HOMOGENEOUS SCHRÖDINGER OPERATORS ON HALF-LINE

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ABSTRACT. The differential expression $L_m = -\partial_x^2 + (m^2 - 1/4)x^{-2}$ defines a self-adjoint operator H_m on $L^2(0,\infty)$ in a natural way when $m^2 \geq 1$. We study the dependence of H_m on the parameter m, show that it has a unique holomorphic extension to the half-plane $\mathrm{Re}\, m > -1$, and analyze spectral and scattering properties of this family of operators.

1. INTRODUCTION

For $m \geq 1$ real, the differential operator $L_m = -\partial_x^2 + (m^2 - 1/4)x^{-2}$ with domain $C_c^\infty = C_c^\infty(0,\infty)$ is essentially self-adjoint and we denote by H_m its closure. Let U_τ be the group of dilations on L^2, that is $(U_\tau f)(x) = e^{\tau/2} f(e^\tau x)$. Then H_m is clearly homogeneous of degree -2, i.e. $U_\tau H_m U_\tau^{-1} = e^{-2\tau} H_m$. The following theorem summarizes the main results of our paper.

Theorem 1.1. *There is a unique holomorphic family $\{H_m\}_{\mathrm{Re}\, m > -1}$ such that H_m coincides with the previously defined operator if $m \geq 1$. The operators H_m are homogeneous of degree -2 and satisfy $H_m^* = H_{\bar{m}}$. In particular, H_m is self-adjoint if m is real. The spectrum and the essential spectrum of H_m are equal to $[0,\infty[$ for each m with $\mathrm{Re}\, m > -1$. On the other hand, for non real m the numerical range of H_m depends on m as follows:*

i) If $0 < \arg m \leq \pi/2$, then $\mathrm{Num}(H_m) = \{z \mid 0 \leq \arg z \leq 2\arg m\}$,
ii) If $-\pi/2 \leq \arg m < 0$, then $\mathrm{Num}(H_m) = \{z \mid 2\arg m \leq \arg z \leq 0\}$,
iii) If $\pi/2 < |\arg m| < \pi$, then $\mathrm{Num}(H_m) = \mathbb{C}$.

If $\mathrm{Re}\, m > -1$, $\mathrm{Re}\, k > -1$ and $\lambda \notin [0,\infty[$, then $(H_m - \lambda)^{-1} - (H_k - \lambda)^{-1}$ is a compact operator.

In the above theorem $\arg\zeta$ is defined for $\zeta \in \mathbb{C}\backslash]-\infty,0]$ by $-\pi < \arg\zeta < \pi$. We note that if $0 \leq m < 1$ the operator L_m is not essentially self-adjoint. If $0 < m < 1$ this operator has exactly two distinct homogeneous extensions which are precisely the operators H_m and H_{-m} defined in the theorem: they are the Friedrichs and Krein extension of L_m respectively. Theorem 1.1 thus shows that we can pass holomorphically from one extension to the other. Note also that L_0 has exactly one homogeneous extension, the operator H_0 which is at the same time the Friedrichs and the Krein extension of L_0. We obtain these results via a rather complete analysis of the extensions (not necessarily self-adjoint) of the operator L_m for complex m.

We are not aware of a similar analysis of the holomorphic family $\{H_m\}_{\mathrm{Re}\, m > -1}$ in the literature. Most of the literature seems to restrict itself to the case of real m and self-adjoint H_m. A detailed study of the case $m > 0$ can be found in [1]. The fact that in this case the operator H_m is the Friedrichs extension of L_m is of course well known. However, even the analysis of the case $-1 \leq m \leq 0$ seems to have been neglected in the literature.

We note that similar results concerning the holomorphic dependence in the parameter α of the operator $H_\alpha = (-\Delta + 1)^{1/2} - \alpha/|x|$ have been obtained in [3] by different techniques.

Besides the results described in Theorem 1.1, we prove a number of other properties of the Hamiltonians H_m. Among other things, we treat the spectral and scattering theory of the operators H_m for real m, see

Date: December 9, 2010.

Sections 5 and 6: we obtain explicit formulas for their spectral representation and the corresponding wave and scattering operators.

Concerning scattering theory, we shall prove that the wave operators $\Omega^{\pm}_{m,k}$ for the pair (H_m, H_k) exist for any real $m, k > -1$. Since both H_m and H_k are homogeneous of the same degree, an easy abstract argument shows that $\Omega^{\pm}_{m,k} = \theta^{\pm}_{m,k}(D)$, where D is the generator of the dilation group and $\theta^{\pm}_{m,k}$ are functions of modulus one, cf. Proposition 2.9. We explicitly compute these functions in Section 6 and obtain

$$\Omega^{\pm}_{m,k} = \mathrm{e}^{\pm \mathrm{i}(m-k)\pi/2} \frac{\Gamma(\frac{k+1+\mathrm{i}D}{2})\Gamma(\frac{m+1-\mathrm{i}D}{2})}{\Gamma(\frac{k+1-\mathrm{i}D}{2})\Gamma(\frac{m+1+\mathrm{i}D}{2})}.$$

Essentially identical formulas in the closely related context of the Aharonov-Bohm Hamiltonians were obtained independently by Pankrashkin and Richard in a recent paper [4] .

The scattering theory for H_m suggests a question, which we were not able to answer. We pose this question as an interesting open problem in Remark 6.5: can the holomorphic family $\{\mathrm{Re}\,(m) > -1\} \ni m \mapsto H_m$ be extended to the whole complex plane? To understand why it is not easy to answer this question let us mention that for $\mathrm{Re}\,(m) > -1$, the resolvent set is non-empty, being equal to $\mathbb{C}\backslash[0,\infty[$. Therefore, to prove that $\{\mathrm{Re}\,(m) > -1\} \ni m \mapsto H_m$ is a holomorphic family, it suffices to show that its resolvent is holomorphic. However, one can show that if an extension of this family to $\mathbb{C}$ exists, then for $\{m \mid \mathrm{Re}\, m = -1, -2, \dots,\ \mathrm{Im}\, m \neq 0\}$ the operator H_m will have an empty resolvent set. Therefore, on this set we cannot use the resolvent of H_m.

Let us describe the organization of the paper. In Section 2 we recall some facts concerning holomorphic families of closed operators and make some general remarks on homogeneous operators and their scattering theory in an abstract setting. Section 3 is devoted to a detailed study of the first order homogeneous differential operators. We obtain there several results, which are then used in Section 4 containing our main results. In Section 5 we give explicitly the spectral representation of H_m for real m and in Section 6 we treat their scattering theory. In the first appendix we recall some technical results on second order differential operators. Finally, as an application of Theorem 1.1, in the second appendix we consider the Aharonov-Bohm Hamiltonian M_λ depending on the magnetic flux λ and describe various holomorphic homogeneous rotationally symmetric extensions of the family $\lambda \to M_\lambda$. For a recent review on Aharonov-Bohm Hamiltonians we refer to [4] and references therein.

To sum up, we believe that the operators H_m are interesting for many reasons.

- They have several interesting physical applications, eg. they appear in the decomposition of the Aharonov-Bohm Hamiltonian.
- They have rather subtle and rich properties, illustrating various concepts of the operator theory in Hilbert spaces (eg. the Friedrichs and Krein extensions, holomorphic families of closed operators). Surprisingly rich is also the theory of the first order homogeneous operators A_α, that we develop in Sect. 3, which is closely related to the theory of H_m.
- Essentially all basic objects related to H_m, such as their resolvents, spectral projections, wave and scattering operators, can be explicitly computed.
- A number of nontrivial identities involving special functions find an appealing operator-theoretical interpretation in terms of the operators H_m. Eg. the Barnes identity (6.4) leads to the formula for wave operators. Let us mention also the Weber-Schafheitlin identity [8], which can be used to describe the distributional kernel of powers of H_m.

Acknowledgement J.D. would like to thank H. Kalf for useful discussions. His research was supported in part by the grant N N201 270135 of the Polish Ministry of Science and Higher Education. The research of L.B. is supported by the ANR project HAM-MARK (ANR-09-BLAN-0098-01) of the French Ministry of Research.

2. PRELIMINARIES

2.1. **Notation.** For an operator A we denote by $\mathcal{D}(A)$ its domain, $\mathrm{sp}(A)$ its spectrum, and $\mathrm{rs}(A)$ its resolvent set. We denote by $\mathrm{Num}(A)$ the (closure of the) numerical range of an operator A, that is

$$\mathrm{Num}(A) := \overline{\{\langle f, Af\rangle \mid f \in \mathcal{D}(A),\ \|f\| = 1\}}.$$

If H is a self-adjoint operator H then $\mathcal{Q}(H)$ will denote its form domain: $\mathcal{Q}(H) = \mathcal{D}(|H|^{1/2})$.

We set $\mathbb{R}_+ =]0, \infty[$. We denote by $\mathbb{1}_+$ the characteristic function of the subset $\mathbb{R}_+$ of $\mathbb{R}$.

We write L^2 for the Hilbert space $L^2(\mathbb{R}_+)$. We abbreviate $C_\mathrm{c}^\infty = C_\mathrm{c}^\infty(\mathbb{R}_+)$, $H^1 = H^1(\mathbb{R}_+)$ and $H_0^1 = H_0^1(\mathbb{R}_+)$. Note that H^1 and H_0^1 are the form domains of the Neumann and Dirichlet Laplacian respectively on $\mathbb{R}_+$. If $-\infty \le a < b \le \infty$ we set $L^2(a,b) = L^2(]a,b[)$ and similarly for $C_c^\infty(a,b)$, etc.

Capital letters decorated with a tilde will denote operators acting on distributions. For instance, let $\tilde{Q}$ and $\tilde{P}$ be the position and momentum operators on $\mathbb{R}_+$, so that $(\tilde{Q}f)(x) = xf(x)$ and $(\tilde{P}f)(x) = -\mathrm{i}\partial_x f(x)$, acting in the sense of distributions on $\mathbb{R}_+$. The operator $\tilde{Q}$ restricted to an appropriate domain becomes a self-adjoint operator on L^2, and then it will be denoted Q. The operator $\tilde{P}$ has two natural restrictions to closed operators on L^2, $P_{\min}$ with domain H_0^1 and its extension $P_{\max}$ with domain H^1. We have $(P_{\min})^* = P_{\max}$.

The differential operator $\tilde{D} := \frac{1}{2}(\tilde{P}\tilde{Q} + \tilde{Q}\tilde{P}) = \tilde{P}\tilde{Q} + \mathrm{i}/2$, when considered as an operator in L^2 with domain C_c^∞, is essentially self-adjoint and its closure D has domain equal to $\{f \in L^2 \mid PQf \in L^2\}$. The unitary group generated by D is the group of dilations on L^2, that is $(\mathrm{e}^{\mathrm{i}\tau D}f)(x) = \mathrm{e}^{\tau/2} f(e^\tau x)$.

We recall the simplest version of the Hardy estimate.

Proposition 2.1. *For any $f \in H_0^1$,*

$$\|P_{\min} f\| \ge \frac{1}{2}\|Q^{-1} f\|.$$

Hence, if $f \in H^1$ then $\tilde{Q}^{-1} f \in L^2$ if and only if $f \in H_0^1$.

Proof. For any $a \in \mathbb{R}$, as a quadratic form on C_c^∞ we have

$$0 \le (\tilde{P} + \mathrm{i}a\tilde{Q}^{-1})^*(\tilde{P} + \mathrm{i}a\tilde{Q}^{-1}) = \tilde{P}^2 + \mathrm{i}a[\tilde{P}, \tilde{Q}^{-1}] + a^2\tilde{Q}^{-2} = \tilde{P}^2 + a(a-1)\tilde{Q}^{-2}.$$

Since $a(a-1)$ attains its minimum for $a = \frac{1}{2}$, one gets $\|\tilde{P}f\| \ge \frac{1}{2}\|\tilde{Q}^{-1}f\|$ for $f \in C_\mathrm{c}^\infty$. By the dominated convergence theorem and Fatou lemma this inequality will remain true for any $f \in H_0^1$. □

2.2. **Holomorphic families of closed operators.** In this subsection we recall the definition of a holomorphic family of closed operators. We refer the reader to [2, Ch. 7] for details.

The definition (or actually a number of equivalent definitions) of a *holomorphic family of bounded operators* is quite obvious and does not need to be recalled. In the case of unbounded operators the situation is more subtle.

Suppose that Θ is an open subset of $\mathbb{C}$, $\mathcal{H}$ is a Banach space, and $\Theta \ni z \mapsto H(z)$ is a function whose values are closed operators on $\mathcal{H}$. We say that this is a *holomorphic family of closed operators* if for each $z_0 \in \Theta$ there exists a neighborhood Θ_0 of z_0, a Banach space $\mathcal{K}$ and a holomorphic family of bounded injective operators $\Theta_0 \ni z \mapsto A(z) \in B(\mathcal{K}, \mathcal{H})$ such that $\mathrm{Ran}\, A(z) = \mathcal{D}(H(z))$ and

$$\Theta_0 \ni z \mapsto H(z)A(z) \in B(\mathcal{K}, \mathcal{H})$$

is a holomorphic family of bounded operators.

We have the following practical criterion:

Theorem 2.2. *Suppose that $\{H(z)\}_{z\in\Theta}$ is a function whose values are closed operators on $\mathcal{H}$. Suppose in addition that for any $z \in \Theta$ the resolvent set of $H(z)$ is nonempty. Then $z \mapsto H(z)$ is a holomorphic family of closed operators if and only if for any $z_0 \in \Theta$ there exists $\lambda \in \mathbb{C}$ and a neighborhood Θ_0 of z_0 such that $\lambda \in \mathrm{rs}(H(z))$ for $z \in \Theta_0$ and $z \mapsto (H(z)-\lambda)^{-1} \in B(\mathcal{H})$ is holomorphic on Θ_0.*

The above theorem indicates that it is more difficult to study holomorphic families of closed operators that for some values of the complex parameter have an empty resolvent set.

To prove the analyticity of the resolvent, the following elementary result is also useful

Proposition 2.3. *Assume $\lambda \in \mathrm{rs}(H(z))$ for $z \in \Theta_0$. If there exists a dense set of vectors $\mathcal{D}$ such that $z \mapsto \langle f, (H(z)-\lambda)^{-1}g\rangle$ is holomorphic on Θ_0 for any $f, g \in \mathcal{D}$ and if $z \mapsto (H(z)-\lambda)^{-1} \in B(\mathcal{H})$ is locally bounded on Θ_0, then it is holomorphic on Θ_0.*

2.3. **Homogeneous operators.** Some of the properties of homogeneous Schrödinger operators follow by general arguments that do not depend on their precise structure. In this and the next subsections we collect such arguments. These two subsections can be skipped, since all the results that are given here will be proven independently.

Let U_τ be a strongly continuous one-parameter group of unitary operators on a Hilbert space $\mathcal{H}$. Let S be an operator on $\mathcal{H}$ and ν a non zero real number. We say that S is *homogeneous (of degree ν)* if $U_\tau S U_\tau^{-1} = \mathrm{e}^{\nu\tau} S$ for all real τ. More explicitly this means $U_\tau \mathcal{D}(S) \subset \mathcal{D}(S)$ and $U_\tau S U_\tau^{-1} f = \mathrm{e}^{\nu\tau} S f$ for all $f \in \mathcal{D}(S)$ and all τ. In particular, we get $U_\tau \mathcal{D}(S) = \mathcal{D}(S)$.

We are really interested only in the case $\mathcal{H} = L^2$ and $U_\tau = \mathrm{e}^{\mathrm{i}\tau D}$ the dilation group but it is convenient to state some general facts in an abstract setting. Then, since we assumed $\nu \neq 0$, there is no loss of generality if we consider only the case $\nu = 1$ (the general case is reduced to this one by working with the group $U_{\tau/\nu}$).

Let T be a homogeneous operator. If T is closable and densely defined then T^* is homogeneous too. If $S \subset T$ then S is homogeneous if and only if its domain is stable under the operators U_τ.

Let S be a homogeneous closed hermitian (densely defined) operator. We are interested in finding homogeneous self-adjoint extensions H of S. Since a self-adjoint extension satisfies $S \subset H \subset S^*$ we see that we need to find subspaces $\mathcal{E}$ with $\mathcal{D}(S) \subset \mathcal{E} \subset \mathcal{D}(S^*)$ such that $U_\tau \mathcal{E} \subset \mathcal{E}$ for all τ. Such subspaces will be called *homogeneous*.

Set $\langle S^* f, g\rangle - \langle f, S^* g\rangle =: \mathrm{i}\{f, g\}$. Then $\{f, g\}$ is a hermitian continuous sesquilinear form on $\mathcal{D}(S^*)$ which is zero on $\mathcal{D}(S)$ and a closed subspace $\mathcal{D}(S) \subset \mathcal{E} \subset \mathcal{D}(S^*)$ is the domain of a closed hermitian extension of S if and only if $\{f, g\} = 0$ for $f, g \in \mathcal{E}$. Such subspaces will be called *hermitian*. Note the following obvious fact: for $f \in \mathcal{D}(S^*)$ we have $\{f, g\} = 0$ for any $g \in \mathcal{D}(S^*)$ if and only if $f \in \mathcal{D}(S)$.

If T is a homogeneous operator and $\lambda \in \mathbb{C}$ is an eigenvalue of T, then $\mathrm{e}^\tau \lambda$ is also an eigenvalue of T for any real τ. In particular, a homogeneous self-adjoint operator cannot have non-zero eigenvalues and its spectrum is $\mathbb{R}$, or $\mathbb{R}_+$, or $-\mathbb{R}_+$, or $\{0\}$. (Note that, since U_τ is a strongly continuous one-parameter group, the least closed subspace which contains an eigenvector and is stable under all the U_τ and all functions of the operator is separable).

The following result, due to von Neumann, is easy to prove:

Proposition 2.4. *Let S be a positive hermitian operator with deficiency indices (n, n) for some finite $n \geq 1$. Then for each $\lambda < 0$ there is a unique self-adjoint extension T_λ of S such that λ is an eigenvalue of multiplicity n of T_λ. Moreover, the negative spectrum of T_λ is equal to $\{\lambda\}$. In particular, if S is homogeneous, then T_λ is not homogeneous, so S has non-homogeneous self-adjoint extensions.*

Proof. It suffices to take $\mathcal{D}(T_\lambda) = \mathcal{D}(S) + \mathrm{Ker}(S^* - \lambda)$. □

Recall that the Friedrichs and Krein extensions of a positive hermitian operator S are positive self-adjoint extensions F and K of S uniquely defined by the following property: any positive self-adjoint extension H of S satisfies $K \leq H \leq F$ (in the sense of quadratic forms). Then a self-adjoint operator H is a positive self-adjoint extension of S if and only if $K \leq H \leq F$.

Proposition 2.5. *If S is as in Proposition 2.4 and if the Friedrichs and Krein extensions of S coincide, then any other self-adjoint extension of S has a strictly negative eigenvalue.*

Proof. Indeed, such an extension will not be positive and its strictly negative spectrum consists of eigenvalues of finite multiplicity. □

It is clear that any homogeneous positive hermitian operator has homogeneous self-adjoint extensions.

Proposition 2.6. *If S is a homogeneous positive hermitian operator then the Friedrichs and Krein extensions of S are homogeneous.*

Proof. For any T we set $T_\tau = \mathrm{e}^{-\tau} U_\tau T U_\tau^{-1}$. Thus homogeneity means $T_\tau = T$. Then from $S \subset T \subset S^*$ we get $S \subset T_\tau \subset S^*$. Clearly, F_τ is a self-adjoint operator and is a positive extension of S, hence $F_\tau \leq F$. Then we also have $F_{-\tau} \leq F$ or $\mathrm{e}^{\tau} U_{-\tau} F U_{-\tau}^{-1} \leq F$ hence $F \leq F_\tau$, i.e. $F = F_\tau$. Similarly $K = K_\tau$. □

2.4. **Scattering theory for homogeneous operators.** In this subsection we continue with the abstract framework of Subsection 2.3.

We shall consider couples of self-adjoint operators (A, H) such that H is homogeneous with respect to the unitary group $U_\tau = \mathrm{e}^{\mathrm{i}\tau A}$ generated by A, i.e. $U_\tau H U_\tau^{-1} = \mathrm{e}^{\tau} H$ for all real τ. We the say that H *is a homogeneous Hamiltonian* (with respect to A). This can be *formally* written as $[\mathrm{i}A, H] = H$. It is clear that H is homogeneous if and only if $U_\tau \varphi(H) U_\tau^{-1} = \varphi(\mathrm{e}^{\tau} H)$ holds for all real τ and all bounded Borel functions $\varphi : \sigma(H) \to \mathbb{C}$. Also, it suffices that this be satisfied for only one function φ which generates the algebra of bounded Borel functions on the spectrum of H, for example for just one continuous injective function. If we set $V_\sigma = \mathrm{e}^{\mathrm{i}\sigma H}$ then another way of writing the homogeneity condition is $U_\tau V_\sigma = V_{\mathrm{e}^{\tau}\sigma} U_\tau$ for all real τ, σ.

We shall call (A, H) a *homogeneous Hamiltonian couple*. We say that this couple is *irreducible* if there are no nontrivial closed subspaces of $\mathcal{H}$ invariant under A and H, or if the von Neumann algebra generated by A and H is $B(\mathcal{H})$. A direct sum (in a natural sense) of homogeneous couples is clearly a homogeneous couple. Below $H > 0$ means that H is positive and injective and similarly for $H < 0$.

