ one has the bound

$$\|(z - B - V)^{-1}\| \le \frac{1}{\mathrm{dist}(z, \sigma(B) \cap \mathbf{R}) - \|V\|}. \tag{3.46}$$

3.2 The Feshbach formula

Let $\mathcal{H}$ be a Hilbert space decomposed into a direct sum $\mathcal{H} = \mathcal{H}^{\mathrm{v}} \oplus \mathcal{H}^{\overline{\mathrm{v}}}$. The projections onto $\mathcal{H}^{\mathrm{v}}$ and $\mathcal{H}^{\overline{\mathrm{v}}}$ we denote by $\mathbf{1}^{\mathrm{vv}}$ and $\mathbf{1}^{\overline{\mathrm{vv}}}$. In this section we study operators of the form

$$H = \begin{bmatrix} H^{\mathrm{vv}} & H^{\mathrm{v}\overline{\mathrm{v}}} \\ H^{\overline{\mathrm{v}}\mathrm{v}} & H^{\overline{\mathrm{vv}}} \end{bmatrix}, \tag{3.47}$$

We assume that H^{vv} and $H^{\overline{\mathrm{vv}}}$ are closed operators on $\mathcal{H}^{\mathrm{v}}$ and $\mathcal{H}^{\overline{\mathrm{v}}}$; moreover we suppose that $H^{\overline{\mathrm{v}}\mathrm{v}} : \mathcal{H}^{\mathrm{v}} \to \mathcal{H}^{\overline{\mathrm{v}}}$ and $H^{\mathrm{v}\overline{\mathrm{v}}} : \mathcal{H}^{\overline{\mathrm{v}}} \to \mathcal{H}^{\mathrm{v}}$ are bounded operators. Clearly H is closed and $\mathcal{D}(H) = \mathcal{D}(H^{\mathrm{vv}}) \oplus \mathcal{D}(H^{\overline{\mathrm{vv}}})$.

For any $z \notin \sigma(H^{\overline{\mathrm{vv}}})$ we define

$$\begin{aligned} W_{\mathrm{v}}(z) &:= H^{\mathrm{v}\overline{\mathrm{v}}}(z\mathbf{1}^{\overline{\mathrm{vv}}} - H^{\overline{\mathrm{vv}}})^{-1} H^{\overline{\mathrm{v}}\mathrm{v}}, \\ G_{\mathrm{v}}(z) &:= z\mathbf{1}^{\mathrm{vv}} - H^{\mathrm{vv}} - W_{\mathrm{v}}(z). \end{aligned} \tag{3.48}$$

In the physics literature, the operator $W_{\mathrm{v}}(z)$ is sometimes called *the self-energy*. For $G_{\mathrm{v}}(z)$ we propose the name *the resonance function*. Note that $W_{\mathrm{v}}(z)$ is an analytic operator-valued function on $\mathbf{C} \setminus \sigma(H^{\overline{\mathrm{vv}}})$, and that $W_{\mathrm{v}}(z)^* = W_{\mathrm{v}}(\overline{z})$.

The following proposition is well known:

Proposition 3.5 *Assume that* $z \notin \sigma(H^{\overline{\mathrm{vv}}})$. *Then,*
(i) $z \in \sigma(H)$ *iff* $0 \in \sigma(G_{\mathrm{v}}(z))$;
(ii) *If* $0 \notin \sigma(G_{\mathrm{v}}(z))$ *then*

$$\begin{aligned} (z-H)^{-1} &= \left(\mathbf{1}^{\mathrm{vv}} + (z\mathbf{1}^{\overline{\mathrm{vv}}} - H^{\overline{\mathrm{vv}}})^{-1} H^{\overline{\mathrm{v}}\mathrm{v}}\right) G_{\mathrm{v}}^{-1}(z) \left(\mathbf{1}^{\mathrm{vv}} + H^{\mathrm{v}\overline{\mathrm{v}}}(z\mathbf{1}^{\overline{\mathrm{vv}}} - H^{\overline{\mathrm{vv}}})^{-1}\right) \\ &\quad + (z\mathbf{1}^{\overline{\mathrm{vv}}} - H^{\overline{\mathrm{vv}}})^{-1}. \end{aligned} \tag{3.49}$$

Proof. Our proof is inspired by [GGK]. Set

$$A(z) := \mathbf{1} + H^{\mathrm{v}\overline{\mathrm{v}}}(z\mathbf{1}^{\overline{\mathrm{vv}}} - H^{\overline{\mathrm{vv}}})^{-1}, \qquad B(z) := \mathbf{1} + (z\mathbf{1}^{\overline{\mathrm{vv}}} - H^{\overline{\mathrm{vv}}})^{-1} H^{\overline{\mathrm{v}}\mathrm{v}}.$$

Both $A(z)$ and $B(z)$ have bounded inverses:

$$A^{-1}(z) = \mathbf{1} - H^{\mathrm{v}\overline{\mathrm{v}}}(z\mathbf{1}^{\overline{\mathrm{vv}}} - H^{\overline{\mathrm{vv}}})^{-1}, \qquad B^{-1}(z) = \mathbf{1} - (z\mathbf{1}^{\overline{\mathrm{vv}}} - H^{\overline{\mathrm{vv}}})^{-1} H^{\overline{\mathrm{v}}\mathrm{v}}.$$

Moreover, both $B(z)$ and $B^{-1}(z)$ are bounded operators on $\mathcal{D}(H)$.

The following identity holds in the sense of operators from $\mathcal{D}(H)$ to $\mathcal{H}$:

$$A(z)(z-H)B(z) = G_{\mathrm{v}}(z) + z\mathbf{1}^{\overline{\mathrm{vv}}} - H^{\overline{\mathrm{vv}}} \tag{3.50}$$

If $z \notin \sigma(H)$, then the left hand side of (3.50) is invertible. Hence so is the right hand side. This implies that $G_{\mathrm{v}}(z)$ is invertible.

Next we will use the identity (also understood in the sense of operators from $\mathcal{D}(H)$ to $\mathcal{H}$):

$$(z - H) = A^{-1}(z)(G_{\rm v}(z) + z\mathbf{1}^{\overline{\rm v}\overline{\rm v}} - H^{\overline{\rm v}\overline{\rm v}})B^{-1}(z). \tag{3.51}$$

If $0 \notin \sigma(G_{\rm v}(z))$, then the right hand side of (3.51) is invertible. Hence so is $z - H$. This completes the proof of (i).

To show (ii) we note that if $z \notin \sigma(H)$, (3.50) or (3.51) implies

$$(z - H)^{-1} = B(z)(G_{\rm v}^{-1}(z) + (z\mathbf{1}^{\overline{\rm v}\overline{\rm v}} - H^{\overline{\rm v}\overline{\rm v}})^{-1})A(z),$$

which, after substituting the expressions defining $A(z)$ and $B(z)$, yields (3.49). □

¿From now on we assume in addition that $H^{\rm vv}$ and $H^{\overline{\rm v}\overline{\rm v}}$ are self-adjoint and $H^{\rm v\overline{\rm v}} = (H^{\overline{\rm v}\rm v})^*$. This clearly implies that H is self-adjoint. Moreover, $W_{\rm v}(z)^* = W_{\rm v}(\overline{z})$, the operator $W_{\rm v}(z)$ is dissipative if $\mathrm{Im} z \geq 0$. If $x \in \mathbf{R} \setminus \sigma(H^{\overline{\rm v}\overline{\rm v}})$, then $W_{\rm v}(x)$ is self-adjoint and

$$\frac{\mathrm{d}}{\mathrm{d}x} W_{\rm v}(x) = -H^{\rm v\overline{\rm v}}(x\mathbf{1}^{\overline{\rm v}\overline{\rm v}} - H^{\overline{\rm v}\overline{\rm v}})^{-2}H^{\overline{\rm v}\rm v} \leq 0. \tag{3.52}$$

The rest of this section is devoted to various refinements of Proposition 3.5. The first result in this direction is

Theorem 3.6 *Assume that* $e \in \mathbf{R} \setminus \sigma(H^{\overline{\rm v}\overline{\rm v}})$. *Then,*
(i) *e is an eigenvalue of H iff 0 is an eigenvalue of $G_{\rm v}(e)$.*
(ii) $\dim \mathbf{1}_{\{e\}}(H) = \dim \mathbf{1}_{\{0\}}(G_{\rm v}(e))$.
(iii) *Set* $p = \mathbf{1}_{\{0\}}(G_{\rm v}(e))$. *Then $pW_{\rm v}'(e)p$ is a negative operator and*

$$\begin{aligned} \mathbf{1}_{\{e\}}(H) &= \left(p + (e\mathbf{1}^{\overline{\rm v}\overline{\rm v}} - H^{\overline{\rm v}\overline{\rm v}})^{-1}H^{\overline{\rm v}\rm v}p\right) \\ &\quad \times (p - pW_{\rm v}'(e)p)^{-1}\left(p + pH^{\rm v\overline{\rm v}}(e\mathbf{1}^{\overline{\rm v}\overline{\rm v}} - H^{\overline{\rm v}\overline{\rm v}})^{-1}\right). \end{aligned} \tag{3.53}$$

(iv) *If*

$$u := (p - pW_{\rm v}'(e)p)^{-\frac{1}{2}}\left(p + pH^{\rm v\overline{\rm v}}(e\mathbf{1}^{\overline{\rm v}\overline{\rm v}} - H^{\overline{\rm v}\overline{\rm v}})^{-1}\right),$$

then u is a partial isometry and

$$uu^* = p, \qquad u^*u = \mathbf{1}_{\{e\}}(H).$$

Proof. Let $H\psi = e\psi$. Then

$$\begin{aligned} H^{\rm vv}\psi^{\rm v} + H^{\rm v\overline{\rm v}}\psi^{\overline{\rm v}} &= e\psi^{\rm v}, \\ H^{\overline{\rm v}\rm v}\psi^{\rm v} + H^{\overline{\rm v}\overline{\rm v}}\psi^{\overline{\rm v}} &= e\psi^{\overline{\rm v}}. \end{aligned} \tag{3.54}$$

The second equation gives (recall that $e \notin \sigma(H^{\overline{\rm v}\overline{\rm v}})$)

$$\psi^{\overline{\rm v}} = (e\mathbf{1}^{\overline{\rm v}\overline{\rm v}} - H^{\overline{\rm v}\overline{\rm v}})^{-1}H^{\overline{\rm v}\rm v}\psi^{\rm v}. \tag{3.55}$$

Inserting this identity into the first equation of (3.54) we get

$$G_{\rm v}(e)\psi^{\rm v} = 0. \tag{3.56}$$

Now, if $\psi^{\rm v} \in \mathcal{D}(H^{\rm vv})$ satisfies (3.56) and $\psi^{\overline{\rm v}}$ is given in terms of $\psi^{\rm v}$ by (3.55), then $\psi = \psi^{\rm v} \oplus \psi^{\overline{\rm v}}$ satisfies $H\psi = e\psi$, which follows by a simple computation. Thus, the projection

$$\psi \mapsto \psi^{\rm v} \tag{3.57}$$

restricted to $\mathrm{Ran}\mathbf{1}_{\{e\}}(H)$ is a bijection onto $\mathrm{Ran}p$. This yields (i) and (ii).

Define $w : \mathcal{H}^{\rm v} \mapsto \mathcal{H}^{\rm v} \oplus \mathcal{H}^{\overline{\rm v}}$ by setting

$$w := p + (e\mathbf{1}^{\overline{\rm v}\overline{\rm v}} - H^{\overline{\rm v}\overline{\rm v}})^{-1} H^{\overline{\rm v}{\rm v}} p.$$

Then w is the inverse of the map (3.57). Besides,

$$w^* w = p + pH^{{\rm v}\overline{\rm v}}(e\mathbf{1}^{\overline{\rm v}\overline{\rm v}} - H^{\overline{\rm v}\overline{\rm v}})^{-2} H^{\overline{\rm v}{\rm v}} p$$

restricted to $\mathrm{Ran}p$ is a positive, invertible operator. With a slight abuse of the notation we denote its inverse by $(w^*w)^{-1}$. One easily checks that $w(w^*w)^{-1}w^*$ is an orthogonal projection on $\mathrm{Ran}w = \mathrm{Ran}\mathbf{1}_{\{e\}}(H)$. Hence

$$\mathbf{1}_{\{e\}}(H) = w(w^*w)^{-1}w^*.$$

This shows (iii). Part (iv) follows from the identity $u = (w^*w)^{-\frac{1}{2}}w^*$. □

Due to the assumption $e \notin \sigma(H^{\overline{\rm v}\overline{\rm v}})$, the proof of Theorem 3.6 was relatively simple. Related results are subtler if $e \in \sigma(H^{\overline{\rm v}\overline{\rm v}})$. As a warm up, we prove

Proposition 3.7 *Let $e \in \mathbf{R}$. Suppose that the limit*

$$\lim_{y\downarrow 0} W_{\rm v}(e + {\rm i}y) =: W_{\rm v}(e + {\rm i}0) \tag{3.58}$$

exists. Then $W_{\rm v}(e + {\rm i}0)$ is dissipative.

Assume moreover that e is not an eigenvalue of $H^{\overline{\rm v}\overline{\rm v}}$. Then

$$\dim \mathrm{Ker}(H - e) \leq \dim \mathrm{Ker} G_{\rm v}(e + {\rm i}0). \tag{3.59}$$

In particular, if e is an eigenvalue of H, then 0 is an eigenvalue of $G_{\rm v}(e + {\rm i}0)$.

Proof. Assume that $\psi \in \mathcal{D}(H)$ and $H\psi = e\psi$. Since e is not an eigenvalue of $H^{\overline{\rm v}\overline{\rm v}}$, $e\mathbf{1}^{\overline{\rm v}\overline{\rm v}} - H^{\overline{\rm v}\overline{\rm v}}$ is injective and

$$\psi^{\overline{\rm v}} = (e\mathbf{1}^{\overline{\rm v}\overline{\rm v}} - H^{\overline{\rm v}\overline{\rm v}})^{-1} H^{\overline{\rm v}{\rm v}} \psi^{\rm v}. \tag{3.60}$$

Moreover,

$$0 = \mathbf{1}_{\{e\}}(H^{\overline{\mathrm{v}}\overline{\mathrm{v}}}) = -\mathrm{s}-\lim_{y\downarrow 0} \mathrm{i}y((e+\mathrm{i}y)\mathbf{1}^{\overline{\mathrm{v}}\overline{\mathrm{v}}} - H^{\overline{\mathrm{v}}\overline{\mathrm{v}}})^{-1}. \tag{3.61}$$

Therefore,

$$\begin{aligned} &\mathrm{s}-\lim_{y\downarrow 0}\left(((e+\mathrm{i}y)\mathbf{1}^{\overline{\mathrm{v}}\overline{\mathrm{v}}} - H^{\overline{\mathrm{v}}\overline{\mathrm{v}}})^{-1} - (e\mathbf{1}^{\overline{\mathrm{v}}\overline{\mathrm{v}}} - H^{\overline{\mathrm{v}}\overline{\mathrm{v}}})^{-1}\right) H^{\overline{\mathrm{v}}\mathrm{v}}\psi^{\mathrm{v}} \\ &= -\mathrm{s}-\lim_{y\downarrow 0}\mathrm{i}y((e+\mathrm{i}y)\mathbf{1}^{\overline{\mathrm{v}}\overline{\mathrm{v}}} - H^{\overline{\mathrm{v}}\overline{\mathrm{v}}})^{-1}(e\mathbf{1}^{\overline{\mathrm{v}}\overline{\mathrm{v}}} - H^{\overline{\mathrm{v}}\overline{\mathrm{v}}})^{-1}H^{\overline{\mathrm{v}}\mathrm{v}}\psi^{\mathrm{v}} \quad = 0. \end{aligned} \tag{3.62}$$

Combining (3.60) and (3.62) we get

$$\psi^{\overline{\mathrm{v}}} = \mathrm{s}-\lim_{y\downarrow 0}((e+\mathrm{i}y)\mathbf{1}^{\overline{\mathrm{v}}\overline{\mathrm{v}}} - H^{\overline{\mathrm{v}}\overline{\mathrm{v}}})^{-1}H^{\overline{\mathrm{v}}\mathrm{v}}\psi^{\mathrm{v}}.$$

Substituting this identity into the first equation in (3.54) we derive that

$$G_{\mathrm{v}}(e+\mathrm{i}0)\psi^{\mathrm{v}} = 0.$$

Now assume that $\psi^{\mathrm{v}} = 0$. Then $\psi \in \mathcal{H}^{\overline{\mathrm{v}}}$ and therefore

$$e\psi = H\psi = H^{\overline{\mathrm{v}}\overline{\mathrm{v}}}\psi.$$

Since $e \notin \sigma_{\mathrm{pp}}(H^{\overline{\mathrm{v}}\overline{\mathrm{v}}})$, $\psi = 0$. Thus, the projection $\psi \mapsto \psi^{\mathrm{v}}$ restricted to $\mathrm{Ran}\mathbf{1}_{\{e\}}(H)$ is an injective map into $\mathrm{Ker}G_{\mathrm{v}}(x+\mathrm{i}0)$. $\square$

The following theorem is the principal result of this section.

Theorem 3.8 *Assume that e is not an eigenvalue of $H^{\overline{\mathrm{v}}\overline{\mathrm{v}}}$, that the limit (3.58) exists, and that the function*

$$\mathbf{C}_+ \cup \{e\} \ni z \mapsto W_{\mathrm{v}}(z)$$

is in the class $C^1(\mathbf{C}_+ \cup \{e\})$ (its derivative at $z = e$ we denote by $W'_{\mathrm{v}}(e+\mathrm{i}0)$). Further, assume that $0 \in \sigma_{\mathrm{disc}}(G_{\mathrm{v}}(e+\mathrm{i}0))$. Then the following holds:
(i) *e is an eigenvalue of H.*
(ii) $\dim \mathbf{1}_{\{e\}}(H) = \dim \mathrm{Ker}G_{\mathrm{v}}(e+\mathrm{i}0)$.
(iii) *Set $p = \mathbf{1}_{\{0\}}(G_{\mathrm{v}}(e+\mathrm{i}0))$. Then, p is the orthogonal projection onto* $\mathrm{Ker}(G_{\mathrm{v}}(e+\mathrm{i}0)$*, the operator*

$$pW_{\mathrm{v}}(e+\mathrm{i}0)p =: pW_{\mathrm{v}}(e)p$$

is self-adjoint, and the operator

$$pW'_{\mathrm{v}}(e+\mathrm{i}0)p =: pW'_{\mathrm{v}}(e)p$$

is negative.
(iv) *The limits*

$$\begin{aligned} &\lim_{y\to 0}((e+\mathrm{i}y)\mathbf{1}^{\overline{\mathrm{v}}\overline{\mathrm{v}}} - H^{\overline{\mathrm{v}}\overline{\mathrm{v}}})^{-1}H^{\overline{\mathrm{v}}\mathrm{v}}p &&=: (e\mathbf{1}^{\overline{\mathrm{v}}\overline{\mathrm{v}}} - H^{\overline{\mathrm{v}}\overline{\mathrm{v}}})^{-1}H^{\overline{\mathrm{v}}\mathrm{v}}p \\ &\lim_{y\to 0} pH^{\mathrm{v}\overline{\mathrm{v}}}((e+\mathrm{i}y)\mathbf{1}^{\overline{\mathrm{v}}\overline{\mathrm{v}}} - H^{\overline{\mathrm{v}}\overline{\mathrm{v}}})^{-1} &&=: pH^{\mathrm{v}\overline{\mathrm{v}}}(e\mathbf{1}^{\overline{\mathrm{v}}\overline{\mathrm{v}}} - H^{\overline{\mathrm{v}}\overline{\mathrm{v}}})^{-1}, \end{aligned} \tag{3.63}$$

exist and

$$\begin{aligned}\mathbf{1}_{\{e\}}(H) &= \left(p + (e\mathbf{1}^{\overline{\mathrm{vv}}} - H^{\overline{\mathrm{vv}}})^{-1}H^{\overline{\mathrm{v}}\mathrm{v}}p\right) \\ &\quad \times (p - pW_{\mathrm{v}}'(e)p)^{-1}\left(p + pH^{\mathrm{v}\overline{\mathrm{v}}}(e\mathbf{1}^{\overline{\mathrm{vv}}} - H^{\overline{\mathrm{vv}}})^{-1}\right).\end{aligned} \tag{3.64}$$

(v) *If*

$$u := (p - pW_{\mathrm{v}}'(e)p)^{-\frac{1}{2}}\left(p + pH^{\mathrm{v}\overline{\mathrm{v}}}(e\mathbf{1}^{\overline{\mathrm{vv}}} - H^{\overline{\mathrm{vv}}})^{-1}\right),$$

then u is a partial isometry and

$$uu^* = p, \qquad u^*u = \mathbf{1}_{\{e\}}(H).$$

Proof. We begin with proofs of Parts (iii) and (iv).

We first observe that the relation $W_{\mathrm{v}}^*(z) = W_{\mathrm{v}}(\overline{z})$ yields that the functions $W_{\mathrm{v}}(z)$ and $G_{\mathrm{v}}(z)$ are also of the class $C^1(\mathbf{C}_- \cup \{e\})$.

Since the operator $-G_{\mathrm{v}}(e+\mathrm{i}0)$ is dissipative and $0 \in \sigma_{\mathrm{disc}}(G_{\mathrm{v}}(e+\mathrm{i}0))$, Proposition 3.2 yields that p is an orthogonal projection. It follows that

$$\mathbf{1}_{\{0\}}(G_{\mathrm{v}}^*(e+\mathrm{i}0)) = p.$$

Since $G_{\mathrm{v}}^*(e+\mathrm{i}0) = G_{\mathrm{v}}(e-\mathrm{i}0)$, the identities $pG_{\mathrm{v}}(e \pm \mathrm{i}0)p = 0$ can be rewritten as

$$p(e\mathbf{1}^{\mathrm{vv}} - H^{\mathrm{vv}})p = pW_{\mathrm{v}}(e \pm \mathrm{i}0)p.$$

This yields the relation

$$pW_{\mathrm{v}}(e+\mathrm{i}0)p = pW_{\mathrm{v}}(e-\mathrm{i}0)p =: pW_{\mathrm{v}}(e)p. \tag{3.65}$$

Clearly, the operator $pW_{\mathrm{v}}(e)p$ is self-adjoint.