Proposition 2.7. *A homogeneous Hamiltonian couple (A, H) is unitarily equivalent to a direct sum of copies of homogeneous couples of the form (P, e^{Q}) or $(P, -\mathrm{e}^{Q})$ or $(A_0, 0)$ with A_0 an arbitrary self-adjoint operator. If $H > 0$ then only couples of the first form appear in the direct sum. A homogeneous Hamiltonian couple is irreducible if and only if it is unitarily equivalent to one of the couples (P, e^{Q}) or $(P, -\mathrm{e}^{Q})$ on $L^2(\mathbb{R})$, or to some $(A_0, 0)$ with A_0 a real number considered as operator on the Hilbert space $\mathbb{C}$. A homogeneous couple is irreducible if and only if one of the operators A or H has simple spectrum (i.e. the von Neumann algebra generated by it is maximal abelian), and in this case both operators have simple spectrum.*

Proof. By taking above φ equal to the characteristic function of the set $\mathbb{R}_+$, then $-\mathbb{R}_+$, and finally $\{0\}$, we see that the closed subspaces $\mathcal{H}_+, \mathcal{H}_-, \mathcal{H}_0$ defined by $H > 0, H < 0, H = 0$ respectively are stable under U_τ. So we have a direct sum decomposition $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_- \oplus \mathcal{H}_0$ which is left invariant by A and H. Hence $A = A_+ \oplus A_- \oplus A_0$ and similarly for H, the operator H_+ being homogeneous with respect to A_+, and so on. Since $H_0 = 0$ the operator A_0 can be arbitrary. The reduction to $\mathcal{H}_-$ is similar to the reduction to $\mathcal{H}_+$, it suffices to replace H_- by $-H_-$.

Thus in order to understand the structure of an arbitrary homogeneous Hamiltonian H it suffices to consider the case when $H > 0$. If we set $S = \ln H$ then by taking $\varphi = \ln$ above we get $U_\tau S U_\tau^{-1} = \tau + S$ for all real τ. Hence the couple (A, S) satisfies the canonical commutation relations, and so we may us the Stone-von

Neumann theorem: $\mathcal{H}$ is a direct sum of subspaces invariant under A and S and the restriction of this couple to each subspace is unitarily equivalent to the couple (P, Q) acting in $L^2(\mathbb{R})$. Since $H = \mathrm{e}^S$ we see that the restriction of (A, H) is unitarily equivalent to the couple (P, e^Q) acting in $L^2(\mathbb{R})$. □

Remark 2.8. Thus an irreducible homogeneous couple with $H > 0$ is unitarily equivalent to the couple (P, e^Q) on $\mathcal{H} = L^2(\mathbb{R})$. A change of variables gives also the unitary equivalence with the couple (D, Q) acting in $L^2(\mathbb{R}_+)$, where $D = (PQ + QP)/2$.

In the next proposition *we fix a self-adjoint operator A with simple spectrum on a Hilbert space $\mathcal{H}$ and assume that there is a homogeneous operator H with $H > 0$.* Then the spectrum of A is purely absolutely continuous and equal to the whole real line by the preceding results. Moreover, the spectrum of H is simple, purely absolutely continuous and equal to $\mathbb{R}_+$. Homogeneity refers always to A.

Proposition 2.9. *Assume that H_1, H_2 are homogeneous hamiltonians such that $H_k > 0$. Then there is a Borel function $\theta : \mathbb{R} \to \mathbb{C}$ with $|\theta(x)| = 1$ for all x such that $H_2 = \theta(A)H_1\theta(A)^{-1}$. If θ' is a second function with the same properties, then there is $\lambda \in \mathbb{C}$ such that $|\lambda| = 1$ and $\theta'(x) = \lambda\theta(x)$ almost everywhere. If the wave operator $\Omega_+ = \mathrm{s-}\lim_{t\to+\infty} \mathrm{e}^{\mathrm{i}tH_2}\mathrm{e}^{-\mathrm{i}tH_1}$ exists, then there is a function θ as above such that $\Omega_+ = \theta(A)$ and this function is uniquely determined almost everywhere. If the wave operator $\Omega_- = \mathrm{s-}\lim_{t\to-\infty} \mathrm{e}^{\mathrm{i}tH_2}\mathrm{e}^{-\mathrm{i}tH_1}$ also exists then there is a uniquely determined complex number ξ such that $\xi\Omega_- = \Omega_+$. In particular, the scattering matrix given by $S = \Omega_-^*\Omega_+ = \xi$ is independent of the energy.*

Proof. As explained above the couples (A, H_1) and (A, H_2) are unitarily equivalent, hence there is a unitary operator V on $\mathcal{H}$ such that $VAV^{-1} = A$ and $VH_1V^{-1} = H_2$. The spectrum of A is simple and V commutes with A so there is a function θ as in the statement of the proposition such that $V = \theta(A)$. If W is another unitary operator with the same properties as V then WV^{-1} commutes with A and H_2. From the irreducibility of (A, H_2) it follows that WV^{-1} is a complex number of modulus one. Uniqueness almost everywhere is a consequence of the fact that the spectrum of A is purely absolutely continuous and equal to $\mathbb{R}$.

Assume that Ω_+ exists. If we denote $\sigma = \mathrm{e}^{-\tau}$ then

$$\mathrm{e}^{\mathrm{i}tH_2}\mathrm{e}^{-\mathrm{i}tH_1}U_\tau = \mathrm{e}^{\mathrm{i}tH_2}U_\tau\mathrm{e}^{-\mathrm{i}\sigma tH_1} = U_\tau\mathrm{e}^{\mathrm{i}\sigma tH_2}\mathrm{e}^{-\mathrm{i}\sigma tH_1},$$

hence $\Omega_+U_\tau = U_\tau\Omega_+$ for all real τ. So the isometric operator Ω_+ belongs to the commutant $\{A\}'$, but $\{A\}''$ is a maximal abelian algebra by hypothesis, so equal to $\{A\}'$. Hence Ω_+ must be a function $\theta(A)$ of A, in particular it must be a normal operator, hence unitary. Now we repeat the arguments above. Since the spectrum of A is equal to $\mathbb{R}$ and is purely absolutely continuous, we see that $|\theta(x)| = 1$ and is uniquely determined almost everywhere. Similarly, if Ω_- exists, then it is a unitary operator in $\{A\}''$. Thus $S = \Omega_-^*\Omega_+$ is a unitary operator in $\{A\}''$, but also has the property $H_1S = SH_1$. Since the couple (A, H_1) is irreducible, we see that S must be a number. □

3. Homogeneous first order operators

In this section we prove some technical results on homogeneous first order differential operators which, besides their own interest, will be needed later on.

For each complex number α, let $\widetilde{A}_\alpha$ be the differential expression

$$\widetilde{A}_\alpha := \tilde{P} + \mathrm{i}\alpha\tilde{Q}^{-1} = -\mathrm{i}\partial_x + \mathrm{i}\frac{\alpha}{x} = -\mathrm{i}x^\alpha\partial_x x^{-\alpha}, \tag{3.1}$$

acting on distributions on $\mathbb{R}_+$, where $x^\alpha := \mathrm{e}^{\alpha\log x}$ with $\log x \in \mathbb{R}$. Its restriction to C_c^∞ is a closable operator in L^2 whose closure will be denoted $A_\alpha^{\min}$. This is the *minimal operator* associated to $\widetilde{A}_\alpha$. The *maximal operator* $A_\alpha^{\max}$ associated to $\widetilde{A}_\alpha$ is defined as the restriction of $\widetilde{A}_\alpha$ to $\mathcal{D}(A_\alpha^{\max}) := \{f \in L^2 \mid \widetilde{A}_\alpha f \in L^2\}$.

The following properties of the operators $A_\alpha^{\min}$ and $A_\alpha^{\max}$ are easy to check:

(i) $A_\alpha^{\min} \subset A_\alpha^{\max}$,
(ii) $(A_\alpha^{\min})^* = A_{-\overline{\alpha}}^{\max}$ and $(A_\alpha^{\max})^* = A_{-\overline{\alpha}}^{\min}$,
(iii) $A_\alpha^{\min}$ and $A_\alpha^{\max}$ are homogeneous of degree -1.

A more detailed description of the domains of the operators $A_\alpha^{\min}$ and $A_\alpha^{\max}$ is the subject of the next proposition. We fix $\xi \in C_{\mathrm{c}}^\infty([0,+\infty[)$ such that $\xi(x) = 1$ for $x \le 1$ and $\xi(x) = 0$ for $x \ge 2$ and set $\xi_\alpha(x) = x^\alpha \xi(x)$.

Proposition 3.1. *(i) We have $A_\alpha^{\min} = A_\alpha^{\max}$ if and only if $|\mathrm{Re}\,\alpha| \ge 1/2$.*
(ii) If $\mathrm{Re}\,\alpha \ne 1/2$, then $\mathcal{D}(A_\alpha^{\min}) = H_0^1$.
(iii) If $\mathrm{Re}\,\alpha = 1/2$, then $H_0^1 \subsetneq H_0^1 + \mathbb{C}\xi_\alpha \subsetneq \mathcal{D}(A_\alpha^{\min})$ and H_0^1 is a core for $A_\alpha^{\min} = A_\alpha^{\max}$.
(iv) If $|\mathrm{Re}\,\alpha| < 1/2$, then $\mathcal{D}(A_\alpha^{\max}) = H_0^1 + \mathbb{C}\xi_\alpha$. In particular, if $|\mathrm{Re}\,\alpha| < 1/2$ and $|\mathrm{Re}\,\beta| < 1/2$ then $\mathcal{D}(A_\alpha^{\max}) \ne \mathcal{D}(A_\beta^{\max})$.

To prove these facts we first need to discuss the resolvent families. Let $\mathbb{C}_\pm = \{\lambda \in \mathbb{C} \mid \pm\mathrm{Im}\,\lambda > 0\}$. The holomorphy of families of unbounded operators is discussed in Subsect. 2.2.

Proposition 3.2. *(1) Let $\mathrm{Re}\,\alpha > -1/2$. Then*

(i) $\mathrm{rs}(A_\alpha^{\max}) = \mathbb{C}_-$.
(ii) If $\mathrm{Im}\,\lambda < 0$, then the resolvent $(A_\alpha^{\max} - \lambda)^{-1}$ is an integral operator with kernel

$$(A_\alpha^{\max} - \lambda)^{-1}(x,y) = -\mathrm{i}\mathrm{e}^{\mathrm{i}\lambda(x-y)} \left(\frac{x}{y}\right)^\alpha 1\!\!1_+(y-x). \tag{3.2}$$

(iii) The map $\alpha \mapsto A_\alpha^{\max}$ is holomorphic in the region $\mathrm{Re}\,\alpha > -1/2$.
(iv) Each complex λ with $\mathrm{Im}\,\lambda > 0$ is a simple eigenvalue of $A_\alpha^{\max}$ with $x^\alpha \mathrm{e}^{\mathrm{i}\lambda x}$ as associated eigenfunction.

(2) Let $\mathrm{Re}\,\alpha < 1/2$. Then

(i) $\mathrm{rs}(A_\alpha^{\min}) = \mathbb{C}_+$.
(ii) If $\mathrm{Im}\,\lambda > 0$ then the resolvent $(A_\alpha^{\min} - \lambda)^{-1}$ is an integral operator with kernel

$$(A_\alpha^{\min} - \lambda)^{-1}(x,y) = \mathrm{i}\mathrm{e}^{\mathrm{i}\lambda(x-y)} \left(\frac{x}{y}\right)^\alpha 1\!\!1_+(x-y). \tag{3.3}$$

(iii) The map $\alpha \mapsto A_\alpha^{\min}$ is holomorphic in the region $\mathrm{Re}\,\alpha < 1/2$.
(iv) The operator $A_\alpha^{\min}$ has no eigenvalues.

In some cases $A_\alpha^{\min}$ and $A_\alpha^{\max}$ are generators of semigroups. We define the generator of a semigroup $\{T_t\}_{t\ge 0}$ such that formally $T_t = \mathrm{e}^{\mathrm{i}tA}$. Note that in (3.5) the function f is extended to $\mathbb{R}$ by the rule $f(y) = 0$ if $y \le 0$.

Proposition 3.3. *If $\mathrm{Re}\,\alpha \ge 0$, then $\mathrm{i}A_\alpha^{\max}$ is the generator of a C^0-semigroup of contractions*

$$(\mathrm{e}^{\mathrm{i}tA_\alpha^{\max}} f)(x) = x^\alpha (x+t)^{-\alpha} f(x+t), \quad t \ge 0, \tag{3.4}$$

whereas if $\mathrm{Re}\,\alpha \le 0$, the operator $-\mathrm{i}A_\alpha^{\min}$ is the generator of a C^0-semigroup of contractions

$$(\mathrm{e}^{-\mathrm{i}tA_\alpha^{\min}} f)(x) = x^\alpha (x-t)^{-\alpha} f(x-t), \quad t \ge 0. \tag{3.5}$$

The operators $\mathrm{i}A_\alpha^{\max}$ for $-1/2 < \mathrm{Re}\,\alpha < 0$ and $-\mathrm{i}A_\alpha^{\min}$ for $0 < \mathrm{Re}\,\alpha < 1/2$ are not generators of C^0-semigroups of bounded operators.

The remaining part of this section is devoted to the proof of these three propositions. We begin with a preliminary fact.

Lemma 3.4. *If R and S are closed operators such that $0 \in \mathrm{rs}(R)$, then the operator RS defined on the domain $\mathcal{D}(RS) := \{f \in \mathcal{D}(S) \mid Sf \in \mathcal{D}(R)\}$ is closed.*

Proof. Let $u_n \in \mathcal{D}(RS)$ such that $u_n \to u$ and $RSu_n \to v$. Then $u_n \in \mathcal{D}(S)$ and $Su_n \in \mathcal{D}(R)$, so that $Su_n = R^{-1}RSu_n \to R^{-1}v$, because R^{-1} is continuous. Since S is closed, we thus get that $u \in \mathcal{D}(S)$ and $Su = R^{-1}v$. Hence $Su \in \mathcal{D}(R)$, i.e. $u \in \mathcal{D}(RS)$, and $v = RSu$. $\square$

Note that the Hardy estimate (Proposition 2.1) gives $\|\widetilde{A}_\alpha f\| \le (1+2|\alpha|)\|Pf\|$ for all $f \in H_0^1$. Since C_c^∞ is dense in H_0^1, we get $H_0^1 \subset \mathcal{D}(A_\alpha^{\min})$ for any α. Our next purpose is to show that $\mathcal{D}(A_\alpha^{\min}) = H_0^1$ if $\operatorname{Re}\alpha \neq 1/2$, which is part (ii) of Proposition 3.1.

Lemma 3.5. *If* $\operatorname{Re}\alpha \neq 1/2$, *then* $\mathcal{D}(A_\alpha^{\min}) = H_0^1$.

Proof. We set $\beta = \mathrm{i}(1/2-\alpha)$ and observe that it suffices to prove that the restriction of $\widetilde{A}_\alpha$ to H_0^1 is a closed operator in L^2 if $\operatorname{Im}\beta \neq 0$. For this we shall use Lemma 3.4 with $R = D - \beta$ and S equal to the self-adjoint operator associated to Q^{-1} in L^2. Then it suffices to show that $\widetilde{A}_\alpha|_{H_0^1} = RS$.

The equality $\widetilde{A}_\alpha = (\widetilde{D} - \beta)Q^{-1}$, where $\widetilde{D} = (PQ+QP)/2$ is the extension to distributions of D, holds on the space of all distributions on $\mathbb{R}_+$, so we only have to check that the domain of the product RS is equal to H_0^1 (because β is not in the spectrum of the self-adjoint operator D). As discussed before, if $f \in H_0^1$ then $Q^{-1}f \in L^2$, so $f \in \mathcal{D}(S)$, and $PQQ^{-1}f = Pf \in L^2$, so $Sf \in \mathcal{D}(D)$. Thus $H_0^1 \subset \mathcal{D}(RS)$. Reciprocally, if $f \in \mathcal{D}(RS)$ then $f \in L^2$, $Q^{-1}f \in L^2$, and $\widetilde{D}Q^{-1}f \in L^2$. But $\widetilde{D}Q^{-1}f \in L^2$ is equivalent to $Pf \in L^2$, so $f \in H^1$. Since $Q^{-1}f \in L^2$ we get $f \in H_0^1$. $\square$

Our next step is the proof of part (1) of Proposition 3.2. Assume $\operatorname{Re}\alpha > -\frac{1}{2}$. The last assertion of part (1) of Proposition 3.2 is obvious, so $\operatorname{sp}(A_\alpha^{\max})$ contains the closure of the upper half plane. We now show that if $\operatorname{Im}\lambda < 0$ then $\lambda \in \operatorname{rs}(A_\alpha^{\max})$ and the resolvent $(A_\alpha^{\max} - \lambda)^{-1}$ is an integral operator with kernel as in (3.2).

The differential equation $(A_\alpha - \lambda)f = g$ is equivalent to $\frac{\mathrm{d}}{\mathrm{d}x}(x^{-\alpha}\mathrm{e}^{-\mathrm{i}\lambda x}f(x)) = \mathrm{i}x^{-\alpha}\mathrm{e}^{-\mathrm{i}\lambda x}g(x)$. Assume $g \in L^2(0,\infty)$. We look for a solution $f \in L^2(0,\infty)$ of the previous equation. Since $\operatorname{Im}(\lambda) < 0$, the function $x^{-\alpha}\mathrm{e}^{-\mathrm{i}\lambda x}g(x)$ is square integrable at infinity. We thus can define an operator $R_\alpha^{\max}$ on L^2 by

$$(R_\alpha^{\max}g)(x) = -\mathrm{i}\int_x^\infty \left(\frac{x}{y}\right)^\alpha \mathrm{e}^{\mathrm{i}\lambda(x-y)}g(y)\mathrm{d}y,$$

i.e. $R_\alpha^{\max}$ is the integral operator with kernel given by (3.2).

Lemma 3.6. $R_\alpha^{\max}$ *is a bounded operator in* L^2.

Proof. For shortness, we write R for $R_\alpha^{\max}$. In the sequel we denote $\lambda = \mu + \mathrm{i}\nu$ and $a = \operatorname{Re}\alpha$. By our assumptions, we have $\nu < 0$ and $a > -1/2$. If $a \ge 0$ then the proof of the lemma is particularly easy, because

$$\int_0^\infty |R(x,y)|\mathrm{d}y = x^a\mathrm{e}^{-\nu x}\int_x^\infty y^{-a}\mathrm{e}^{\nu y}\mathrm{d}y \le \mathrm{e}^{-\nu x}\int_x^\infty \mathrm{e}^{\nu y}\mathrm{d}y = -\nu^{-1},$$

and similarly $\int_0^\infty |R(x,y)|\mathrm{d}x \le -\nu^{-1}$. Then the boundedness of R follows from the Schur criterion. To treat the case $-1/2 < a < 0$ we split the integral operator R in two parts R_0 and R_1 with kernels

$$R_0(x,y) = 1\!\!1_{]0,1[}(x)R(x,y), \quad R_1(x,y) = 1\!\!1_{[1,\infty[}(x)R(x,y).$$

We shall prove that R_1 is bounded and R_0 is Hilbert-Schmidt. For R_1 we use again the Schur criterion. If $x < 1$, then $\int_0^\infty |R_1(x,y)|\mathrm{d}y = 0$ while if $x \ge 1$ then

$$\int_0^\infty |R_1(x,y)|\mathrm{d}y = x^a\mathrm{e}^{-\nu x}\int_x^\infty y^{-a}\mathrm{e}^{\nu y}\mathrm{d}y.$$

We then integrate by parts twice to get

$$\int_0^\infty |R_1(x,y)|\mathrm{d}y = -\nu^{-1} - \frac{a}{\nu^2 x} + \frac{a(a+1)}{\nu^2}x^a\mathrm{e}^{-\nu x}\int_x^\infty \mathrm{e}^{\nu y}y^{-a-2}\mathrm{d}y. \tag{3.6}$$

Then, using $a > -1/2$, we estimate

$$x^a \mathrm{e}^{-\nu x} \int_x^\infty \mathrm{e}^{\nu y} y^{-a-2} \mathrm{d}y \le x^a \int_x^\infty y^{-a-2} \mathrm{d}y = \frac{1}{(a+1)x},$$

which, together with (3.6), proves that $\sup_{x \ge 1} \int_0^\infty |R_1(x,y)| \mathrm{d}y < +\infty$. Similarly $\int_0^\infty |R_1(x,y)| \mathrm{d}x = 0$ if $y < 1$, and for $y \ge 1$

$$\int_0^\infty |R_1(x,y)| \mathrm{d}x = y^{-a} \mathrm{e}^{\nu y} \int_1^y x^a \mathrm{e}^{-\nu x} \mathrm{d}y$$

is estimated similarly. We now prove that the operator R_0 is Hilbert-Schmidt. We have

$$\int_0^\infty \mathrm{d}x \int_0^\infty \mathrm{d}y |R_0(x,y)|^2 = \int_0^1 \mathrm{d}x\, x^{2a} \mathrm{e}^{-2\nu x} \int_x^\infty \mathrm{d}y\, y^{-2a} \mathrm{e}^{2\nu y}.$$

Since a and ν are strictly negative, the integral $\int_0^\infty y^{-2a} \mathrm{e}^{2\nu y} \mathrm{d}y$ converges. Hence

$$\int_0^\infty \mathrm{d}x \int_0^\infty \mathrm{d}y |R_0(x,y)|^2 \le C \int_0^1 x^{2a} \mathrm{e}^{-2\nu x} \mathrm{d}x,$$

which is convergent because $a > -1/2$. □

So we proved that for $\mathrm{Im}\,(\lambda) < 0$ the operator R defines a bounded operator on L^2 such that $(\widetilde{A}_\alpha - \lambda)Rg = g$ for all $g \in L^2$. Hence, $R : L^2 \to \mathcal{D}(A_\alpha^{\max})$ and $(A_\alpha^{\max} - \lambda)R = 1\!\!1_{L^2}$.

Reciprocally, let $f \in \mathcal{D}(A_\alpha^{\max})$ and set $g := (A_\alpha^{\max} - \lambda)f \in L^2$. The preceeding argument shows that $(A_\alpha^{\max} - \lambda)(f - Rg) = 0$. But $A_\alpha^{\max} - \lambda$ is injective. Indeed, if $(A_\alpha^{\max} - \lambda)h = 0$, then there exists $C \in \mathbb{C}$ such that $h(x) = Cx^\alpha \mathrm{e}^{\mathrm{i}\lambda x}$ which is not in L^2 near infinity unless $C = 0$ (recall that $\mathrm{Im}\,\lambda < 0$).

We have therefore proven that each $\lambda \in \mathbb{C}_-$ belongs to the resolvent set of $A_\alpha^{\max}$ and that $(A_\alpha^{\max} - \lambda)^{-1} = R$. If we fix such a λ and look at $R = R(\alpha)$ as an operator valued function of α defined for $\mathrm{Re}\,\alpha > -1/2$, then from the preceding estimates on the kernel of R it follows that $\|R(\alpha)\|$ is a locally bounded function of α. On the other hand, it is clear that if $f, g \in C_\mathrm{c}^\infty$, then $\alpha \mapsto \langle f, R(\alpha)g\rangle$ is a holomorphic function. Thus, by Proposition 2.3, $\alpha \mapsto (A_\alpha^{\max} - \lambda)^{-1}$ is holomorphic on $\mathrm{Re}\,\alpha > -1/2$. This finishes the proof of point (1) of Proposition 3.2. The second part of the proposition follows from the first part by using the relation $A_\alpha^{\min} = (A_{-\overline{\alpha}}^{\max})^*$.

We now complete the proof of Proposition 3.1 and consider first the most difficult case when $\mathrm{Re}\,(\alpha) = 1/2$. The function ξ_α is of class C^∞ on $\mathbb{R}_+$, is equal to zero on $x > 2$, we have $\xi_\alpha \in L^2$, and $\widetilde{A}_\alpha \xi_\alpha = 0$ on $x < 1$. Hence $\xi_\alpha \in \mathcal{D}(A_\alpha^{\max})$. On the other hand $\xi_\alpha' \notin L^2$ (it is not square integrable at the origin) so $\xi_\alpha \notin H_0^1$.

Lemma 3.7. *Let* $\mathrm{Re}\,(\alpha) \ge 1/2$. *Then* $\xi_\alpha \in \mathcal{D}(A_\alpha^{\min})$.

Proof. The case $\mathrm{Re}\,\alpha > 1/2$ is obvious since $\xi_\alpha \in H_0^1$. Now for $\mathrm{Re}\,\alpha = 1/2$ we prove that ξ_α belongs to the closure of H_0^1 in $\mathcal{D}(A_\alpha^{\max})$ which is precisely $\mathcal{D}(A_\alpha^{\min})$. For $0 < \varepsilon < 1/2$ we define $\xi_{\alpha,\varepsilon}$ as

$$\xi_{\alpha,\varepsilon}(x) = \begin{cases} \frac{x}{\varepsilon} x^\alpha & \text{if} \quad x < \varepsilon, \\ \xi_\alpha(x) & \text{if} \quad x \ge \varepsilon. \end{cases}$$

For $x < \varepsilon$ one has $\xi_{\alpha,\varepsilon}'(x) = \frac{\alpha+1}{\varepsilon} x^\alpha$. Hence $\xi_{\alpha,\varepsilon}' \in L^2$ so that $\xi_{\alpha,\varepsilon} \in H_0^1$. Moreover $\|\xi_{\alpha,\varepsilon} - \xi_\alpha\|_{L^2} \to 0$ as $\varepsilon \to 0$. We then have

$$\widetilde{A}_\alpha \xi_{\alpha,\varepsilon}(x) = \begin{cases} -\frac{\mathrm{i}}{\varepsilon} x^\alpha & \text{if} \quad x < \varepsilon, \\ 0 & \text{if} \quad \varepsilon \le x < 1, \end{cases} \qquad \text{and} \quad \widetilde{A}_\alpha \xi_\alpha(x) = 0 \quad \text{if}\ x < 1,$$

while $\widetilde{A}_\alpha \xi_{\alpha,\varepsilon}(x) = \widetilde{A}_\alpha \xi_\alpha(x)$ if $x \ge 1$. Therefore

$$\|\widetilde{A}_\alpha \xi_{\alpha,\varepsilon}\|_{L^2}^2 = \int_0^\varepsilon \left| \frac{x^\alpha}{\varepsilon} \right|^2 \mathrm{d}x + \|\widetilde{A}_\alpha \xi_\alpha\|_{L^2}^2 = \frac{1}{2} + \|\widetilde{A}_\alpha \xi_\alpha\|_{L^2}^2.$$

Thus $\xi_{\alpha,\varepsilon} \to \xi_\alpha$ in L^2, $\xi_{\alpha,\varepsilon} \in H_0^1 \subset \mathcal{D}(A_\alpha^{\max})$, and there is $C > 0$ such that $\|\widetilde{A}_\alpha \xi_{\alpha,\varepsilon}\|_{L^2} \le C$ for any ε. Since $A_\alpha^{\max}$ is closed, this proves that ξ_α belongs to the closure of H_0^1 in $\mathcal{D}(A_\alpha^{\max})$, i.e. $\xi_\alpha \in \mathcal{D}(A_\alpha^{\min})$.

Lemma 3.8. *Let* $\mathrm{Re}\,(\alpha) \ge 1/2$. *Then* $\mathcal{D}(A_\alpha^{\min}) = \mathcal{D}(A_\alpha^{\max})$.

Fix $\lambda \in \mathbb{C}$ such that $\mathrm{Im}\,(\lambda) < 0$, e.g. $\lambda = -\mathrm{i}$, and let $R = (A_\alpha^{\max} + \mathrm{i})^{-1}$. R is continuous from L^2 onto $\mathcal{D}(A_\alpha^{\max})$, hence $R(C_\mathrm{c}^\infty)$ is dense in $\mathcal{D}(A_\alpha^{\max})$. Let now $g \in C_\mathrm{c}^\infty$ and $0 < c < d < \infty$ such that $\mathrm{supp}\, g \subset [c, d]$. Then for any $x < c$,

$$\begin{aligned} f(x) = (Rg)(x) &= -\mathrm{i}x^\alpha \mathrm{e}^x \int_c^d y^{-\alpha}\mathrm{e}^{-y} g(y)\mathrm{d}y \\ &\sim \; Cx^\alpha + Cx^\alpha(\mathrm{e}^x - 1) \;\sim\; Cx^\alpha + Dx^{\alpha+1} \end{aligned}$$

as $x \to 0$. Hence $f \in \mathbb{C}\xi_\alpha + H_0^1$. Therefore $R(C_c^\infty) \subset \mathbb{C}\xi_\alpha + H_0^1 \subset \mathcal{D}(A_\alpha^{\min})$. Since $R(C_c^\infty)$ is dense in $\mathcal{D}(A_\alpha^{\max})$, the same is true for $\mathcal{D}(A_\alpha^{\min})$. But $A_\alpha^{\min}$ is a closed operator, and so $\mathcal{D}(A_\alpha^{\min}) = \mathcal{D}(A_\alpha^{\max})$. □

Lemma 3.9. *If* $\mathrm{Re}\,\alpha = 1/2$, *then* $\mathbb{C}\xi_\alpha + H_0^1 \neq \mathcal{D}(A_\alpha^{\max})$.

Proof. Let R be as above and let $g(y) = y^{-\bar\alpha}|\ln(y)|^{-\gamma}\mathbb{1}_{]0,\frac12[}(y)$ where $\gamma > 1/2$. Then $g \in L^2$, hence $Rg \in \mathcal{D}(A_\alpha^{\max})$. On the other hand, for $x \le 1/2$ we have

$$Rg(x) = -\mathrm{i}x^\alpha\mathrm{e}^x \int_x^{\frac12} \frac{\mathrm{e}^{-y}}{y|\ln(y)|^\gamma}\mathrm{d}y \sim Cx^\alpha|\ln(x)|^{1-\gamma}$$

as $x \to 0$. In particular, if $\gamma < 1$, then $Rg \notin \mathbb{C}\xi_\alpha + H_0^1$. □

All the assertions related to the case $\mathrm{Re}\,\alpha = 1/2$ of Proposition 3.1 have been proved. Since

$$A_\alpha^{\min} = A_\alpha^{\max} \Longrightarrow A_{-\bar\alpha}^{\min} = A_{-\bar\alpha}^{\max} \tag{3.7}$$

holds for any α, we get $A_\alpha^{\min} = A_\alpha^{\max}$, and so $\mathcal{D}(A_\alpha^{\max}) = H_0^1$ if $\mathrm{Re}\,\alpha = -1/2$. We now turn to the case $|\mathrm{Re}\,(\alpha)| > 1/2$ and show $\mathcal{D}(A_\alpha^{\max}) = \mathcal{D}(A_\alpha^{\min}) = H_0^1$. Due to (3.7) it suffices to consider the case $\mathrm{Re}\,\alpha > 1/2$, which is precisely the statements of Lemmas 3.5 and 3.8. Now we prove (iv) of Proposition 3.1.

Lemma 3.10. *If* $|\mathrm{Re}\,\alpha| < 1/2$, *then* $\mathbb{C}\xi_\alpha + H_0^1 = \mathcal{D}(A_\alpha^{\max})$.

Proof. Clearly, $\xi_\alpha \notin H_0^1$. We easily show that $\xi_\alpha \in \mathcal{D}(A_\alpha^{\max})$.

Once again, let $R = (A_\alpha^{\max} + \mathrm{i})^{-1}$ and let $f \in \mathcal{D}(A_\alpha^{\max})$. There exists $g \in L^2$ such that $f = Rg$, or

$$f(x) = -\mathrm{i}x^\alpha\mathrm{e}^x \int_x^\infty \mathrm{e}^{-y}y^{-\alpha}g(y)\mathrm{d}y.$$

We show that $f \in \mathbb{C}\xi_\alpha + H_0^1$. Clearly, only the behaviour at the origin matters. For $x < 1$ decompose f as

$$f(x) = -\mathrm{i}x^\alpha\mathrm{e}^x \int_0^\infty \mathrm{e}^{-y}y^{-\alpha}g(y)\mathrm{d}y + \mathrm{i}x^\alpha\mathrm{e}^x \int_0^x \mathrm{e}^{-y}y^{-\alpha}g(y)\mathrm{d}y =: f_0(x) + f_1(x).$$

Note that the first integral makes sense because $|\mathrm{Re}\,(\alpha)| < 1/2$, so $\mathrm{e}^{-y}y^{-\alpha}$ is square integrable. Clearly

$$f_0(x) = Cx^\alpha\mathrm{e}^x = Cx^\alpha + Cx^\alpha(\mathrm{e}^x - 1) \in \mathbb{C}\xi_\alpha + H_0^1$$

near the origin. We then prove that $f_1 \in H_0^1$ near the origin. By construction, $(A_\alpha + \mathrm{i})f_1 = g \in L^2$, so if we prove that $Q^{-1}f_1$ is in L^2 near the origin, we will get $f_1 \in H^1$ near the origin, and hence $f_1 \in H_0^1$ near the origin.

For any $0 < x < 1$ we can estimate (with $a = \mathrm{Re}\,\alpha$ as before)

$$\frac{1}{x}|f_1(x)| = \frac{1}{x}\left|\int_0^x \mathrm{e}^{x-y}\left(\frac{x}{y}\right)^\alpha g(y)\mathrm{d}y\right| \le C\int_1^{+\infty} t^{a-2}|g(\frac{x}{t})|\mathrm{d}t. \tag{3.8}$$

For any $t \geq 1$ let τ_t be the map in L^2 defined by $(\tau_t g)(x) = g(x/t)$, and let $T = \int_1^\infty t^{a-2}\tau_t \mathrm{d}t$. We have $\|\tau_t\|_{L^2\to L^2} = \sqrt{t}$ hence T is a bounded operator on L^2 with $\|T\| \leq \int_1^\infty t^{a-3/2}\mathrm{d}t$ which converges since $a < 1/2$. Together with (3.8), this proves that $\frac{1}{x}f_1(x)$ is square integrable on $]0,1[$. This completes the proof of Proposition 3.1.

It remains to prove Proposition 3.3. Since this is just a computation, we shall only sketch the argument. Note that it suffices to consider the case of $A_\alpha^{\max}$, because then we get the result concerning $A_\alpha^{\min}$ by taking adjoints. Let us denote $A_0^{\max} = P_{\max}$, so $P_{\max}$ is the restriction to the Sobolev space H^1 of the operator P. It is well-known and easy to check that $P_{\max}$ is the generator of the contraction semigroup $(\mathrm{e}^{\mathrm{i}tP_{\max}}f)(x) = f(x+t)$ for $t \geq 0$ and $f \in L^2$. Now if we write (3.1) as $\widetilde{A}_\alpha = Q^\alpha P Q^{-\alpha}$, then (3.4) is formally obvious, because it is equivalent to

$$\mathrm{e}^{\mathrm{i}tA_\alpha^{\max}} = Q^\alpha \mathrm{e}^{\mathrm{i}tP_{\max}} Q^{-\alpha}.$$

For a rigorous justification, we note that the right hand side here or in (3.4) clearly defines a C_0-semigroup of contractions if (and only if) $\mathrm{Re}\,\alpha \geq 0$, and then a straightforward computation shows that its generator is $A_\alpha^{\max}$. One may note that $C_\mathrm{c}^\infty + \mathbb{C}\xi_\alpha$ is a core for $A_\alpha^{\max}$ for all such α.

4. Homogeneous second order operators

4.1. Formal operators. For an arbitrary complex number m we introduce the differential expression

$$\tilde{L}_m = \tilde{P}^2 + (m^2 - 1/4)\tilde{Q}^{-2} = -\partial_x^2 + \frac{m^2 - 1/4}{x^2} \tag{4.1}$$

acting on distributions on $\mathbb{R}_+$. Let $L_m^{\min}$ and $L_m^{\max}$ be the minimal and maximal operators associated to it in L^2 (see Appendix A). It is clear that they are homogeneous operators (of degree -2, we shall not specify this anymore). The operator $L_m^{\min}$ is hermitian if and only if m^2 is a real number, i.e. m is either real or purely imaginary, and then $(L_m^{\min})^* = L_m^{\max}$. In general we have

$$(L_m^{\min})^* = L_{\bar{m}}^{\max}.$$

Note that (4.1) does not make any difference between m and $-m$. We will however see that m, not m^2, is the natural parameter. In particular this will be clear in the construction of other L^2 realizations of L_m, i.e. operators H such that $L_m^{\min} \subset H \subset L_m^{\max}$.

Observe also that one can factorize $\tilde{L}_m$ as

$$\tilde{L}_m = \left(\tilde{P} + \mathrm{i}\frac{\bar{m} + \frac{1}{2}}{\tilde{Q}}\right)^* \left(\tilde{P} + \mathrm{i}\frac{m + \frac{1}{2}}{\tilde{Q}}\right) = \widetilde{A}^*_{\bar{m}+\frac{1}{2}} \widetilde{A}_{m+\frac{1}{2}} \tag{4.2}$$

where $\widetilde{A}^*_{\bar{m}+\frac{1}{2}}$ is the formal adjoint of the differential expression $\widetilde{A}_{\bar{m}+\frac{1}{2}}$. The above expression makes a priori a difference between m and $-m$, since $\tilde{L}_m$ does not depend on the sign of m, whereas the factorizations corresponding to m and $-m$ are different. These factorizations provide one of the methods to distinguish between the various homogeneous extensions of $L_m^{\min}$. However, as we have seen in the previous section, one has to be careful in the choice of the realization of $\widetilde{A}_{m+\frac{1}{2}}$.

4.2. Homogeneous holomorphic family. If m is a complex number we set

$$\zeta_m(x) = x^{1/2+m} \text{ if } m \neq 0 \quad \text{and} \quad \zeta_0(x) \equiv \zeta_{+0}(x) = \sqrt{x}, \quad \zeta_{-0}(x) = \sqrt{x}\ln x. \tag{4.3}$$

The notation is chosen in such a way that for any m the functions $\zeta_{\pm m}$ are linearly independent solutions of the equation $L_m u = 0$. Note that $\zeta_{\pm m}$ are both square integrable at the origin if and only if $|\mathrm{Re}\, m| < 1$.

We also choose $\xi \in C^\infty(\mathbb{R}_+)$ such that $\xi = 1$ on $[0,1]$ and 0 on $[2,\infty[$.

Definition 4.1. *For* $\mathrm{Re}\,(m) > -1$*, we define* H_m *to be the operator* $L_m^{\max}$ *restricted to* $\mathcal{D}(L_m^{\min}) + \mathbb{C}\xi\zeta_m$.

Clearly, H_m does not depend on the choice of ξ. Our first result concerning the family of operators H_m is its analyticity with respect to the parameter m.

Theorem 4.2. *$\{H_m\}_{\mathrm{Re}\, m>-1}$ is a holomorphic family of operators. More precisely, the number -1 belongs to the resolvent set of H_m for any such m and $m \mapsto (H_m+1)^{-1} \in \mathcal{B}(L^2)$ is a holomorpic map.*

Before we prove the above theorem, let us analyze the eigenvalue problem for $\widetilde{L}_m$. The latter is closely related to Bessel's equation. In the sequel, J_m will denote the Bessel functions of the first kind, i.e.

$$J_m(x) := \sum_{j=0}^{\infty} \frac{(-1)^j (x/2)^{2j+m}}{j!\Gamma(j+m+1)}, \tag{4.4}$$

and I_m and K_m the modified Bessel functions [6]

$$I_m(x) = \mathrm{i}^{-m} J_m(\mathrm{i}x), \qquad K_m(x) = \frac{\pi}{2} \frac{I_{-m}(x) - I_m(x)}{\sin(m\pi)}. \tag{4.5}$$

Lemma 4.3. *For any m such that $\mathrm{Re}\,(m) > -1$, the functions $\sqrt{x} I_m(x)$, $\sqrt{x} K_m(x)$ form a basis of solutions of the differential equation $-\partial_x^2 u + (m^2 - \frac{1}{4})\frac{1}{x^2} u = -u$ such that $\sqrt{x} I_m(x) \in L^2(]0,1[)$ and $\sqrt{x} K_m(x) \in L^2(]1,+\infty[)$. Besides, the Wronskian of these two solutions equals 1.*

Proof. If we introduce $w(x) = x^{-1/2} v(x)$, then v satisfies $\widetilde{L}_m v = -v$ iff w satisfies

$$x^2 w''(x) + x w(x) - (x^2 + m^2) w = 0,$$

which is modified Bessel's differential equation. Linearly independent solutions of this equation are (I_m, K_m). Therefore, a basis of solution for the equation $\widetilde{L}_m u = -u$ is $(\sqrt{x} I_m(x), \sqrt{x} K_m(x)) =: (u_0, u_\infty)$.

One has $I_m'(x) K_m(x) - I_m(x) K_m'(x) = -\frac{1}{x}$ (see [6]), and hence $W = u_0' u_\infty - u_0 u_\infty' = 1$. Moreover, $I_m(x) \sim \frac{1}{\Gamma(m+1)} \left(\frac{x}{2}\right)^m$ as $x \to 0$ [6]. Therefore, $u_0(x)$ is square integrable near the origin iff $\mathrm{Re}\,(m) > -1$. On the other hand, $K_m(x) \sim \sqrt{\frac{\pi}{2x}} \mathrm{e}^{-x}$ as $x \to \infty$, so that u_∞ is always square integrable near ∞. □

Note that $\sqrt{x} I_m(x)$ belongs to the domain of H_m for all $\mathrm{Re}\,(m) > -1$. Therefore, the candidate for the inverse of the operator $H_m + 1$ has kernel (cf. Proposition A.1)

$$G_m(x,y) = \begin{cases} \sqrt{xy} I_m(x) K_m(y) & \text{if} \quad x < y, \\ \sqrt{xy} I_m(y) K_m(x) & \text{if} \quad x > y. \end{cases}$$

We still need to prove that G_m is bounded, which will be proven in the next lemma.