Let $\tau \mapsto \gamma(\tau) \in \mathbf{C}_+ \cup \{e\}$, $\gamma(0) = e$, be a smooth curve such that $\gamma(0) = e$ and γ is tangent to $\mathbf{R}$ at $\tau = 0$. We may assume that $\gamma'(0) = 1$. Then,

$$\tau \mapsto \mathrm{Im} pW_{\mathrm{v}}(\gamma(\tau))p,$$

is a function with values in dissipative operators such that

$$\mathrm{Im} pW_{\mathrm{v}}(\gamma(0))p = \mathrm{Im} pW_{\mathrm{v}}(e)p = 0.$$

It follows that

$$0 = \frac{\mathrm{d}}{\mathrm{d}\tau}\mathrm{Im} pW_{\mathrm{v}}(\gamma(\tau))p|_{\tau=0} = \mathrm{Im} pW_{\mathrm{v}}'(e+\mathrm{i}0)p.$$

This shows that

$$pW_{\mathrm{v}}'(e+\mathrm{i}0)p =: pW_{\mathrm{v}}'(e)p \tag{3.66}$$

is a self-adjoint operator. This proves (iii) except for the part asserting that $pW_{\rm v}'(e)p$ is a negative operator. Using $p = p^*$, $W_{\rm v}'(e - \mathrm{i}0) = W_{\rm v}'(e + \mathrm{i}0)^*$ and (3.66), we also have that

$$pW_{\rm v}'(e - \mathrm{i}0)p = pW_{\rm v}'(e + \mathrm{i}0)p. \tag{3.67}$$

We now show that the limit

$$(e\mathbf{1}^{\overline{\rm v}\overline{\rm v}} - H^{\overline{\rm v}\overline{\rm v}})^{-1}H^{\overline{\rm v}{\rm v}}p := \lim_{y\to 0, y\neq 0}((e+\mathrm{i}y)\mathbf{1}^{\overline{\rm v}\overline{\rm v}} - H^{\overline{\rm v}\overline{\rm v}})^{-1}H^{\overline{\rm v}{\rm v}}p, \tag{3.68}$$

exists. Denote the expression inside the limit by $L(y)$. Then, the resolvent identity yields

$$\begin{aligned}(L^*(y_1) - L^*(y_2))(L(y_1) - L(y_2)) \;&= \tfrac{1}{2\mathrm{i}y_1}\left(pW_{\rm v}(e - \mathrm{i}y_1)p - pW_{\rm v}(e + \mathrm{i}y_1)p\right)\\ &+ \tfrac{1}{2\mathrm{i}y_2}\left(pW_{\rm v}(e - \mathrm{i}y_2)p - pW_{\rm v}(e + \mathrm{i}y_2)p\right)\\ &- \tfrac{1}{\mathrm{i}(y_2+y_1)}\left(pW_{\rm v}(e - \mathrm{i}y_1)p - pW_{\rm v}(e + \mathrm{i}y_2)p\right)\\ &- \tfrac{1}{\mathrm{i}(y_1+y_2)}\left(pW_{\rm v}(e - \mathrm{i}y_2)p - pW_{\rm v}(e + \mathrm{i}y_1)p\right).\end{aligned} \tag{3.69}$$

Note that it follows from (3.65) and (3.67) that the function

$$\mathbf{C}_+ \cup \mathbf{C}_- \cup \{e\} \ni z \mapsto pW_{\rm v}(z)p, \tag{3.70}$$

is continuously differentiable. This observation and identity (3.69) yield that the sequence $L(y_n)$ is Cauchy whenever $y_n \to 0$. Thus, the limit (3.68) exists and this implies the existence of the limits (3.63) (for the second limit take the adjoint of (3.68)). Since

$$pW_{\rm v}'(e)p = -((e\mathbf{1}^{\overline{\rm v}\overline{\rm v}} - H^{\overline{\rm v}\overline{\rm v}})^{-1}H^{\overline{\rm v}{\rm v}}p)^*(e\mathbf{1}^{\overline{\rm v}\overline{\rm v}} - H^{\overline{\rm v}\overline{\rm v}})^{-1}H^{\overline{\rm v}{\rm v}}p,$$

it follows that the operator $pW_{\rm v}'(e)p$ is negative. This completes the proof of Part (iii).

Set $w := p + (e\mathbf{1}^{\overline{\rm v}\overline{\rm v}} - H^{\overline{\rm v}\overline{\rm v}})^{-1}H^{\overline{\rm v}{\rm v}}p$ and $w(z) := p + (z\mathbf{1}^{\overline{\rm v}\overline{\rm v}} - H^{\overline{\rm v}\overline{\rm v}})^{-1}H^{\overline{\rm v}{\rm v}}p$. By (3.63) we have

$$\lim_{y\downarrow 0} w(e+\mathrm{i}y) = w. \tag{3.71}$$

We easily compute that

$$Hw(z) = zw(z) - G_{\rm v}(z)p.$$

Hence

$$\lim_{y\downarrow 0} Hw(e+\mathrm{i}y) = ew. \tag{3.72}$$

(3.71) and (3.72) imply that $\mathrm{Ran}w \subset \mathrm{Ker}(H-e)$. Thus w maps $\mathrm{Ker}G_{\rm v}(e+\mathrm{i}0)$ into $\mathrm{Ker}(H-e)$. Clearly, the inverse of w is $\mathbf{1}^{\rm vv}$ restricted to $\mathrm{Ker}(H-e)$. Therefore, w is a bijective map from $\mathrm{Ker}G_{\rm v}(e+\mathrm{i}0)$ to $\mathrm{Ker}(H-e)$. Similarly as at the end of the proof of Theorem 3.6, we note that

$$\begin{aligned}w^*w \;&= p + ((e\mathbf{1}^{\overline{\rm v}\overline{\rm v}} - H^{\overline{\rm v}\overline{\rm v}})^{-1}H^{\overline{\rm v}{\rm v}}p)^*(e\mathbf{1}^{\overline{\rm v}\overline{\rm v}} - H^{\overline{\rm v}\overline{\rm v}})^{-1}H^{\overline{\rm v}{\rm v}}p\\ &= p - pW_{\rm v}'(e)p,\end{aligned}$$

and

$$\mathbf{1}_{\{e\}}(H) = w(w^*w)^{-1}w^*.$$

This proves Relation (3.64) and completes the proof of Part (iv). Part (v) follows from the identity

$$u = (w^*w)^{-\frac{1}{2}}w^*.$$

□

3.3 Counting the eigenvalues

Let $[e_-, e_+] \ni x \mapsto G(x)$ be a function with values in bounded self-adjoint operators on a Hilbert space $\mathcal{H}$. We say that the function G is strictly increasing if the following holds: If $x > y$, then there is $\epsilon > 0$ such that, for all $\psi \in \mathcal{H}$,

$$(\psi|G(x)\psi) > (\psi|G(y)\psi) + \epsilon\|\psi\|^2.$$

Proposition 3.9 *Let $[e_-, e_+] \ni x \mapsto G(x) \in \mathcal{B}(\mathcal{H})$ be a function such that:*
(a) *For all $x \in [e_-, e_+]$, $G(x)$ is self-adjoint.*
(b) *$[e_-, e_+] \ni x \mapsto G(x)$ is continuous and strictly increasing.*
(c) $\dim \mathbf{1}_{[0,\infty[}(G(e_+)) < \infty$.
Then:
(i) The set $\{x \in [e_-, e_+] \ : \ 0 \in \sigma(G(x))\}$ is a finite subset of $[e_-, e_+]$ and, for $x \in [e_-, e_+]$, we have $0 \in \sigma(G(x))$ iff $0 \in \sigma_{\rm disc}(G(x))$.
(ii)

$$\sum_{x\in[e_-,e_+]} \dim \operatorname{Ker} G(x) = \dim \mathbf{1}_{[0,\infty[}(G(e_+)) - \dim \mathbf{1}_{]0,\infty[}(G(e_-)).$$

Proof. We will use the Courant–Weyl (min-max) principle ([RS4], Theorem XIII.1, [We]). Let

$$f_n(x) := \inf_{\psi_1,\dots,\psi_{n-1}} \sup_{\substack{\|\psi\|=1 \\ \psi\in\{\psi_1,\dots,\psi_{n-1}\}^\perp}} (\psi|G(x)\psi).$$

We set $\Sigma(x) := -\infty$ if $\mathcal{H}$ is finite dimensional, $\Sigma(x) := \inf_n f_n(x)$ otherwise. It follows from the min-max principle that $\Sigma(x) = \sup \sigma_{\rm ess}(G(x))$ (note that $\sup \emptyset = -\infty$). Furthermore, $f_n(x)$ is a non-increasing sequence such that if $\Sigma(x) < f_n(x)$ then $f_n(x)$ is the n-th eigenvalue of $G(x)$ (in the non-increasing order) counted with multiplicites.

Clearly, since $G(x)$ is a strictly increasing continuous function, the functions $f_n(x)$ are also strictly increasing and continuous in x. In particular, it follows that $\Sigma(x)$ is an increasing function. By (c), $\Sigma(e_+) < 0$, hence $\Sigma(x) < 0$ for all $x \in [e_-, e_+]$. Therefore,

$$\dim \mathbf{1}_{[0,\infty[}(G(x)) = \#\{n \ : \ f_n(x) \geq 0\},$$

$$\dim \operatorname{Ker} G(x) = \#\{n \ : \ f_n(x) = 0\},$$

and the result follows from elementary properties of strictly increasing continuous functions. □

The following two theorems follow by combining the last proposition with Theorems 3.6 and 3.8. We consider a self-adjoint operator H which has the same form as in Section 3.2. We will assume in addition that $H^{\rm vv}$ is a bounded operator. The first theorem describes how to count eigenvalues outside $\sigma(H^{\overline{\rm vv}})$.

Theorem 3.10 *Assume that* $[e_-,e_+]\cap\sigma(H^{\overline{\rm vv}})=\emptyset$ *and that* $\dim \mathbf{1}_{[0,\infty[}(G_{\rm v}(e_+))<\infty$. *Then,*

$$\dim \mathbf{1}_{[e_-,e_+]}^{\rm pp}(H)=\dim \mathbf{1}_{[e_-,e_+]}(H)=\dim \mathbf{1}_{[0,\infty[}(G_{\rm v}(e_+))-\dim \mathbf{1}_{]0,\infty[}(G_{\rm v}(e_-)).$$

Proof. We know by Proposition 3.5 (i) that

$$[e_-,e_+]\cap\sigma(H)\subset\{x\in[e_-,e_+]\ :\ 0\in\sigma(G_{\rm v}(x))\}. \tag{3.73}$$

Since $G_{\rm v}'(x)\geq \mathbf{1}^{\rm vv}$ we see that $G_{\rm v}(x)$ is strictly increasing. Thus we easily see that the function $G(x)$ satisfies the assumptions of Proposition 3.9, which implies that the set (3.73) is finite. Hence

$$\mathbf{1}_{[e_-,e_+]}(H)=\mathbf{1}_{[e_-,e_+]}^{\rm pp}(H)=\sum_{x\in[e_-,e_+]}\mathbf{1}_{\{x\}}(H).$$

It follows from Theorem 3.6 that for all $x\in[e_-,e_+]$,

$$\dim \mathbf{1}_{\{x\}}(H)=\dim {\rm Ker}\, G_{\rm v}(x),$$

and the result follows from Proposition 3.9. □

Corollary 3.11 *Assume that* $[e_-,e_+]\cap\sigma(H^{\overline{\rm vv}})=\emptyset$. *Then*

$$\dim \mathbf{1}_{[e_-,e_+]}(H)\leq \dim\mathcal{H}^{\rm v}. \tag{3.74}$$

If the self-energy is differentiable, one can also count eigenvalues of H inside $\sigma(H^{\overline{\rm vv}})$.

Theorem 3.12 *Assume that* $\sigma_{\rm pp}(H^{\overline{\rm vv}})\cap[e_-,e_+]=\emptyset$, *and that the following holds:*
(a) *The limit*

$$W_{\rm v}(x+{\rm i}0):=\lim_{y\downarrow 0}W_{\rm v}(x+{\rm i}y) \tag{3.75}$$

exists for all $x\in[e_-,e_+]$.
(b) *The function* ${\rm Re}W_{\rm v}(x+{\rm i}0)$ *is differentiable and for some* $\epsilon>0$ *and all* $x\in[e_-,e_+]$ *it satisfies*

$${\rm Re}G_{\rm v}'(x+{\rm i}0)=\mathbf{1}^{\rm vv}-{\rm Re}W_{\rm v}'(x)\geq\epsilon\mathbf{1}^{\rm vv}.$$

(c) $\dim \mathbf{1}_{[0,\infty[}({\rm Re}\, G_{\rm v}(e_++{\rm i}0))<\infty$.
Then,

$$\dim \mathbf{1}_{[e_-,e_+]}^{\rm pp}(H)\leq\dim \mathbf{1}_{[0,\infty[}({\rm Re}G_{\rm v}(e_++{\rm i}0))-\dim \mathbf{1}_{]0,\infty[}({\rm Re}G_{\rm v}(e_-+{\rm i}0)).$$

Proof. Assumptions (a) and Theorem 3.8, and then Proposition 3.2, yield that for $x \in [e_-, e_+]$,

$$\dim \mathbf{1}_{\{x\}}(H) \leq \dim \operatorname{Ker} G_{\mathrm{v}}(x + \mathrm{i}0) \leq \dim \operatorname{Ker}\operatorname{Re} G_{\mathrm{v}}(x + \mathrm{i}0). \tag{3.76}$$

Assumption (b) yields that the function $[e_-, e_+] \ni x \mapsto \operatorname{Re} G_{\mathrm{v}}(x + \mathrm{i}0)$ is continuous and strictly increasing. Using also (c) we see that the function $\operatorname{Re} G_{\mathrm{v}}(x + \mathrm{i}0)$ satisfies the conditions of Proposition 3.9. This proposition and Relation (3.76) yield the statement of the theorem. □

Corollary 3.13 *Assume that* $\sigma_{\mathrm{pp}}(H^{\overline{\mathrm{vv}}}) \cap [e_-, e_+] = \emptyset$, *and that the following holds:*
(a) *The limit*

$$W_{\mathrm{v}}(x + \mathrm{i}0) := \lim_{y \downarrow 0} W_{\mathrm{v}}(x + \mathrm{i}y), \tag{3.77}$$

exists for all $x \in [e_-, e_+]$.
(b) *The function* $[e_-, e_+] \ni x \mapsto \operatorname{Re} W_{\mathrm{v}}(x + \mathrm{i}0)$ *is differentiable, and for some* $\epsilon > 0$ *and all* $x \in [e_-, e_+]$,

$$\operatorname{Re} G_{\mathrm{v}}'(x + \mathrm{i}0) = \mathbf{1}^{\mathrm{vv}} - \operatorname{Re} W_{\mathrm{v}}'(x + \mathrm{i}0) \geq \epsilon \mathbf{1}^{\mathrm{vv}}.$$

Then

$$\dim \mathbf{1}^{\mathrm{pp}}_{[e_-, e_+]}(H) \leq \dim \mathcal{H}^{\mathrm{v}}. \tag{3.78}$$

4 Fock spaces and all that

In this section we describe in an abstract setting some Hilbert spaces and operators of the quantum field theory. We have attempted to give an essentially self-contained presentation of the topics we will need. For additional information, the reader may consult [BSZ, BR, De, DG, GJ, RS2].

Let $\mathfrak{h}$ be a Hilbert space. We set $\mathfrak{h}^{0\otimes} := \mathbf{C}$ and $\mathfrak{h}^{n\otimes} := \mathfrak{h} \otimes \ldots \otimes \mathfrak{h}$. If A is a closed operator on $\mathfrak{h}$, we denote by $A^{\otimes n}$ the closed operator on $\mathfrak{h}^{n\otimes}$ defined by $A \otimes \ldots \otimes A$ (if $n = 0$, $A^{\otimes 0} = 1$). Let S_n be the group of permutations of n elements. For each $\sigma \in S_n$ we define an operator (which we also denote by σ) on the basis elements of $\mathfrak{h}^{n\otimes}$ by

$$\sigma(f_{i_1} \otimes \ldots \otimes f_{i_n}) = f_{\sigma(i_1)} \otimes \ldots \otimes f_{\sigma(i_n)},$$

where $\{f_k\}$ is a basis of $\mathfrak{h}$. σ extends by linearity to a unitary operator on $\mathfrak{h}^{\otimes n}$, which does not depend on a basis. We set

$$P_n := \frac{1}{n!} \sum_{\sigma \in S_n} \sigma.$$

The operator P_n is an orthogonal projection. Let

$$\Gamma_n(\mathfrak{h}) := \mathrm{Ran} P_n.$$

This Hilbert space is commonly called the n-particle bosonic space.

The symmetric (or boson) Fock space over $\mathfrak{h}$ is defined by

$$\Gamma(\mathfrak{h}) := \overset{\infty}{\underset{n=0}{\oplus}} \Gamma_n(\mathfrak{h}).$$

The vector $\Omega = (1, 0, 0, \ldots)$ plays a special role and is called *the vacuum*. A vector $\Psi = (\psi_0, \psi_1, \ldots)$ is called a finite particle vector if $\psi_n = 0$ for all but finitely many n. The set of all finite particle vectors we denote by $\Gamma_{\mathrm{fin}}(\mathfrak{h})$.

If A is an operator on $\mathfrak{h}$, we define $\Gamma(A)$ to be the operator which is equal to $A^{\otimes n}$ on $\Gamma_n(\mathfrak{h})$. If ω is a self-adjoint operator on $\mathcal{H}$ then $\Gamma(\exp(it\omega))$ is a strongly continuous unitary group on $\Gamma(\mathfrak{h})$, and we denote its generator by $\mathrm{d}\Gamma(\omega)$. Note that

$$\Gamma(\exp(it\omega)) = \exp(it\mathrm{d}\Gamma(\omega)),$$

and $\mathrm{d}\Gamma(\omega)\Omega = 0$. $\mathrm{d}\Gamma(\omega)$ preserves the n-particle subspaces, and on $\mathcal{D}(\mathrm{d}\Gamma(\omega))) \cap \Gamma_n(\mathfrak{h})$ it acts as

$$\omega \otimes \mathbf{1} \ldots \otimes \mathbf{1} + \mathbf{1} \otimes \omega \ldots \otimes \mathbf{1} + \ldots \mathbf{1} \otimes \ldots \otimes \mathbf{1} \otimes \omega.$$

The number operator is defined by $N = \mathrm{d}\Gamma(\mathbf{1})$.

In the models we will study, the boson Fock space will represent only a part of the system, usually referred to as a "radiation field" or a "heat bath". The other part is an "atom" or a "small system", to which we associate a Hilbert space $\mathcal{K}$. The interaction of these two sub-systems is described by a suitable self-adjoint operator on $\mathcal{K} \otimes \Gamma(\mathfrak{h})$. To describe these interaction operators in a sufficient generality, it is convenient to extend the notion of the usual creation and annihilation operators on the Fock space. For the conventional definitions of these operators we refer the reader to [RS2], Section X.7. The definitions we will use are also discussed in [DG].

Notation. In the rest of the paper, whenever the meaning is clear within the context, we will write A for the operators of the form $\mathbf{1} \otimes A$ and $A \otimes \mathbf{1}$.

Let

$$v \in \mathcal{B}(\mathcal{K}, \mathcal{K} \otimes \mathfrak{h}).$$

Such operators will be called form-factors. We define a linear operator $b(v)$ on $\mathcal{K} \otimes (\oplus_{n=0}^{\infty} \mathfrak{h}^{n\otimes})$ as follows:

$$b(v) : \mathcal{K} \otimes \mathfrak{h}^{0\otimes} \mapsto 0,$$

$$b(v) : \mathcal{K} \otimes \mathfrak{h}^{n\otimes} \mapsto \mathcal{K} \otimes \mathfrak{h}^{(n-1)\otimes},$$

$$b(v)(\psi \otimes \phi_1 \otimes \ldots \otimes \phi_n) := (v^* \psi \otimes \phi_1) \otimes \phi_2 \otimes \ldots \otimes \phi_n.$$

It is not difficult to show that $b(v)$ is a bounded operator which takes $\mathcal{K}\otimes\Gamma(\mathfrak{h})$ into itself. Note that $\|b(v)\| = \|v\|_{\mathcal{B}(\mathcal{K},\mathcal{K}\otimes\mathfrak{h})}$.

We define the annihilation operator $a(v)$ on $\mathcal{K}\otimes\Gamma(\mathfrak{h})$ with domain $\mathcal{K}\otimes\Gamma_{\mathrm{fin}}(\mathfrak{h})$ by

$$a(v) = (N+1)^{\frac{1}{2}} b(v).$$

The operator $a(v)$ is closable, and we denote its closure with the same letter.

The adjoint of $a(v)$ we denote by $a^*(v)$ and call it the creation operator. To describe this operator, note that on $\mathcal{K}\otimes\Gamma_{\mathrm{fin}}(\mathfrak{h})$ we have

$$a^*(v) = Pb^*(v)(N+1)^{\frac{1}{2}}, \tag{4.79}$$

where $P = \sum_{n=0}^{\infty} P_n$. Moreover, $b^*(v)$ acts on $\mathcal{K}\otimes(\oplus_{n=0}^{\infty}\mathfrak{h}^{n\otimes})$ as follows:

$$b^*(v) : \mathcal{K}\otimes\mathfrak{h}^{n\otimes} \mapsto \mathcal{K}\otimes\mathfrak{h}^{(n+1)\otimes},$$

$$b^*(v)(\psi\otimes\phi_1\otimes\ldots\phi_n) = (v\psi)\otimes\phi_1\otimes\ldots\otimes\phi_n.$$

Here $\psi\in\mathcal{K}$, $\phi_k\in\mathfrak{h}$.

We remark that the map $v\mapsto a(v)$ is anti-linear while the map $v\mapsto a^*(v)$ is linear. In the sequel $a^{\#}(v)$ stands either for $a(v)$ or $a^*(v)$.

The (Segal) field operator is defined by

$$\varphi(v) := \frac{1}{\sqrt{2}}(a(v)+a^*(v)).$$

The operator $\varphi(v)$ is essentially self-adjoint on $\mathcal{K}\otimes\mathcal{D}(N^{\frac{1}{2}})$. The following two elementary estimates will be often used in the rest of the paper:

$$\|(N+1)^{-\frac{1}{2}}a^{\#}(v)\| \le \|v\|, \qquad \|(N+1)^{-\frac{1}{2}}\varphi(v)\| \le \sqrt{2}\|v\|. \tag{4.80}$$

Note that if v acts as $v\psi = (q\psi)\otimes f$, where f is a fixed vector in $\mathfrak{h}$ and $q\in\mathcal{B}(\mathcal{K})$, then

$$a^*(v) = q\otimes a^*(f), \qquad a(v) = q^*\otimes a(f).$$

Here $a^{\#}(f)$ are the usual creation and annihilation operators on $\Gamma(\mathfrak{h})$. Such form-factors we will call simple.

More generally, if $\{f_n\}$ is an orthogonal basis of $\mathfrak{h}$, for any $v\in B(\mathcal{K},\mathcal{K}\otimes\mathfrak{h})$, there are operators $v_n\in\mathcal{B}(\mathcal{K})$ such that

$$v\psi = \sum_n (v_n\psi)\otimes f_n, \qquad \|v\psi\|^2 = \sum_n \|v_n\psi\|^2, \quad \psi\in\mathcal{K}.$$

Thus, every form-factor can be decomposed into a sum of simple form-factors. One can alternatively use this fact to define the operators $a^{\#}(v)$, etc.

If U is a unitary operator on $\mathfrak{h}$, then

$$\Gamma(U)\varphi(v)\Gamma(U^{-1}) = \varphi(Uv). \tag{4.81}$$

¿From this relation it follows that if ω is a self-adjoint operator on $\mathfrak{h}$ such that $\omega v \in \mathcal{B}(\mathcal{K}, \mathcal{K}\otimes\mathfrak{h})$, then

$$\mathrm{i}[\mathrm{d}\Gamma(\omega), \varphi(v)] = \varphi(\mathrm{i}\omega v). \tag{4.82}$$

We now describe several technical results which will be used in the sequel.

Proposition 4.1 *Let $v \in B(\mathcal{K}, \mathcal{K}\otimes\mathfrak{h})$ and ω an operator on $\mathfrak{h}$. Assume that $\omega \geq 0$ and that ω is invertible on the range of v. Set $K_0 := v^*v$. Then the following estimates hold for any Ψ in the quadratic form domain of $\mathrm{d}\Gamma(\omega)$:*

$$\begin{aligned}
\|a(v)\Psi\|^2 &\leq \|v^*\omega^{-1}v\|(\Psi|\mathrm{d}\Gamma(\omega)\Psi),\\
\|a^*(v)\Psi\|^2 &\leq (\Psi|K_0\Psi) + \|v^*\omega^{-1}v\|(\Psi|\mathrm{d}\Gamma(\omega)\Psi),\\
\|\varphi(v)\Psi\|^2 &\leq (\Psi|K_0\Psi) + 2\|v^*\omega^{-1}v\|(\Psi|\mathrm{d}\Gamma(\omega)\Psi).
\end{aligned} \tag{4.83}$$

Remark. Results of this genre go back to the N_τ-estimates of Glimm and Jaffe [GJ], see also [Ar, BFS1].