Lemma 4.4. *The map $m \mapsto G_m$ is a holomorphic family of bounded operators and it does not have a holomorphic extension to a larger subset of the complex plane.*

Proof. We prove that G_m is locally bounded and that $m \mapsto \langle f, G_m g\rangle$ is analytic for f, g in a dense set of L^2, so that the result follows from Proposition 2.3.

The modified Bessel functions depend analytically in m. Therefore, the Green function $G_m(x,y)$ is an analytic function of the parameter m, and it is easy to see that for any $f, g \in C_\mathrm{c}^\infty(]0,+\infty[)$, the quantity $\langle f, (H_m+1)^{-1} g\rangle = \int \bar{f}(x) G_m(x,y) g(y) \mathrm{d}x \mathrm{d}y$ is analytic in m. Since $C_\mathrm{c}^\infty(]0,+\infty[)$ is dense in $L^2(0,+\infty)$, it remains to prove that $(H_m+1)^{-1}$ is locally bounded in m.

We shall split this resolvent as $G_m = G_m^{--} + G_m^{-+} + G_m^{+-} + G_m^{++}$, where $G_m^{\pm\pm}$ is the operator that has kernel $G_m^{\pm\pm}(x,y)$ with

$$\begin{aligned} G_m^{--}(x,y) &= G(x,y) \mathbb{1}_{]0,1]}(x) \mathbb{1}_{]0,1](y)}, \\ G_m^{-+}(x,y) &= G(x,y) \mathbb{1}_{]0,1]}(x) \mathbb{1}_{]1,\infty[(y)}, \\ G_m^{+-}(x,y) &= G(x,y) \mathbb{1}_{]1,\infty[}(x) \mathbb{1}_{]0,1](y)}, \\ G_m^{++}(x,y) &= G(x,y) \mathbb{1}_{]1,\infty[}(x) \mathbb{1}_{]1,\infty[(y)}. \end{aligned}$$

We control the norm of G_m^{++} using Schur's Theorem (see [7]), whereas for the other terms, we estimate the L^2 norm of the kernel. (This means in particular that G_m^{--}, G_m^{-+} and G_m^{+-} are actually Hilbert-Schmidt).

For that purpose, we use the explicit expression given in Lemma 4.3 together with the following estimates on the modified Bessel functions (see e.g. [6]):

- as $x \to 0$,

$$\begin{aligned} I_m(x) &\sim \frac{1}{\Gamma(m+1)}\left(\frac{x}{2}\right)^m, \qquad m \neq -1,-2,\dots, &(4.6)\\ K_m(x) &\sim \begin{cases} \mathrm{Re}\,\left(\Gamma(m)\left(\frac{2}{x}\right)^m\right) & \text{if} \quad \mathrm{Re}\, m = 0,\ m \neq 0,\\ -\ln\left(\frac{x}{2}\right) - \gamma & \text{if} \quad m = 0,\\ \frac{\Gamma(m)}{2}\left(\frac{2}{x}\right)^m & \text{if} \quad \mathrm{Re}\, m > 0,\\ \frac{\Gamma(-m)}{2}\left(\frac{x}{2}\right)^m & \text{if} \quad \mathrm{Re}\, m < 0. \end{cases} &(4.7)\end{aligned}$$

- as $x \to \infty$,

$$\begin{aligned} I_m(x) &\sim \frac{1}{\sqrt{2\pi x}}\mathrm{e}^x, &(4.8)\\ K_m(x) &\sim \sqrt{\frac{\pi}{2x}}\mathrm{e}^{-x}. &(4.9)\end{aligned}$$

The various constants which appear in (4.6)-(4.9) are locally bounded in m, except $\Gamma(m)$ as m goes to zero, so that we may estimate the $G_m^{\pm\pm}(x,y)$ by

$$\begin{aligned} |G_m^{--}(x,y)| &\leq C_m|\Gamma(m)|\Big(x^{1/2-|\nu|}y^{1/2+\nu}\mathbb{1}_{0<y<x<1}(x,y) &(4.10)\\ &\qquad\qquad + x^{1/2+\nu}y^{1/2-|\nu|}\mathbb{1}_{0<x<y<1}(x,y)\Big),\\ |G_m^{+-}(x,y)| &\leq C_m\mathrm{e}^{-x}y^{\nu+1/2}\mathbb{1}_{]1,\infty[}(x)\mathbb{1}_{]0,1](y)},\\ |G_m^{-+}(x,y)| &\leq C_m x^{\nu+1/2}\mathrm{e}^{-y}\mathbb{1}_{]0,1]}(x)\mathbb{1}_{]1,\infty[(y)},\\ |G_m^{++}(x,y)| &\leq C_m\mathrm{e}^{-|x-y|}\mathbb{1}_{]1,\infty[}(x)\mathbb{1}_{]1,\infty[(y)}, \end{aligned}$$

where $\nu = \mathrm{Re}\,(m)$ and C_m are constants which depend on m but are locally bounded in m. The only problem is when $m = 0$, where we shall replace (4.10) by

$$|G_0^{--}(x,y)| \leq C\left(y^{1/2}|\ln(x)|\mathbb{1}_{0<y<x<1}(x,y) + x^{1/2}|\ln(y)|\mathbb{1}_{0<x<y<1}(x,y)\right). \tag{4.11}$$

Note also that the constant appearing in (4.10) blows up as m goes to zero due to the factor $|\Gamma(m)|$.

Straightforward computations lead to the following bounds:

$$\begin{aligned} \|G_m^{--}\|_{L^2}^2 &\leq \frac{C_m|\Gamma(m)|}{(\nu+1)(4+2\nu-2|\nu|)}, \qquad m \neq 0,\\ \|G_m^{-+}\|_{L^2}^2 &\leq \frac{C_m}{4(1+\nu)},\\ \|G_m^{+-}\|_{L^2}^2 &\leq \frac{C_m}{4(1+\nu)},\\ \|G_m^{++}\|_{L_x^\infty(L_y^1)} &\leq 2C_m,\\ \|G_m^{++}\|_{L_y^\infty(L_x^1)} &\leq 2C_m. \end{aligned}$$

This proves that G_m^{--}, G_m^{-+} and G_m^{+-} are Hilbert-Schmidt operators whose norm is locally bounded in m (except maybe for G_m^{--} near 0). Using Schur's Theorem G_m^{++} is bounded with $\|G_m^{++}\| \leq 2C(m)$.

It remains to prove that G_m^{--} is locally bounded around 0. To this end we use $|K_m(z)| < C\frac{|x^m - x^{-m}|}{|m|}$ and estimate the Hilbert-Schmidt norm, where we set $\nu := \mathrm{Re}\, m$:

$$\begin{aligned}\int_{0<x<y<1} |G_m^{--}(x,y)|^2 \mathrm{d}x\mathrm{d}y &\leq \frac{C}{|m|^2}\int_{0<x<y<1} xy|x^m|^2|y^m - y^{-m}|^2 \\ &\leq \frac{C'}{|m|^2}\left(\frac{1}{4\nu+2} + \frac{1}{4} - \frac{2}{2\nu+4}\right) = \frac{C'}{4(\nu+1)(\nu+2)}.\end{aligned}$$

As a conclusion, G_m is locally bounded in m for all m such that $\mathrm{Re}\,(m) > -1$.

We finally prove that G_m does not extend to a holomorphic family of bounded operators beyond the axis $\mathrm{Re}\, m = -1$. Fix $g \in C_{\mathrm{c}}^\infty(]0,\infty[)$. The function $m \mapsto G_m g$ with values in $L^2_{\mathrm{loc}}(]0,\infty[)$ is entire analytic. If G_m could be extended to a holomorphic family of bounded operators, when applied to the function g this extension should coincide with $G_m g$. For x below the support of g we clearly have

$$(G_m g)(x) = \sqrt{x} I_m(x)\int_0^\infty \sqrt{y} K_m(y) g(y)\mathrm{d}y = C_m\sqrt{x} I_m(x)$$

which is not in L^2 if $\mathrm{Re}\, m \leq -1$. □

This proves that for $\mathrm{Re}\,(m) > -1$ the number -1 belongs to the resolvent set of H_m, we have $G_m = (H_m + 1)^{-1}$, and H_m is a holomorphic family of operators, cf. Proposition 2.3. This proves Theorem 4.2.

The next theorem gives more properties of the operators H_m. The main technical point is that the differences of the resolvents $R_{m'}(\lambda) - R_{m''}(\lambda)$ are compact operators, where we set $R_m(\lambda) = (H_m - \lambda)^{-1}$ for λ in the resolvent set of H_m. For the proof we need the following facts.

Lemma 4.5. *Let Ω be an open connected complex set, X a Banach space, Y a closed linear subspace of X, and $F : \Omega \to X$ a holomorphic map. If $F(z) \in Y$ for $z \in \omega$, where $\omega \subset \Omega$ has an accumulation point in Ω, then $F(z) \in Y$ for $z \in \Omega$.*

Proof. All the derivatives of F at an accumulation point of ω in Ω can be computed in terms of $F|_\omega$, hence belong to the closed subspace generated by the $F(z)$ with $z \in \omega$. □

Lemma 4.6. *Let S, T be two closed operators on a Banach space $\mathcal{H}$ and let $K(\lambda) = (S-\lambda)^{-1} - (T-\lambda)^{-1}$. If $K(\lambda)$ is compact for some $\lambda \in \mathrm{rs}(S) \cap \mathrm{rs}(T)$ then $K(\lambda)$ is compact for all $\lambda \in \mathrm{rs}(S) \cap \mathrm{rs}(T)$.*

Proof. We denote $S_\lambda = (S-\lambda)^{-1}$ and $S_{\lambda\mu} = (S-\lambda)(S-\mu)^{-1}$ and use similar notation when S is replaced by T. Then $S_\lambda = S_\mu S_{\mu\lambda}$, hence $K(\lambda) = K(\mu)S_{\mu\lambda} + T_\mu(S_{\mu\lambda} - T_{\mu\lambda})$. If $K(\mu)$ is compact then the first term on the right hand side is compact. For the second term we note that

$$S_{\mu\lambda} - T_{\mu\lambda} = S_{\lambda\mu}^{-1} - T_{\lambda\mu}^{-1} = (1 + (\mu-\lambda)S_\mu)^{-1} - (1 + (\mu-\lambda)T_\mu)^{-1} = (\mu-\lambda)S_{\mu\lambda}K(\mu)T_{\mu\lambda},$$

and the last expression is a compact operator. □

Theorem 4.7. *For any $\mathrm{Re}\,(m) > -1$ we have $\mathrm{sp}(H_m) = \bar{\mathbb{R}}_+$, and if $\lambda \in \mathbb{C}\backslash\bar{\mathbb{R}}_+$ then $R_m(\lambda) - R_{1/2}(\lambda)$ is a compact operator. If $R_m(\lambda; x, y)$ is the integral kernel of the operator $R_m(\lambda)$, then for $\mathrm{Re}\, k > 0$ we have*

$$R_m(-k^2; x, y) = \begin{cases} \sqrt{xy} I_m(kx) K_m(ky) & \text{if} \quad x < y, \\ \sqrt{xy} I_m(ky) K_m(kx) & \text{if} \quad x > y. \end{cases} \tag{4.12}$$

Proof. We first show that $G_m - G_{1/2}$ is compact for all m. From Lemma 4.5 it follows that it suffices to prove this for $0 < m < 1/2$ (take X the space of bounded operators, Y the subspace of compact operators, $\omega =]0, 1/2[$ and $\Omega = \{z \in \mathbb{C}, \mathrm{Re}\, z > -1\}$). In this case H_m is a positive operator and we have $H_m = H_{1/2} + V$ in the form sense, where $V(x) = ax^{-2}$ with $a = m^2 - 1/4$, hence $-1/4 < a < 0$. The Hardy estimate (Proposition 2.1) implies $\pm V \leq 4|a|H_{1/2}$, and $4|a| < 1$, so if we set $S = (H_{1/2} + \lambda)^{-1/2}$ with $\lambda > 0$ we get

$$\pm SVS \leq 4|a|H_{1/2}(H_{1/2} + \lambda)^{-1} \leq 4|a| < 1.$$

Thus $\|SVS\| < 1$. From $H_m + \lambda = S^{-1}(1 + SVS)S^{-1}$ we obtain

$$(H_m + \lambda)^{-1} = S(1 + SVS)^{-1}S = (H_{1/2} + \lambda)^{-1/2} + \sum_{n>0}(-1)^n S(SVS)^{n-1}SVS^2$$

where the series is norm convergent. Hence $R_m(-\lambda) - R_{1/2}(-\lambda)$ is compact if SVS^2 is compact (recall that we assume $0 < m < 1/2$).

We now prove that SVS^2 is a compact operator. Note that $S^2 = (H_{1/2} + \lambda)^{-1}$ and $H_{1/2}$ is the Dirichlet Laplacian, so that $S^2L^2 = H_0^1 \cap H^2$ and $SL^2 = H_0^1$. Thus we have to show that V when viewed as operator $H_0^1 \cap H^2 \to H^{-1}$ is compact. Clearly this operator is continuous, in fact V is continuous as operator $H_0^1 \to H^{-1}$. Moreover, H_0^2 is the subspace of $H_0^1 \cap H^2$ defined by $f'(0) = 0$, hence is a closed subspace of codimension one of $H_0^1 \cap H^2$. Thus it suffices to prove that $V : H_0^2 \to H^{-1}$ is compact. Let θ be a C^∞ function which is equal to one for $x < 1$ and equal to zero for $x > 2$. Clearly $(1 - \theta)V$ is a compact operator $H_0^2 \to L^2$, and so it suffices to prove that $\theta V : H_0^2 \to H^{-1}$ is compact. Again it is clear that $\theta : L^2 \to H^{-1}$ is compact, so it suffices to show that $V : H_0^2 \to L^2$ is continuous. If $f \in C_0^\infty$, then

$$V(x)f(x) = x^2V(x)\int_0^x \frac{x-y}{x^2} f''(y)\mathrm{d}y = x^2V(x)\int_0^1 (1-t)f''(tx)\mathrm{d}t.$$

So if $c = \sup_x |x^2V(x)|$, then

$$\|Vf\| \le c\int_0^1 (1-t)\|f''(t\cdot)\|\mathrm{d}t = c\int_0^1 (1-t)t^{-1/2}\mathrm{d}t\|f''\| = \frac{4c}{3}\|f''\|,$$

hence $V : H_0^2 \to L^2$ is continuous.

Thus we proved that $R_m(-1) - R_{1/2}(-1)$ is a compact operator if $\mathrm{Re}\,(m) > -1$. From Lemma 4.6 it follows that $R_m(\lambda) - R_{1/2}(\lambda)$ is compact if λ is in the resolvent set of H_m and of $H_{1/2}$. We have $\mathrm{sp}(H_{1/2}) = \bar{\mathbb{R}}_+$ and we now show that $\mathrm{sp}(H_m) = \bar{\mathbb{R}}_+$. Clearly the operator $G_{1/2}$ is self-adjoint, its spectrum is the interval $[0, 1]$, and we have $G_m = G_{1/2} + K$ for some compact operator K. Thus if $z \notin [0, 1]$, we have

$$G_m - z = (G_{1/2} - z)\left[1 + (G_{1/2} - z)^{-1}K\right] \equiv (G_{1/2} - z)\left[1 + K(z)\right]$$

where $K(\cdot)$ is a holomorphic compact operator valued function on $\mathbb{C} \setminus [0, 1]$ such that $\|K(z)\| \to 0$ as $z \to \infty$. From the analytic Fredholm alternative it follows that there is a discrete subset N of $\mathbb{C} \setminus [0, 1]$ such that $1 + K(z)$ is a bijective map $L^2 \to L^2$ if $z \notin [0, 1] \cup N$. Thus $G_m - z$ is a bijective map in L^2 if $z \notin N \cup [0, 1]$. The function $z \mapsto \lambda = z^{-1} - 1$ is a homeomorphism of $\mathbb{C} \setminus \{0\}$ onto $\mathbb{C} \setminus \{-1\}$ which sends $]0, 1]$ onto $\bar{\mathbb{R}}_+$, hence the image of N through it is a set M whose accumulation points belong to $\bar{\mathbb{R}}_+ \cup \{-1\}$. If $\lambda \notin \bar{\mathbb{R}}_+ \cup \{-1\} \cup M$, then

$$(\lambda + 1)^{-1} - (H_m + 1)^{-1} = (\lambda + 1)^{-1}(H_m - \lambda)(H_m + 1)^{-1}$$

and the left hand side is a bijection in L^2, hence $H_m - \lambda$ is a bijective map $\mathcal{D}(H_m) \to L^2$. So λ belongs to the resolvent set of H_m. Thus the spectrum of H_m is included in $\bar{\mathbb{R}}_+ \cup \{-1\} \cup M$. But H_m is homogeneous, so $\mathrm{sp}(H_m)$ must be a union of half-lines. Since it is not empty, it has to be equal to $\bar{\mathbb{R}}_+$.

The explicit form of the kernel of $R_m(\lambda)$ given in (4.12) can be proven by a minor variation of the arguments of the proof of Theorem 4.2 based on more refined estimates for the modified Bessel functions. Since we shall not need this formula, we do not give the details. □

Remark 4.8. We describe here in more abstract terms the main fact behind the preceding proof. Let H_0 be a self-adjoint operator on a Hilbert space $\mathcal{H}$ with form domain $\mathcal{K} = \mathcal{D}(|H_0|^{1/2})$ and let V be a continuous symmetric sesquilinear form on $\mathcal{K}$. If V, when viewed as operator $\mathcal{K} \to \mathcal{K}^*$, is compact, then it is easy to prove that the form sum $H = H_0 + V$ is well defined, and that $(H - z)^{-1} - (H_0 - z)^{-1}$ is a compact operator on $\mathcal{H}$ (in fact, also as operator $\mathcal{K}^* \to \mathcal{K}$). This compactness condition on V is never satisfied if H_0 and V are homogeneous of the same orders, so this criterion is useless in our context. But our argument requires only that V be compact as operator $\mathcal{D}(H_0) \to \mathcal{K}^*$, and this property holds in the case of interest here.

4.3. **Domain of the minimal and maximal operator.** In this subsection we analyze the operators $L_m^{\min}$ and $L_m^{\max}$.

Proposition 4.9. *If $|\mathrm{Re}\, m| < 1$, then $L_m^{\min} \subsetneq L_m^{\max}$ and $\mathcal{D}(L_m^{\min})$ is a closed subspace of codimension two of $\mathcal{D}(L_m^{\max})$.*

Proof. In this case, we have two solutions of $L_m u = 0$ that are in L^2 around 0. Hence, the result follows from Proposition A.5. □

Proposition 4.10. *If $|\mathrm{Re}\, m| \geq 1$, then $L_m^{\min} = L_m^{\max}$. Hence, for $\mathrm{Re}\,(m) \geq 1$, $H_m = L_m^{\min} = L_m^{\max}$.*

Proof. We use the notation of the proof of Lemma 4.3. We know that the operator G_m is continuous in L^2, that the functions u_0 and u_∞ are uniquely defined modulo constant factors, and there are no solutions in L^2 of the equation $(\tilde{L}_m + 1)u = 0$. Lemma A.1 says that $(\tilde{L}_m + 1)G_m g = g$ for all $g \in L^2$, hence $(L_m^{\max} + 1)G_m = 1$ on L^2. In particular, $G_m : L^2 \to \mathcal{D}(L_m^{\max})$ is continuous. More explicitly, we have

$$(G_m g)(x) = u_0(x) \int_x^\infty u_\infty(y) g(y) \mathrm{d}y + u_\infty(x) \int_0^x u_0(y) g(y) \mathrm{d}y.$$

Now we shall use the following easily proven fact.

Let E be a normed space and let φ, ψ be linear functionals on E such that a linear combination $a\varphi + b\psi$ is not continuous unless it is zero. Then $\mathrm{Ker}\varphi \cap \mathrm{Ker}\psi$ is dense in E.

We take $E = C_c^\infty$ equipped with the L^2 norm and $\varphi(g) = \int_0^\infty u_0(x) g(x) \mathrm{d}x$, $\psi(g) = \int_0^\infty u_\infty(x) g(x) \mathrm{d}x$. The linear combination $a\varphi + b\psi$ is given by a similar expression with $u = au_0 + bu_\infty$ as integrating function. Since $(\tilde{L}_m + 1)u = 0$ we have $u \in L^2$ only if $u = 0$. Thus $E_0 = \mathrm{Ker}\varphi \cap \mathrm{Ker}\psi$ is dense in E. It is clear that $G_m E_0 \subset C_c^\infty$. Hence by continuity we get $G_m L^2 \subset \mathcal{D}(L_m^{\min})$, and thus $(L_m^{\min} + 1)G_m = 1$ on L^2. On the other hand it is easy to show that $G_m(\tilde{L}_m + 1)f = f$ if $f \in C_c^\infty$, hence $G_m(L_m^{\min} + 1) = 1$ on $\mathcal{D}(L_m^{\min})$. Thus $L_m^{\min} + 1 : \mathcal{D}(L_m^{\min}) \to L^2$ is a bijective map. Since $L_m^{\max} + 1$ is an extension of $L_m^{\min} + 1$ and is injective, we must have $L_m^{\min} = L_m^{\max}$. □

If $m = 1/2$, then clearly $\mathcal{D}(L_m^{\min}) = H_0^2$. If $m \neq 1/2$ then $\mathcal{D}(L_m^{\min}) \neq H_0^2$. However, the functions from $\mathcal{D}(L_m^{\min})$ behave at zero as if they were in H_0^2 with the exception of the case $m = 0$.

Proposition 4.11. *Let $f \in \mathcal{D}(L_m^{\min})$.*

(i) If $m \neq 0$, then $f(x) = o(x^{3/2})$ and $f'(x) = o(x^{1/2})$ as $x \to 0$.
(ii) If $m = 0$, then $f(x) = o(x^{3/2} \ln x)$ and $f'(x) = o(x^{1/2} \ln x)$ as $x \to 0$.
(iii) For any m, $\mathcal{D}(L_m^{\min}) \subset H_0^1$.

Proof. Since $\tilde{L}_m$ does not make any difference between m and $-m$, we may assume $\mathrm{Re}\, m \geq 0$.

Assume first $\mathrm{Re}\, m \geq 1$. If $f \in \mathcal{D}(L_m^{\min})$ and $g = (L_m^{\min} + 1)f$, then $f = G_m g$, and hence $f = u_0 g_\infty + u_\infty g_0$ and $f' = u_0' g_\infty - u_\infty' g_0$ with $g_0(x) = \int_0^x u_0(y) g(y) \mathrm{d}y$ and $g_\infty(x) = \int_x^\infty u_\infty(y) g(y) \mathrm{d}y$. The functions u_0 and u_∞ are of Bessel type and their behaviour at zero is known, see (4.7). More precisely, if we set $\mu = \mathrm{Re}\, m$, then we have

$$u_0(x) = O(x^{\mu+1/2}), \quad u_0'(x) = O(x^{\mu-1/2}), \quad u_\infty(x) = O(x^{-\mu+1/2}), \quad u_\infty'(x) = O(x^{-\mu-1/2}).$$

Then for $x < 1$ we have

$$\begin{aligned} |u_0(x) g_\infty(x)| &\leq Cx^{\mu+1/2} \left(\int_x^1 y^{-\mu+1/2} |g(y)| \mathrm{d}y + \int_1^\infty |u_\infty(y) g(y)| \mathrm{d}y \right) \\ &\leq Cx^{\mu+1/2} \left(\left(\frac{x^{2-2\mu} - 1}{2\mu - 2} \right)^{1/2} + \|u_\infty\|_{L^2(1,\infty)} \right) \|g\|, \end{aligned}$$

which is $O(x^{3/2})$. We have $u_\infty g_0 = o(x^{3/2})$ by a simpler argument. Let F be the Banach space consisting of continuous functions on $I =]0,1[$ such that $\|h\|_F \equiv \sup_{x\in I} x^{-3/2}|h(x)| < \infty$. For $g \in L^2$ let Tg be the restriction of $G_m g$ to I. By what we have shown we have $TL^2 \subset F$, hence, by the closed graph theorem, $T : L^2 \to F$ is a continuous operator. With the notation of the proof of Proposition 4.10, if $g \in E_0$, then Tg is equal to zero near zero, so T sends the dense subspace E_0 of L^2 into the closed subspace F_0 of F consisting of functions such that $x^{-3/2}h(x) \to 0$ as $x \to 0$. By continuity, we get $TL^2 \subset F_0$, hence $f(x) = o(x^{3/2})$. A similar argument based on the representation $f' = u_0' g_\infty - u_\infty' g_0$ gives $f'(x) = o(x^{1/2})$.

We treat now the case $0 \le \operatorname{Re} m < 1$. Now all the solutions of the equation $L_m u = 0$ are square integrable at the origin, hence we may use Proposition A.7 with $v_\pm$ proportional to $\zeta_{\pm m}$. A straightforward computation gives for $m \neq 0$

$$|v_+(x)|\|v_-\|_x + |v_-(x)|\|v_+\|_x \le Cx^{3/2}, \quad |v_+'(x)|\|v_-\|_x + |v_-'(x)|\|v_+\|_x \le Cx^{1/2}$$

while if $m = 0$ then

$$|v_+(x)|\|v_-\|_x+|v_-(x)|\|v_+\|_x \le Cx^{3/2}(|\ln x|+1), \quad |v_+'(x)|\|v_-\|_x+|v_-'(x)|\|v_+\|_x \le Cx^{1/2}(|\ln x|+1).$$

This finishes the proof. □

We describe now some consequences of the representations (A.5) and (A.6) in the present context. We say that a function h *is in* $\mathcal{D}(L_m^{\min})$ *near the origin* if for some (hence any) function $\xi \in C_c^\infty(\mathbb{R})$ which is one on a neighbourhood of the origin we have $\xi h \in \mathcal{D}(L_m^{\min})$. Assume $|\operatorname{Re} m| < 1$ and let $f \in \mathcal{D}(L_m^{\max})$. Then there are constants a, b and a function f_0 which is in $\mathcal{D}(L_m^{\min})$ near the origin such that

$$f(x) = ax^{1/2-m} + bx^{1/2+m} + f_0(x) \quad \text{if } m \neq 0, \tag{4.13}$$

$$f(x) = ax^{1/2}\ln x + bx^{1/2} + f_0(x) \quad \text{if } m = 0. \tag{4.14}$$

These relations give by differentiation representations of f'. By Proposition 4.11, it is clear that f_0 decays more rapidly at zero than the other two terms, in particular the constants a, b and the function f_0 are uniquely determined by f. This allows one to state assertions converse to that of Proposition 4.11, for example:

Proposition 4.12. *We have the following characterization of the domain of the minimal operator:*

$$\begin{aligned}
0 < \operatorname{Re} m \le \mu \le 1 \quad \Rightarrow \quad \mathcal{D}(L_m^{\min}) &= \{f \in \mathcal{D}(L_m^{\max}) \mid f(x) = o(x^{\mu+1/2})\} \\
&= \{f \in \mathcal{D}(L_m^{\max}) \mid f'(x) = o(x^{\mu-1/2})\}, \\
0 \le \operatorname{Re} m < \mu \le 1 \quad \Rightarrow \quad \mathcal{D}(L_m^{\min}) &= \{f \in \mathcal{D}(L_m^{\max}) \mid f(x) = O(x^{\mu+1/2})\} \\
&= \{f \in \mathcal{D}(L_m^{\max}) \mid f'(x) = O(x^{\mu-1/2})\}.
\end{aligned}$$

4.4. **Strict extensions of $L_m^{\min}$.** Now we study the closed extensions of $L_m^{\min}$ for $|\operatorname{Re} m| < 1$. The first result is a particular case of Proposition A.5. We recall that by a *strict extension* of $L_m^{\min}$ we mean an operator H such that $L_m^{\min} \subsetneq H \subsetneq L_m^{\max}$. We denote by $W_x(f,g) := f(x)g'(x) - f'(x)g(x)$ the Wronskian of two functions f and g at point x, and take ξ as in Section 3.

Proposition 4.13. *Assume that* $|\operatorname{Re} m| < 1$*. Let* u *be a non-zero solution of* $\tilde{L}_m u = 0$*. Then* $W_0(u,f) = \lim_{x\to 0} W_x(u,f)$ *exists for each* $f \in \mathcal{D}(L_m^{\max})$*, and the operator* L_m^u *defined as the restriction of* $L_m^{\max}$ *to the set of* $f \in \mathcal{D}(L_m^{\max})$ *such that* $W_0(u,f) = 0$ *is a strict extension of* $L_m^{\min}$*. Reciprocally, each strict extension of* $L_m^{\min}$ *is of the form* L_m^u *for some non-zero solution* u *of* $\tilde{L}_m u = 0$*, which is uniquely defined modulo a constant factor. We have* $\mathcal{D}(L_m^u) = \mathcal{D}(L_m^{\min}) + \mathbb{C}\xi u$*.*

We shall describe now the homogeneous strict extensions of $L_m^{\min}$. The case $|\operatorname{Re} m| \ge 1$ is trivial because $L_m^{\min} = L_m^{\max}$ is homogeneous.

Proposition 4.14. *If* $-1 < \operatorname{Re} m < 1$*, then* H_m *is the restriction of* $L_m^{\max}$ *to the subspace defined by*

$$\lim_{x\to 0} x^{m+1/2}\left(f'(x) - \frac{m+1/2}{x} f(x)\right) = 0. \tag{4.15}$$

Proof. Observe that

$$W_x(\zeta_m, f) = x^{m+1/2} f'(x) - (m+1/2)x^{m-1/2} f(x) = x^{m+1/2}\left(f'(x) - \frac{m+1/2}{x} f(x)\right),$$

so the limit from the left hand side of (4.15) exists for all $f \in \mathcal{D}(L_{\max})$ if $|\mathrm{Re}\, m| < 1$. Hence, with the notation of Proposition 4.13, we have $H_m = L_m^{\zeta_m}$, where ζ_m is defined in (4.3). □

Proposition 4.15. *If $|\mathrm{Re}\, m| < 1$ and $m \neq 0$, then $L_m^{\min}$ has exactly two homogeneous strict extensions, namely the operators $H_{\pm m}$. If $m = 0$ then the operator H_0 is the unique homogeneous strict extension of $L_m^{\min}$.*

Proof. Thanks to Proposition 4.13 it suffices to see when the extension L_m^u is homogeneous. If $(U_t f)(x) = \mathrm{e}^{t/2} f(\mathrm{e}^t x)$, then it is clear that L_m^u is homogeneous if and only if its domain is stable under the action of U_t for each real t. We have

$$\begin{aligned} W_0(u, U_t f) &= \lim_{x\to 0}\left(u(x)\mathrm{e}^{t/2}\frac{d}{dx}f(\mathrm{e}^t x) - u'(x)\mathrm{e}^{t/2} f(\mathrm{e}^t x)\right) \\ &= \mathrm{e}^{t/2}\lim_{x\to 0}\left(\mathrm{e}^t u(x) f'(\mathrm{e}^t x) - u'(x) f(\mathrm{e}^t x)\right) \\ &= \mathrm{e}^{3t/2}\lim_{x\to 0}\left(u(\mathrm{e}^{-t}x) f'(x) - \mathrm{e}^{-t}u'(\mathrm{e}^{-t}x) f(x)\right). \end{aligned}$$

Thus we obtain

$$W_0(u, U_t f) = \mathrm{e}^{2t} W_0(U_{-t}u, f).$$

Let $u_t = \mathrm{e}^{2t} U_{-t} u$. From Proposition 4.13 we see that $\mathcal{D}(L_u) = \mathcal{D}(L_{u_t})$ for all real t if and only if u_t is proportional to u for all t, which means that the function u is homogeneous. Thus it remains to see which are the homogeneous solutions of the equation $L_m u = 0$. Clearly $u_{\pm m}$ are both homogeneous and only they are so if $m \neq 0$, and if $m = 0$ then only u_{+0} is homogeneous. □

Proposition 4.16. *For $\mathrm{Re}\, m > 0$, we have the following alternative characterizations of the domain of H_m:*

$$\begin{aligned} 0 < \mu \le \mathrm{Re}\, m < 1 \quad &\Rightarrow \quad \mathcal{D}(H_m) = \{f \in \mathcal{D}(L_m^{\max}) \mid f(x) = o(x^{-\mu+1/2})\}, \\ 0 \le \mu < \mathrm{Re}\, m < 1 \quad &\Rightarrow \quad \mathcal{D}(H_m) = \{f \in \mathcal{D}(L_m^{\max}) \mid f(x) = O(x^{-\mu+1/2})\}. \end{aligned}$$

Proof. We use Propositions 4.11, and the representations (4.13) and (4.14). □

4.5. **The hermitian case.** We shall consider now the particular case when $L_m^{\min}$ is hermitian, i.e. m^2 is a real number. Everything follows immediately from the preceding propositions and from the last assertion of Proposition A.5. If m is real or $m = \mathrm{i}\mu$ with μ real it suffices to consider the cases $m \ge 0$ and $\mu > 0$, because $L_m^{\min} = L_{-m}^{\min}$.

Proposition 4.17. *The operator $H_m = L_m^{\min}$ is self-adjoint and homogeneous for $m^2 \ge 1$. When $m^2 < 1$ the operator $L_m^{\min}$ has deficiency indices $(1,1)$ and therefore admits a one-parameter family of self-adjoint extensions.*

(1) If $0 < m < 1$ and $0 \le \theta < \pi$, let u_θ be the function on $\mathbb{R}_+$ defined by

$$u_\theta(x) = x^{1/2-m}\cos\theta + x^{1/2+m}\sin\theta. \tag{4.16}$$

Then each self-adjoint extension of $L_m^{\min}$ is of the form $H_m^\theta = L_m^{u_\theta}$ for a unique θ. There are exactly two homogeneous strict extensions, namely the self-adjoint operators $H_m = H_m^{\pi/2}$ and $H_{-m} = H_m^0$.

(2) If $m = 0$ and $0 \le \theta < \pi$, let u_θ be the function on $\mathbb{R}_+$ defined by

$$u_\theta(x) = x^{1/2}\ln x\cos\theta + x^{1/2}\sin\theta. \tag{4.17}$$

Then each self-adjoint extension of $L_0^{\min}$ is of the form $H_0^\theta = L_0^{u_\theta}$ for a unique θ. The operator $L_0^{\min}$ has exactly one homogeneous strict extension: this is the self-adjoint operator $H_0 = H_0^{\pi/2}$.

(3) *Let $m^2 < 0$ so that $m = \mathrm{i}\mu$ with $\mu > 0$. For $0 \le \theta < \pi$ let u_θ be the function given by*

$$u_\theta(x) = x^{1/2}\cos(\mu \ln x)\cos\theta + x^{1/2}\sin(\mu \ln x)\sin\theta. \tag{4.18}$$

Then each self-adjoint extension of $L_m^{\min}$ is of the form $H_m^\theta = L_m^{u_\theta}$ for a unique θ. The operator $L_m^{\min}$ does not have homogeneous self-adjoint extensions but has two homogeneous strict extensions, namely the operators H_m and H_{-m}.

We shall now study the quadratic forms associated to the self-adjoint operators H_m^θ for $0 < m < 1$.

We recall that $A_{1/2+m}^{\min} = A_{1/2+m}^{\max}$ if $\operatorname{Re} m \ge 0$, and $A_{1/2-m}^{\min} = A_{1/2-m}^{\max}$ if $\operatorname{Re} m \ge 1$, see Proposition 3.1. Let us abbreviate $A_\alpha = A_\alpha^{\min} = A_\alpha^{\max}$ when the minimal and maximal realizations of $\widetilde{A}_\alpha$ coincide.

Recall also that for $0 < m < 1$,

$$\mathcal{D}(A_{1/2-m}^{\max}) = H_0^1 + \mathbb{C}\xi\zeta_{-m}.$$

By Proposition 3.1, the operator $A_{1/2-m}^{\max}$ is closed in L^2 and H_0^1 is a closed subspace of its domain (for the graph topology), because $A_{1/2-m}^{\max}\restriction_{H_0^1} = A_{1/2-m}^{\min}$ is also a closed operator. Note that for $f \in H_0^1$ we have $f(x) = o(\sqrt{x})$, because

$$|f(x)| \le \int_0^x |f'(x)|\mathrm{d}x \le \sqrt{x}\|f'\|_{L^2(0,x)}.$$

Thus $\xi\zeta_{-m} \notin H_0^1$ and the sum $H_0^1 + \mathbb{C}\xi\zeta_m$ is a topological direct sum in $\mathcal{D}(A_{1/2-m}^{\max})$. Hence each $f \in \mathcal{D}(A_{1/2-m}^{\max})$ can be uniquely written as a sum $f = f_0 + c\xi\zeta_{-m}$, and the map $f \mapsto c$ is a continuous linear form on $\mathcal{D}(A_{1/2-m}^{\max})$. We shall denote $\varkappa_m$ this form and observe that

$$\varkappa_m(f) = \lim_{x\to 0} x^{m-1/2} f(x), \quad f \in \mathcal{D}(A_{1/2-m}^{\max}).$$

Note also that from Proposition 3.1 we get $(A_{1/2-m}^{\max})^* = A_{m-1/2}^{\min}$, in particular $\mathcal{D}\left((A_{1/2-m}^{\max})^*\right) = H_0^1$.

Proposition 4.18. *Let $0 < m < 1$ and $0 \le \theta < \pi$.*

(1) *If $\theta = \pi/2$, then $\mathcal{D}(H_m^{\pi/2})$ is a dense subspace of H_0^1 and if $f \in \mathcal{D}(H_m^{\pi/2})$, then*

$$\langle f, H_m^{\pi/2} f\rangle = \|A_{1/2+m} f\|^2 = \|A_{1/2-m}^{\max} f\|^2. \tag{4.19}$$

Thus $\mathcal{Q}(H_m^{\pi/2}) = H_0^1$. Moreover, we have $H_m^{\pi/2} = (A_{1/2+m})^* A_{1/2+m} = (A_{1/2-m}^{\min})^* A_{1/2-m}^{\min}$.

(2) *Assume $\theta \ne \pi/2$. Then $\mathcal{D}(H_m^\theta)$ is a dense subspace of $\mathcal{D}(A_{1/2-m}^{\max})$, and for each $f \in \mathcal{D}(H_m^\theta)$ we have*

$$\langle f, H_m^\theta f\rangle = \|A_{1/2-m}^{\max} f\|^2 + m\sin(2\theta)|\varkappa_m(f)|^2. \tag{4.20}$$

Thus $\mathcal{Q}(H_m^\theta) = \mathcal{D}(A_{1/2-m}^{\max})$, and the right hand side of (4.20) *is equal to the quadratic form of H_m^θ.*

Proof. From Proposition 4.11, the definition of H_m and (4.16) we get

$$\mathcal{D}(H_m^\theta) = \mathcal{D}(L_m^{\min}) + \mathbb{C}\xi u_\theta \subset H_0^1 + \mathbb{C}\xi u_\theta = H_0^1 + \mathbb{C}\cos\theta\,\xi\zeta_{-m},$$

because $\xi\zeta_m \in H_0^1$ if $m > 0$. But $C_c^\infty \subset \mathcal{D}(L_m^{\min})$, so $\mathcal{D}(H_m^\theta)$ is a dense subspace of $H_0^1 + \mathbb{C}\cos\theta\,\xi\zeta_{-m}$. Thus if $\theta = \pi/2$ we get $\mathcal{D}(H_m^{\pi/2}) \subset H_0^1$, and if $\theta \ne \pi/2$, then $\mathcal{D}(H_m^\theta) \subset \mathcal{D}(A_{1/2-m}^{\max})$ densely in both cases.

The relation $\|A_{1/2-m}^{\max} f\|^2 = \|A_{1/2+m} f\|^2$ for $f \in H_0^1$ holds, because both terms are continuous on H_0^1 by Hardy inequality and they are equal to $\langle f, \tilde{L}_m f\rangle$ if $f \in C_c^\infty$.

It remains to establish (4.20). Let $f = f_0 + c\xi u_\theta$ with $f_0 \in \mathcal{D}(L_m^{\min})$ and $c \in \mathbb{C}$. Then $A_{1/2-m}^{\max} f \in L^2$ and

$$H_m^\theta f = \tilde{L}_m f = \widetilde{A}_{1/2-m}^* A_{1/2-m}^{\max} f \in L^2$$

due to (4.2). Denote $\langle\cdot,\cdot\rangle_a$ the scalar product in $L^2(a,\infty)$. Then $\langle f, H_m^\theta f\rangle = \lim_{a\to 0}\langle f, H_m^\theta f\rangle_a$ and

$$\langle f, H_m^\theta f\rangle_a = \langle f, -\mathrm{i}(\partial_x + (1/2-m)Q^{-1})A_{1/2-m}^{\max} f\rangle_a = \langle A_{1/2-m}^{\max} f, A_{1/2-m}^{\max} f\rangle_a + \mathrm{i}\bar{f}(a)A_{1/2-m}^{\max} f(a).$$

On a neighborhood of the origin we have

$$\begin{aligned}
\mathrm{i}A_{1/2-m}^{\max} f(x) &= \left(\partial_x + \frac{m-1/2}{x}\right)\left(cx^{1/2-m}\cos\theta + cx^{1/2+m}\sin\theta + f_0(x)\right) \\
&= \left(\partial_x + \frac{m-1/2}{x}\right)\left(cx^{1/2+m}\sin\theta + f_0(x)\right) = 2mc\sin\theta x^{m-1/2} + o(\sqrt{x})
\end{aligned}$$

by Proposition 4.11. Then by the same proposition we get

$$\begin{aligned}
\mathrm{i}\bar{f}(x)A_{1/2-m}^{\max} f(x) &= (\bar{f}_0(x) + \bar{c}u_\theta(x))(2mc\sin\theta x^{m-1/2} + o(\sqrt{x})) \\
&= 2m|c|^2\sin\theta\left(x^{1/2-m}\cos\theta + x^{1/2+m}\sin\theta\right)x^{m-1/2} + o(\sqrt{x}) \\
&= 2m|c|^2\sin\theta\cos\theta + o(1).
\end{aligned}$$

Hence $\lim_{a\to 0}\mathrm{i}\bar{f}(a)A_{1/2-m}^{\max} f(a) = m|c|^2\sin 2\theta$. □

Proposition 4.19. *Let $0 < m < 1$. Then $L_m^{\min}$ is a positive hermitian operator with deficiency indices $(1,1)$. The operators $H_m = H_m^{\pi/2}$ and $H_{-m} = H_m^0$ are respectively the Friedrichs and the Krein extensions of $L_m^{\min}$. If $0 \le \theta \le \pi/2$, then H_m^θ is a positive self-adjoint extension of $L_m^{\min}$. If $\pi/2 < \theta < \pi$ then the self-adjoint extension H_m^θ of $L_m^{\min}$ has exactly one strictly negative eigenvalue and this eigenvalue is simple.*

Proof. We have, by Hardy inequality and Proposition 4.11, $L_m^{\min} \ge m^2 Q^{-2}$ as quadratic forms on H_0^1, so $L_m^{\min}$ is positive. The operators H_m^θ have the same form domain if $\theta \ne \pi/2$, namely $\mathcal{D}(A_{1/2-m}^{\max})$, and $H_m^{\pi/2}$ has H_0^1 as form domain, which is strictly smaller.