Proof. Before we start, we remark that $v^*\omega^{-1}v \in \mathcal{K}$ iff $v^*\omega^{-\frac{1}{2}} \in \mathcal{B}(\mathcal{K}\otimes\mathfrak{h}, \mathfrak{h})$, and that

$$\|v^*\omega^{-1}v\| = \|v^*\omega^{-\frac{1}{2}}\|^2 = \|\omega^{-\frac{1}{2}}v\|^2.$$

Let $\mathcal{Q}$ denote the quadratic form domain of $\mathrm{d}\Gamma(\omega)$. Since

$$[N, a^*(v)a(v)] = [N, a(v)a^*(v)] = 0,$$

it suffices to establish the first two relations for

$$\Psi \in \mathcal{Q} \cap (\mathcal{K}\otimes\Gamma_n(\mathfrak{h})). \tag{4.84}$$

Note also that if $\Psi_i \in \mathcal{K}\otimes\Gamma_{n_i}(\mathfrak{h})$, $i = 1, 2$, then

$$(\Psi_2|a^*(v)\Psi_1) = \begin{cases} 0 & \text{if } n_2 - n_1 \neq 1,\\ \sqrt{n_1+1}(\Psi_2|v\otimes\mathbf{1}^{\otimes n_1}\Psi_1) & \text{if } n_2 - n_1 = 1. \end{cases} \tag{4.85}$$

Using Relation (4.85) twice, we derive

$$\begin{aligned}
(\Psi|a^*(v)a(v)\Psi) &= n(\Psi|vv^*\otimes\mathbf{1}^{\otimes(n-1)}\Psi)\\
&= n(\mathbf{1}_\mathcal{K}\otimes\omega^{\frac{1}{2}}\otimes\mathbf{1}^{\otimes(n-1)}\Psi|(v^*(\mathbf{1}_\mathcal{K}\otimes\omega^{-\frac{1}{2}}))^*\\
&\qquad\times(v^*(\mathbf{1}_\mathcal{K}\otimes\omega^{-\frac{1}{2}}))(\mathbf{1}_\mathcal{K}\otimes\omega^{\frac{1}{2}}\otimes\mathbf{1}^{\otimes(n-1)}\Psi)\\
&\leq \|v^*(\mathbf{1}_\mathcal{K}\otimes\omega^{-\frac{1}{2}})\|^2\, n(\Psi|\mathbf{1}_\mathcal{K}\otimes\omega\otimes\mathbf{1}^{\otimes(n-1)}\Psi)\\
&= \|v^*\omega^{-1}v\|(\Psi|\mathrm{d}\Gamma(\omega)\Psi).
\end{aligned}$$

This proves the first relation in (4.83). Before we prove the second, it is convenient to introduce some additional notation.

For $i = 1, \dots, n$, let $\tau_i^{(n)}$ denote the transposition of 1 and i. Recall that $\tau_i^{(n)}$ defines a unitary operator (which we denote by the same letter) on $\mathfrak{h}^{\otimes n}$. Clearly, $(\tau_i^{(n)})^2 = \mathbf{1}$. For any $h \in \mathcal{B}(\mathfrak{h})$ we define

$$h_i^{(n)} := \mathbf{1}^{\otimes(i-1)} \otimes h \otimes \mathbf{1}^{\otimes(n-i)} = \tau_i^{(n)}(h \otimes \mathbf{1}^{\otimes(n-1)})(\tau_i^{(n)})^{-1}.$$

Assume that Ψ satisfies (4.84). Then, using (4.79) and (4.85), we derive that

$$\begin{aligned}(a^*(v)\Psi|a^*(v)\Psi) &= (n+1)(\Psi|(v^* \otimes \mathbf{1}^{\otimes n})P_{n+1}(v \otimes \mathbf{1}^{\otimes n})\Psi) \\ &= \sum_{i=1}^{n+1}(\Psi|(v^* \otimes \mathbf{1}^{\otimes n})(\mathbf{1}_{\mathcal{K}} \otimes \tau_i^{(n+1)})(v \otimes \mathbf{1}^{\otimes n})\Psi) \\ &= (\Psi|K_0\Psi) + \textstyle\sum_{i=2}^{n}(\Psi|(v^* \otimes \mathbf{1}^{\otimes n})(\mathbf{1}_{\mathcal{K}} \otimes \tau_i^{(n+1)})(v \otimes \mathbf{1}^{\otimes n})\Psi).\end{aligned} \tag{4.86}$$

Note that for $i > 1$,

$$\begin{aligned}&(\Psi|(v^* \otimes \mathbf{1}^{\otimes n})(\mathbf{1}_{\mathcal{K}} \otimes \tau_i^{(n+1)})(v \otimes \mathbf{1}^{\otimes n})\Psi) = \\ &= ((\omega^{\frac12})_{i-1}^{(n)}\Psi|(v^* \otimes \mathbf{1}^{\otimes n})(\omega^{-\frac12})_i^{(n+1)}(\mathbf{1}_{\mathcal{K}} \otimes \tau_i^{(n+1)})(\omega^{-\frac12})_i^{(n+1)})(v \otimes \mathbf{1}^{\otimes n})(\omega^{\frac12})_{i-1}^{(n)}\Psi) \\ &= ((\omega^{\frac12})_{i-1}^{(n)}\Psi|(v^*(\mathbf{1}_{\mathcal{K}} \otimes \omega^{-\frac12}) \otimes \mathbf{1}^{\otimes n})(\mathbf{1}_{\mathcal{K}} \otimes \tau_i^{(n+1)})(\mathbf{1}_{\mathcal{K}} \otimes \omega^{-\frac12})v \otimes \mathbf{1}^{\otimes n})(\omega^{\frac12})_{i-1}^{(n)}\Psi) \\ &\le \|v^*(\mathbf{1}_{\mathcal{K}} \otimes \omega^{-\frac12})\|\,\|(\mathbf{1}_{\mathcal{K}} \otimes \omega^{-\frac12})v\|\,\|(\omega^{\frac12})_{i-1}^{(n)}\Psi\|^2 \\ &= \|v^*\omega^{-1}v\|(\Psi|\omega \otimes \mathbf{1}^{\otimes(n-1)}\Psi).\end{aligned}$$

Thus,

$$\begin{aligned}\textstyle\sum_{i=2}^{n+1}(\Psi|(v^* \otimes \mathbf{1}^{\otimes n})(\mathbf{1}_{\mathcal{K}} \otimes \tau_i^{(n+1)})(v \otimes \mathbf{1}^{\otimes n})\Psi) &\le n\|v^*\omega^{-1}v\|(\Psi|\omega \otimes \mathbf{1}^{\otimes(n-1)}\Psi) \\ &= \|v^*\omega^{-1}v\|(\Psi|\mathrm{d}\Gamma(\omega)\Psi).\end{aligned}$$

Combining this inequality with the identity (4.86), we derive the second relation in (4.83).

Finally, the third relation follows from the first two and the simple estimate

$$\|\varphi(v)\Psi\|^2 \le \|a(v)\Psi\|^2 + \|a^*(v)\Psi\|^2.$$

□

We will also make use of the following estimate.

Lemma 4.2 *Let $v \in B(\mathcal{K}, \mathcal{K} \otimes \mathfrak{h})$, $1 \ge \delta \ge 0$ and $N_\delta := 1 + \delta N$. Then, for any β,*

$$\|(\varphi(v) - N_\delta^{-\beta}\varphi(v)N_\delta^{\beta})\| \le C_\beta\sqrt{\delta}\|v\|.$$

Proof. It suffices to consider the case $\beta \geq 0$. For $\Psi \in \mathcal{K} \otimes \Gamma_n(\mathfrak{h})$ we have

$$\left(a^*(v) - N_\delta^{-\beta} a^*(v) N_\delta^\beta\right) \Psi = \sqrt{n+1}\left(1 - (\tfrac{1+\delta n}{1+\delta(n+1)})^\beta\right) P_{n+1} v \otimes \mathbf{1}^{\otimes n} \Psi.$$

For $0 < x \leq 1$ we have $|1 - (1-x)^\beta| \leq C_\beta x$. Hence

$$\begin{aligned}
\left\| a^*(v) - N_\delta^{-\beta} a^*(v) N_\delta^\beta \right\| &\leq \sup_{n \geq 0} \sqrt{n+1} \left| \left(1 - (\tfrac{1+\delta n}{1+\delta(n+1)})^\beta\right) \right| \|v\| \\
&\leq \sup_{n \geq 0} C_\beta \sqrt{n+1} \tfrac{\delta}{1+\delta(n+1)} \|v\| \\
&\leq C_\beta \sqrt{\delta} \|v\|.
\end{aligned} \tag{4.87}$$

After taking adjoints, (4.87) yields

$$\left\| a(v) - N_\delta^{-\beta} a(v) N_\delta^\beta \right\| \leq C_\beta \sqrt{\delta} \|v\|.$$

Clearly, the above two estimates yield the statement. □

The final results which we need is the exponential law for bosonic systems, see e.g. [BSZ], Section 3.2.

Theorem 4.3 *Let $\mathfrak{h}_1$ and $\mathfrak{h}_2$ be Hilbert spaces. There exist a unitary mapping*

$$U : \Gamma(\mathfrak{h}_1) \otimes \Gamma(\mathfrak{h}_2) \mapsto \Gamma(\mathfrak{h}_1 \oplus \mathfrak{h}_2),$$

with the following properties:
(i) *If A_1 and A_2 are operators on $\mathfrak{h}_1$ and $\mathfrak{h}_2$ then*

$$U(\Gamma(A_1) \otimes \Gamma(A_2))U^{-1} = \Gamma(A_1 \oplus A_2).$$

(ii) *If Ω denotes the vacuum on $\Gamma(\mathfrak{h}_1 \oplus \mathfrak{h}_2)$ and Ω_1, Ω_2 the vacua on $\Gamma(\mathfrak{h}_1)$, $\Gamma(\mathfrak{h}_2)$, then*

$$U(\Omega_1 \otimes \Omega_2) = \Omega.$$

(iii) *If $f_1 \in \mathfrak{h}_1$, $f_2 \in \mathfrak{h}_2$, then*

$$U \exp(\mathrm{i}\varphi(f_1) \otimes \exp(\mathrm{i}\varphi(f_2))U^{-1} = \exp(\mathrm{i}\varphi(f_1 \oplus f_2)).$$

(iv) *Let $\mathcal{K}$ be a Hilbert space. Assume that $v_1 \in B(\mathcal{K}, \mathcal{K} \otimes \mathfrak{h}_1)$, $f_2 \in \mathfrak{h}_2$ and $v_2 = \mathbf{1}_\mathcal{K} \otimes f_2$. Then $v_1 \oplus v_2$ can be viewed as an element of $B(\mathcal{K}, \mathcal{K} \otimes (\mathfrak{h}_1 \oplus \mathfrak{h}_2))$ and*

$$(\mathbf{1}_\mathcal{K} \otimes U) \exp(\mathrm{i}\varphi(v_1)) \otimes \exp(\mathrm{i}\varphi(v_2))(\mathbf{1}_\mathcal{K} \otimes U)^{-1} = \exp(\mathrm{i}\varphi(v_1 \oplus v_2)).$$

Remark. The properties (ii), (iii) specify U uniquely.

5 Pauli-Fierz operators

In this section we define operators which we will study.

In quantum physics such operators are used to describe systems which consist of two parts: the "small system" $\mathcal{A}$ and the "radiation field" $\mathcal{R}$. The system $\mathcal{A}$ is described by a Hilbert space $\mathcal{K}$ and a self-adjoint operator K on $\mathcal{K}$. The "radiation field" $\mathcal{R}$ is described by a bosonic Fock space. Its 1-particle space is $\mathfrak{h} := L^2(\mathbf{R}) \otimes \mathfrak{g}$, where $\mathfrak{g}$ is an auxiliary Hilbert space. We denote by ω the operator of multiplication by $\omega \in \mathbf{R}$. The Hilbert space of the combined system is $\mathcal{H} := \mathcal{K} \otimes \Gamma(\mathfrak{h})$ and its free Hamiltonian is

$$H_{\mathrm{fr}} := K \otimes \mathbf{1} + \mathbf{1} \otimes \mathrm{d}\Gamma(\omega).$$

Let $\alpha \in \mathcal{B}(\mathcal{K}, \mathcal{K} \otimes \mathfrak{h})$ be a given form factor. The Hamiltonian of the coupled system is formally given by

$$H := H_{\mathrm{fr}} + \lambda\varphi(\alpha). \tag{5.88}$$

We make the following hypothesis:

Hypothesis A. H is essentially self-adjoint on $\mathcal{D} := \mathcal{D}(H_{\mathrm{fr}}) \cap \mathcal{D}(\varphi(\alpha))$.

If we equip $\mathcal{D}$ with the norm

$$\|\Phi\|_{\mathcal{D}} := \|\Phi\| + \|H_{\mathrm{fr}}\Phi\| + \|\varphi(\alpha)\Phi\|,$$

then $\mathcal{D}$ becomes a Banach space. It follows from Hypothesis A and an easy abstract argument (the same that is needed to show Lemma 2.5) that any vector space dense in $\mathcal{D}$ is a core for H. Thus, for instance, $\mathcal{D}_{\mathrm{fin}} := \mathcal{D}(K) \otimes (\Gamma_{\mathrm{fin}}(\mathfrak{h}) \cap \mathcal{D}(\mathrm{d}\Gamma(|\omega|)))$ is a core of H.

Below we give two explicit conditions that imply Hypothesis A.

5.1 "Positive-temperature systems"

Proposition 5.1 *Assume that the operators α, $|\omega|\alpha$ and $(|K| \otimes \mathbf{1}_{\mathfrak{h}})\alpha - \alpha|K|$ are bounded. Let*

$$\hat{N} := |K| + \mathrm{d}\Gamma(|\omega| + 1).$$

Then, for any $\lambda \in \mathbf{R}$, H is essentially self-adjoint on any core of $\hat{N}$. Moreover, Hypothesis A *is satisfied.*

Proof. Clearly, H is a well defined symmetric operator on $\mathcal{D}(\hat{N})$. We will prove the proposition by invoking Nelson's commutator theorem ([RS2], Theorem X.37). We must show that there is a constant $d > 0$ such that the following estimates hold for any $\Psi \in \mathcal{D}(\hat{N})$:

$$\begin{aligned} \|H\Psi\| &\leq d\|\hat{N}\Psi\|, \\ |(H\Psi|\hat{N}\Psi) - (\hat{N}\Psi|H\Psi)| &\leq d\|\hat{N}^{\frac{1}{2}}\Psi\|^2. \end{aligned} \tag{5.89}$$

By (4.80), $\lambda\varphi(\alpha)$ is an infinitesimal perturbation of N and therefore of $\hat{N}$. Obviously, $\mathrm{d}\Gamma(\omega)$ is bounded with respect to $\mathrm{d}\Gamma(|\omega|)$. These observation yield the first relation.

Since

$$|(H\Psi|\hat{N}\Psi) - (\hat{N}\Psi|H\Psi)| = |\lambda||(\Psi|[\hat{N}, \varphi(\alpha)]\Psi)|,$$

and

$$\mathrm{i}[\hat{N}, \varphi(\alpha)] = \varphi(\mathrm{i}(|\omega| + 1)\alpha) + \varphi(\beta),$$

where $\beta = \mathrm{i}(|K| \otimes \mathbf{1}_{\mathfrak{h}})\alpha - \mathrm{i}\alpha|K|$, the second relation in (5.89) follows from the estimates (4.80).

Finally, since $\mathcal{D}_{\mathrm{fin}}$ is a core for $\hat{N}$, Hypothesis A is satisfied. □

5.2 "Zero-temperature systems"

Let $\tilde{\mathfrak{h}} := L^2(\mathbf{R}_+) \otimes \mathfrak{g}$, where $\mathfrak{g}$ is an auxiliary Hilbert space. We denote by $\tilde{\omega}$ the operator of multiplication by $\omega \in \mathbf{R}_+$. Consider the Hilbert space $\tilde{\mathcal{H}} := \mathcal{K} \otimes \Gamma(\tilde{\mathfrak{h}})$ and the free Hamiltonian

$$\tilde{H}_{\mathrm{fr}} := K \otimes \mathbf{1} + \mathbf{1} \otimes \mathrm{d}\Gamma(\tilde{\omega}). \tag{5.90}$$

Let $\tilde{\alpha} \in \mathcal{B}(\mathcal{K}, \mathcal{K} \otimes \tilde{\mathfrak{h}})$. The Hamiltonian of the coupled system is formally given by

$$\tilde{H} := \tilde{H}_{\mathrm{fr}} + \lambda\varphi(\tilde{\alpha}), \tag{5.91}$$

where λ is a real constant.

Proposition 5.2 *Assume that the operator K is bounded from below and that*

$$\tilde{\alpha}^*\tilde{\omega}^{-1}\tilde{\alpha} \in \mathcal{B}(\mathcal{K}). \tag{5.92}$$

Then $\varphi(\tilde{\alpha})$ is infinitesimally small with respect to $\tilde{H}_{\mathrm{fr}}$. In particular, for any $\lambda \in \mathbf{R}$, the operator $\tilde{H}$ is self-adjoint on $\mathcal{D}(\tilde{H}_{\mathrm{fr}})$.

Proof. Without loss of generality we may assume that K is strictly positive. By (4.83) there is a constant $c > 0$ such that for any $\Psi \in \mathcal{D}(H_{\mathrm{fr}})$ and any $\epsilon > 0$,

$$\|\varphi(\tilde{\alpha})\Psi\|^2 \le c\|\Psi\|^2 + c(\Psi|\tilde{H}_0\Psi)) \le c(1 + \epsilon^{-1})\|\Psi\|^2 + c\epsilon\|\tilde{H}_{\mathrm{fr}}\Psi\|^2.$$

It follows that $\varphi(\tilde{\alpha})$ is an infinitesimal perturbation of $\tilde{H}_{\mathrm{fr}}$. The other conclusions of the proposition follow from the Kato-Rellich theorem. □

The operator $\tilde{H}$ has a different form than the operator (5.88). Nevertheless, we will show below that by studying operators of the form (5.88) one can obtain information on the "zero-temperature" Hamiltonian $\tilde{H}$.

Consider an operator of the form (5.91) and assume that (5.92) holds. This operator can be extended to act on the Hilbert space $\tilde{\mathcal{H}} \otimes \Gamma(\tilde{\mathfrak{h}})$ as

$$\tilde{H}_{\mathrm{fr}}^{\mathrm{ext}} := \tilde{H}_{\mathrm{fr}} \otimes \mathbf{1} - \mathbf{1} \otimes \mathrm{d}\Gamma(\tilde{\omega}),$$
$$\tilde{H}^{\mathrm{ext}} := \tilde{H} \otimes \mathbf{1} - \mathbf{1} \otimes \mathrm{d}\Gamma(\tilde{\omega}).$$

Since $\tilde{\mathcal{H}} \otimes \Gamma_0(\tilde{\mathfrak{h}})$ is an invariant subspace of $\tilde{H}^{\mathrm{ext}}$ and

$$\tilde{H} = \tilde{H}^{\mathrm{ext}}\Big|_{\tilde{\mathcal{H}}\otimes\Gamma_0(\tilde{\mathfrak{h}})},$$

the spectral properties of $\tilde{H}$ can be inferred from the spectral properties of $\tilde{H}^{\mathrm{ext}}$ (note in particular that $\sigma_{\mathrm{pp}}(\tilde{H}) = \sigma_{\mathrm{pp}}(\tilde{H}^{\mathrm{ext}})$ and $\sigma_{\mathrm{sc}}(\tilde{H}) = \sigma_{\mathrm{sc}}(\tilde{H}^{\mathrm{ext}})$). Let us show that $\tilde{H}^{\mathrm{ext}}$ is unitarily equivalent to an operator of the form (5.88) satisfying Hypothesis A.

Let U be the map from $\Gamma(\tilde{\mathfrak{h}}) \otimes \Gamma(\tilde{\mathfrak{h}})$ to $\Gamma(\tilde{\mathfrak{h}} \oplus \tilde{\mathfrak{h}})$ defined in Theorem 4.3. Clearly,

$$\mathbf{1}_{\mathcal{K}} \otimes U : \tilde{\mathcal{H}} \otimes \Gamma(\mathfrak{h}) \to \mathcal{K} \otimes \Gamma(\tilde{\mathfrak{h}} \oplus \tilde{\mathfrak{h}})$$

is a unitary map. Next, we have the unitary map

$$L^2(\mathbf{R}_+) \oplus L^2(\mathbf{R}_+) \ni (f_1, f_2) \mapsto f \in L^2(\mathbf{R}), \tag{5.93}$$

where

$$f(\omega) = \begin{cases} f_1(\omega) & \text{if } \omega \geq 0 \\ f_2(\omega) & \text{if } \omega < 0, \end{cases}$$

which induces the unitary map

$$w : \tilde{\mathfrak{h}} \oplus \tilde{\mathfrak{h}} = \Big(L^2(\mathbf{R}_+) \otimes \mathfrak{g}\Big) \oplus \Big(L^2(\mathbf{R}_+) \otimes \mathfrak{g}\Big) \mapsto \mathfrak{h} = L^2(\mathbf{R}) \otimes \mathfrak{g}.$$

Set

$$W := \mathbf{1}_{\mathcal{K}} \otimes (\Gamma(w)U).$$

Clearly,

$$W : \tilde{\mathcal{H}} \otimes \Gamma(\tilde{\mathfrak{h}}) \mapsto \mathcal{H},$$

is a unitary map. Let $\alpha \in B(\mathcal{K}, \mathcal{K} \otimes \mathfrak{h})$ be given by

$$\alpha := \tilde{\alpha} \oplus 0.$$

We have

$$\begin{aligned} w(\tilde{\omega}, -\tilde{\omega})w^* &= \omega, \\ W(\mathrm{d}\Gamma(\tilde{\omega})) \otimes \mathbf{1} - \mathbf{1} \otimes \mathrm{d}\Gamma(\tilde{\omega}))W^* &= \mathrm{d}\Gamma(\omega), \\ W\varphi(\tilde{\alpha}) \otimes \mathbf{1}W^* &= \varphi(\alpha). \end{aligned}$$

Thus

$$W\tilde{H}_{\rm fr}^{\rm ext}W^* = K\otimes\mathbf{1} + \mathbf{1}\otimes {\rm d}\Gamma(\omega),$$

$$W\tilde{H}^{\rm ext}W^* = K\otimes\mathbf{1} + \mathbf{1}\otimes {\rm d}\Gamma(\omega) + \lambda\varphi(\alpha).$$

Hence $W\tilde{H}_{\rm fr}^{\rm ext}W^*$ and $W\tilde{H}^{\rm ext}W^*$ have the form of the operators $H_{\rm fr}$, H. Furthermore, it follows from Proposition 5.2 that the operator $W\tilde{H}^{\rm ext}W^*$ is self-adjoint on

$$\mathcal{D}(W\tilde{H}_{\rm fr}^{\rm ext}W^*) = \mathcal{D}(W\tilde{H}_{\rm fr}^{\rm ext}W^*)\cap\mathcal{D}(\varphi(\alpha)).$$

Thus the operator $W\tilde{H}^{\rm ext}W^*$ satisfies Hypothesis A. □

6 Main results

The Hilbert space $\mathcal{H} = \mathcal{K}\otimes\Gamma(\mathfrak{h})$ has a natural decomposition $\mathcal{H} = \mathcal{H}^{\rm v}\oplus\mathcal{H}^{\overline{\rm v}}$, where

$$\begin{array}{ll} \mathcal{H}^{\rm v} & := \mathcal{K}\otimes\Gamma_0(\mathfrak{h}), \\ \mathcal{H}^{\overline{\rm v}} & := \bigoplus_{n=1}^{\infty}\mathcal{K}\otimes\Gamma_n(\mathfrak{h}). \end{array}$$

(v stands for the vacuum). Note that $H_{\rm fr}^{\rm vv} = K$, $\mathcal{D}(H_{\rm fr}) = \mathcal{D}(K)\oplus\mathcal{D}(H_{\rm fr}^{\overline{\rm v}\overline{\rm v}})$ and $\mathcal{D}(\varphi(\alpha)) = \mathcal{K}\oplus\mathcal{D}(\varphi(\alpha)^{\overline{\rm v}\overline{\rm v}})$. Hence, if $\mathcal{D}$ is as in Hypothesis A, then $\mathcal{D} = \mathcal{D}(K)\oplus\mathcal{D}(H_{\rm fr}^{\overline{\rm v}\overline{\rm v}})\cap\mathcal{D}(\varphi(\alpha)^{\overline{\rm v}\overline{\rm v}})$. Using Hypothesis A, the fact that $H^{\overline{\rm v}{\rm v}} = \lambda\varphi(\alpha)^{\overline{\rm v}{\rm v}}$ is a bounded operator and the Kato–Rellich theorem we see that $H^{\rm vv} + H^{\overline{\rm v}\overline{\rm v}}$ is essentially self-adjoint on $\mathcal{D}$. Therefore, $H^{\overline{\rm v}\overline{\rm v}}$ is essentially self-adjoint on $\mathcal{D}(H_{\rm fr}^{\overline{\rm v}\overline{\rm v}})\cap\mathcal{D}(\varphi(\alpha)^{\overline{\rm v}\overline{\rm v}})$. Thus the formalism and results of Chapter 3 can be applied to the operator H. We will use the notation introduced in Section 3.2. In particular, we recall that the self-energy is defined by

$$\begin{aligned} W_{\rm v}(z) &= H^{{\rm v}\overline{\rm v}}(z\mathbf{1}^{\overline{\rm v}\overline{\rm v}} - H^{\overline{\rm v}\overline{\rm v}})^{-1}H^{\overline{\rm v}{\rm v}} \\ &= \tfrac{1}{2}\lambda^2 a(\alpha)^{{\rm v}\overline{\rm v}}(z\mathbf{1}^{\overline{\rm v}\overline{\rm v}} - H^{\overline{\rm v}\overline{\rm v}})^{-1}a^*(\alpha)^{\overline{\rm v}{\rm v}}. \end{aligned}$$

We define a self-adjoint operator s on the Hilbert space $\mathfrak{h}$ by $s := -{\rm i}\partial_\omega\otimes\mathbf{1}_{\mathfrak{g}}$, so that $[s,\omega] = -{\rm i}$. The conjugate operator is defined by

$$S := \mathbf{1}_{\mathcal{K}}\otimes{\rm d}\Gamma(s). \tag{6.94}$$

For any $\nu\geq 0$ we introduce the following hypothesis:

Hypothesis $S(\nu)$. $\quad\langle s\rangle^\nu\alpha\in\mathcal{B}(\mathcal{K},\mathcal{K}\otimes\mathfrak{h})$.

We will compare $W_{\rm v}(z)$ with its second-order approximation $\lambda^2 w(z)$, where

$$\begin{aligned} w(z) &:= \varphi(\alpha)^{{\rm v}\overline{\rm v}}(z\mathbf{1}^{\overline{\rm v}\overline{\rm v}} - H_{\rm fr}^{\overline{\rm v}\overline{\rm v}})^{-1}\varphi(\alpha)^{\overline{\rm v}{\rm v}} \\ &= \tfrac{1}{2}\Big(a(\alpha)(z - H_{\rm fr})^{-1}a^*(\alpha)\Big)^{\rm vv} \\ &= \tfrac{1}{2}\alpha^*(z - K\otimes\mathbf{1} - \mathbf{1}\otimes\omega)^{-1}\alpha. \end{aligned}$$

We now state an auxiliary result on the regularity properties of the function $w(z)$. Some of these properties will be used in the statement of our main theorems, but we remark that they are of independent interest.

Theorem 6.1 *Assume that Hypothesis $S(\nu)$ holds with $\nu > \frac{1}{2}$. Let $n \in \mathbf{N}$ and $0 < \theta \leq 1$ be such that $\nu = n + \frac{1}{2} + \theta$. Then, the function $\mathbf{C}_+ \ni z \mapsto w(z)$ extends by continuity to $\overline{\mathbf{C}}_+$ and is in the class $C_{\mathrm{u}}^{n,\theta}(\overline{\mathbf{C}}_+)$.*

The next three theorems describe our main results. In our model, the spectrum of K plays a role of the threshold set. The first theorem asserts that, away from an $O(\lambda^2)$ neighborhood of $\sigma(K)$, the Limiting Absorption Principle holds.

Theorem 6.2 *Assume that Hypotheses* A *and $S(\nu)$ hold with $\nu > 1$. Let $\mu > \frac{1}{2}$ and $0 < \Lambda_1 < (\sqrt{2}\|s\alpha\|)^{-1}$. Then, there exists a constant $\beta_1 > 0$ such that for $|\lambda| \leq \Lambda_1$ the following holds:*
(i) *Set for shortness*

$$\Theta := \mathbf{R} \setminus I(\sigma(K), \lambda^2\beta_1).$$

Then, the function

$$z \mapsto \langle S\rangle^{-\mu}(z - H)^{-1}\langle S\rangle^{-\mu} \tag{6.95}$$

extends by continuity to a function on $\mathbf{C}_+ \cup \Theta$. In particular, the spectrum of H on the set Θ is absolutely continuous.
(ii) *Let $n \in \mathbf{N}$, $0 < \theta \leq 1$, and μ be such that $\nu \geq \mu + \frac{1}{2} = 1 + n + \theta$. Then, the function (6.95) is of the class $C_{\mathrm{u}}^{n,\theta}$ of the set*

$$\overline{\mathbf{C}}_+ \setminus B(\sigma(K), \lambda^2\beta_1).$$

Our last two theorems describe the structure of the spectrum near an isolated eigenvalue k of K. They incorporate the notion of Fermi's Golden Rule. We remark that in our approach the eigenvalue k may have an infinite multiplicity. Let $p_k = \mathbf{1}_{\{k\}}(K)$. If $S(\nu)$ holds for some $\nu > \frac{1}{2}$ then it follows from Theorem 6.1 that

$$w_k := p_k w(k + \mathrm{i}0)p_k \tag{6.96}$$

is a bounded dissipative operator. We will always consider w_k as an operator on the Hilbert space $\mathrm{Ran}p_k$. In the standard description of atomic radiation, the spectrum of $\mathrm{Im}w_k$ captures the emission and absorption processes and radiative life-time of energy level k (of order λ^2). The spectrum of $\mathrm{Re}w_k$ captures the line shift of this energy level (of order λ^2). If $\sigma(w_k) \cap \mathbf{R} = \emptyset$, that is, if $\mathrm{Im}w_k < 0$, one expects that the energy level k has dissolved into the continuum, and that the spectrum of H in a neighborhood of k is purely absolutely continuous. Among other things, our next theorem justifies rigorously this heuristic expectation.

Theorem 6.3 *Assume that Hypotheses* A *and* $S(\nu)$ *hold with* $\nu > 1$ *and let* $\mu > \frac{1}{2}$. *Let* k *be an isolated eigenvalue of* K. *Assume that*

$$\mathcal{T}_k := \sigma(w_k) \cap \mathbf{R} \subset \sigma_{\mathrm{disc}}(w_k).$$

Let β_1 *be the constant from the previous theorem and* $\kappa := 1 - \nu^{-1}$. *Then there exist constants* $\Lambda_2 > 0$ *and* $\beta_2 > 0$ *such that for* $|\lambda| \leq \Lambda_2$ *the following holds:*
(i) *Set for shortness*

$$\Theta(k) := \overline{I}(k, \lambda^2\beta_1) \setminus I(\{k\} + \lambda^2\mathcal{T}_k, |\lambda|^{2+\kappa}\beta_2).$$

Then, the function

$$z \mapsto \langle S\rangle^{-\mu}(z - H)^{-1}\langle S\rangle^{-\mu} \tag{6.97}$$

extends by continuity to a function on $\mathbf{C}_+ \cup \Theta(k)$. *In particular, the spectrum of* H *on the set* $\Theta(k)$ *is absolutely continuous.*
(ii) *Let* $n \in \mathbf{N}$, $0 < \theta \leq 1$ *and* μ *be such that* $\nu \geq \mu + \frac{1}{2} = n + 1 + \theta$. *Then, the function (6.97) is in the class* $C_{\mathrm{u}}^{n,\theta}$ *of the set*

$$\overline{\mathbf{C}}_+ \cap \left(\overline{B}(k, \lambda^2\beta_1) \setminus B(\{k\} + \lambda^2\mathcal{T}_k, |\lambda|^{2+\kappa}\beta_2\right).$$

The next theorem is perhaps our deepest result. It concerns the situation where $\mathcal{T}_k \neq \emptyset$, and describes the structure of the spectrum of H around a point $m \in \sigma_{\mathrm{disc}}(w_k) \cap \mathbf{R}$. We set $p_{k,m} = \mathbf{1}_{\{m\}}(w_k)$. It follows from Proposition 3.2 that $p_{k,m}$ is an orthogonal projection. We emphasize that in the following theorem we need a stronger assumption on the interaction, namely we need $S(\nu)$ with $\nu > 2$.

Theorem 6.4 *Assume that Hypotheses* A *and* $S(\nu)$ *hold with* $\nu > 2$. *Let* k *be an isolated eigenvalue of* K,

$$\mathcal{T}_k := \sigma(w_k) \cap \mathbf{R} \subset \sigma_{\mathrm{disc}}(w_k).$$

and let $m \in \mathcal{T}_k$. *Let* β_1, β_2 *and* κ *be as in the previous theorems. Then there exists a constant* $\Lambda_3 > 0$ *such that for* $0 < |\lambda| \leq \Lambda_3$ *the following holds:*
(i) *Set for shortness*

$$\Theta(k, m) := \overline{I}(k + \lambda^2 m, |\lambda|^{2+\kappa}\beta_2) \cap \overline{I}(k, \lambda^2\beta_1).$$

Then $\dim \mathbf{1}^{\mathrm{pp}}_{\Theta(k,m)} \leq \dim p_{k,m}$. *In particular,* $\sigma_{\mathrm{pp}}(H) \cap \Theta(k, m)$ *is a finite set consisting of eigenvalues of finite multiplicity.*
(ii) *If* $\mu > \frac{1}{2}$, *then the function*

$$z \mapsto \langle S\rangle^{-\mu}(z - H)^{-1}\langle S\rangle^{-\mu} \tag{6.98}$$

extends by continuity to a function on $\mathbf{C}_+ \cup (\Theta(k,m) \setminus \sigma_{\mathrm{pp}}(H))$*. In particular, the spectrum of* H *on* $\Theta(k,m) \setminus \sigma_{\mathrm{pp}}(H)$ *is absolutely continuous.*
(iii) *Let* $n \in \mathbf{N}$, $0 < \theta \leq 1$ *and* $\nu \geq \mu + \frac{1}{2} = n + 1 + \theta$*. Then, for any* $\epsilon > 0$*, the function (6.98) is in the class* $C_{\mathrm{u}}^{n,\theta}$ *of the set*

$$\overline{\mathbf{C}}_+ \cap \left(\overline{B}(k + \lambda^2 m, |\lambda|^{2+\kappa}\beta_2) \cap \overline{B}(k, \lambda^2\beta_1) \setminus B(\sigma_{\mathrm{pp}}(H), \epsilon) \right).$$

7 Mourre theory on the radiation sector

In this chapter we prove the Limiting Absorption Principle for the operator H reduced to the radiation sector. In Section 7.1 we prove the basic form and in Section 7.2 more refined versions of the Limiting Absorption Principle. As we have remarked in the introduction, the Limiting Absorption Principle for $H^{\overline{\mathrm{vv}}}$ will hold uniformly on $\mathbf{R}$. In Section 7.3 we prove Theorem 6.1. In Section 7.4 we prove an estimate on the difference $(z\mathbf{1}^{\overline{\mathrm{vv}}} - H^{\overline{\mathrm{vv}}})^{-1} - (z\mathbf{1}^{\overline{\mathrm{vv}}} - H_{\mathrm{fr}}^{\overline{\mathrm{vv}}})^{-1}$.

7.1 Limiting Absorption Principle

This section is devoted to the proof of the following theorem:

Theorem 7.1 *Assume that Hypotheses* A *and* $S(\nu)$ *hold with* $\nu > 1$*. Let* $\mu > \frac{1}{2}$ *and* $0 < \Lambda_1 < (\sqrt{2}\|s\alpha\|)^{-1}$*. Then,*

$$\sup_{(\lambda,z)\in[-\Lambda_1,\Lambda_1]\times\mathbf{C}_+} \left\| \langle S^{\overline{\mathrm{vv}}} \rangle^{-\mu} (z\mathbf{1}^{\overline{\mathrm{vv}}} - H^{\overline{\mathrm{vv}}})^{-1} \langle S^{\overline{\mathrm{vv}}} \rangle^{-\mu} \right\| < \infty.$$

Notation. In this section we will always work in the space $\mathcal{H}^{\overline{\mathrm{v}}}$. Henceforth, until the end of the section we will drop the superscripts $\overline{\mathrm{vv}}$. Thus, we write H, S, N for the operators $H^{\overline{\mathrm{vv}}}$, $S^{\overline{\mathrm{vv}}}$, $N^{\overline{\mathrm{vv}}}$, etc.

We start by assembling some preliminary definitions and facts. Let

$$\mathbf{S}A := [S, A],$$

$$\mathrm{e}^{\mathrm{i}\tau\mathbf{S}}A := \mathrm{e}^{\mathrm{i}\tau S} A \mathrm{e}^{-\mathrm{i}\tau S}.$$

Note that

$$\begin{aligned} \mathrm{e}^{\mathrm{i}\tau\mathbf{S}}\mathrm{d}\Gamma(\omega) &= \mathrm{d}\Gamma(\omega) + \tau N, \\ \mathrm{e}^{\mathrm{i}\tau\mathbf{S}} N &= N, \\ \mathrm{e}^{\mathrm{i}\tau\mathbf{S}}(a(\alpha)) &= a(\mathrm{e}^{\mathrm{i}\tau s}\alpha), \\ \mathrm{e}^{\mathrm{i}\tau\mathbf{S}}(a^*(\alpha)) &= a^*(\mathrm{e}^{\mathrm{i}\tau s}\alpha). \end{aligned}$$

We choose a real function $\zeta \in C_0^\infty(\mathbf{R})$ such that $\zeta(t) = 1$ in a neighborhood of 0 and set

$$\xi(t) := \mathrm{e}^t \zeta(t). \tag{7.99}$$

Let

$$\mathfrak{D} := \mathcal{D}(H_{\mathrm{fr}}) \cap \mathcal{D}(N). \tag{7.100}$$

For $\epsilon \in \mathbf{R}$ we define

$$\begin{aligned} H_{\epsilon,\mathrm{fr}} &:= H_{\mathrm{fr}} - \mathrm{i}\epsilon N, \\ V_\epsilon &:= \tfrac{1}{\sqrt{2}}(a(\xi(-\epsilon s)\alpha) + a^*(\xi(\epsilon s)\alpha)), \\ H_\epsilon &:= H_{\epsilon,\mathrm{fr}} + \lambda V_\epsilon. \end{aligned} \tag{7.101}$$

where for $\epsilon = 0$ we have $H_{0,\mathrm{fr}} = H_{\mathrm{fr}}$, $H_0 = H$ and for $\epsilon \neq 0$ the domain is chosen as $\mathcal{D}(H_{\epsilon,\mathrm{fr}}) = \mathcal{D}(H_\epsilon) = \mathfrak{D}$.
Remark. We remark that the following identity holds on $\mathfrak{D}$:

$$H_\epsilon = \frac{1}{2\pi} \int \hat{\xi}(\tau) \mathrm{e}^{\mathrm{i}\tau\epsilon\mathbf{S}} H \mathrm{d}\tau. \tag{7.102}$$

Thus, formally, we could write $H_\epsilon = \xi(\epsilon\mathbf{S})H$, following the notation of [BG] and [BGS]. In [BG], a functional calculus was developed for expressions similar to (7.102). In our case, strictly speaking, this calculus does not apply, because H is an unbounded operator. Nevertheless, this calculus certainly motivates the definition of H_ϵ and the algebraic computations of this section.

The basic properties of the operators $H_{\epsilon,\mathrm{fr}}$ are summarized in the following lemma:

Lemma 7.2 *For any $\epsilon \in \mathbf{R}$, $H_{\epsilon,\mathrm{fr}}$ is a normal operator such that $H_{\epsilon,\mathrm{fr}}^* = H_{-\epsilon,\mathrm{fr}}$,*

$$\begin{aligned} \|H_{\epsilon,\mathrm{fr}}\Psi\|^2 = \|H_{\mathrm{fr}}\Psi\|^2 + \epsilon^2\|N\Psi\|^2, \quad \Psi \in \mathcal{D}(H_{\epsilon,\mathrm{fr}}), \\ \sigma(H_{\epsilon,\mathrm{fr}}) = \{-\mathrm{i}n\epsilon + \mathbf{R} \; : \; n = 1, 2, \ldots\}. \end{aligned} \tag{7.103}$$

The next lemma gives the basic properties of the operator H_ϵ.

Lemma 7.3 *Assume that Hypothesis $S(0)$ holds. Then the following is true:*
(i) *For any $\epsilon \neq 0$, V_ϵ is an infinitesimal perturbation of $H_{\epsilon,\mathrm{fr}}$. In particular, H_ϵ is a closed operator with domain $\mathfrak{D}$.*
(ii) *For any ϵ we have $H_\epsilon^* = H_{-\epsilon}$.*

Proof. It follows from estimate (4.80) that

$$\|N^{-\frac{1}{2}}V_\epsilon\| \leq 2\|\xi\|_\infty\|\alpha\|, \tag{7.104}$$

and thus V_ϵ is an infinitesimal perturbation of N. This observation and (7.103) yield that V_ϵ is also an infinitesimal perturbation of $H_{\epsilon,\mathrm{fr}}$. This proves (i).

Let us show (ii). Clearly, we can assume that $\epsilon \neq 0$. It is easy to see that we can split V_ϵ as $V_\epsilon = V_{\epsilon,1} + V_{\epsilon,2}$ where $V_{\epsilon,2}$ is bounded and $\|N^{-1}V_{\epsilon,1}\| < 1$, $\|V_{\epsilon,1}N^{-1}\| < 1$. Therefore, we can apply Proposition 2.11. □

For $z \notin \sigma(H_\epsilon)$ we set

$$G_\epsilon(z) := (z - H_\epsilon)^{-1}.$$

In the next 3 lemmas we describe some properties of $G_\epsilon(z)$ that hold under the assumption S(0) and $\epsilon \neq 0$. Although they are formally obvious, they require a proof due to the unboundedness of some of the operators.

For $n = 1, 2, \ldots$ we define closed operators $H_\epsilon^{(n)}$ by the formula

$$H_\epsilon^{(n)} = -\mathrm{i}\delta_{1n}N + \frac{\lambda}{\sqrt{2}}\left((-1)^k a(s^k\xi^{(n)}(-\epsilon s)\alpha) + a^*(s^k\xi^{(n)}(\epsilon s)\alpha)\right), \tag{7.105}$$

where $\delta_{1n} = 1$ if $n = 1$ and 0 otherwise. Here, of course, $\xi^{(n)}$ is the n-th derivative of the function ξ. Note that for any $\Psi \in \mathfrak{D}$

$$\frac{\mathrm{d}^n}{\mathrm{d}\epsilon^n}H_\epsilon\Psi = H_\epsilon^{(n)}\Psi.$$

Lemma 7.4 *Assume that Hypothesis S(0) holds and let $z \notin \sigma(H_\epsilon)$. Then the function*

$$\mathbf{R}_+ \ni \epsilon \mapsto G_\epsilon(z), \tag{7.106}$$

is infinitely differentiable and

$$\frac{\mathrm{d}^n}{\mathrm{d}\epsilon^n}G_\epsilon(z) = \sum_{n_1+\cdots+n_l=n} G_\epsilon(z)H_\epsilon^{(n_1)}G_\epsilon(z)\cdots G_\epsilon(z)H_\epsilon^{(n_l)}G_\epsilon(z). \tag{7.107}$$

Remark. In this section we will deal only with the first derivative of the function (7.106). The higher derivatives will be used in Section 7.2.

Proof. Let $\epsilon > 0$ be fixed and $z \notin \sigma(H_\epsilon)$. First note that

$$\|NG_\epsilon(z)\| < C. \tag{7.108}$$

In fact, by Lemma 7.3 $\|(H_{\epsilon,\mathrm{fr}} + \mathrm{i})G_\epsilon(z)\|$ is bounded and hence (7.108) follows from the bound

$$\|NG_\epsilon(z)\| \leq \|N(H_{\epsilon,\mathrm{fr}} + \mathrm{i})^{-1}\|\|(H_{\epsilon,\mathrm{fr}} + \mathrm{i})G_\epsilon(z)\|.$$

Next we note that for $|h| \leq \frac{\epsilon}{2}$ we have

$$\|(V_{\epsilon+h} - V_\epsilon)N^{-1}\| \leq C\frac{|h|}{|\epsilon|}\|\alpha\| \sup_t |\xi'(t)|. \tag{7.109}$$

Hence, for $|h| \leq \frac{\epsilon}{2}$

$$\|(H_{\epsilon+h} - H_\epsilon)G_\epsilon(z)\| = \|(-\mathrm{i}hN + V_{\epsilon+h} - V_\epsilon)G_\epsilon(z)\| \leq C_1 h.$$

Therefore, for small enough h, $z \notin \sigma(H_{\epsilon+h})$ and $h \mapsto G_{\epsilon+h}(z)$ is norm continuous at $h = 0$.

We will now show that the function (7.106) is differentiable and that Relation (7.107) holds for $n = 1$. An easy inductive argument then yields that this function is infinitely differentiable and that Relation (7.107) holds for all n.

We have

$$G_{\epsilon+h}(z) - G_\epsilon(z) - hG_\epsilon(z)H_\epsilon^{(1)}G_\epsilon(z) = \mathrm{I} + \mathrm{II}, \tag{7.110}$$

where

$$\begin{aligned}
\mathrm{I} \; &:= \left(G_{\epsilon+h}(z) - G_\epsilon(z)\right)\left(H_{\epsilon+h} - H_\epsilon\right)G_\epsilon(z) \\
&= G_{\epsilon+h}(z)(-\mathrm{i}hN + \lambda V_{\epsilon+h} - \lambda V_\epsilon)G_\epsilon(z)(-\mathrm{i}hN + \lambda V_{\epsilon+h} - \lambda V_\epsilon)G_\epsilon(z), \\
\mathrm{II} \; &:= G_\epsilon(z)(H_{\epsilon+h} - H_\epsilon - hH_\epsilon^{(1)})G_\epsilon(z) \\
&= G_\epsilon(z)(\lambda V_{\epsilon+h} - \lambda V_\epsilon - h\lambda V_\epsilon^{(1)})G_\epsilon(z)
\end{aligned}$$

For $|h| \leq \frac{\epsilon}{2}$, we have

$$\|(V_{\epsilon+h} - V_\epsilon - hV_\epsilon^{(1)})N^{-1}\| \leq C\frac{|h|^2}{\epsilon^2}|\lambda|\|\alpha\| \sup_t |\xi''(t)|. \tag{7.111}$$

Using (7.108), (7.109) and (7.111) we see that I and II are less than Ch^2. This ends the proof of the lemma for $n = 1$. □

We proceed to derive an alternative expression for $\frac{\mathrm{d}}{\mathrm{d}\epsilon}G_\epsilon(z)$ which will play an important role in the sequel.