Thus to finish the proof it suffices to show the last assertion of the proposition. Recall the modified Bessel function K_m (see (4.5)). It is easy to see that $u_{m,k} := \sqrt{kx}K_m(kx)$ solves $L_m^{\max} u_{m,k} = k^2 u_{m,k}$. Using (4.6), one gets that

$$u_{m,k} \sim \frac{\pi}{2\sin\pi m}\left(\frac{1}{\Gamma(1-m)}(kx/2)^{-m+1/2} - \frac{1}{\Gamma(1+m)}(kx/2)^{m+1/2}\right),$$

so that if $(k/2)^2 m = -\tan\theta\Gamma(1+m)/\Gamma(1-m)$, then $u_{m,k} \in \mathcal{D}(L_m^\theta)$. This proves that L_m^θ has a negative eigenvalue for $\pi/2 < \theta < \pi$. It cannot have more eigenvalues, since $L_m^{\min}$ is positive and its deficiency indices are just $(1,1)$. □

Remark 4.20. The fact that $H_{\pm m}$ are the Friedrichs and the Krein extensions of $L_m^{\min}$ also follows from Proposition 2.6, because we know that these are the only homogeneous extensions of $L_m^{\min}$.

Proposition 4.21. *$L_0^{\min}$ is a positive hermitian operator with deficiency indices $(1,1)$. Its Friedrichs and Krein extensions coincide and are equal to $H_0 = H_0^{\pi/2}$. The domain of H_0 is a dense subspace of $\mathcal{D}(A_{1/2})$, and for $f \in \mathcal{D}(H_0)$ we have $\langle f, H_0 f\rangle = \|A_{1/2}f\|^2$. Thus the quadratic form of H_0 equals $A_{1/2}^* A_{1/2}$. If $0 \le \theta < \pi$ and $\theta \ne \pi/2$, then the self-adjoint extension H_0^θ of $L_0^{\min}$ has exactly one strictly negative eigenvalue.*

Proof. Since $L_0^{\min}$ has only one homogeneous self-adjoint extension, this follows from Proposition 2.6 and Remark 2.5. For the assertions concerning the quadratic form, it suffices to apply Proposition 3.1. □

We can summarize our results in the following theorem:

Theorem 4.22. *Let $m > -1$. Then the operators H_m are positive, self-adjoint, homogeneous of degree 2 with $\mathrm{sp}H_m = \bar{\mathbb{R}}_+$. Besides we have the following table:*

$m \geq 1$: $\quad H_m = A^*_{1/2+m}A_{1/2+m} = A^*_{1/2-m}A_{1/2-m}$, $\quad H_0^1 = \mathcal{Q}(H_m)$, $H_m = L_m^{\min} = L_m^{\max}$;

$0 < m < 1$: $\quad H_m = A^*_{1/2+m}A_{1/2+m} = \left(A^{\min}_{1/2-m}\right)^* A^{\min}_{1/2-m}$ $\quad H_0^1 = \mathcal{Q}(H_m)$, *$H_m$ is the Friedrichs ext. of $L_m^{\min}$;*

$m = 0$: $\quad H_0 = A^*_{1/2}A_{1/2}$, $\quad H_0^1 + \mathbb{C}\xi\zeta_0$ *dense in* $\mathcal{Q}(H_0)$, *H_0 is the Friedrichs and Krein ext. of $L_0^{\min}$;*

$-1 < m < 0$: $\quad H_m = \left(A^{\max}_{1/2+m}\right)^* A^{\max}_{1/2+m}$, $\quad H_0^1 + \mathbb{C}\xi\zeta_m = \mathcal{Q}(H_m)$, *$H_m$ is the Krein ext. of $L_m^{\min}$.*

In the region $-1 < m < 1$ (which is the most interesting one), it is quite remarkable that for strictly positive m one can factorize H_m in two different ways, whereas for $m \leq 0$ only one factorization appears.

As an example, let us consider the case of the Laplacian $-\partial_x^2$, i.e. $m^2 = 1/4$. The operators $H_{1/2}$ and $H_{-1/2}$ coincide with the Dirichlet and Neumann Laplacian respectively. One usually factorizes them as $H_{1/2} = P^*_{\min}P_{\min}$ and $H_{-1/2} = P^*_{\max}P_{\max}$, where $P_{\min}$ and $P_{\max}$ denote the usual momentum operator on the half-line with domain H_0^1 and H^1 respectively. The above analysis says that, whereas for the Neumann Laplacian this is the only factorization of the form S^*S with S homogeneous, in the case of the Dirichlet Laplacian one can also factorize it in a rather unusual way:

$$H_{1/2} = \left(P_{\min} + \mathrm{i}Q^{-1}\right)^* \left(P_{\min} + \mathrm{i}Q^{-1}\right).$$

Proposition 4.23. *The family H_m has the following property:*

$$\begin{aligned} 0 \leq m \leq m' \quad &\Rightarrow \quad H_m \leq H_{m'}, \\ 0 \leq m < 1 \quad &\Rightarrow \quad H_{-m} \leq H_m. \end{aligned}$$

4.6. **The non hermitian case: numerical range and dissipativeness.** In this section we come back to the non hermitian case. We study the numerical range of the operators H_m in terms of the parameter m. As a consequence we obtain dissipative properties of H_m.

Proposition 4.24. *Let $m \neq 0$.*

i) If $0 \leq \arg m \leq \pi/2$, then $\mathrm{Num}(H_m) = \{z \mid 0 \leq \arg z \leq 2\arg m\}$. Hence H_m is maximal sectorial and $\mathrm{i}H_m$ is dissipative.
ii) If $-\pi/2 \leq \arg m \leq 0$, then $\mathrm{Num}(H_m) = \{z \mid 2\arg m \leq \arg z \leq 0\}$. Hence H_m is maximal sectorial and $-\mathrm{i}H_m$ is dissipative.
iii) If $|\arg m| \leq \pi/4$, then $-H_m$ is dissipative.
iv) If $\pi/2 < |\arg m| < \pi$, then $\mathrm{Num}(H_m) = \mathbb{C}$.

Remark 4.25. For $m = 0$ and $\arg m = \pi$, H_m is selfadjoint so that $\mathrm{Num}(H_m) = \mathrm{sp}(H_m) = [0, +\infty[$.

Proof. First note that since H_m is homogeneous, if a point z is in the numerical range $\mathbb{R}_+ z$ is included in the numerical range. Thus the numerical range is a closed convex cone. Moreover, since $H_m^* = H_{\bar{m}}$ it suffices to consider the case $\mathrm{Im}\,(m) > 0$.

Let us recall that for $\mathrm{Re}\, m > -1$ the operator H_m is defined by

$$H_m f = -f'' + (m^2 - 1/4)x^{-2}f, \quad f \in \mathcal{D}(H_m) = \mathcal{D}(L_m^{\min}) + \mathbb{C}\xi\zeta_m.$$

Thus $C_{\rm c}^\infty + \mathbb{C}\xi\zeta_m$ is a core for H_m. Let $0 < a < 1$, $c \in \mathbb{C}$, and f a function of class C^2 on $\mathbb{R}_+$ such that $f(x) = cx^{m+1/2}$ for $x < a$ and $f(x) = 0$ for large x. By what we just said the set of functions of this form is a core for H_m. We set $V(x) = (m^2 - 1/4)x^{-2}$ and note that for any $f \in \mathcal{D}(H_m)$

$$\begin{aligned}\langle f, H_m f\rangle &= \lim_{b\to 0} \int_b^\infty \big(-(\bar{f}f')' + |f'|^2 + V|f|^2\big)\mathrm{d}x \\ &= \lim_{b\to 0} \left(\bar{f}(b)f'(b) + \int_b^\infty \big(|f'|^2 + V|f|^2\big)\mathrm{d}x \right).\end{aligned}$$

If f is of the form indicated above, we have $\bar{f}(b) = \bar{c}b^{\bar{m}+1/2}$ and $f'(b) = (m+1/2)cb^{m-1/2}$ for $b < a$, hence $\bar{f}(b)f'(b) = |c|^2(m+1/2)b^{2\mathrm{Re}\, m}$. To simplify notations we set $m = \mu + \mathrm{i}\nu$ with μ, ν real. Thus we get

$$\begin{aligned}\langle f, H_m f\rangle &= \lim_{b\to 0} \left(|c|^2(m+1/2)b^{2\mu} + \int_b^\infty \big(|f'|^2 + V|f|^2\big)\mathrm{d}x \right) \\ &= \lim_{b\to 0} \left(|c|^2(m+1/2)b^{2\mu} + \int_b^a \big(|f'|^2 + V|f|^2\big)\mathrm{d}x \right) + \int_a^\infty \big(|f'|^2 + V|f|^2\big)\mathrm{d}x.\end{aligned}$$

But for $b < a$ we have

$$\begin{aligned}\int_b^a \big(|f'|^2 + V|f|^2\big)\mathrm{d}x &= |c|^2 \int_b^a \big(|m+1/2|^2 x^{2\mu-1} + (m^2-1/4)x^{-2}x^{2\mu+1}\big)\mathrm{d}x \\ &= |c|^2(m+1/2) \int_b^a (\bar{m} + 1/2 + m - 1/2)x^{2\mu-1}\mathrm{d}x \\ &= |c|^2(m+1/2) \int_b^a (x^{2\mu})'\mathrm{d}x \ = \ |c|^2(m+1/2)\big(a^{2\mu} - b^{2\mu}\big).\end{aligned}$$

Thus we get

$$\langle f, H_m f\rangle = |c|^2(m+1/2)a^{2\mu} + \int_a^\infty \big(|f'|^2 + V|f|^2\big)\mathrm{d}x =: \Psi(a,c,f). \tag{4.21}$$

So the numerical range of H_m coincides with the closure of the set of numbers of the form $\Psi(a,c,f)$ with $0 < a < 1$, $c \in \mathbb{C}$, and f a function of class C^2 on $x \geq a$ which vanishes for large x and such that the derivatives $f^{(i)}(a)$ coincide with the corresponding derivatives of $cx^{m+1/2}$ at $x = a$ for $0 \leq i \leq 2$. The map $f \mapsto \int_a^\infty \big(|f'|^2 + V|f|^2\big)\mathrm{d}x$ is continuous on $H^1(]a,+\infty[)$, the functions of class C^2 on $[a,\infty[$ vanishing for large x are dense in this space, and the functionals $f \mapsto f'(a)$ and $f \mapsto f''(a)$ are not continuous in the H^1 topology. Hence we can consider in the definition of $\Psi(a,c,f)$ functions $f \in H^1(]a,+\infty[)$ such that $f(a) = ca^{m+1/2}$ without extending the numerical range.

Let $\gamma < \frac{1}{2}$, $\delta < -\frac{1}{2}$ and $R > a$, and let

$$f(x) = \begin{cases} x^{m+1/2} & \text{if} \quad x < a, \\ a^{m+1/2-\gamma}x^\gamma & \text{if} \quad a \leq x < R, \\ a^{m+1/2-\gamma}R^{\gamma-\delta}x^\delta & \text{if} \quad R \leq x. \end{cases}$$

Then one can explicitly compute

$$\begin{aligned}&(m+1/2)a^{2\mu} + \int_a^\infty \big(|f'|^2 + V|f|^2\big)\mathrm{d}x \\ &= \quad \frac{a^{2\mu}}{1-2\gamma}(m+1/2-\gamma)^2 + a^{2\mu+1-2\gamma}R^{2\gamma-1}\left(\frac{\delta^2 + m^2 - 1/4}{1-2\delta} - \frac{\gamma^2 + m^2 - 1/4}{1-2\gamma}\right).\end{aligned}$$

For $\gamma < \frac{1}{2}$, the argument of the first term is $2\arg(m + \frac{1}{2} - \gamma)$ and the second term vanishes as $R \to +\infty$. Using the fact that the numerical range is a convex cone, we thus have

(1) If $\mu \geq 0$, then $\{z \mid 0 \leq \arg z \leq 2\arg m\} \subset \mathrm{Num}(H_m)$,
(2) If $-1 < \mu < 0$, then $\mathrm{Num}(H_m) = \mathbb{C}$.

It remains to prove the reverse inclusion of 1.

We first consider the case $\mu > 0$. Observe that in (4.21) a can be taken as small as we wish. Hence we can make $a \to 0$, and we get

$$\langle f, H_m f\rangle = \int_0^\infty \big(|f'|^2 + V|f|^2\big)\mathrm{d}x = \|Pf\|^2 + (m^2 - 1/4)\|Q^{-1}f\|^2,$$

and the result follows from Proposition 2.1.

On the other hand, if $\mu = 0$, then the formula is different:

$$\langle f, H_m f\rangle = (m + 1/2)|c(f)|^2 + \|Pf\|^2 + (m^2 - 1/4)\|Q^{-1}f\|^2,$$

where $c(f) = \lim_{x\to 0} x^{-(m+1/2)} f(x)$ is a continuous linear functional on $\mathcal{D}(H_m)$ which is nontrivial except in the case $m = 0$, cf. (4.13) and (4.14). In particular we have

$$\operatorname{Im}\langle f, H_m f\rangle = \nu \left(|c|^2 a^{2\mu} + 2\mu \int_a^\infty x^{-2}|f|^2 \mathrm{d}x \right) \geq 0.$$

Since we have established the last two identities for f in a core of H_m, they remain valid on $\mathcal{D}(H_m)$. $\square$

As a last result, let us mention that the factorization obtained in Theorem 4.22 can be extended to the complex case (see also (4.2)), and can thus be used as an alternative definition of H_m:

Proposition 4.26. *For* $\operatorname{Re} m > -1$ *we have*

$$\begin{aligned} \mathcal{D}(H_m) \quad &:= \quad \left\{ f \in \mathcal{D}(A^{\max}_{m+\frac12}) \mid A^{\max}_{m+\frac12} f \in \mathcal{D}(A^{\max *}_{\overline{m}+\frac12}) \right\}, \\ H_m f \quad &:= \quad A^{\max *}_{\overline{m}+\frac12} A^{\max}_{m+\frac12} f, \quad f \in \mathcal{D}(H_m). \end{aligned}$$

Proof. Using Proposition 3.1 and 4.12 we have $\mathcal{D}(H_m) \subset \left\{ f \in \mathcal{D}(A^{\max}_{m+\frac12}) \mid A^{\max}_{m+\frac12} f \in \mathcal{D}(A^{\max *}_{\overline{m}+\frac12}) \right\}$. One then prove the reverse inclusion using Proposition 3.1 and 4.14. $\square$

5. Spectral projections of H_m and the Hankel transformation

In this section, we provide an explicit spectral representation of the operator H_m in terms of Bessel functions.

Recall that the (unmodified) Bessel equation reads

$$x^2 w''(x) + x w'(x) + (x^2 - m^2) w = 0.$$

It is well known that the Bessel function of the first kind, J_m and J_{-m} (see (4.4)), solve this equation. Other solutions of the Bessel equations are the so-called Bessel functions of the third kind ([6]) or the Hankel functions:

$$H_m^{\pm}(z) = \frac{J_{-m}(z) - \mathrm{e}^{\mp \mathrm{i} m\pi} J_m(z)}{\pm \mathrm{i}\sin(m\pi)}.$$

(When m is an integer, one replaces the above expression by their limits). We have the relations

$$J_m(x) = \mathrm{e}^{\pm \mathrm{i}\pi\frac{m}{2}} I_m(\mp \mathrm{i}x), \quad H^{\pm}(x) = \mp\frac{2\mathrm{i}}{\pi}\mathrm{e}^{\mp\mathrm{i}\pi\frac{m}{2}} K_m(\mp\mathrm{i}x).$$

We know that H_m has no point spectrum. Hence, for any $a < b$ the Stone formula says

$$1\!\!1_{[a,b]}(H_m) = \mathrm{s-}\lim_{\epsilon\searrow 0} \frac{1}{2\pi\mathrm{i}} \int_a^b \left(G_m(\lambda + \mathrm{i}\epsilon) - G_m(\lambda - \mathrm{i}\epsilon)\right)\mathrm{d}\lambda. \tag{5.1}$$

Using (4.12) we can express the boundary values of the integral kernel of the resolvent at $\lambda \in]0, \infty[$ by solutions of the standard Bessel equation:

$$G_m(\lambda \pm \mathrm{i}0; x, y) \;:=\; \lim_{\epsilon \searrow 0} G_m(\lambda \pm \mathrm{i}\epsilon; x, y) \;=\; \begin{cases} \pm\frac{\pi \mathrm{i}}{2}\sqrt{xy} J_m(\sqrt{\lambda}x) H_m^{\pm}(\sqrt{\lambda}y) & \text{if} \quad x < y, \\ \pm\frac{\pi \mathrm{i}}{2}\sqrt{xy} J_m(\sqrt{\lambda}y) H_m^{\pm}(\sqrt{\lambda}x) & \text{if} \quad x > y. \end{cases}$$

Now

$$\begin{aligned} & \frac{1}{2\pi \mathrm{i}}\left(G_m(\lambda + \mathrm{i}0; x, y) - G_m(\lambda - \mathrm{i}0; x, y)\right) \\ = & \begin{cases} \frac{1}{4}\sqrt{xy} J_m(\sqrt{\lambda}x)\left(H_m^{+}(\sqrt{\lambda}y) + H_m^{-}(\sqrt{\lambda}y)\right) & \text{if} \quad x < y, \\ \frac{1}{4}\sqrt{xy} J_m(\sqrt{\lambda}y)\left(H_m^{+}(\sqrt{\lambda}y) + H_m^{-}(\sqrt{\lambda}y)\right) & \text{if} \quad x > y; \end{cases} \\ = & \frac{1}{2} J_m(\sqrt{\lambda}x) J_m(\sqrt{\lambda}y). \end{aligned}$$

Together with (5.1), this gives an expression for the integral kernel of the spectral projection of H_m, valid, say, as a quadratic form on $C_{\mathrm{c}}^{\infty}(\mathbb{R})$.

Proposition 5.1. *For $0 < a < b < \infty$, the integral kernel of $1\!\!1_{[a,b]}(H_m)$ is*

$$\begin{aligned} 1\!\!1_{[a,b]}(H_m)(x, y) &= \int_a^b \frac{1}{2}\sqrt{xy} J_m(\sqrt{\lambda}x) J_m(\sqrt{\lambda}y) \mathrm{d}\lambda \\ &= \int_{\sqrt{a}}^{\sqrt{b}} \sqrt{xy} J_m(kx) J_m(ky) k \mathrm{d}k. \end{aligned}$$

Let $\mathcal{F}_m$ be the operator on $L^2(0, \infty)$ given by

$$\mathcal{F}_m : f(x) \mapsto \int_0^{\infty} J_m(kx)\sqrt{kx} f(x) \mathrm{d}x. \tag{5.2}$$

Up to an inessential factor, $\mathcal{F}_m$ is the so-called Hankel transformation.

Theorem 5.2. *$\mathcal{F}_m$ is a unitary involution on $L^2(0, \infty)$ diagonalizing H_m, more precisely*

$$\mathcal{F}_m H_m \mathcal{F}_m^{-1} = Q^2.$$

It satisfies $\mathcal{F}_m \mathrm{e}^{\mathrm{i}tD} = \mathrm{e}^{-\mathrm{i}tD} \mathcal{F}_m$ for all $t \in \mathbb{R}$.

Proof. Obviously, $\mathcal{F}_m$ is hermitian. Proposition 5.1 can be rewritten as

$$1\!\!1_{[a,b]}(H_m) = \mathcal{F}_m 1\!\!1_{[a,b]}(Q^2) \mathcal{F}_m^{*}.$$

Letting $a \to 0$ and $b \to \infty$ we obtain $1\!\!1 = \mathcal{F}_m \mathcal{F}_m^{*}$. This implies that $\mathcal{F}_m$ is isometric. Using again the fact that it is hermitian we see that it is unitary. □

6. SCATTERING THEORY OF H_m

Let us now give a short and self-contained description of the scattering theory for the operators H_m with real m.

Theorem 6.1. *If $m, k > -1$ are real then the wave operators associated to the pair H_m, H_k exist and*

$$\Omega_{m,k}^{\pm} := \lim_{t \to \pm\infty} \mathrm{e}^{\mathrm{i}tH_m} \mathrm{e}^{-\mathrm{i}tH_k} = \mathrm{e}^{\pm \mathrm{i}(m-k)\pi/2} \mathcal{F}_m \mathcal{F}_k.$$

In particular the scattering operator $S_{m,k}$ for the pair (H_m, H_k) is a scalar operator: $S_{m,k} = \mathrm{e}^{\mathrm{i}\pi(m-k)} 1\!\!1$.