The commutator $[S, H_\epsilon]$, defined as a quadratic form on $\mathcal{D}(S) \cap \mathcal{D}(H_\epsilon)$ is equal to

$$[S, H_\epsilon] = -\mathrm{i}N + \frac{\lambda}{\sqrt{2}}\left(-a(s\xi(-\epsilon s)\alpha) + a^*(s\xi(\epsilon s)\alpha)\right). \tag{7.112}$$

If we assume S(0), then it is easy to show that $\frac{\lambda}{\sqrt{2}}\left(-a(s\xi(-\epsilon s)\alpha) + a^*(s\xi(\epsilon s)\alpha)\right)$ is an infinitesimal perturbation of $-\mathrm{i}N$. Hence the right hand side of (7.112) defines a closed operator with domain $\mathcal{D}(N)$. By a slight abuse of notation, this operator will be also denoted by $[S, H_\epsilon]$.

Lemma 7.5 *Assume that Hypothesis S(0) holds. Let $\epsilon \neq 0$, $z \notin \sigma(H_\epsilon)$ and $m \geq 1$. Then, $[S, G_\epsilon^m(z)]$, defined as a quadratic form on $\mathcal{D}(S)$, extends by continuity to a bounded operator on $\mathcal{H}$ equal to*

$$[S, G_\epsilon^m(z)] = \sum_{k=0}^{m-1} G_\epsilon^{k+1}(z)[S, H_\epsilon]G_\epsilon^{m-k}(z).$$

Remark. In this section, we will use Lemmas 7.5 and 7.6 with $m = 1$. The cases $m > 1$ will be used in the next section.

Proof. For t real we define

$$\begin{aligned} H_{\epsilon,t} &:= \mathrm{e}^{\mathrm{i}tS} H_\epsilon \mathrm{e}^{-\mathrm{i}tS} \\ &= H_{\epsilon,\mathrm{fr}} + tN + \tfrac{\lambda}{\sqrt{2}}(a(\mathrm{e}^{\mathrm{i}ts}\xi(-\epsilon s)\alpha) + a^*(\mathrm{e}^{\mathrm{i}ts}\xi(\epsilon s)\alpha)). \end{aligned}$$

Let

$$\begin{aligned} G_{\epsilon,t} &:= \mathrm{e}^{\mathrm{i}tS}(z - H_\epsilon)^{-1}\mathrm{e}^{-\mathrm{i}tS} \\ &= (z - H_{\epsilon,t})^{-1}. \end{aligned}$$

Arguing as in the proof of Lemma 7.4, one shows that the function

$$\mathbf{R} \ni t \mapsto G_{\epsilon,t}$$

is differentiable and that

$$\frac{\mathrm{d}}{\mathrm{d}t} G_{\epsilon,t}\big|_{t=0} = G_\epsilon(z)\mathrm{i}[S, H_\epsilon]G_\epsilon(z).$$

It follows that

$$\frac{\mathrm{d}}{\mathrm{d}t} G_{\epsilon,t}^m\Big|_{t=0} = \sum_{k=0}^{m-1} G_\epsilon^{k+1}(z)\mathrm{i}[S, H_\epsilon]G_\epsilon^{m-k}(z). \tag{7.113}$$

On the other hand, in the quadratic form sense on $\mathcal{D}(S)$,

$$\frac{\mathrm{d}}{\mathrm{d}t} G_{\epsilon,t}^m\Big|_{t=0} = \mathrm{i}[S, G_\epsilon^m(z)]. \tag{7.114}$$

Combining (7.113) and (7.114) we derive the statement. □

Let ζ be as in (7.99) and let

$$\eta(t) := \mathrm{e}^t t\zeta'(t) = t(\xi'(t) - \xi(t)). \tag{7.115}$$

We set

$$K_\epsilon := \frac{\lambda\epsilon^{-1}}{\sqrt{2}} \left(a(\eta(-\epsilon s)\alpha) + a^*(\eta(\epsilon s)\alpha)\right). \tag{7.116}$$

Note that $0 \notin \operatorname{supp}\eta$ (this fact will play an important role in the sequel) and that

$$H_\epsilon^{(1)} - [S, H_\epsilon] = K_\epsilon. \tag{7.117}$$

Lemma 7.6 *Assume that Hypothesis S(0) holds and let $z \notin \sigma(H_\epsilon)$ be fixed. Then, for any $m \geq 1$,*

$$\frac{\mathrm{d}}{\mathrm{d}\epsilon} G_\epsilon^m(z) = [S, G_\epsilon^m(z)] + \sum_{k=0}^{m-1} G_\epsilon^{k+1}(z) K_\epsilon G_\epsilon^{m-k}(z). \tag{7.118}$$

Proof. Note that

$$\begin{aligned} \tfrac{\mathrm{d}}{\mathrm{d}\epsilon} G_\epsilon^m(z) &= \textstyle\sum_{k=0}^{m-1} G_\epsilon^k(z) \left(\tfrac{\mathrm{d}}{\mathrm{d}\epsilon} G_\epsilon(z)\right) G_\epsilon^{m-1-k}(z) \\ &= \textstyle\sum_{k=0}^{m-1} G_\epsilon^{k+1}(z) H_\epsilon^{(1)} G_\epsilon^{m-k}(z). \end{aligned}$$

The result follows from the identity (7.117) and Lemma 7.5. □

¿From now on we strengthen the hypothesis on the interaction and we will assume S(1). Note that the following inequality plays the role of the Mourre estimate.

Lemma 7.7 *Assume that Hypothesis S(1) holds and let $\epsilon > 0$. Then, for any $\Psi \in \mathfrak{D}$,*

$$-\frac{1}{2\mathrm{i}\epsilon}(\Psi|(H_\epsilon - H_\epsilon^*)\Psi) \geq (1 - \sqrt{2}|\lambda|\|\xi'\|_\infty \|s\alpha\|)(\Psi|N\Psi).$$

Remark. Since $\xi'(t) = \mathrm{e}^t$ around 0, we have that $\|\xi'\|_\infty \geq 1$. On the other hand, by an appropriate choice of the function ζ, we can make $\|\xi'\|_\infty$ as close to 1 as we wish.

Proof. Let

$$\xi_1(t) = \frac{1}{2\mathrm{i}t}(\xi(-t) - \xi(t)).$$

Then on $\mathfrak{D}$,

$$\begin{aligned} -\tfrac{1}{2\mathrm{i}\epsilon}(H_\epsilon - H_\epsilon^*) &= N + \tfrac{\lambda}{2\mathrm{i}\epsilon}(V_{-\epsilon} - V_\epsilon) \\ &= N + \lambda\varphi(\xi_1(\epsilon s)s\alpha) \\ &= N^{\frac{1}{2}} \left(1 + \lambda N^{-\frac{1}{2}} \varphi(\xi_1(\epsilon s)s\alpha) N^{-\frac{1}{2}}\right) N^{\frac{1}{2}}. \end{aligned}$$

The result follows from this identity and the elementary estimate

$$\begin{aligned} \|N^{-\frac{1}{2}} \varphi(\xi_1(\epsilon s)s\alpha) N^{-\frac{1}{2}}\| &\leq \sqrt{2}\|\xi_1(\epsilon s)s\alpha\| \\ &\leq \sqrt{2}\|\xi_1\|_\infty \|s\alpha\| \leq \sqrt{2}\|\xi'\|_\infty \|s\alpha\|. \end{aligned}$$

□

In the sequel we choose $\Lambda_1 > 0$ and $C_0 > 0$ such that

$$\begin{aligned} \Lambda_1 &< (\sqrt{2}\|s\alpha\|)^{-1}, \\ C_0 &< 1 - \sqrt{2}\Lambda_1\|s\alpha\|. \end{aligned} \tag{7.119}$$

It follows from Lemma 7.7 that we can choose ζ in (7.99) so that for $|\lambda| \leq \Lambda_1$ and $\Psi \in \mathfrak{D}$,

$$-\frac{1}{2\mathrm{i}\epsilon}(\Psi|(H_\epsilon - H_\epsilon^*)\Psi) \geq C_0(\Psi|N\Psi). \tag{7.120}$$

All the results in the sequel will hold for real λ such that $|\lambda| \leq \Lambda_1$, uniformly in λ.

Lemma 7.8 *Assume that Hypothesis S(1) holds and let $\epsilon > 0$. If* $\mathrm{Im}z > -C_0\epsilon$ *then* $z \notin \sigma(H_\epsilon)$ *and*

$$\|G_\epsilon(z)\| \leq \frac{1}{\mathrm{Im}z + C_0\epsilon}.$$

Proof. It follows from Relation (7.120) that the numerical range of the operator H_ϵ is contained in the region $\mathrm{Im}z \leq -C_0\epsilon$. Since $\mathcal{D}(H_\epsilon) = \mathcal{D}(H_\epsilon^*)$, the statement follows from Proposition 2.9. □

Before we make use of the last result, we need one additional lemma.

Lemma 7.9 *Assume that Hypothesis S(1) holds and let $\epsilon > 0$ and $z \in \mathbf{C}_+$ be given. Then, for any $\Psi \in \mathcal{H}$,*

$$\begin{aligned} \|N^{\frac{1}{2}}G_\epsilon(z)\Psi\| &\leq (C_0\epsilon)^{-\frac{1}{2}}|(\Psi|G_\epsilon(z)\Psi)|^{\frac{1}{2}}, \\ \|N^{\frac{1}{2}}G_\epsilon^*(z)\Psi\| &\leq (C_0\epsilon)^{-\frac{1}{2}}|(\Psi|G_\epsilon(z)\Psi)|^{\frac{1}{2}}. \end{aligned}$$

Proof. We prove the first relation. A similar argument yields the second. We have

$$\begin{aligned} \|N^{\frac{1}{2}}G_\epsilon(z)\Psi\|^2 &= (\Psi|G_\epsilon^*(z)NG_\epsilon(z)\Psi)| \\ &\leq (C_0\epsilon)^{-1}(\Psi|G_\epsilon^*(z)(\mathrm{Im}H_\epsilon + \mathrm{Im}z)G_\epsilon(z)\Psi) \\ &= (\mathrm{i}2C_0\epsilon)^{-1}(\Psi|(G_\epsilon^*(z) - G_\epsilon(z))\Psi) \\ &\leq (C_0\epsilon)^{-1}|(\Psi|G_\epsilon(z)\Psi)|, \end{aligned}$$

where in the first estimate we used (7.120) □

¿From now on ρ will denote a Schwartz function. Set

$$\begin{aligned} \langle S\rangle_{\epsilon,\rho}^{-\mu} &:= \langle S\rangle^{-\mu}\rho(\epsilon S), \\ F_{\epsilon,\rho}(z) &:= \langle S\rangle_{\epsilon,\rho}^{-\mu}G_\epsilon(z)\langle S\rangle_{\epsilon,\rho}^{-\mu}. \end{aligned} \tag{7.121}$$

Note that Lemma 7.4 yields that the function $\mathbf{R}_+ \ni \epsilon \mapsto F_{\epsilon,\rho}(z)$ is infinitely differentiable. We are now ready to prove one of the key technical results of this section.

Lemma 7.10 *Assume that Hypothesis $S(\nu)$ holds with $\nu \geq 1$ and let $\mu > 0$. Set $\gamma(\mu) = \min(\mu, 1)$. Then, for any $z \in \mathbf{C}_+$,*

$$\left\| \frac{\mathrm{d}}{\mathrm{d}\epsilon} F_{\epsilon,\rho}(z) \right\| \leq C_1 \epsilon^{-\frac{3}{2}+\gamma(\mu)} \|F_{\epsilon,\rho}(z)\|^{\frac{1}{2}} + |\lambda| C_2 \epsilon^{-2+\nu} \|F_{\epsilon,\rho}(z)\|. \tag{7.122}$$

Proof. It follows from Lemma 7.6 that

$$\frac{\mathrm{d}}{\mathrm{d}\epsilon} F_{\epsilon,\rho}(z) = \mathrm{I} + \mathrm{II} + \mathrm{III},$$

where

$$\begin{aligned} \mathrm{I} &:= \left(\tfrac{\mathrm{d}}{\mathrm{d}\epsilon} \langle S \rangle_{\epsilon,\rho}^{-\mu}\right) G_\epsilon(z) \langle S \rangle_{\epsilon,\rho}^{-\mu} + \langle S \rangle_{\epsilon,\rho}^{-\mu} G_\epsilon(z) \left(\tfrac{\mathrm{d}}{\mathrm{d}\epsilon} \langle S \rangle_{\epsilon,\rho}^{-\mu}\right), \\ \mathrm{II} &:= \langle S \rangle_{\epsilon,\rho}^{-\mu} [S, G_\epsilon(z)] \langle S \rangle_{\epsilon,\rho}^{-\mu}, \\ \mathrm{III} &:= \langle S \rangle_{\epsilon,\rho}^{-\mu} G_\epsilon(z) K_\epsilon G_\epsilon(z) \langle S \rangle_{\epsilon,\rho}^{-\mu}. \end{aligned}$$

Note that

$$\left\| \frac{\mathrm{d}}{\mathrm{d}\epsilon} \langle S \rangle_{\epsilon,\rho}^{-\mu} \right\| \leq \epsilon^{-1+\gamma(\mu)} \|\rho'\|_\infty.$$

Note also that Lemma 7.9 yields the estimates

$$\begin{aligned} \|G_\epsilon^*(z) \langle S \rangle_{\epsilon,\rho}^{-\mu}\| &\leq \|N^{\frac{1}{2}} G_\epsilon^*(z) \langle S \rangle_{\epsilon,\rho}^{-\mu}\| \leq (C_0\epsilon)^{-\frac{1}{2}} \|F_{\epsilon,\rho}(z)\|^{\frac{1}{2}}, \\ \|G_\epsilon(z) \langle S \rangle_{\epsilon,\rho}^{-\mu}\| &\leq \|N^{\frac{1}{2}} G_\epsilon(z) \langle S \rangle_{\epsilon,\rho}^{-\mu}\| \leq (C_0\epsilon)^{-\frac{1}{2}} \|F_{\epsilon,\rho}(z)\|^{\frac{1}{2}}. \end{aligned} \tag{7.123}$$

Thus, the term I is estimated as follows:

$$\begin{aligned} \|\mathrm{I}\| &\leq \epsilon^{-1+\gamma(\mu)} \|\rho'\|_\infty \left(\|G_\epsilon(z) \langle S \rangle_{\epsilon,\rho}^{-\mu}\| + \|G_\epsilon^*(z) \langle S \rangle_{\epsilon,\rho}^{-\mu}\| \right) \\ &\leq 2C_0^{-\frac{1}{2}} \|\rho'\|_\infty \epsilon^{-\frac{3}{2}+\gamma(\mu)} \|F_{\epsilon,\rho}(z)\|^{\frac{1}{2}}. \end{aligned} \tag{7.124}$$

Since for any $\epsilon > 0$,

$$\|S \langle S \rangle_{\epsilon,\rho}^{-\mu}\| = \|\langle S \rangle_{\epsilon,\rho}^{-\mu} S\| \leq \epsilon^{-1+\gamma(\mu)} \|\rho\|_\infty,$$

the estimates (7.123) yield

$$\begin{aligned} \|\mathrm{II}\| &\leq \epsilon^{-1+\gamma(\mu)} \|\rho\|_\infty \left(\|G_\epsilon(z) \langle S \rangle_{\epsilon,\rho}^{-\mu}\| + \|G_\epsilon^*(z) \langle S \rangle_{\epsilon,\rho}^{-\mu}\| \right) \\ &\leq 2C_0^{-\frac{1}{2}} \|\rho\|_\infty \epsilon^{-\frac{3}{2}+\gamma(\mu)} \|F_{\epsilon,\rho}(z)\|^{\frac{1}{2}}. \end{aligned} \tag{7.125}$$

The term III is estimated as

$$\begin{aligned} \|\mathrm{III}\| &\leq \|N^{\frac{1}{2}} G_\epsilon^*(z) \langle S \rangle_{\epsilon,\rho}^{-\mu}\| \|N^{\frac{1}{2}} G_\epsilon(z) \langle S \rangle_{\epsilon,\rho}^{-\mu}\| \|N^{-\frac{1}{2}} K_\epsilon N^{-\frac{1}{2}}\| \\ &\leq C_0^{-1} |\lambda| \epsilon^{-2} (\|\eta(-\epsilon s)\alpha\| + \|\eta(\epsilon s)\alpha\|) \|F_{\epsilon,\rho}(z)\| \\ &\leq 2C_0^{-1} |\lambda| \|s^\nu \alpha\| \sup_t |t^{-\nu} \eta(t)| \epsilon^{-2+\nu} \|F_{\epsilon,\rho}(z)\|. \end{aligned} \tag{7.126}$$

Note that since $0 \notin \operatorname{supp}\eta$, $\sup_t |t^{-\nu}\eta(t)| < \infty$. Combining the estimates (7.124), (7.125) and (7.126) we derive that Relation (7.122) holds with

$$\begin{aligned} C_1 &= 2C_0^{-\frac{1}{2}}(\|\rho\|_\infty + \|\rho'\|_\infty), \\ C_2 &= 2C_0^{-1}\|s^\nu \alpha\| \sup_t |t^{-\nu}\eta(t)|. \end{aligned} \tag{7.127}$$

□

Lemma 7.11 *Assume that Hypothesis $S(\nu)$ holds for some $\nu > 1$ and let $\mu > \frac{1}{2}$. Then,*
(i) $\sup_{(\epsilon,z)\in \mathbf{R}_+\times \mathbf{C}_+} \|F_{\epsilon,\rho}(z)\| \le C$.
(ii) *For any $z \in \mathbf{C}_+$, the norm-limit $\lim_{\epsilon\downarrow 0} F_{\epsilon,\rho}(z)$ exist.*

Proof. Since

$$\|F_{\epsilon,\rho}(z)\|^{\frac{1}{2}} \le 1 + \|F_{\epsilon,\rho}(z)\|,$$

it follows from (7.122) that

$$\left\| \frac{\mathrm{d}}{\mathrm{d}\epsilon} F_{\epsilon,\rho}(z) \right\| \le a_\epsilon \|F_{\epsilon,\rho}(z)\| + b_\epsilon, \tag{7.128}$$

where

$$\begin{aligned} a_\epsilon &:= C_1 \epsilon^{-\frac{3}{2}+\gamma(\mu)} + |\lambda| C_2 \epsilon^{-2+\nu}, \\ b_\epsilon &:= C_1 \epsilon^{-\frac{3}{2}+\gamma(\mu)}. \end{aligned}$$

Since $\mu > \frac{1}{2}$, $\nu > 1$, we have

$$\int_0^1 a_\tau \mathrm{d}\tau < \infty, \qquad \int_0^1 b_\tau \mathrm{d}\tau < \infty.$$

Note also that Lemma 7.8 yields that for $\epsilon \ge 1$, $\|F_{\epsilon,\rho}(z)\| \le C_0^{-1}$ for all $z \in \mathbf{C}_+$. Thus by the Gronwall inequality (see e.g. [DG], Proposition A.1.1) we obtain for all $z \in \mathbf{C}_+$ and $\epsilon > 0$,

$$\|F_{\epsilon,\rho}(z)\| \le C, \tag{7.129}$$

where

$$C := \exp\left(\int_0^1 a_\tau d\tau\right)\left(C_0^{-1}\int_0^1 a_\tau \mathrm{d}\tau + \int_0^1 b_\tau d\tau\right). \tag{7.130}$$

This yields Part (i) of the lemma.

To establish Part (ii), note that Relations (7.128) and (7.129) yield that for any $0 < \epsilon_1 < \epsilon_2$,

$$\begin{aligned} \|F_{\epsilon_2,\rho}(z) - F_{\epsilon_1,\rho}(z)\| &\le \textstyle\int_{\epsilon_1}^{\epsilon_2} \left\| \frac{\mathrm{d}}{\mathrm{d}\tau} F_{\tau,\rho}(z) \right\| \mathrm{d}\tau \\ &\le \textstyle\int_{\epsilon_1}^{\epsilon_2} (Ca_\tau + b_\tau)\mathrm{d}\tau. \end{aligned}$$

Thus, if $\epsilon_n \to 0$ then the sequence $F_{\epsilon_n,\rho}(z)$ is Cauchy and this yields the statement. □

Assume now that $\rho(0) = 1$. Clearly,

$$F_{0,\rho}(z) = \langle S\rangle^{-\mu}(z - H)^{-1}\langle S\rangle^{-\mu}.$$

Thus, to finish the proof of Theorem 7.1 it suffices to show that

$$\lim_{\epsilon\downarrow 0} F_{\epsilon,\rho}(z) = F_{0,\rho}(z). \tag{7.131}$$

(Note that Part (ii) of Lemma 7.11 guarantees the existence of the limit in (7.131) but says nothing about its value). The proof of Relation (7.131) resolves the infrared problem we have discussed in the introduction. In the physics language, there is no infrared problem as long as $\epsilon > 0$ – this constant plays a role of a "complex boson mass". In our approach, the infrared problem appears in the limit $\epsilon \downarrow 0$ and is due to the fact that domains of H and H_ϵ with $\epsilon > 0$ are different. This difficulty is resolved below. We remark that an argument similar to ours has been used in [JP1]. The reader may consult [JP3] for additional discussion of this point.

The following technical result plays a key role in the resolution of the infrared problem. Recall that $G_0(z) = (z - H)^{-1}$.

Lemma 7.12 *Assume that Hypothesis $S(1)$ holds and let $z \in \mathbf{C}_+$ be given. Let $\beta = \pm\frac{1}{2}$. Then,*
(i) *For any $\epsilon > 0$,*

$$\|N^{-\beta}G_\epsilon(z)N^\beta\| \le \frac{2}{\mathrm{Im}z}\left(1 + \frac{\sqrt{2}\|\alpha\||\lambda|}{\mathrm{Im}z}\right). \tag{7.132}$$

(ii) *Assume in addition that Hypothesis* A *is satisfied. Then, (7.132) is true also for $\epsilon = 0$.*

Proof. Let N_δ be as in Lemma 4.2. For any $\epsilon \ge 0$ consider the operator

$$H_{\epsilon,\delta,\beta} := H_\epsilon + \lambda N_\delta^{-\beta}V_\epsilon N_\delta^\beta - \lambda V_\epsilon.$$

It follows from Lemma 4.2 that

$$\|N_\delta^{-\beta}V_\epsilon N_\delta^\beta - V_\epsilon\| \le C\sqrt{\delta},$$

where we use the shorthand $C := \sqrt{2}\|\alpha\|$. Therefore, if $\epsilon > 0$, $H_{\epsilon,\delta,\beta}$ is a closed operator on $\mathfrak{D}$. If $\epsilon = 0$, Hypothesis A yields that $H_{\epsilon,\delta,\beta}$ is essentially self-adjoint on $\mathfrak{D}$. Note also that for any $\epsilon \ge 0$,

$$\mathfrak{N}(H_\epsilon) \subset \{z : \ \mathrm{Im}z \le -C_0\epsilon\},$$

(recall the estimate (7.120)), therefore

$$\mathfrak{N}(H_{\epsilon,\delta,\beta}) \subset \{z : \ \mathrm{Im}z \le -C_0\epsilon + C|\lambda|\sqrt{\delta}\}.$$

Thus, if $\mathrm{Im}z > -C_0\epsilon + C|\lambda|\sqrt{\delta}$ then the operator $z - H_{\epsilon,\delta,\beta}$ is invertible and

$$\|(z - H_{\epsilon,\delta,\beta})^{-1}\| \leq \left(\mathrm{Im}z + C_0\epsilon - C|\lambda|\sqrt{\delta}\right)^{-1}.$$

¿From now on we fix $z \in \mathbf{C}_+$ and choose δ such that

$$C|\lambda|\sqrt{\delta} < \mathrm{Im}z.$$

Let

$$\mathfrak{D}_{\mathrm{fin}} := \mathfrak{D} \cap (\mathcal{K} \otimes \Gamma_{\mathrm{fin}}).$$

For any $\Psi \in \mathfrak{D}_{\mathrm{fin}}$ we have

$$H_{\epsilon,\mathrm{fr}}\Psi = N_\delta^{-\beta} H_{\epsilon,\mathrm{fr}} N_\delta^{\beta}\Psi.$$

Similarly, for $\Psi \in \mathfrak{D}_{\mathrm{fin}}$ we have

$$(N_\delta^{-\beta} V_\epsilon N_\delta^{\beta} - V_\epsilon)\Psi = N_\delta^{-\beta} V_\epsilon N_\delta^{\beta}\Psi - V_\epsilon\Psi.$$

Thus, on $\mathfrak{D}_{\mathrm{fin}}$,

$$\begin{aligned} H_{\epsilon,\delta,\beta} &= H_{\epsilon,\mathrm{fr}} + N_\delta^{-\beta} V_\epsilon N_\delta^{\beta} \\ &= N_\delta^{-\beta} H_\epsilon N_\delta^{\beta}. \end{aligned} \tag{7.133}$$

One easily shows that $\mathfrak{D}_{\mathrm{fin}}$ is a core for $H_{\epsilon,\mathrm{fr}}$. If $\epsilon > 0$, V_ϵ is an infinitesimal perturbation of $H_{\epsilon,\mathrm{fr}}$, and therefore $\mathfrak{D}_{\mathrm{fin}}$ is also a core for H_ϵ and $H_{\epsilon,\delta,\beta}$. It follows from Hypothesis A that $\mathfrak{D}_{\mathrm{fin}}$ is a core of H_0. Therefore, for $\epsilon \geq 0$,

$$\tilde{\mathfrak{D}}_{\mathrm{fin}} := (z - H_{\epsilon,\delta,\beta})\mathfrak{D}_{\mathrm{fin}}, \tag{7.134}$$

is dense in $\mathcal{H}$ and on $\tilde{\mathfrak{D}}_{\mathrm{fin}}$,

$$(z - H_{\epsilon,\delta,\beta})^{-1} = N_\delta^{-\beta}(z - H_\epsilon)^{-1} N_\delta^{\beta}.$$

Next, we note that

$$\|N_\delta^{\beta} N^{-\beta}\| = \|N^{-\beta} N_\delta^{\beta}\| \leq \begin{cases} (1+\delta)^{\beta}, & \beta > 0, \\ \delta^{\beta}, & \beta \leq 0. \end{cases}$$

Therefore, for any $\epsilon \geq 0$ and $\Psi \in \tilde{\mathfrak{D}}_{\mathrm{fin}}$,

$$\|N^{-\beta}(z - H_\epsilon)^{-1} N^{\beta}\Psi\| \leq \left(\frac{1+\delta}{\delta}\right)^{|\beta|} \left(\mathrm{Im}z + C_0\epsilon - C|\lambda|\sqrt{\delta}\right)^{-1} \|\Psi\|.$$

Taking $\delta = \left(\frac{\mathrm{Im}z}{2C|\lambda|}\right)^2$, we derive the statements of the lemma. □

Lemma 7.13 *Assume that Hypotheses* A *and* $S(1)$ *hold. Then*

$$\lim_{\epsilon\downarrow 0} N^{-\frac{1}{2}} G_\epsilon(z) N^{-\frac{1}{2}} = N^{-\frac{1}{2}}(z-H)^{-1}N^{-\frac{1}{2}}. \tag{7.135}$$

Proof. Let $\epsilon > 0$. We have

$$N^{-\frac{1}{2}}(G_\epsilon(z) - G_0(z))N^{-\frac{1}{2}} = N^{-\frac{1}{2}}G_0(z)(H_\epsilon - H)G_\epsilon(z)N^{-\frac{1}{2}} \tag{7.136}$$

$$= N^{-\frac{1}{2}}G_0(z)N^{\frac{1}{2}}\left(-\mathrm{i}\epsilon + \lambda N^{-\frac{1}{2}}(V_\epsilon - V)N^{-\frac{1}{2}}\right)N^{\frac{1}{2}}G_\epsilon(z)N^{-\frac{1}{2}}. \tag{7.137}$$

This identity and Lemma 7.12 yield that

$$\|N^{-\frac{1}{2}}(G_\epsilon(z) - G_0(z))N^{-\frac{1}{2}}\| \le C_z^2\epsilon\left(1 + 2|\lambda|\|\alpha\|\sup_{t_1,t_2}|t_1\xi'(t_1+t_2)|\right), \tag{7.138}$$

where C_z is the constant on the right-hand side in (7.132). Clearly, this estimate yields (7.135). □

We are now ready to finish

Proof of Theorem 7.1. As we have already remarked, it follows from Lemma 7.11 that to prove Theorem 7.1 it suffices to show that for any $z \in \mathbf{C}_+$,

$$\lim_{\epsilon\downarrow 0} F_{\epsilon,\rho}(z) = F_{0,\rho}(z). \tag{7.139}$$

Since we know that the limit on the right hand side exists, it suffices to show that

$$\mathrm{w-}\lim_{\epsilon\downarrow 0} F_{\epsilon,\rho}(z) = F_{0,\rho}(z). \tag{7.140}$$

This relation follows from (7.135). □

7.2 Hölder continuity

This section is devoted to the proof of the following theorem:

Theorem 7.14 *Assume that Hypotheses* A *and* $S(\nu)$ *hold with* $\nu > 1$. *Let* $n \in \mathbf{N}$, $0 < \theta \le 1$, *and* $\mu \ge \frac{1}{2}$ *be such that* $\nu \ge \mu + \frac{1}{2} = 1 + n + \theta$. *Let* $0 < \Lambda_1 < (\sqrt{2}\|s\alpha\|)^{-1}$. *Then, for* $|\lambda| \le \Lambda_1$, *the function*

$$\mathbf{C}_+ \ni z \mapsto \langle S^{\overline{\mathrm{vv}}}\rangle^{-\mu}(z\mathbf{1}^{\overline{\mathrm{vv}}} - H^{\overline{\mathrm{vv}}})^{-1}\langle S^{\overline{\mathrm{vv}}}\rangle^{-\mu} \tag{7.141}$$

extends by continuity to $\overline{\mathbf{C}}_+$ *and is of the class* $C_{\mathrm{u}}^{n,\theta}(\overline{\mathbf{C}}_+)$ *uniformly in* λ.

The rest of this section is devoted to the proof of Theorem 7.14. We will freely use the results and notation of Section 7.1. In particular, we drop superscripts $\overline{\mathrm{vv}}$ until the end of this section. We fix $\Lambda_1 > 0$, $C_0 > 0$ and ζ such that Relations (7.119) and (7.120) hold. All the results in the sequel will hold for real λ such that $|\lambda| \le \Lambda_1$, uniformly in λ.

For any $\epsilon \ge 0$, the function

$$\mathbf{C}_+ \ni z \mapsto G_\epsilon(z),$$

is analytic and

$$\partial_z^l G_\epsilon(z) = (-1)^l l! G_\epsilon^{l+1}(z).$$

It follows from Lemma 7.4 that for any l and m the mixed derivatives

$$\partial_z^l \partial_\epsilon^m G_\epsilon(z)$$

exists on $\mathbf{C}_+ \times \mathbf{R}_+$, and that they are linear combinations of the terms

$$G_\epsilon^{l_1}(z) H_\epsilon^{(m_1)} G_\epsilon^{l_2}(z) \dots G_\epsilon^{l_k}(z) H_\epsilon^{(m_k)} G_\epsilon^{l_{k+1}}(z), \tag{7.142}$$

where l_k's and m_k's are positive integers such that

$$\sum_{j=1}^{k} m_j = m, \qquad \sum_{j=1}^{k+1} l_j = l + k + 1. \tag{7.143}$$

We proceed to study in some detail these mixed derivatives.

Lemma 7.15 *Assume that Hypothesis $S(\nu)$ holds with $\nu > 1$ and let $\mu > \frac{1}{2}$. Then,*

$$\begin{aligned} \sup_{z\in\mathbf{C}_+} \left\| \langle S\rangle_{\epsilon,\rho}^{-\mu} \left(\partial_z^l \partial_\epsilon^m G_\epsilon(z) \right) N^{\frac{1}{2}} \right\| &\le C\epsilon^{-\frac{1}{2}-l-m}, \\ \sup_{z\in\mathbf{C}_+} \left\| N^{\frac{1}{2}} \left(\partial_z^l \partial_\epsilon^m G_\epsilon(z) \right) \langle S\rangle_{\epsilon,\rho}^{-\mu} \right\| &\le C\epsilon^{-\frac{1}{2}-l-m}. \end{aligned} \tag{7.144}$$

Proof. We will prove the first relation. A similar argument yields the second. We write $\partial_z^l \partial_\epsilon^m G_\epsilon(z)$ as a linear combinations of the terms (7.142). After inserting $\langle S\rangle_{\epsilon,\rho}^{-\mu}$ and $N^{\frac{1}{2}}$, we estimate the norm of each term by

$$\begin{aligned} &\|\langle S\rangle_{\epsilon,\rho}^{-\mu} G_\epsilon(z) N^{\frac{1}{2}}\| \|N^{\frac{1}{2}} G_\epsilon(z) N^{\frac{1}{2}}\|^{l_1-1} \|N^{-\frac{1}{2}} H_\epsilon^{(m_1)} N^{-\frac{1}{2}}\| \|N^{\frac{1}{2}} G_\epsilon(z) N^{\frac{1}{2}}\|^{l_2} \dots \\ &\dots \|N^{\frac{1}{2}} G_\epsilon(z) N^{\frac{1}{2}}\|^{l_k} \|N^{-\frac{1}{2}} H_\epsilon^{(m_k)} N^{-\frac{1}{2}}\| \|N^{\frac{1}{2}} G_\epsilon(z) N^{\frac{1}{2}}\|^{l_{k+1}}. \end{aligned} \tag{7.145}$$

It follows from Lemmas 7.9 and 7.11 that

$$\begin{aligned} \|\langle S\rangle_{\epsilon,\rho}^{-\mu} G_\epsilon(z) N^{\frac{1}{2}}\| &\le C_0^{-\frac{1}{2}} \epsilon^{-\frac{1}{2}} \|F_{\epsilon,\rho}(z)\| \le C_0^{\frac{1}{2}} \epsilon^{-\frac{1}{2}} C^{\frac{1}{2}}, \\ \|N^{\frac{1}{2}} G_\epsilon(z) N^{\frac{1}{2}}\| &\le C_0^{-1} \epsilon^{-1}. \end{aligned} \tag{7.146}$$

Furthermore, for $m_j \geq 1$,

$$\|N^{-\frac{1}{2}} H_\epsilon^{(m_j)} N^{-\frac{1}{2}}\| \leq c_{m_j} \epsilon^{1-m_j}, \tag{7.147}$$

where

$$c_{m_j} = \delta_{1 m_j} + 2|\lambda| \sup_t |t^{m_j - 1} \xi^{(m_j)}(t)| \|s\alpha\|. \tag{7.148}$$

Combining these estimates and using (7.143) we bound (7.145) with

$$\epsilon^{-\frac{1}{2}-l-m} C_0^{\frac{1}{2}-l-k} C^{\frac{1}{2}} \prod c_{m_j},$$

Summing over the terms (7.142) we derive the Relation (7.144). □

Lemma 7.16 *Assume that Hypothesis $S(\nu)$ holds with $\nu > 1$ and let $\mu > \frac{1}{2}$. Let $k = k_1 + k_2$. Then*

$$\sup_{z \in \mathbf{C}_+} \left\| \langle S \rangle_{\epsilon,\rho_1}^{-\mu} S^{k_1} \left(\partial_z^l \partial_\epsilon^m G_\epsilon(z) K_\epsilon G_\epsilon(z) \right) S^{k_2} \langle S \rangle_{\epsilon,\rho_2}^{-\mu} \right\| \leq C \epsilon^{-l-m-k-2+\nu}. \tag{7.149}$$

Proof. Note that

$$\partial_z^l \partial_\epsilon^m G_\epsilon(z) K_\epsilon G_\epsilon(z)$$

is a linear combination of terms

$$(\partial_z^{l_1} \partial_\epsilon^{m_1} G_\epsilon(z))(\partial_\epsilon^k K_\epsilon)(\partial_z^{l_2} \partial_\epsilon^{m_2} G_\epsilon(z)),$$

where $l = l_1 + l_2$, $m = m_1 + m_2 + k$. After inserting $\langle S \rangle_{\epsilon,\rho_1}^{-\mu}$ and $\langle S \rangle_{\epsilon,\rho_2}^{-\mu}$, we bound the norms of these terms by

$$\left\| \langle S \rangle_{\epsilon,\rho_1}^{-\mu} \left(\partial_z^{l_1} \partial_\epsilon^{m_1} G_\epsilon(z) \right) N^{\frac{1}{2}} \right\| \|N^{-\frac{1}{2}} \partial_\epsilon^k K_\epsilon N^{-\frac{1}{2}}\| \left\| N^{\frac{1}{2}} \left(\partial_z^{l_2} \partial_\epsilon^{m_2} G_\epsilon(z) \right) \langle S \rangle_{\epsilon,\rho_2}^{-\mu} \right\|. \tag{7.150}$$

We estimate

$$\begin{aligned} \|N^{-\frac{1}{2}} \partial_\epsilon^k K_\epsilon N^{-\frac{1}{2}}\| &\leq |\lambda| \epsilon^{-1} \left(\|s^k \eta^{(k)}(\epsilon s)\alpha\| + \|s^k \eta^{(k)}(-\epsilon s)\alpha\| \right) \\ &\leq 2|\lambda| \sup_t |t^{k-\nu} \eta^{(k)}(t)| \|s^\nu \alpha\| \epsilon^{-1+\nu-k}. \end{aligned} \tag{7.151}$$

Note that since $0 \notin \operatorname{supp} \eta$, the constant $\sup_t |t^{k-\nu} \eta^{(k)}(t)|$ is finite. Combining the estimate (7.151) with Lemma 7.15, we obtain

$$\sup_{z \in \mathbf{C}_+} \|\langle S \rangle_{\epsilon,\rho_1}^{-\mu} \left(\partial_z^l \partial_\epsilon^m G_\epsilon(z) K_\epsilon G_\epsilon(z) \right) \langle S \rangle_{\epsilon,\rho_2}^{-\mu}\| \leq C \epsilon^{-l-m-2+\nu}. \tag{7.152}$$

Next note that we can find Schwartz functions γ_i, $\tilde{\gamma}_i$ such that $\rho_i = \gamma_i \tilde{\gamma}_i$, for $i = 1, 2$. Clearly

$$\langle S \rangle_{\epsilon,\rho_i}^{-\mu} = \tilde{\gamma}_i(\epsilon S) \langle S \rangle_{\epsilon,\gamma_i}^{-\mu}, \quad i = 1, 2.$$

Therefore, we can estimate the left-hand side of (7.149) by

$$\|\tilde\gamma_1(\epsilon S)S^{k_1}\| \left\|\langle S\rangle_{\epsilon,\gamma_2}^{-\mu}\left(\partial_z^l\partial_\epsilon^m G_\epsilon(z)K_\epsilon G_\epsilon(z)\right)\langle S\rangle_{\epsilon,\gamma_2}^{-\mu}\right\| \|S^{k_2}\tilde\gamma_2(\epsilon S)\|.$$

Thus, Relation (7.149) follows from (7.152) and the estimates

$$\|\tilde\gamma_i(\epsilon S)S^{k_i}\| \quad \le \epsilon^{-k_i}\sup_t |t^{k_i}\tilde\gamma_i(t)|, \quad i=1,2. \tag{7.153}$$

□

Lemma 7.17 *Assume that Hypothesis $S(\nu)$ holds with $\nu > 1$. Let $\mu > \frac{1}{2}$, $k \ge 2\mu$, $k = k_1 + k_2$. Then,*

$$\sup_{z\in\mathbf{C}_+} \|\partial_z^l \langle S\rangle_{\epsilon,\rho_1}^{-\mu} S^{k_1} G_\epsilon(z) S^{k_2}\langle S\rangle_{\epsilon,\rho_2}^{-\mu}\| \le C\epsilon^{-k-l-\frac{1}{2}+\mu}. \tag{7.154}$$

Proof. We use the splitting $\rho_2 = \tilde\gamma_2\gamma_2$, as in the proof of the previous lemma. If $k_1 \ge k_2$, then $k_1 \ge \mu$ and we use the estimates

$$\begin{aligned}
\|\langle S\rangle_{\epsilon,\rho_1}^{-\mu} S^{k_1}\| &\le \epsilon^{-k_1+\mu}\sup_t |t^{k_1-\mu}\rho_1(t)|,\\
\|S^{k_2}\tilde\gamma_2(\epsilon S)\| &\le \epsilon^{-k_2}\sup_t |t^{k_2}\tilde\gamma_2(t)|,\\
\|\partial_z^l G_\epsilon(z)\langle S\rangle_{\epsilon,\gamma_2}^{-\mu}\| &\le C\epsilon^{-l-\frac{1}{2}},
\end{aligned}$$

where we used Lemma 7.15 in the last estimate. If $k_1 \le k_2$, one interchanges the roles of k_1 and k_2 and argues similarly. □

We recall that for any operator A,

$$\mathbf{S}A = [S, A],$$

is the quadratic form defined on $\mathcal{D}(S)\cap\mathcal{D}(A)$. If A is bounded, one can define the multiple commutators $\mathbf{S}^p A$ for any positive integer p as quadratic forms on $\mathcal{D}(S^p)$.

In the following lemma it will be convenient to use the following function:

$$\omega(\nu,\epsilon) := \begin{cases} \epsilon^\nu, & \nu < 0,\\ 1+\log(1+\epsilon^{-1}), & \nu = 0,\\ 1, & \nu > 0. \end{cases} \tag{7.155}$$

Let us note the following properties of this function:

$$\begin{aligned}
&\textstyle\int_\epsilon^1 \omega(\nu,\tau)\mathrm{d}\tau \le C\omega(\nu+1,\epsilon),\\
&\omega(\theta-1,\epsilon) = \epsilon^{-1}\ell_\theta(\epsilon), \qquad 0 < \theta \le 1,
\end{aligned} \tag{7.156}$$

where the function $\ell_\theta(\epsilon)$ was defined in (2.34).

Lemma 7.18 *Assume that Hypothesis $S(\nu)$ holds with $\nu > 1$ and let $\mu > \frac{1}{2}$. Then, for some constants C_1 and C_2,*

$$\begin{aligned} &\sup_{z\in \mathbf{C}_+} \|\partial_z^l \partial_\epsilon^k \langle S\rangle_{\epsilon,\rho}^{-\mu} G_\epsilon(z) \langle S\rangle_{\epsilon,\rho}^{-\mu}\| \\ &\leq C_1\omega(-k-l-\tfrac{1}{2}+\mu,\epsilon) + C_2\omega(-k-l-1+\nu,\epsilon). \end{aligned} \tag{7.157}$$

Proof. Note that

$$\partial_\epsilon^k \langle S\rangle_{\epsilon,\rho}^{-\mu} G_\epsilon(z) \langle S\rangle_{\epsilon,\rho}^{-\mu},$$

is the sum of the terms

$$\left(\partial_\epsilon^{k_1}\langle S\rangle_{\epsilon,\rho}^{-\mu}\right)\left(\partial_\epsilon^{k_2} G_\epsilon(z)\right)\left(\partial_\epsilon^{k_3}\langle S\rangle_{\epsilon,\rho}^{-\mu}\right), \tag{7.158}$$

where $k_1+k_2+k_3=k$. ¿From the formula

$$\partial_\epsilon G_\epsilon(z) = \mathbf{S}G_\epsilon(z) + G_\epsilon(z)K_\epsilon G_\epsilon(z),$$

(recall Lemma 7.5) one easily shows by induction that

$$\partial_\epsilon^{k_2} G_\epsilon(z) = \mathbf{S}^{k_2} G_\epsilon(z) + \sum_{p+r+1=k_2} \mathbf{S}^p \partial_\epsilon^r G_\epsilon(z) K_\epsilon G_\epsilon(z), \tag{7.159}$$

as a quadratic form on $\mathcal{D}(S^k)$.

Using (7.118) and the identity $\partial_\epsilon^{k_1}\langle S\rangle_{\epsilon,\rho}^{-\mu} = S^{k_1}\langle S\rangle_{\epsilon,\rho^{(k_1)}}^{-\mu}$, we can write (7.158) as a linear combination of the terms

$$\langle S\rangle_{\epsilon,\rho^{(k_1)}}^{-\mu} S^{k_1+j_1} G_\epsilon(z) S^{j_2+k_3} \langle S\rangle_{\epsilon,\rho^{(k_3)}}^{-\mu}, \tag{7.160}$$

where $j_1+j_2=k_2$, and of the terms

$$\langle S\rangle_{\epsilon,\rho^{(k_1)}}^{-\mu} S^{k_1+l_1} \left(\partial_\epsilon^r G_\epsilon(z) K_\epsilon G_\epsilon(z)\right) S^{l_2+k_3} \langle S\rangle_{\epsilon,\rho^{(k_3)}}^{-\mu}, \tag{7.161}$$

where $l_1+l_2+r+1=k_2$. After applying ∂_z^l to terms (7.160) and (7.161) we estimate them with the help of Lemmas 7.17 and 7.16. This yields (7.157) if $k \geq 2\mu$.

Next note that (7.142), (7.146) and (7.147) yield that for any k and l there is a constant C such that

$$\sup_{z\in\mathbf{C}_+,\epsilon=1} \|\partial_z^l \partial_\epsilon^k \langle S\rangle_{\epsilon,\rho}^{-\mu} G_\epsilon(z) \langle S\rangle_{\epsilon,\rho}^{-\mu}\| \leq C.$$

Therefore, if Relation (7.157) holds for $k+1$, integrating the function

$$\tau \mapsto \partial_z^l \partial_\tau^{k+1} \langle S\rangle_{\tau,\rho}^{-\mu} G_\tau(z) \langle S\rangle_{\tau,\rho}^{-\mu},$$

over $[\epsilon, 1[$ and using (7.156) we derive that Relation (7.157) also holds for k. The proof of Lemma 7.18 is complete. □

We are now ready to finish

Proof of Theorem 7.14. Assume that $\rho(0)=1$. Let $m=0,\dots,n$. We have

$$\|\partial_\epsilon\partial_z^m\langle S\rangle_{\epsilon,\rho}^{-\mu}G_\epsilon(z)\langle S\rangle_{\epsilon,\rho}^{-\mu}\|\leq C\omega(-1+\theta+n-m,\epsilon). \tag{7.162}$$

Since $-1<-1+\theta+n-m$, the right hand side of (7.162) is integrable around 0. This implies the uniform convergence of

$$\lim_{\epsilon\downarrow 0}\partial_z^m\langle S\rangle_{\epsilon,\rho}^{-\mu}G_\epsilon(z)\langle S\rangle_{\epsilon,\rho}^{-\mu},\quad m=0,\dots,n.$$

Hence, by the well known calculus lemma we can interchange the order of the limit and the differentiation in the following formula:

$$\begin{aligned}\partial_z^n\langle S\rangle^{-\mu}(z-H)^{-1}\langle S\rangle^{-\mu} &= \partial_z^n\lim_{\epsilon\downarrow 0}\langle S\rangle_{\epsilon,\rho}^{-\mu}G_\epsilon(z)\langle S\rangle_{\epsilon,\rho}^{-\mu}\\ &= \lim_{\epsilon\downarrow 0}\partial_z^n\langle S\rangle_{\epsilon,\rho}^{-\mu}G_\epsilon(z)\langle S\rangle_{\epsilon,\rho}^{-\mu},\end{aligned}$$

where in the first step we used (7.139).