Proof. Note that $\Omega^\pm_{m,k} := \mathrm{e}^{\pm\mathrm{i}(m-k)\pi/2}\mathcal{F}_m\mathcal{F}_k$ is a unitary operator in L^2 such that $\mathrm{e}^{-\mathrm{i}tH_m}\Omega^\pm_{m,k} = \Omega^\pm_{m,k}\mathrm{e}^{-\mathrm{i}tH_k}$ for all t. Thus to prove the theorem it suffices to show that $(\Omega^\pm_{m.k} - 1)\mathrm{e}^{-\mathrm{i}tH_k} \to 0$ strongly as $t \to \pm\infty$. Let π_a be the operator of multiplication by the characteristic function of the interval $]0, a[$ and $\pi_a^\perp = 1 - \pi_a$. Then from Theorem 5.2 it follows easily that $\pi_a\mathrm{e}^{-\mathrm{i}tH_m} \to 0$ and $\pi_a\mathrm{e}^{-\mathrm{i}tH_k} \to 0$ strongly as $t \to \pm\infty$ for any $a > 0$. Thus we are reduced to proving

$$\lim_{a\to\infty}\sup_{\pm t>0}\|\pi_a^\perp(\Omega^\pm_{m,k} - 1)\mathrm{e}^{-\mathrm{i}tH_k}f\| = 0 \quad \text{for all } f \in L^2.$$

By using again Theorem 5.2 we get

$$(\Omega^\pm_{m,k} - 1)\mathrm{e}^{-\mathrm{i}tH_k} = \mathrm{e}^{\mp\mathrm{i}k\pi/2}(\mathrm{e}^{\pm\mathrm{i}m\pi/2}\mathcal{F}_m - \mathrm{e}^{\pm\mathrm{i}k\pi/2}\mathcal{F}_k)\mathrm{e}^{-\mathrm{i}tQ^2}\mathcal{F}_k,$$

hence it will be sufficient to show that

$$\lim_{a\to\infty}\sup_{\pm t>0}\|\pi_a^\perp(\mathrm{e}^{\pm\mathrm{i}k\pi/2}\mathcal{F}_k - \mathrm{e}^{\pm\mathrm{i}m\pi/2}\mathcal{F}_m)\mathrm{e}^{-\mathrm{i}tQ^2}g\| = 0 \quad \text{for all } g \in C_{\mathrm{c}}^\infty(\mathbb{R}_+). \tag{6.1}$$

Let us set $j_m(x) = \sqrt{x}J_m(x)$ and $\tau_m = m\pi/2 + \pi/4$. Then $(\mathcal{F}_m h)(x) = \int_0^\infty j_m(xp)h(p)\mathrm{d}p$, and from the asymptotics of the Bessel functions we get

$$\sqrt{\tfrac{\pi}{2}}j_m(y) = \cos(y - \tau_m) + j_m^\circ(y) \quad \text{where } j_m^\circ(y) \sim O(y^{-1}). \tag{6.2}$$

If we set $g_t(p) = (\pi/2)^{1/2}\mathrm{e}^{-\mathrm{i}tp^2}g(p)$ and $G_t^\pm = (\mathrm{e}^{\pm\mathrm{i}k\pi/2}\mathcal{F}_k - \mathrm{e}^{\pm\mathrm{i}m\pi/2}\mathcal{F}_m)g_t$, then

$$G_t^\pm(x) = \int(\mathrm{e}^{\pm\mathrm{i}k\pi/2}\cos(xp - \tau_k) - \mathrm{e}^{\pm\mathrm{i}m\pi/2}\cos(xp - \tau_m))g_t(p)\mathrm{d}p + \int(j_k^\circ(xp) - j_m^\circ(xp))g_t(p)\mathrm{d}p.$$

The second contribution to this expression is obviously bounded by a constant time $|x|^{-1}\int|g_t(p)/p|\mathrm{d}p$, and the $L^2(\mathrm{d}x)$ norm of this quantity over $[a, \infty[$ is less than $Ca^{-1/2}$ for some number C independent of t. Thus we may forget this term in the proof of (6.1).

Finally, we consider the first contribution to G_t^+, for example. Since

$$\mathrm{e}^{\mathrm{i}k\pi/2}\cos(xp - \tau_k) - \mathrm{e}^{\mathrm{i}m\pi/2}\cos(xp - \tau_m) = \mathrm{e}^{-\mathrm{i}xp+\mathrm{i}\pi/4}(\mathrm{e}^{\mathrm{i}k\pi} - \mathrm{e}^{\mathrm{i}m\pi})/2,$$

we get an integral of the form $\int \mathrm{e}^{-\mathrm{i}p(xp+tp)}g(p)\mathrm{d}p$, which is rapidly decaying in x uniformly in $t > 0$, because $g \in C_{\mathrm{c}}(\mathbb{R}_+)$ and there are no points of stationary phase. This finishes the proof of (6.1). □

Since H_m and H_k are homogeneous of degree -2 with respect to the operator D, which has simple spectrum, we can apply Proposition 2.9 with $A = D$ and deduce that the wave operators are functions of D. Our next goal is to give explicit formulas for these functions.

Let $\mathcal{J} : L^2 \to L^2$ be the unitary involution

$$\mathcal{J}f(x) = \frac{1}{x}f(\frac{1}{x}).$$

Clearly $\mathcal{J}\mathrm{e}^{\mathrm{i}\tau D} = \mathrm{e}^{-\mathrm{i}\tau D}\mathcal{J}$ for all $\tau \in \mathbb{R}$, and $\mathcal{J}Q^2\mathcal{J} = Q^{-2}$. In particular, the operator

$$\mathcal{G}_m := \mathcal{J}\mathcal{F}_m \tag{6.3}$$

is a unitary operator on L^2 which commutes with all the $\mathrm{e}^{\mathrm{i}\tau D}$. Hence there exists $\Xi_m : \mathbb{R} \to \mathbb{C}$, $|\Xi_m(x)| = 1$ a.e. and $\mathcal{G}_m = \Xi_m(D)$. Moreover, we have

$$\mathcal{F}_m\mathcal{F}_k = \mathcal{F}_m\mathcal{J}\mathcal{J}\mathcal{F}_k = \mathcal{G}_m^*\mathcal{G}_k,$$

so that

$$\Omega^\pm_{m,k} = \mathrm{e}^{\pm\mathrm{i}(m-k)\pi/2}\mathcal{G}_m^*\mathcal{G}_k = \mathrm{e}^{\pm\mathrm{i}(m-k)\pi/2}\frac{\Xi_k(D)}{\Xi_m(D)}.$$

Note that $\mathcal{G}_mH_m\mathcal{G}_m^* = \mathcal{J}Q^2\mathcal{J} = Q^{-2}$.

Theorem 6.2. *For $m > -1$,*

$$\mathcal{G}_m = \mathrm{e}^{\mathrm{i}\ln(2)D} \frac{\Gamma(\frac{m+1+\mathrm{i}D}{2})}{\Gamma(\frac{m+1-\mathrm{i}D}{2})}.$$

Therefore, for $m, k > -1$, the wave operators for the pair (H_m, H_k) are equal to

$$\Omega^{\pm}_{m,k} = \mathrm{e}^{\pm\mathrm{i}(m-k)\pi/2} \frac{\Gamma(\frac{k+1+\mathrm{i}D}{2})\Gamma(\frac{m+1-\mathrm{i}D}{2})}{\Gamma(\frac{k+1-\mathrm{i}D}{2})\Gamma(\frac{m+1+\mathrm{i}D}{2})}.$$

For the proof we need the following representation of Bessel functions:

Lemma 6.3. *For any m such that $\mathrm{Re}\,(m) > -1$ the following identity holds in the sense of distributions:*

$$J_m(x) = \frac{1}{4\pi} \int_{-\infty}^{+\infty} \frac{\Gamma(\frac{m+\mathrm{i}t+1}{2})}{\Gamma(\frac{m-\mathrm{i}t+1}{2})} \left(\frac{x}{2}\right)^{-\mathrm{i}t-1} \mathrm{d}t.$$

Proof. If $\mathrm{Re}\,(m) > 0$ one has the following representation of the Bessel function $J_m(x)$, cf. [6, ch. VI.5]:

$$\begin{aligned} J_m(x) &= \frac{1}{2\pi\mathrm{i}} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \frac{\Gamma(z)}{\Gamma(m-z+1)} \left(\frac{x}{2}\right)^{m-2z} \mathrm{d}z \\ &= \frac{1}{4\pi} \int_{-\infty}^{+\infty} \frac{\Gamma(c+\mathrm{i}\frac{t}{2})}{\Gamma(m+1-c-\mathrm{i}\frac{t}{2})} \left(\frac{x}{2}\right)^{m-2c-\mathrm{i}t} \mathrm{d}t, \end{aligned} \tag{6.4}$$

where $c \in \left]0, \dfrac{\mathrm{Re}\,m}{2}\right[$. Note that the subintegral function is everywhere analytic except for the poles at $z = 0, -1, -2, \dots$, all of them on the left hand side of the contour. By the Stirling asymptotic formula, the subintegral function can be estimated by $|z|^{-1-\mathrm{Re}\,m+2c}$ at infinity, hence it is integrable.

We shall extend the formula (6.4) for $\mathrm{Re}\,m > -1$ and $c \in]0, \mathrm{Re}\,(m)+1[$. For that purpose we have to understand it in the distributional sense, that is after smearing it with a function of x belonging to C_{c}^{∞}.

Let $\varphi \in C_{\mathrm{c}}^{\infty}$ and $\phi(z) := \dfrac{1}{4\pi} \displaystyle\int_0^{+\infty} \left(\frac{x}{2}\right)^z \varphi(x)\mathrm{d}x$. For $\mathrm{Re}\,m > 0$ and $0 < c < \frac{\mathrm{Re}\,m}{2}$ we thus have

$$\int_0^{\infty} J_m(x)\varphi(x)\mathrm{d}x = \int_{-\infty}^{+\infty} \frac{\Gamma(c+\mathrm{i}\frac{t}{2})}{\Gamma(m+1-c-\mathrm{i}\frac{t}{2})} \phi(m-2c-\mathrm{i}t)\mathrm{d}t. \tag{6.5}$$

Since $\varphi \in C_{\mathrm{c}}^{\infty}$, the function ϕ is holomorphic and for any $K \subset \mathbb{C}$ compact and $n \in \mathbb{N}$ there exists $C_{K,n}$ such that

$$|\phi(z+\mathrm{i}t)| \leq C_{K,n}\langle t\rangle^{-n}, \quad \forall z \in K,\, \forall t \in \mathbb{R}, \tag{6.6}$$

where $\langle t\rangle = \sqrt{1+t^2}$. Likewise, the function $z \mapsto \theta(z) = \dfrac{\Gamma(z)}{\Gamma(m+1-z)}$ is holomorphic in the strip $0 < \mathrm{Re}\,(z) < \mathrm{Re}\,(m)+1$, and for any compact $K \subset \mathbb{C}$ there exists $C_K > 0$ such that

$$|\theta(z+\mathrm{i}t)| \leq C_k \langle t\rangle^{2\mathrm{Re}\,(z) - \mathrm{Re}\,(m) - 1}, \quad \forall z \in K,\, \forall t \in \mathbb{R}. \tag{6.7}$$

Combining (6.6)-(6.7), this proves that the function

$$c \mapsto \int_{-\infty}^{+\infty} \frac{\Gamma(c+\mathrm{i}\frac{t}{2})}{\Gamma(m+1-c-\mathrm{i}\frac{t}{2})} \phi(m-2c-\mathrm{i}t)\mathrm{d}t$$

is holomorphic in the strip $0 < \mathrm{Re}\,(c) < \mathrm{Re}\,(m)+1$. Moroever, (6.5) shows that this function is constant equal to $\displaystyle\int_0^{\infty} J_m(x)\varphi(x)\mathrm{d}x$ for $c \in \left]0, \frac{\mathrm{Re}\,m}{2}\right[$. Hence (6.5) extends to any c such that $0 < \mathrm{Re}\,(c) < \mathrm{Re}\,(m)+1$. In particular, if we chose $c = \frac{\mathrm{Re}\,(m)+1}{2}$, we get for any m with $\mathrm{Re}\,(m) > 0$

$$\int_0^{\infty} J_m(x)\varphi(x)\mathrm{d}x = \frac{1}{4\pi} \int_0^{\infty} \mathrm{d}x \int_{-\infty}^{+\infty} \mathrm{d}t\, \frac{\Gamma(\frac{m+\mathrm{i}t+1}{2})}{\Gamma(\frac{m-\mathrm{i}t+1}{2})} \left(\frac{x}{2}\right)^{-\mathrm{i}t-1} \varphi(x). \tag{6.8}$$

Using (6.6)-(6.7) once more, one gets that the right-hand side of the above identity is holomorphic for $\mathrm{Re}\,(m) > -1$. Since the Bessel function J_m also depends on m in an holomorphic way, the left-hand side is holomorphic as well, and hence (6.8) extends to any m such that $\mathrm{Re}\,(m) > -1$, which ends the proof of the lemma. □

The next lemma will also be needed.

Lemma 6.4. *For a given distribution ψ, the operator $\psi(D)$ from C_c^∞ to $(C_c^\infty)'$ has integral kernel*

$$\psi(D)(x,y) = \frac{1}{2\pi\sqrt{xy}} \int_{-\infty}^{+\infty} \psi(t) \frac{y^{-\mathrm{i}t}}{x^{-\mathrm{i}t}} \mathrm{d}t.$$

Proof. We use the Mellin transformation $\mathcal{M} : L^2(0,\infty) \to L^2(\mathbb{R})$. We recall the formula for $\mathcal{M}$ and $\mathcal{M}^{-1}$:

$$\begin{aligned}(\mathcal{M}f)(s) &:= \frac{1}{\sqrt{2\pi}} \int_0^\infty \mathrm{d}x\, x^{-\frac{1}{2}-\mathrm{i}s} f(x) \\ (\mathcal{M}^{-1}g)(x) &:= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty \mathrm{d}s\, x^{-\frac{1}{2}+\mathrm{i}s} g(s).\end{aligned}$$

The Mellin transformation diagonalizes the operator of dilations, so that $\mathcal{M}\psi(D)\mathcal{M}^{-1}$ is the operator of multiplication by $\psi(s)$. □

Proof of Theorem 6.2. Using (5.2), (6.3) and Lemma 6.3 we get that the operator $\mathcal{G}_m$ has the integral kernel

$$\begin{aligned}\mathcal{G}_m(x,y) &= \frac{1}{x} J_m\left(\frac{y}{x}\right)\sqrt{\frac{y}{x}} \\ &= \frac{1}{2\pi\sqrt{xy}} \int_{-\infty}^{+\infty} \frac{\Gamma(\frac{m+\mathrm{i}t+1}{2})}{\Gamma(\frac{m-\mathrm{i}t+1}{2})} \left(\frac{1}{2}\right)^{-\mathrm{i}t} \frac{y^{-\mathrm{i}t}}{x^{-\mathrm{i}t}} \mathrm{d}t.\end{aligned}$$

Hence by Lemma 6.4, the unitary operator $\mathcal{G}_m$ coincides with $\Xi_m(D)$ on C_c^∞, where

$$\Xi_m(t) = \mathrm{e}^{\mathrm{i}\ln(2)t} \frac{\Gamma(\frac{m+1+\mathrm{i}t}{2})}{\Gamma(\frac{m+1-\mathrm{i}t}{2})}.$$

Since $|\Xi_m(t)| = 1$ for $m \in \mathbb{R}$, the operator $\Xi_m(D)$ is a unitary operator on L^2 which coincides with $\mathcal{G}_m$ on the dense subspace C_c^∞, and hence $\mathcal{G}_m = \Xi_m(D)$ on L^2.

□

Remark 6.5. It is interesting to note that $\Xi_m(D)$ is a unitary operator for all real values of m and

$$\Xi_m^{-1}(D) Q^{-2} \Xi_m(D) \tag{6.9}$$

is a function with values in self-adjoint operators for all real m. $\Xi_m(D)$ is bounded and invertible also for all m such that $\mathrm{Re}\, m \neq -1, -2, \ldots$. Therefore, the formula (6.9) defines an operator for all $\{m \mid \mathrm{Re}\, m \neq -1, -2, \ldots\} \cup \mathbb{R}$. Clearly, for $\mathrm{Re}\, m > -1$, this operator function coincides with the operator H_m studied in this paper. Its spectrum is always equal to $[0, \infty[$ and it is analytic in the interior of its domain.

One can then pose the following question: does this operator function extend to a holomorphic function of closed operators (in the sense of the definition of Subsec. 2.2) on the whole complex plane?

Appendix A. Second order differential operators

To make this paper self-contained we summarize in this appendix some facts on second order differential operators. We are especially interested in the case when the potential is complex and/or singular at the origin.

A.1. **Green functions.** We consider an arbitrary complex potential $V \in L^2_{\rm loc}$ and a complex number λ. Let $\tilde{L}$ be the distribution valued operator defined on $L^2_{\rm loc}$ by

$$\tilde{L} = -\partial_x^2 + V(x). \tag{A.1}$$

We recall that the Wronskian of two functions f, g of class C^1 on $\mathbb{R}_+$ is the function $W(f,g)$ whose value at a point $x > 0$ is given by $W_x(f,g) = f(x)g'(x) - f'(x)g(x)$. If f, g are solutions of an equation $u'' = Vu$ then $W(f,g)$ is a constant which is not zero if and only if f, g are linearly independent.

We recall a standard method for constructing the Green function of a differential operator. An elementary computation gives

Proposition A.1. *Suppose that u_0 and u_∞ are solutions of $\tilde{L}u = \lambda u$, which are square integrable near 0 and ∞ respectively, and such that $W(u_\infty, u_0) = 1$. Let $g \in L^2$, and define*

$$f_0 = u_0 g_\infty + u_\infty g_0 \quad \textit{with} \quad g_0(x) = \textstyle\int_0^x u_0(y)g(y)\mathrm{d}y, \quad g_\infty(x) = \textstyle\int_x^\infty u_\infty(y)g(y)\mathrm{d}y.$$

Then the function f_0 satisfies $(\tilde{L} - \lambda)f_0 = g$ and $f_0' = u_0' g_\infty - u_\infty' g_0$. The general solution of the equation $(\tilde{L} - \lambda)f = g$ can be written as $f = c_0 u_0 + c_\infty u_\infty + f_0$ with $c_0, c_\infty \in \mathbb{C}$. We have

$$f_0(x) = \int_0^\infty G(x,y)g(y)\mathrm{d}y \quad \textit{with} \quad G(x,y) = \begin{cases} u_0(x)u_\infty(y) & \text{if} \quad 0 < x < y, \\ u_0(y)u_\infty(x) & \text{if} \quad 0 < y < x. \end{cases}$$

A.2. **Maximal and minimal operators.** We denote $L_{\rm min}$ and $L_{\rm max}$ the minimal and maximal operator associated to the differential expression (A.1). More precisely, $L_{\rm max}$ is the restriction of $\tilde{L}$ to the space $\mathcal{D}(L_{\rm max}) := \{f \in L^2 \mid \tilde{L}f \in L^2\}$ considered as operator in L^2, and $L_{\rm min}$ is the closure of the restriction of $L_{\rm max}$ to $C_{\rm c}^\infty$. $L_{\rm max}$ is a closed operator on L^2, because it is a restriction of the continuous operator $\tilde{L} : L^2_{\rm loc} \to \mathcal{D}'(\mathbb{R}_+)$.

From now on we assume that $\sup_{b>a} \int_b^{b+1} |V(x)|\,\mathrm{d}x < \infty$ for each $a > 0$. Then the following is true (cf. [5]):

Proposition A.2. *If $f \in \mathcal{D}(L_{\rm max})$, then f and f' are continuous functions on $\mathbb{R}_+$ which tend to zero at infinity. For $f, g \in \mathcal{D}(L_{\rm max})$,*

$$\lim_{x \to 0} W_x(f,g) =: W_0(f,g) \tag{A.2}$$

exists and we have

$$\int_0^\infty (L_{\rm max}fg - fL_{\rm max}g)dx = -W_0(f,g). \tag{A.3}$$

In particular, W_0 is a continuous bilinear antisymmetric form on $\mathcal{D}(L_{\rm max})$ (equipped with the graph topology), and if one of the functions f or g belongs to $\mathcal{D}(L_{\rm min})$, then $W_0(f,g) = 0$.

Remark A.3. Note that the so defined $W_0(f,g)$ depends only on the restriction of f and g to an arbitrary neighborhood of zero. Hence *if f, g are continuous square integrable functions on an interval $]0,a[$ such that the distributions Lf and Lg are square integrable on $]0,a[$, then the limit in* (A.3) *exists and defines $W_0(f,g)$.*

If V is a real function, the operator $L_{\rm min}$ is hermitian and $L_{\rm min}^* = L_{\rm max}$. From (A.3) we get

$$\langle L_{\rm max}f, g\rangle - \langle f, L_{\rm max}g\rangle = -W_0(\bar{f}, g) \equiv \{f, g\}$$

for all $f, g \in \mathcal{D}(L_{\rm max})$. Here $\{f,g\}$ is a continuous hermitian sesquilinear form on $\mathcal{D}(L_{\rm max})$ which is zero on $\mathcal{D}(L_{\rm min})$. Moreover, an element $f \in \mathcal{D}(L_{\rm max})$ belongs to $\mathcal{D}(L_{\rm min})$ if and only if $\{f,g\} = 0$ for all $g \in \mathcal{D}(L_{\rm max})$. A subspace $\mathcal{E} \subset \mathcal{D}(L_{\rm max})$ will be called *hermitian* if it is closed, contains $\mathcal{D}(L_{\rm min})$, and the restriction of $\{\cdot,\cdot\}$ to it is zero. It is clear that H is a closed hermitian extension of $L_{\rm min}$ if and only if H is the restriction of $L_{\rm max}$ to a hermitian subspace.