It follows from Lemma 7.18 that for some constant C,

$$\sup_{z\in\mathbf{C}_+}\|\partial_\epsilon^k\partial_z^l\partial_z^n\langle S\rangle_{\epsilon,\rho}^{-\mu}G_\epsilon(z)\langle S\rangle_{\epsilon,\rho}^{-\mu}\|\leq C\omega(-1+\theta,\epsilon)=C\epsilon^{-1}\ell_\theta(\epsilon),\quad k+l\leq 1.$$

Therefore, the function

$$[0,1[\times\mathbf{C}_+\ni(\epsilon,z)\mapsto\partial_z^n\langle S\rangle_{\epsilon,\rho}^{-\mu}G_\epsilon(z)\langle S\rangle_{\epsilon,\rho}^{-\mu},$$

satisfies all the conditions of Proposition 2.3. It follows that the function

$$\mathbf{C}_+\ni z\mapsto\partial_z^n\langle S\rangle^{-\mu}(z-H)^{-1}\langle S\rangle^{-\mu}, \tag{7.163}$$

is in the class $C_{\mathrm{u}}^{0,\theta}(\mathbf{C}_+)$. Therefore the function (7.163) with $n=0$ satisfies the conditions of Proposition 2.4. The proof of Theorem 7.14 is complete. □

7.3 Properties of $w(z)$

In this section we prove Theorem 6.1. Let us first state a version of Theorem 7.14 for the free operator. Setting $\lambda=0$ in Theorem 7.14 we derive

Theorem 7.19 *Let $n\in\mathbf{N}$, $0<\theta\leq 1$, $\nu=\frac{1}{2}+n+\theta$. Then, the function*

$$\mathbf{C}_+\ni z\mapsto\langle S^{\overline{\mathrm{v}}\overline{\mathrm{v}}}\rangle^{-\nu}(z\mathbf{1}^{\overline{\mathrm{v}}\overline{\mathrm{v}}}-H_{\mathrm{fr}}^{\overline{\mathrm{v}}\overline{\mathrm{v}}})^{-1}\langle S^{\overline{\mathrm{v}}\overline{\mathrm{v}}}\rangle^{-\nu} \tag{7.164}$$

extends by continuity to $\overline{\mathbf{C}}_+$ and is in the class $C_{\mathrm{u}}^{n,\theta}(\overline{\mathbf{C}}_+)$.

Proof of Theorem 6.1. Note that if Hypothesis $S(\nu)$ holds then the operators

$$\varphi(\alpha)^{\mathrm{v}\overline{\mathrm{v}}}\langle S^{\overline{\mathrm{v}}\overline{\mathrm{v}}}\rangle^{\nu}\qquad\text{and}\qquad\langle S^{\overline{\mathrm{v}}\overline{\mathrm{v}}}\rangle^{\nu}\varphi(\alpha)^{\overline{\mathrm{v}}\mathrm{v}},$$

are bounded and their norms are less than or equal to $\|\langle s\rangle^\nu\alpha\|/\sqrt{2}$. Since

$$w(z)=\varphi(\alpha)^{\mathrm{v}\overline{\mathrm{v}}}\langle S^{\overline{\mathrm{v}}\overline{\mathrm{v}}}\rangle^{\nu}\left(\langle S^{\overline{\mathrm{v}}\overline{\mathrm{v}}}\rangle^{-\nu}(z\mathbf{1}^{\overline{\mathrm{v}}\overline{\mathrm{v}}}-H_{\mathrm{fr}}^{\overline{\mathrm{v}}\overline{\mathrm{v}}})^{-1}\langle S^{\overline{\mathrm{v}}\overline{\mathrm{v}}}\rangle^{-\nu}\right)\langle S^{\overline{\mathrm{v}}\overline{\mathrm{v}}}\rangle^{\nu}\varphi(\alpha)^{\overline{\mathrm{v}}\mathrm{v}}, \tag{7.165}$$

we derive the result from Theorem 7.19. □

7.4 Comparison with the free resolvent

In this section we estimate the difference of the free and the full resolvent on the radiation sector.

Theorem 7.20 *Assume that Hypotheses* A *and* $S(\nu)$ *hold with* $\nu > 1$. *Let* $\mu = \nu - \frac{1}{2}$ *and* $0 < \Lambda_1 < (\sqrt{2}\|s\alpha\|)^{-1}$. *Then*

$$\sup_{(\lambda,z)\in[-\Lambda_1,\Lambda_1]\times\overline{\mathbf{C}}_+} \left\|\langle S^{\overline{\mathrm{v}}\overline{\mathrm{v}}}\rangle^{-\mu}\Big((z\mathbf{1}^{\overline{\mathrm{v}}\overline{\mathrm{v}}} - H^{\overline{\mathrm{v}}\overline{\mathrm{v}}})^{-1} - (z\mathbf{1}^{\overline{\mathrm{v}}\overline{\mathrm{v}}} - H_{\mathrm{fr}}^{\overline{\mathrm{v}}\overline{\mathrm{v}}})^{-1}\Big)\langle S^{\overline{\mathrm{v}}\overline{\mathrm{v}}}\rangle^{-\mu}\right\| \leq C|\lambda|^{(\nu-1)/\nu} \tag{7.166}$$

For notational simplicity, in the sequel we drop the superscripts $\overline{\mathrm{v}}\overline{\mathrm{v}}$. It follows from Theorem 7.14 that it suffices to take in (7.166) supremum over $z \in \mathbf{C}_+$. We choose again $\Lambda_1 > 0$, $C_0 > 0$ and ζ such that Relations (7.119) and (7.120) hold. All the results in the sequel will hold for real λ such that $|\lambda| \leq \Lambda_1$.

In what follows we will denote by the same letter C various constants which depend only on the constants introduced in the previous section, but do not depend on λ. The values of these constants may change from estimate to estimate.

Lemma 7.21 *Assume that Hypotheses* A *and* $S(\nu)$ *hold with* $\nu > 1$. *Let* $\mu > \frac{1}{2}$. *Then*

$$\sup_{z\in\mathbf{C}_+} \|\partial_\epsilon^k \langle S\rangle_{\epsilon,\rho_1}^{-\mu}(G_\epsilon(z) - G_{\mathrm{fr},\epsilon}(z))\langle S\rangle_{\epsilon,\rho_2}^{-\mu}\| \leq C|\lambda|\epsilon^{-k-1}.$$

Proof. Using

$$G_\epsilon(z) - G_{\mathrm{fr},\epsilon}(z) = \lambda G_\epsilon(z) V_\epsilon G_{\mathrm{fr},\epsilon}(z)$$

we can write

$$\langle S\rangle_{\epsilon,\rho_1}^{-\mu}\left(\partial_\epsilon^m(G_\epsilon(z) - G_{\mathrm{fr},\epsilon}(z))\right)\langle S\rangle_{\epsilon,\rho_2}^{-\mu}$$

as a linear combination the terms

$$\begin{aligned}&\langle S\rangle_{\epsilon,\rho_1}^{-\mu} G_\epsilon(z) H_\epsilon^{(m_1)} G_\epsilon(z)\cdots G_\epsilon(z) H_\epsilon^{(m_{k-1})} G_\epsilon(z)\\ &\times \lambda V_\epsilon^{(m_k)} G_{\mathrm{fr},\epsilon}(z) H_{\mathrm{fr},\epsilon}^{(m_{k+1})} G_{\mathrm{fr},\epsilon}(z)\cdots G_{\mathrm{fr},\epsilon}(z) H_{\mathrm{fr},\epsilon}^{(m_l)} G_{\mathrm{fr},\epsilon}(z)\langle S\rangle_{\epsilon,\rho_2}^{-\mu},\end{aligned} \tag{7.167}$$

where $\sum_{j=1}^{l} m_j = m$, $m_j \geq 1$ for $j \neq k$ and $m_k \geq 0$.. Using

$$\|N^{-\frac{1}{2}} H_\epsilon^{(m_j)} N^{-\frac{1}{2}}\| \leq C\epsilon^{1-m_j}, \quad \|N^{-\frac{1}{2}} H_{\mathrm{fr},\epsilon}^{(m_j)} N^{-\frac{1}{2}}\| \leq C\epsilon^{1-m_j}, \quad m_j \geq 1,$$

$$\|N^{-\frac{1}{2}} V_\epsilon^{(m_k)} N^{-\frac{1}{2}}\| \leq C\epsilon^{-m_k}, \quad m_k \geq 0,$$

$$\|\langle S\rangle_{\epsilon,\rho_1}^{-\mu} G_\epsilon(z) N^{\frac{1}{2}}\| \leq C\epsilon^{-\frac{1}{2}}, \qquad \|N^{\frac{1}{2}} G_{\mathrm{fr},\epsilon}(z)\langle S\rangle_{\epsilon,\rho_2}^{-\mu}\| \leq C\epsilon^{-\frac{1}{2}},$$

$$\|N^{\frac{1}{2}}G_\epsilon(z)N^{\frac{1}{2}}\| \le C\epsilon^{-1}, \qquad \|N^{\frac{1}{2}}G_{\mathrm{fr},\epsilon}(z)N^{\frac{1}{2}}\| \le C\epsilon^{-1},$$

we see that (7.167) can be estimated by $|\lambda|\epsilon^{-m-1}$. This shows

$$\sup_{z\in\mathbf{C}_+} \left\| \langle S\rangle_{\epsilon,\rho_1}^{-\mu} \left(\partial_\epsilon^m (G_\epsilon(z) - G_{\mathrm{fr},\epsilon}(z))\right) \langle S\rangle_{\epsilon,\rho_2}^{-\mu} \right\| \le C|\lambda|\epsilon^{-m-1}. \tag{7.168}$$

Next we note that

$$\partial_\epsilon^k \langle S\rangle_{\epsilon,\rho}^{-\mu}(G_\epsilon(z) - G_{\mathrm{fr},\epsilon}(z))\langle S\rangle_{\epsilon,\rho}^{-\mu},$$

is a linear combination of the terms

$$S^{k_1}\langle S\rangle_{\epsilon,\rho^{(k_1)}}^{-\mu}(\partial_\epsilon^{k_2}(G_\epsilon(z) - G_{\mathrm{fr},\epsilon}(z)))\langle S\rangle_{\epsilon,\rho^{(k_3)}}^{-\mu}S^{k_3}, \tag{7.169}$$

where $k_1 + k_2 + k_3 = k$. Then we write $\rho^{(k_i)} = \gamma_i\tilde{\gamma}_i$, $i = 1, 2$ for some Schwartz functions $\gamma_i\tilde{\gamma}_i$ and we rewrite (7.169) as

$$\tilde{\gamma}_1(\epsilon S)S^{k_1}\langle S\rangle_{\epsilon,\gamma_1}^{-\mu}(\partial_\epsilon^{k_2}(G_\epsilon(z) - G_{\mathrm{fr},\epsilon}(z)))\langle S\rangle_{\epsilon,\gamma_2}^{-\mu}S^{k_3}\tilde{\gamma}_2(\epsilon S). \tag{7.170}$$

Now (7.168) combined with (7.153) yields the statement. □

We are now ready to finish

Proof of Theorem 7.20. Let n be the integer such that $n+1 > \nu$, and let ρ be a fixed Schwartz function such that $\rho(0) = 1$. We will use the shorthand

$$R(\epsilon) = \langle S\rangle_{\epsilon,\rho}^{-\mu}G_\epsilon(z)\langle S\rangle_{\epsilon,\rho}^{-\mu}.$$

It follows from Lemma 7.18 and the choice of n and μ that

$$\sup_{z\in\mathbf{C}_+} \|\partial_\epsilon^n R(\epsilon)\| \le C\epsilon^{-n-1+\nu}. \tag{7.171}$$

Furthermore, it follows from Theorem 7.14 and Taylor's formula that for any $\epsilon > 0$

$$R(0) = \sum_{k=0}^{n-1}(-1)^k\frac{\epsilon^k}{k!}\partial_\epsilon^k R(\epsilon) + \frac{(-1)^n}{(n-1)!}\int_0^\epsilon (\epsilon-\tau)^{n-1}\partial_\tau^n R(\tau)\mathrm{d}\tau.$$

(This formula is derived using Taylor's expansion of the function $R(\epsilon - \delta)$ in the variable δ and then taking $\delta \uparrow \epsilon$.) Setting $\lambda = 0$, we get a similar expansion for $R_{\mathrm{fr}}(0)$:

$$R_{\mathrm{fr}}(0) = \sum_{k=0}^{n-1}(-1)^k\frac{\epsilon^k}{k!}\partial_\epsilon^k R_{\mathrm{fr}}(\epsilon) + \frac{(-1)^n}{(n-1)!}\int_0^\epsilon (\epsilon-\tau)^{n-1}\partial_\tau^n R_{\mathrm{fr}}(\tau)\mathrm{d}\tau.$$

It follows from (7.171) that the error terms in both expansions are estimated by $C\epsilon^{\nu-1}$. Combining this estimate with Lemma 7.21 we derive that

$$\begin{aligned} \|R(0) - R_{\mathrm{fr}}(0)\| &\le \textstyle\sum_{k=0}^{n-1}\frac{\epsilon^k}{k!}\|\partial_\epsilon^k(R(\epsilon) - R_{\mathrm{fr}}(\epsilon))\| + C\epsilon^{\nu-1} \\ &\le C(|\lambda|\epsilon^{-1} + \epsilon^{\nu-1}). \end{aligned} \tag{7.172}$$

This estimate is optimized for $\epsilon = (|\lambda|/\nu - 1)^{1/\nu}$. Substituting this value into (7.172) we complete the proof of Theorem 7.20. □

8 Proofs of the main theorems

Throughout this chapter we assume that Hypotheses A and $S(\nu)$ hold with $\nu > 1$ and that $n \in \mathbf{N}$, $0 < \theta \leq 1$ and $\mu > \frac{1}{2}$ satisfy $\nu \geq \mu + \frac{1}{2} = 1 + n + \theta$.
Notation. In this and the next section we adopt the following shorthand. Let I be an interval, $\Omega \subset \mathbf{C}$, $\Omega \times I \ni (z, \lambda) \mapsto A_\lambda(z)$ an operator-valued function, and $f(\lambda)$ a positive function on I. We will write $A_\lambda(z) = O(f(\lambda))$ *for* $z \in \Omega$ if there exist constant C such that $\forall (z, \lambda) \in \Omega \times I$, $\|A_\lambda(z)\| \leq C f(\lambda)$. As customary, we will suppress the variable λ in the operator-valued functions, and write $A(z)$ for $A_\lambda(z)$, etc.

8.1 Proof of Limiting Absorption Principle away from $\sigma(K)$

In this section we prove Theorem 6.2. We fix $\Lambda_1 > 0$ such that $\Lambda_1 < (\sqrt{2}\|s\alpha\|)^{-1}$.

In what follows we assume that $|\lambda| \leq \Lambda_1$. Recall that the self-energy $W_{\mathrm{v}}(z)$ and the resonance function $G_{\mathrm{v}}(z)$ are defined by (3.48).

Lemma 8.1 *The function $\mathbf{C}_+ \ni z \mapsto W_{\mathrm{v}}(z)$ extends by continuity to $\overline{\mathbf{C}}_+$ and is of the class $C_{\mathrm{u}}^{n,\theta}(\overline{\mathbf{C}}_+)$ uniformly in λ. Furthermore, there exist β_1 such that*

$$\sup \lambda^{-2} \|W_{\mathrm{v}}(z)\| < \beta_1. \tag{8.173}$$

where the supremum is taken over $|\lambda| \leq \Lambda_1$ and $z \in \overline{\mathbf{C}}_+$.

Proof. Since

$$W_{\mathrm{v}}(z) = \lambda^2 \varphi(\alpha)^{\overline{\mathrm{vv}}} \langle S^{\overline{\mathrm{vv}}} \rangle^{\mu} \left(\langle S^{\overline{\mathrm{vv}}} \rangle^{-\mu} (z \mathbf{1}^{\overline{\mathrm{vv}}} - H^{\overline{\mathrm{vv}}})^{-1} \langle S^{\overline{\mathrm{vv}}} \rangle^{-\mu} \right) \langle S^{\overline{\mathrm{vv}}} \rangle^{\mu} \varphi(\alpha)^{\overline{\mathrm{v}}\mathrm{v}},$$

the result follows from Theorem 7.14. □

An immediate consequence of the previous lemma is that for $z \in \overline{\mathbf{C}}_+$, $G_{\mathrm{v}}(z)$ is a well-defined closed operator with domain $\mathcal{D}(K)$.

Lemma 8.2 *The operators $G_{\mathrm{v}}(z)$ are invertible for z in $\overline{\mathbf{C}}_+ \setminus B(\sigma(K), \lambda^2\beta_1)$ and the function $G_{\mathrm{v}}^{-1}(z)$ is of the class $C_{\mathrm{u}}^{n,\theta}$ of this set.*

Proof. Since $G_{\mathrm{v}}(z) = z\mathbf{1}^{\mathrm{vv}} - K - W_{\mathrm{v}}(z)$, the estimate (8.173) and Proposition 2.8 yield that $G_{\mathrm{v}}(z)$ is invertible and

$$\|G_{\mathrm{v}}^{-1}(z)\| = O(\lambda^{-2}). \tag{8.174}$$

for $z \in \overline{\mathbf{C}}_+ \setminus B(\sigma(K), \lambda^2\beta_1)$. The regularity properties of $G_{\mathrm{v}}(z)$ are inferred by induction from the identity

$$G_{\mathrm{v}}^{-1}(z_1) - G_{\mathrm{v}}^{-1}(z_2) = G_{\mathrm{v}}^{-1}(z_1) \left((z_2 - z_1) \mathbf{1}^{\mathrm{vv}} - (W_{\mathrm{v}}(z_2) - W_{\mathrm{v}}(z_1)) \right) G_{\mathrm{v}}^{-1}(z_2), \tag{8.175}$$

and Lemma 8.1. □

Proof of Theorem 6.2. The theorem follows from Theorem 7.14 and Lemma 8.2 after sandwiching the Feshbach formula

$$\begin{aligned}(z-H)^{-1} &= (z\mathbf{1}^{\overline{\mathrm{vv}}} - H^{\overline{\mathrm{vv}}})^{-1} \\ &+(\mathbf{1}^{\mathrm{vv}} + (z\mathbf{1}^{\overline{\mathrm{vv}}} - H^{\overline{\mathrm{vv}}})^{-1}H^{\overline{\mathrm{v}}\mathrm{v}})G_{\mathrm{v}}^{-1}(z)(H^{\mathrm{v}\overline{\mathrm{v}}}(z\mathbf{1}^{\overline{\mathrm{vv}}} - H^{\overline{\mathrm{vv}}})^{-1} + \mathbf{1}^{\mathrm{vv}}),\end{aligned} \tag{8.176}$$

with $\langle S\rangle^{-\mu}$, and after inserting $\langle S\rangle^{-\mu}\langle S\rangle^{\mu}$ in front of (after) $H^{\overline{\mathrm{v}}\mathrm{v}}$ ($H^{\mathrm{v}\overline{\mathrm{v}}}$). (Note that $\langle S\rangle^{-\mu} = \mathbf{1}^{\mathrm{vv}} \oplus \langle S^{\overline{\mathrm{vv}}}\rangle^{-\mu}$.) □

In the next two sections, similar elementary arguments based on the insertion of various powers of $\langle S\rangle$ at appropriate places will be skipped.

8.2 Proof of Limiting Absorption Principle around $k \in \sigma(K)$

In this section we prove Theorem 6.3. We introduce new splittings of the Hilbert space $\mathcal{H}$,

$$\mathcal{H} = \mathcal{H}^{k} \oplus \mathcal{H}^{\overline{k}} = \mathcal{H}^{k} \oplus \mathcal{H}^{\underline{k}} \oplus \mathcal{H}^{\overline{\mathrm{v}}}, \tag{8.177}$$

where

$$\mathcal{H}^{k} := \mathrm{Ran} p_k, \quad \mathcal{H}^{\overline{k}} := \mathrm{Ran}(\mathbf{1} - p_k), \quad \mathcal{H}^{\underline{k}} := \mathcal{H}^{\overline{k}} \cap \mathcal{H}^{\mathrm{v}}.$$

In our argument we will apply several times the Feshbach formula with respect to these decompositions. To that end we introduce some additional notation. The matrix form of the operator H with respect to the decomposition $\mathcal{H} = \mathcal{H}^{k} \oplus \mathcal{H}^{\overline{k}}$ is denoted by

$$H = \begin{bmatrix} H^{kk} & H^{k\overline{k}} \\ H^{\overline{k}k} & H^{\overline{kk}} \end{bmatrix}. \tag{8.178}$$

The operator $H^{\overline{kk}}$ acts on $\mathcal{H}^{\underline{k}} \oplus \mathcal{H}^{\overline{\mathrm{v}}}$, and its matrix form is denoted by

$$H^{\overline{kk}} = \begin{bmatrix} H^{\underline{kk}} & H^{\underline{k}\overline{\mathrm{v}}} \\ H^{\overline{\mathrm{v}}\underline{k}} & H^{\overline{\mathrm{vv}}} \end{bmatrix}. \tag{8.179}$$

(Arguing as in the beginning of Chapter 6 one easily shows that these matrix representations are well-defined and that the formalism and results of Chapter 3 can be applied.) We employ the same notation for other operators. Note that $p_k = \mathbf{1}^{kk}$. Note also that

$$H^{\overline{k}k} = H^{\overline{\mathrm{v}}k}, \qquad H^{k\overline{k}} = H^{k\overline{\mathrm{v}}}. \tag{8.180}$$

For $z \in \overline{\mathbf{C}}_{+}$ we set

$$\begin{aligned} W_{\underline{k}}(z) &:= H^{\underline{k}\overline{\mathrm{v}}}(z\mathbf{1}^{\overline{\mathrm{vv}}} - H^{\overline{\mathrm{vv}}})^{-1}H^{\overline{\mathrm{v}}\underline{k}}, \\ G_{\underline{k}}(z) &:= z\mathbf{1}^{\underline{kk}} - H^{\underline{kk}} - W_{\underline{k}}(z). \end{aligned} \tag{8.181}$$

Note that if $W_{\rm v}(z)$ and $G_{\rm v}(z)$ are the usual self-energy and resonance function, then

$$W_{\underline{k}}(z) = W_{\rm v}^{\underline{kk}}(z), \qquad G_{\underline{k}}(z) = G_{\rm v}^{\underline{kk}}(z). \tag{8.182}$$

In the next lemma we assume that $|\lambda| < \Lambda_1$.

Lemma 8.3 *The function $W_{\underline{k}}(z)$ belongs to $C_{\rm u}^{n,\theta}(\overline{\mathbf{C}}_+)$ uniformly in λ and we have*

$$\sup \lambda^{-2} \|W_{\underline{k}}(z)\| < \beta_1, \tag{8.183}$$

where the supremum is taken over $|\lambda| \leq \Lambda_1$ and $z \in \overline{\mathbf{C}}_+$.

Proof. We apply Lemma 8.1 and (8.182). □

Let $\delta_1 > 0$ be such that

$$\delta_1 < \operatorname{dist}(k, \sigma(K) \setminus \{k\}).$$

We choose $\Lambda_2 > 0$ such that

$$\Lambda_2 \leq \Lambda_1, \qquad \beta_1 \Lambda_2^2 < \delta_1. \tag{8.184}$$

Until the end of this section we assume that $|\lambda| \leq \Lambda_2$.

Lemma 8.4 *The operators $G_{\underline{k}}(z)$ are invertible for $z \in \overline{\mathbf{C}}_+ \cap \overline{B}(k, \delta_1)$ and the function $G_{\underline{k}}^{-1}(z)$ is in the class $C_{\rm u}^{n,\theta}$ of this set uniformly in λ.*

Proof. The invertibility of $G_{\underline{k}}(z)$ follows from Proposition 2.8, Lemma 8.3 and the choice of Λ_2. The regularity properties of $G_{\underline{k}}^{-1}(z)$ follow by induction from Lemma 8.3 and an identity similar to (8.176). □

For $z \in \mathbf{C}_+$ we set

$$\begin{aligned} W_k(z) &:= H^{k\overline{k}}(z\mathbf{1}^{\overline{kk}} - H^{\overline{kk}})^{-1} H^{\overline{k}k}, \\ G_k(z) &:= z\mathbf{1}^{kk} - H^{kk} - W_k(z). \end{aligned} \tag{8.185}$$

Lemma 8.5 *The function*

$$\mathbf{C}_+ \ni z \mapsto \langle S \rangle^{-\mu} (z\mathbf{1}^{\overline{kk}} - H^{\overline{kk}})^{-1} \langle S \rangle^{-\mu}, \tag{8.186}$$

extends by continuity to $\overline{\mathbf{C}}_+ \cap \overline{B}(k, \delta_1)$ and is in the class $C_{\rm u}^{n,\theta}$ of this set uniformly in λ. The same result holds for the function $W_k(z)$.

Proof. The Feshbach formula yields that for $z \in \mathbf{C}_+$,

$$\begin{aligned}(z\mathbf{1}^{\overline{k}\overline{k}} - H^{\overline{k}\overline{k}})^{-1} &= (z\mathbf{1}^{\overline{\mathrm{v}}\overline{\mathrm{v}}} - H^{\overline{\mathrm{v}}\overline{\mathrm{v}}})^{-1} \\ &+(\mathbf{1}^{\underline{k}\underline{k}} + (z\mathbf{1}^{\overline{\mathrm{v}}\overline{\mathrm{v}}} - H^{\overline{\mathrm{v}}\overline{\mathrm{v}}})^{-1}H^{\overline{\mathrm{v}}\underline{k}})G_{\underline{k}}^{-1}(z)(\mathbf{1}^{\underline{k}\underline{k}} + H^{\underline{k}\overline{\mathrm{v}}}(z\mathbf{1}^{\overline{\mathrm{v}}\overline{\mathrm{v}}} - H^{\overline{\mathrm{v}}\overline{\mathrm{v}}})^{-1}).\end{aligned} \tag{8.187}$$

We derive the result sandwiching this identity with $\langle S\rangle^{-\mu}$ and invoking the previous lemma and Theorem 7.14. □

We will now make use of the Feshbach formula with respect to the decomposition $\mathcal{H} = \mathcal{H}^k \oplus \mathcal{H}^{\overline{k}}$. Note that in the notation we have adopted, $w^{kk}(z) = p_k w(z) p_k$. In the sequel we set $\kappa := (\nu-1)/\nu$. Recall that in (6.96) we defined

$$w_k := w^{kk}(k + \mathrm{i}0).$$

Lemma 8.6 *There is a constant β_2 such that if $z \in \overline{\mathbf{C}}_+ \cap \overline{B}(k, \lambda^2\beta_1)$, then*

$$\sup |\lambda|^{-2-\kappa}\|W_k(z) - \lambda^2 w_k\| < \beta_2, \tag{8.188}$$

where the supremum is taken over $z \in \overline{\mathbf{C}}_+ \cap \overline{B}(k, \lambda^2\beta_1)$, $|\lambda| \le \Lambda_2$.

Proof. It follows from (8.187) that

$$\begin{aligned}W_k(z) &= H^{k\overline{k}}(z\mathbf{1}^{\overline{\mathrm{v}}\overline{\mathrm{v}}} - H^{\overline{\mathrm{v}}\overline{\mathrm{v}}})^{-1}H^{\overline{k}k} \\ &+H^{k\overline{k}}(z\mathbf{1}^{\overline{\mathrm{v}}\overline{\mathrm{v}}} - H^{\overline{\mathrm{v}}\overline{\mathrm{v}}})^{-1}H^{\overline{\mathrm{v}}\underline{k}}G_{\underline{k}}^{-1}(z)H^{\underline{k}\overline{\mathrm{v}}}(z\mathbf{1}^{\overline{\mathrm{v}}\overline{\mathrm{v}}} - H^{\overline{\mathrm{v}}\overline{\mathrm{v}}})^{-1}H^{\overline{k}k}.\end{aligned} \tag{8.189}$$

Since $\beta_1\Lambda_2^2 < \delta_1$, Lemma 8.4 yields that for $z \in \mathbf{C}_+ \cap \overline{B}(k, \lambda^2\beta_1)$ the second term on the right-hand side is $O(\lambda^4)$. It follows from Theorem 7.20 that the first term equals

$$H^{k\overline{k}}(z\mathbf{1}^{\overline{\mathrm{v}}\overline{\mathrm{v}}} - H^{\overline{\mathrm{v}}\overline{\mathrm{v}}})^{-1}H^{\overline{k}k} = H^{k\overline{k}}(z\mathbf{1}^{\overline{\mathrm{v}}\overline{\mathrm{v}}} - H_{\mathrm{fr}}^{\overline{\mathrm{v}}\overline{\mathrm{v}}})^{-1}H^{\overline{k}k} + O(|\lambda|^{2+\kappa}).$$

Recall that in (2.34) we defined the function $\ell_\theta(\tau)$. We extend the definition of this function for $\theta \in [0, \infty[$ as follows:

$$\ell_\theta(\tau) := \begin{cases} \tau^\theta, & 0 < \theta < 1, \\ \tau\,(1 + \ln(1+\tau^{-1})) & \theta = 1, \\ \tau, & \theta > 1. \end{cases}$$

Then, by Theorem 7.19,

$$H^{k\overline{k}}(z\mathbf{1}^{\overline{\mathrm{v}}\overline{\mathrm{v}}} - H_{\mathrm{fr}}^{\overline{\mathrm{v}}\overline{\mathrm{v}}})^{-1}H^{\overline{k}k} = \lambda^2 w^{kk}(z) = \lambda^2 w_k^{kk} + R(z),$$

where

$$\begin{aligned}\|R(z)\| &\le C_1\lambda^2\ell_{\nu-\frac{1}{2}}(|z-k|) \\ &\le C_2\lambda^2\ell_{\nu-\frac{1}{2}}(\beta_1\lambda^2) \le C_3|\lambda|^{2+\kappa},\end{aligned}$$

where we used $2\nu - 1 > \kappa$ and $2 > \kappa$ in the last step. □

The operator w_k is dissipative and by the assumption,

$$\mathcal{T}_k = \sigma(w_k) \cap \mathbf{R} \subset \sigma_{\mathrm{disc}}(w_k).$$

Therefore, Propositions 3.3 and 3.4 can be applied. Let

$$c := \sup_{z \in \overline{\mathbf{C}}_+} \|(z - w_k)^{-1} \mathbf{1}_{\sigma(w_k) \setminus \mathbf{R}}(w_k)\|, \tag{8.190}$$

(which is the constant c in Proposition 3.3 applied to w_k). In addition to (8.184), in the sequel we assume that Λ_2 satisfies

$$\beta_2 \Lambda_2^\kappa c < 1. \tag{8.191}$$

Lemma 8.7 *The operator $G_k(z)$ is invertible for*

$$z \in \overline{\mathbf{C}}_+ \cap \left(\overline{B}(k, \lambda^2 \beta_1) \setminus B(k + \lambda^2 \mathcal{T}_k, |\lambda|^{2+\kappa} \beta_2) \right),$$

and the function $G_k^{-1}(z)$ is in the class $C_{\mathrm{u}}^{n,\theta}$ of this set.

Proof. It follows from the definition of $G_k(z)$ and Lemma 8.6 that if $z \in \overline{\mathbf{C}}_+ \cap \overline{B}(k, \lambda^2 \beta_1)$, then

$$\begin{aligned} G_k(z) &= (z - k)\mathbf{1}^{kk} - W_k(z) \\ &= (z - k)\mathbf{1}^{kk} - \lambda^2 w_k + R(z) \\ &= \lambda^2 \left(\lambda^{-2}(z - k)\mathbf{1}^{kk} - w_k + \lambda^{-2} R(z) \right), \end{aligned} \tag{8.192}$$

where $\|R(z)\| < \beta_2 |\lambda|^{2+\kappa}$. If c is defined by (8.190), then $\|\lambda^{-2} R(z)\| c < 1$ and

$$\sup \mathrm{dist}(\sigma(\lambda^{-2} k + w_k) \cap \mathbf{R}, \lambda^{-2} z) > |\lambda|^\kappa \beta_2,$$

where the supremum is taken over $z \in \overline{\mathbf{C}}_+ \setminus B(k + \lambda^2 \mathcal{T}_k, |\lambda|^{2+\kappa} \beta_2)$. Therefore, it follows from Proposition 3.4 that $G_k(z)$ is invertible for

$$z \in \overline{\mathbf{C}}_+ \cap \left(\overline{B}(k, \lambda^2 \beta_1) \setminus B(k + \lambda^2 \mathcal{T}_k, |\lambda|^{2+\kappa} \beta_2) \right).$$

The regularity properties of $G_k^{-1}(z)$ are inferred from Lemma 8.5 and an identity similar to (8.176). □

Proof of Theorem 6.3. Since (ii) ⇒ (i), we have only to prove (ii). We choose Λ_2 such that (8.184) and (8.191) hold. The Feshbach formula yields

$$\begin{aligned} (z - H)^{-1} &= (z \mathbf{1}^{\overline{kk}} - H^{\overline{kk}})^{-1} \\ &\quad + (\mathbf{1}^{kk} + (z \mathbf{1}^{\overline{kk}} - H^{\overline{kk}})^{-1} H^{\overline{k}k}) G_k^{-1}(z) (\mathbf{1}^{kk} + H^{k\overline{k}} (z \mathbf{1}^{\overline{kk}} - H^{\overline{kk}})^{-1}). \end{aligned}$$

Sandwiching this formula with $\langle S \rangle^{-\mu}$, we derive (ii) from Lemmas 8.5 and 8.7. □

8.3 Proof of Limiting Absorption Principle around $k + \lambda^2 m$

In this section we prove Theorem 6.4. We will freely use the notation and the results of the previous section. We will indicate the place in our argument where we require that Hypothesis $S(\nu)$ with $\nu > 2$ holds.

In addition to (8.177) we introduce the following splittings of the Hilbert space $\mathcal{H}$:

$$\mathcal{H} = \mathcal{H}^m \oplus \mathcal{H}^{\overline{m}} = \mathcal{H}^m \oplus \mathcal{H}^{\underline{m}} \oplus \mathcal{H}^{\overline{k}},$$

where

$$\mathcal{H}^m := \mathrm{Ran} p_{k,m}, \quad \mathcal{H}^{\overline{m}} := \mathrm{Ran}(\mathbf{1} - p_{k,m}), \quad \mathcal{H}^{\underline{m}} := \mathcal{H}^k \cap \mathcal{H}^{\overline{m}}.$$

Note that by the definition of $p_{k,m}$, $\mathcal{H}^m \subset \mathcal{H}^k$, so the above splittings are well-defined. Note also that

$$H^{\overline{m}m} = H^{\overline{k}m} = H^{\bar{\mathrm{v}}m}, \qquad H^{m\overline{m}} = H^{m\overline{k}} = H^{m\bar{\mathrm{v}}}. \tag{8.193}$$

For these splittings we adopt the notation analogous to (8.178) and (8.179).

For $z \in \overline{\mathbf{C}}_+ \cap \overline{B}(k, \delta_1)$ we set

$$W_{\underline{m}}(z) := H^{\underline{m}\overline{k}}(z\mathbf{1}^{\overline{k}\overline{k}} - H^{\overline{k}\overline{k}})^{-1} H^{\overline{k}\underline{m}},$$

$$G_{\underline{m}}(z) := z\mathbf{1}^{\underline{m}\underline{m}} - H^{\underline{m}\underline{m}} - W_{\underline{m}}(z).$$

Note that if $W_k(z)$ and $G_k(z)$ are given by (8.185) then

$$W_{\underline{m}}(z) = W_k^{\overline{m}\overline{m}}(z), \qquad G_{\underline{m}}(z) = G_k^{\overline{m}\overline{m}}(z). \tag{8.194}$$

We assume that $|\lambda| \leq \Lambda_2$.

Lemma 8.8 *Assume that $z \in \overline{\mathbf{C}}_+ \cap B(k, \lambda^2 \beta_1)$. Then*

$$\begin{aligned} H^{\underline{m}\overline{k}}(z\mathbf{1}^{\overline{k}\overline{k}} - H^{\overline{k}\overline{k}})^{-1} H^{\overline{k}m} &= O(|\lambda|^{2+\kappa}), \\ H^{m\overline{k}}(z\mathbf{1}^{\overline{k}\overline{k}} - H^{\overline{k}\overline{k}})^{-1} H^{\overline{k}\underline{m}} &= O(|\lambda|^{2+\kappa}). \end{aligned} \tag{8.195}$$

Moreover, we have

$$\sup |\lambda|^{-2-\kappa} \| W_{\underline{m}}(z) - \lambda^2 w_k^{\overline{m}\overline{m}} \| < \beta_2, \tag{8.196}$$

where the supremum is taken over $z \in \overline{\mathbf{C}}_+ \cap \overline{B}(k, \lambda^2 \beta_1)$, $|\lambda| \leq \Lambda_2$.

Proof. To prove (8.195) we use $w_k^{\overline{m}m} = (p_k - p_{k,m}) w_k p_{k,m} = 0$ and Lemma 8.6. (8.196) follows from Relation (8.194) and Lemma 8.6. □

Let

$$\delta_2 < \mathrm{dist}\left(\{m\}, \sigma(w_k) \setminus \{m\}\right).$$

We introduce $\Lambda_3 > 0$ such that

$$\Lambda_3 \leq \Lambda_2, \qquad \beta_2 \Lambda_3^\kappa \leq \delta_2. \tag{8.197}$$

¿From now on we assume that $|\lambda| \leq \Lambda_3$.

Lemma 8.9 *For* $z \in \overline{\mathbf{C}}_+ \cap \overline{B}(k, \lambda^2\beta_1) \cap \overline{B}(k + \lambda^2 m, \lambda^{2+\kappa}\beta_2)$ *the operators* $G_{\underline{m}}(z)$ *are invertible and the function* $G_{\underline{m}}^{-1}(z)$ *is in the class* $C_{\mathrm{u}}^{n,\theta}$ *of this set. Furthermore,*

$$\frac{\mathrm{d}^j}{\mathrm{d}z^j} G_{\underline{m}}^{-1}(z) = O(\lambda^{-2j-2}), \quad j = 0, \dots, n. \tag{8.198}$$

Proof. Clearly,

$$\begin{aligned} G_{\underline{m}}(z) &= (z-k)\mathbf{1}^{\underline{mm}} - W_{\underline{m}}(z) \\ &= (z-k)\mathbf{1}^{\underline{mm}} - \lambda^2 w_k^{\underline{mm}} + R^{\underline{mm}}(z) \\ &= \lambda^2 \left(\lambda^{-2}(z-k)\mathbf{1}^{\underline{mm}} - w_k^{\underline{mm}} + \lambda^{-2} R^{\underline{mm}}(z)\right), \end{aligned}$$

where $R(z) = W_k(z) - \lambda^2 w_k$ is the same as in (8.192). By Lemma 8.6,

$$\|R^{\underline{mm}}(z)\| \leq \|R(z)\| \leq \beta_2 |\lambda|^{2+\kappa},$$

if $z \in \overline{\mathbf{C}}_+ \cap \overline{B}(k, \lambda^2\beta_1)$. Moreover,

$$w_k^{\underline{mm}} \mathbf{1}_{\sigma(w_k^{\underline{mm}})\setminus\mathbf{R}}(w_k^{\underline{mm}}) = w_k \mathbf{1}_{\sigma(w_k)\setminus\mathbf{R}}(w_k).$$

Therefore,

$$c = \sup_{z \in \overline{\mathbf{C}}_+} \|(z - w_k^{\underline{mm}})^{-1} \mathbf{1}_{\sigma(w_k^{\underline{mm}})\setminus\mathbf{R}}(w_k^{\underline{mm}})\|$$

is the same as in (8.190). Thus, we can apply Proposition 3.4, which implies that $G_{\underline{m}}(z)$ is invertible and satisfies the bound (8.198) with $j = 0$.

The regularity properties of $G_{\underline{m}}^{-1}(z)$ are inferred from the formula analogous to (8.176) and Lemma 8.5. The bound (8.198) for $j = 0$ and induction yield (8.198) for all j. □

For $z \in \mathbf{C}_+$ we set

$$\begin{aligned} W_m(z) &:= H^{m\overline{m}}(z\mathbf{1}^{\overline{mm}} - H^{\overline{mm}})^{-1} H^{\overline{m}m}, \\ G_m(z) &:= z\mathbf{1}^{mm} - H^{mm} - W_m(z). \end{aligned}$$

Lemma 8.10 *The function*

$$\mathbf{C}_+ \ni z \mapsto \langle S \rangle^{-\mu} (z\mathbf{1}^{\overline{mm}} - H^{\overline{mm}})^{-1} \langle S \rangle^{-\mu},$$

extends by continuity to $z \in \overline{\mathbf{C}}_+ \cap \overline{B}(k, \lambda^2\beta_1) \cap \overline{B}(k + \lambda^2 m, \lambda^{2+\kappa}\beta_2)$ *and is in the class* $C_{\mathrm{u}}^{n,\theta}$ *of this set. The same result holds for the function* $W_m(z)$*.*

Proof. The Feshbach formula yields

$$
\begin{aligned}
(z\mathbf{1}^{\overline{mm}} - H^{\overline{mm}})^{-1} &= (z\mathbf{1}^{\overline{kk}} - H^{\overline{kk}})^{-1} \\
&\quad + (\mathbf{1}^{\underline{mm}} + (z\mathbf{1}^{\overline{kk}} - H^{\overline{kk}})^{-1}H^{\overline{k}\underline{m}})G_{\underline{m}}^{-1}(z)(z\mathbf{1}^{\underline{mm}} + H^{\underline{m}\overline{k}}(z\mathbf{1}^{\overline{kk}} - H^{\overline{kk}})^{-1}),
\end{aligned}
$$

and the result follows from Lemmas 8.6 and 8.8. □

Lemma 8.11 *Assume that Hypothesis $S(\nu)$ holds with $\nu > 2$. There exist a constant γ such that for $z \in \overline{\mathbf{C}}_+ \cap \overline{B}(k, \lambda^2\beta_1) \cap \overline{B}(k + \lambda^2 m, \lambda^{2+\kappa}\beta_2)$*

$$
\left\| \frac{\mathrm{d}}{\mathrm{d}z} W_m(z) \right\| \leq \gamma |\lambda|^{2\kappa}. \tag{8.199}
$$

Proof. It follows from the Feshbach formula that

$$
\begin{aligned}
W_m(z) &= H^{m\overline{m}}(z\mathbf{1}^{\overline{kk}} - H^{\overline{kk}})^{-1}H^{\overline{m}m} \\
&\quad + H^{m\overline{m}}(z\mathbf{1}^{\overline{kk}} - H^{\overline{kk}})^{-1}H^{\overline{k}\underline{m}}G_{\underline{m}}^{-1}(z)H^{\underline{m}\overline{k}}(z\mathbf{1}^{\overline{kk}} - H^{\overline{kk}})^{-1}H^{\overline{m}m}.
\end{aligned}
\tag{8.200}
$$

The derivative of the first term in (8.200) is $O(\lambda^2)$ by Lemma 8.8. When we differentiate the second term and the derivative hits $(z\mathbf{1}^{\overline{kk}} - H^{\overline{kk}})^{-1}$, then we get by Lemmas 8.8, 8.9 and (8.198)

$$
O(\lambda^2)O(\lambda^{-2})O(|\lambda|^{2+\kappa}) = O(|\lambda|^{2+\kappa}).
$$

When the derivative hits $G_{\underline{m}}^{-1}(z)$, then we get by the same lemmas and (8.198)

$$
O(|\lambda|^{2+\kappa})O(\lambda^{-4})O(|\lambda|^{2+\kappa}) = O(|\lambda|^{2\kappa}).
$$

Then we use $2\kappa \leq 2 + \kappa$. □

We are now ready to finish

Proof of Theorem 6.4. Let γ be given by (8.199). Recall that so far Λ_3 has to satisfy (8.197). We demand in addition that

$$
\Lambda_3^{2\kappa} < 1/\gamma. \tag{8.201}
$$

Assumption (8.201) and Lemma 8.11 imply that

$$
\sup_{x \in \Theta(k,m)} \|W_m'(x + \mathrm{i}0)\| < 1.
$$

Thus, the conditions of Corollary 3.13 are satisfied on the interval $\Theta(k, m)$ with respect to the decomposition $\mathcal{H} = \mathcal{H}^m \oplus \mathcal{H}^{\overline{m}}$. By Corollary 3.13,

$$
\dim \mathbf{1}_{\Theta(k,m)}^{\mathrm{pp}} \leq \dim \mathcal{H}^m = \dim p_{k,m}.
$$

This completes the proof of Part (i).

Since (iii) $\Rightarrow$ (ii), it remains to prove (iii). First note that, by Lemma 8.10, $\Theta(k,m)\cap\sigma_{\rm pp}(H^{\overline{mm}})=\emptyset$. Therefore, by Proposition 3.7 and Theorem 3.8, $\sigma_{\rm pp}(H)\cap\Theta(k,m)$ coincides with $\{x\in\Theta(k,m)\ :\ 0\in\sigma(G_m(x+{\rm i}0))\}$. Let $\epsilon>0$. Since $G_m(z)$ is a continuous function and $\overline{\mathbf{C}}_+\cap\overline{B}(k,\lambda^2\beta_1)\cap\overline{B}(k+\lambda^2m,\beta_2\lambda^{2+\kappa})\setminus B(\sigma_{\rm pp}(H),\epsilon)$ is compact,

$$\|G_m^{-1}(z)\|\leq C.$$

on this set. A formula analogous to (8.176) and Lemma 8.10 yield that $G_m^{-1}(z)\in C_{\rm u}^{n,\theta}$ of this set.

The Feshbach formula yields that for $z\in\mathbf{C}_+$,

$$\begin{aligned}(z-H)^{-1}\ &=(z\mathbf{1}^{\overline{mm}}-H^{\overline{mm}})^{-1}\\ &+(\mathbf{1}^{mm}+(z\mathbf{1}^{\overline{mm}}-H^{\overline{mm}})^{-1}H^{\overline{m}m})G_m^{-1}(z)(\mathbf{1}^{mm}+H^{m\overline{m}}(z\mathbf{1}^{\overline{mm}}-H^{\overline{mm}})^{-1}).\end{aligned}$$

Sandwiching this formula with $\langle S\rangle^{-\mu}$, we derive (iii) from Lemma 8.10 and the regularity properties of $G_m^{-1}(z)$ proven above. The proof of Theorem 6.4 is complete. □

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