Now we consider the case of complex V.

Lemma A.4. *Let $f \in \mathcal{D}(L_{\max})$. Then $f \in \mathcal{D}(L_{\min})$ if and only if $W_0(f,g) = 0$ for all $g \in \mathcal{D}(L_{\max})$.*

Proof. One implication is obvious. To prove the inverse assertion let us denote $\bar{\tilde{L}} = -\partial_x^2 + \bar{V}$ acting on continuous functions, and let $\bar{L}_{\min}, \bar{L}_{\max}$ be the minimal and maximal operators associated to $\bar{\tilde{L}}$. It is trivial to show that $L_{\min}^* = \bar{L}_{\max}$, hence $L_{\min} = \bar{L}_{\max}^*$ because $L_{\min}$ is closed. Thus $f \in L^2$ belongs to $\mathcal{D}(L_{\min})$ if and only if there is $h \in L^2$ such that $\langle \bar{L}_{\max} g, f\rangle = \langle g, h\rangle$ for all $g \in \mathcal{D}(\bar{L}_{\max})$. But $g \in \mathcal{D}(\bar{L}_{\max})$ if and only if $\bar{g} \in \mathcal{D}(L_{\max})$, so for $f \in \mathcal{D}(L_{\max})$ we get from (A.3)

$$\langle \bar{L}_{\max} g, f\rangle = \int_0^\infty \tilde{L}\bar{g} f \mathrm{d}x = \int_0^\infty \bar{g}\tilde{L} f \mathrm{d}x - W_0(\bar{g}, f) = \langle g, \tilde{L} f\rangle - W_0(\bar{g}, f).$$

Hence if $W_0(\bar{g}, f) = 0$ for all $g \in \mathcal{D}(\bar{L}_{\max})$, then $f \in \mathcal{D}(L_{\min})$. □

We denote $\mathcal{L} = \{u \mid \tilde{L}u = 0\}$, this is a two dimensional subspace of $\in C^1(\mathbb{R}_+)$ and if $u, v \in \mathcal{L}$ then $W(f,g)$ is a constant which is not zero if and only if u, v are linearly independent. By the preceding comments, if $u \in \mathcal{L}$ and $\int_0^1 |u|^2 dx < \infty$ then $f \mapsto W_0(u,f)$ defines a linear continuous form ℓ_u on $\mathcal{D}(L_{\max})$ which vanishes on $\mathcal{D}(L_{\min})$. Let L_u be the restriction of $L^{\max}$ to $\operatorname{Ker}\ell_u$. Clearly L_u is a closed operator on L^2 such that $L_{\min} \subset L_u \subset L_{\max}$.

A.3. **Extensions of $L_{\min}$.** Below by *strict extension* of $L_{\min}$ we mean an operator T such that $L_{\min} \subsetneq T \subsetneq L_{\max}$. We denote ξ a function in C_{c}^∞ such that $\xi(x) = 1$ for $x \le 1$ and $\xi(x) = 0$ for $x \ge 2$.

Until the end of the subsection we assume that all the solutions of the equation $\tilde{L}u = 0$ are square integrable at the origin.

Proposition A.5. *$\mathcal{D}(L_{\min})$ is a closed subspace of codimension two of $\mathcal{D}(L_{\max})$ and*

$$\mathcal{D}(L_{\min}) = \{f \in \mathcal{D}(L_{\max}) \mid W_0(u,f) = 0\ \forall u \in \mathcal{L}\} = \textstyle\bigcap_{u\in\mathcal{L}} \operatorname{Ker}\ell_u. \tag{A.4}$$

If $u \neq 0$ then L_u is a strict extension of $L_{\min}$ and, reciprocally, each strict extension of $L_{\min}$ is of this form. More explicitly, $\mathcal{D}(L_u) = \mathcal{D}(L_{\min}) + \mathbb{C}\xi u$. We have $L_u = L_v$ if and only if $v = cu$ with $c \in \mathbb{C}\setminus\{0\}$. If V is real, then the operator $L_{\min}$ is hermitian, has deficiency indices $(1,1)$, and if $u \in \mathcal{L}\setminus\{0\}$ then L_u is hermitian (hence self-adjoint) if and only if u is real (modulo a constant factor).

Proof. We first show that $\ell_u = 0$ if and only if $u = 0$. Indeed, if $u \neq 0$ then, the equation $Lv = 0$ has a solution linearly independent from u, so that $W(u,v) \neq 0$. But there is $g \in \mathcal{D}(L_{\max})$ such that $g = v$ on a neighborhood of zero, and then $\ell_u(g) = W(u,v) \neq 0$. This also proves the last assertion of the proposition.

Assume for the moment that (A.4) is known. If u, v are linearly independent elements of $\mathcal{L}$, then they are a basis of the vector space $\mathcal{L}$, hence we have $\mathcal{D}(L_{\min}) = \operatorname{Ker}\ell_u \cap \operatorname{Ker}\ell_v$, and so $\mathcal{D}(L_{\min})$ is of codimension two in $\mathcal{D}(L_{\max})$. Moreover, if $u \neq 0$, then $\mathcal{D}(L_{\min})$ is of codimension one in $\operatorname{Ker}\ell_u$, we have $\xi u \in \mathcal{D}(L_{\max}) \setminus \mathcal{D}(L_{\min})$ and $\xi u \in \operatorname{Ker}\ell_u$, hence $\mathcal{D}(L_u) = \mathcal{D}(L_{\min}) + \mathbb{C}\xi u$.

If V is real, the deficiency indices of $L_{\min}$ are $(1,1)$, because $\mathcal{D}(L_{\min})$ has codimension two in $\mathcal{D}(L_{\max})$. The space $\operatorname{Ker}\ell_u$ is hermitian if and only if $\{f,f\} = 0$ for all $f \in \operatorname{Ker}\ell_u$. But $\operatorname{Ker}\ell_u = \mathcal{D}(L_{\min}) + \mathbb{C}\xi u$, so we may write $f = f_0 + \lambda\xi u$, and then clearly $\{f,f\} = \{\lambda\xi u, \lambda\xi u\} = |\lambda|^2\{u,u\} = -|\lambda|^2 W_0(\bar{u}, u)$. So $\operatorname{Ker}\ell_u$ is hermitian if and only if $W_0(\bar{u}, u) = 0$. But $\bar{u}$ and u are solutions of the same equation $Lf = 0$, and $W(\bar{u}, u) = W_0(\bar{u}, u) = 0$. Thus $\bar{u}$ and u must be proportional, i.e. there is a complex number c such that $\bar{u} = cu$. Clearly $|c| = 1$, so we may write $c = \mathrm{e}^{2i\theta}$, and then we see that the function $\mathrm{e}^{i\theta}u$ is real.

Thus it remains to prove (A.4), and for this we need some preliminary considerations which will be useful in another context later on. Let $v_\pm \in \mathcal{L}$ such that $W(v_+, v_-) = 1$. If g is a function on $\mathbb{R}_+$ such that $\int_0^a |g|^2 dx < \infty$ for all $a > 0$, we set $g_\pm(x) = \int_0^x v_\pm(y) g(y) dy$. It is easy to check that if $Lf = g$, then there is a unique pair of complex numbers $a_\pm$ such that

$$f = (a_+ + g_-)v_+ + (a_- - g_+)v_- \tag{A.5}$$

and, reciprocally, if f is defined by (A.5), then $Lf = g$. Since $g_\pm' = v_\pm g$, we also have

$$f' = (a_+ + g_-)v_+' + (a_- - g_+)v_-'. \tag{A.6}$$

Now assume $h \in \mathcal{D}(L_{\max})$ and $W_0(u,h) = 0$ for all $u \in \mathcal{L}$. This is equivalent to $\ell_{v_\pm}(h) = 0$. We shall prove that $W_0(f,h) = 0$ for all $f \in \mathcal{D}(L_{\max})$, and this will imply $h \in \mathcal{D}(L_{\min})$ by Lemma A.4. If we set $v = a_+v_+ + a_-v_-$ and $f_0 = g_-v_+ - g_+v_-$, then we get $W_0(f,h) = W_0(f_0,h)$. Then

$$W_0(f_0,h) = W_0(g_-v_+ - g_+v_-, h) = \lim_{x\to 0}\left((g_-v_+ - g_+v_-)(x)h'(x) - (g_-v_+ - g_+v_-)'(x)h(x)\right).$$

For a fixed x we rearrange the last expression as follows:

$$g_-v_+h' - (g_-v_+)'h - g_+v_-h' + (g_+v_-)'h = g_-W_x(v_+,h) - g_+W_x(v_-,h) - g_-'v_+h + g_+'v_-h.$$

When $x \to 0$ the first two terms on the right hand side clearly converge to zero. The last two become $-gv_-v_+h + gv_+v_-h = 0$. This finishes the proof. □

Remark A.6. If zero is a regular endpoint, i.e. $\int_0^1 |V(x)|\mathrm{d}x < \infty$, then for each $f \in \mathcal{D}(L_{\max})$ the limits $\lim_{x\to 0} f(x) \equiv f(0)$ and $\lim_{x\to 0} f'(x) \equiv f'(0)$ exist. If V is real we easily get the classification of the self-adjoint realizations of L in terms of boundary conditions of the form $f(0)\sin\theta - f'(0)\cos\theta = 0$.

We point out now some consequences of the preceding proof. We denote $\|h\|_x$ the L^2 norm of a function h on the interval $]0,x[$. Then we get $|g_\pm(x)| \le \|v_\pm\|_x\|g\|_x$ for all $x > 0$, where the numbers $\|v_\pm\|_x$ are finite and tend to zero as $x \to 0$. Note that in general $\|v_\pm'\|_x = \infty$ for all x for at least one of the indices $\pm$. Anyway, we have

$$\begin{aligned}
|f(x) - (a_+v_+(x) + a_-v_-(x))| &\le \Big(|v_+(x)|\|v_-\|_x + |v_-(x)|\|v_+\|_x\Big)\|g\|_x,\\
|f'(x) - (a_+v_+'(x) + a_-v_-'(x))| &\le \Big(|v_+'(x)|\|v_-\|_x + |v_-'(x)|\|v_+\|_x\Big)\|g\|_x.
\end{aligned}$$

In other terms: if f is a solution of $Lf = g$, then there are complex numbers $a_\pm$ such that, as $x \to 0$,

$$f(x) = a_+v_+(x) + a_-v_-(x) + o(1)\Big(|v_+(x)|\|v_-\|_x + |v_-(x)|\|v_+\|_x\Big), \tag{A.7}$$

$$f'(x) = a_+v_+'(x) + a_-v_-'(x) + o(1)\Big(|v_+'(x)|\|v_-\|_x + |v_-'(x)|\|v_+\|_x\Big), \tag{A.8}$$

In the next proposition we continue to assume that all the solutions of the equation $Lu = 0$ are square integrable at the origin and keep the notations introduced in the proof of Proposition A.5.

Proposition A.7. *A function $f \in \mathcal{D}(L_{\max})$ belongs to $\mathcal{D}(L_{\min})$ if and only if $f = v_+g_- - v_-g_+$ with $g = Lf$. In particular, if $f \in D(L_{\min})$, then for $x \to 0$ we have*

$$f(x) = o(1)\Big(|v_+(x)|\|v_-\|_x + |v_-(x)|\|v_+\|_x\Big), \quad f'(x) = o(1)\Big(|v_+'(x)|\|v_-\|_x + |v_-'(x)|\|v_+\|_x\Big).$$

Proof. We take above $g = Lf$ and we get the relations (A.5), (A.6), (A.7) and (A.8) for some uniquely determined numbers $a_\pm$. If we set $v = a_+v_+ + a_-v_-$ and $f_0 = v_+g_- - v_-g_+$, then $f = v + f_0$. We know that $f \in D(L_{\min})$ if and only if $W_0(u,f) = 0$ for all $u \in \mathcal{L}$. Since $v_\pm$ form a basis in $\mathcal{L}$, it suffices in fact to have this only for $u = v_\pm$. We have $W_0(v_\pm, f_0) = 0$ because $f_0' = v_+'g_- - v_-'g_+$, so that

$$v_\pm f_0' - v_\pm' f_0 = v_\pm(v_+'g_- - v_-'g_+) - v_\pm'(v_+g_- - v_-g_+) = -g_\pm,$$

and $g_\pm(x) \to 0$ as $x \to 0$. Hence $W_0(v_\pm, f) = W_0(v_\pm, v) + W_0(v_\pm, f_0) = W_0(v_\pm, v) = \pm a_\mp$, and so $f \in D(L_{\min})$ if and only if $a_\pm = 0$, or if and only if $f = v_+g_- - v_-g_+$ with $g = Tf$. Thus, if $f \in D(L_{\min})$, then we have the relations (A.7) and (A.8) with $a_\pm = 0$, so we have the required asymptotic behaviours of f and f'. □

APPENDIX B. AHARONOV-BOHM HAMILTONIAN

Consider the Hilbert space $L^2(\mathbb{R}^2)$. We will use simultaneously the polar coordinates, r, ϕ, which identify this Hilbert space with $L^2(0,\infty) \otimes L^2(-\pi,\pi)$ by the unitary transformation

$$L^2(\mathbb{R}^2) \ni f \mapsto Uf \in L^2(0,\infty) \otimes L^2(-\pi,\pi)$$

given by $Uf(r,\phi) = \sqrt{r} f(r\cos\phi, r\sin\phi)$.

Let $\lambda \in \mathbb{R}$. We consider the magnetic hamiltonian associated to the magnetic potential $(\frac{\lambda y}{x^2+y^2}, -\frac{\lambda x}{x^2+y^2})$. The curl of this potential equals zero away from the origin of coordinates and the corresponding Hamiltonian (at least for real λ) is called the Aharonov-Bohm Hamiltonian. More precisely, let M_λ denote the minimal operator associated to the differential expression

$$M_\lambda := -\left(-i\partial_x - \frac{\lambda y}{x^2+y^2}\right)^2 - \left(-i\partial_y + \frac{\lambda x}{x^2+y^2}\right)^2, \tag{B.1}$$

a priori defined on $C_c^\infty(\mathbb{R}^2 \backslash \{0\})$. Clearly, M_λ is a positive hermitian operator, homogeneous of degree -2. In polar coordinates, M_λ becomes

$$M_\lambda = -\partial_r^2 + \frac{1}{r^2}\left[(-i\partial_\phi + \lambda)^2 - \frac{1}{4}\right]. \tag{B.2}$$

Let $L := -\mathrm{i}x\partial_y + \mathrm{i}y\partial_x$ be the angular momentum. $L = -\mathrm{i}\partial_\phi$ in polar coordinates. Then L commutes with M_λ (or equivalently, M_λ is rotation symmetric). L is a self-adjoint operator with the spectrum $\mathrm{sp}(L) = \mathbb{Z}$. Therefore, we have a direct sum decomposition $L^2(\mathbb{R}^2) = \underset{l\in\mathbb{Z}}{\oplus} \mathcal{H}_l$ where $\mathcal{H}_l$ is the spectral subspace of L for the eigenvalue l. With the help of U we can identify $\mathcal{H}_l$ with $L^2(\mathbb{R}_+)$.

Using (B.2), one immediately gets that

$$UM_\lambda U^* = \underset{l\in\mathbb{Z}}{\oplus} L_{l+\lambda}^{\min}. \tag{B.3}$$

Using general arguments, see Proposition 2.6, one easily gets that the Friedrichs and the Krein extensions of M_λ, denoted M_λ^{FF} and M_λ^{KK} respectively, are also homogeneous and rotation symmetric. (The reason for the double superscript will become apparent later).

Proposition B.1. *(i) If $\lambda \in \mathbb{Z}$, then M_λ has deficiency indices $(1,1)$. We have $M_\lambda^{FF} = M_\lambda^{KK}$, and M_λ has no other homogeneous extension.*

(ii) If $\lambda \notin \mathbb{Z}$, then M_λ has deficiency indices $(2,2)$. We have $M_\lambda^{FF} \neq M_\lambda^{KK}$, and M_λ has two other (distinct) homogeneous and rotation symmetric self-adjoint extensions M_λ^{FK} and M_λ^{KF}.

Remark B.2. *When $\lambda \notin \mathbb{Z}$, M_λ has also many homogeneous self-adjoint extensions which are not rotation symmetric.*

Remark B.3. *If V denotes the unitary operator such that $V = \mathrm{e}^{i\phi}$ in polar coordinates, then*

$$V^* M_\lambda V = M_{\lambda+1}. \tag{B.4}$$

Proof. Using (B.3), the deficiency indices of M_λ are (n,n) where $n = \sum_{l\in\mathbb{Z}} n_l$, and (n_l, n_l) are the deficiency indices of $L_{l+\lambda}^{\min}$. By Proposition 4.17, we have $n_l = 0$ unless $|l+\lambda| < 1$, in which case $n_l = 1$. Thus, if $\lambda \in \mathbb{Z}$, only the term with $l = -\lambda$ has nonzero deficiency indices, namely $n_{-\lambda} = 1$, and if $\lambda \notin \mathbb{Z}$, then $n_l = 1$ only when $l = -[\lambda] - 1$ and $l = -[\lambda]$, where $[\lambda]$ denotes the integer part of λ. This proves the assertions concerning the deficiency indices.

Using (B.4), we can then restrict ourselves to the case $0 \leq \lambda < 1$. The result follows from the analysis of Section 4.4. If $\lambda = 0$, the only term which is not self-adjoint in the decomposition of M_0 is $L_0^{\min}$. Using Proposition 4.15 we see that M_0 has a unique homogeneous self-adjoint extension. Since M_0^{FF} and M_0^{KK} are both homogeneous, they necessarily coincide.

We then turn to the case $0 < \lambda < 1$. Only the terms $L_{\lambda-1}^{\min}$ and $L_{\lambda}^{\min}$ are not self-adjoint. Using Proposition 4.15 again, each of these term has exactly two homogeneous extensions $H_{\pm(\lambda-1)}$ and $H_{\pm\lambda}$ respectively, those with a $+$ sign corresponding to the Friedrichs extension and those with a $-$ sign to the Krein extension. Hence M_λ has 4 distinct homogeneous *and* rotation symmetric self-adjoint extensions. The super-indices FF, KK, FK and KF correspond to the choice of the two extensions (the first index for the extension of $L_{\lambda-1}^{\min}$). □

We can then apply the results of Section 4.2 to study the analiticity properties of the various homogeneous extensions of M_λ.

Theorem B.4. *Let $n \in \mathbb{Z}$. For any $\# \in \{FF, KK, FK, KF\}$ the map $]n, n+1[\, \ni \lambda \mapsto M_\lambda^\#$ extends to a holomorphic family $M_z^\#$ on the strip $\{n < \mathrm{Re}\,(z) < n+1\}$. Moreover,*

(i) the family $z \mapsto M_z^{FF}$ can be extended to a holomorphic family on the strip $\{n-1 < \mathrm{Re}\,(z) < n+2\}$.
(ii) the family $z \mapsto M_z^{FK}$ can be extended to a holomorphic family on the strip $\{n-2 < \mathrm{Re}\,(z) < n+1\}$.
(iii) the family $z \mapsto M_z^{KF}$ can be extended to a holomorphic family on the strip $\{n < \mathrm{Re}\,(z) < n+3\}$.

Proof. Using Proposition B.1, for any $\lambda \in\,]n, n+1[$, we have

$$M_\lambda^\# = \underset{l \leq -n-2}{\oplus} H_{-l-\lambda} \oplus H_{\pm(\lambda-n-1)} \oplus H_{\pm(\lambda-n)} \underset{l \geq -n+1}{\oplus} H_{l+\lambda}. \tag{B.5}$$

Using Theorem 4.2, the components $H_{-l-\lambda}$ (for $l \leq -n-2$) have an analytic extension to the half-plane $\mathrm{Re}\,(z) < -l+1$, the components $H_{l+\lambda}$ (for $l \geq -n+1$) have an analytic extension to the half-plane $\mathrm{Re}\,(z) > -l-1$. Similarly, $H_{\lambda-n-1}$ (the Krein extension of $L_{\lambda-n-1}^{\min}$) has an extension to the half-plane $\mathrm{Re}\,(z) > n$, $H_{-\lambda+n+1}$ to the half-plane $\mathrm{Re}\,(z) < n+2$, $H_{\lambda-n}$ to the half-plane $\mathrm{Re}\,(z) > n-1$ and $H_{-\lambda+n}$ to the half-plane $\mathrm{Re}\,(z) < n+1$. The result then easily follows. □

Remark B.5. *The value at $z = n$ of both families M_z^{FK} and M_z^{FF} coincides with the unique homogeneous extension of M_n.*

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