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Level shift operator and second order perturbation theory

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We give approximate formulas for spectrum and the corresponding spectral projections of perturbed linear operators. The main tool is the so-called *level shift operator*, which expresses the effects of second order perturbation theory on the point spectrum. © *2005 American Institute of Physics.* [DOI: 10.1063/1.1850833]

I. INTRODUCTION

One of the main tools of quantum mechanics is perturbation theory for eigenvalues of family of linear operators of the form $L_\lambda := L_0 + \lambda Q$. This theory is particularly simple if one considers an isolated eigenvalue of L_0 of finite degeneracy and one assumes that L_0 and Q are self-adjoint. In this case, both the eigenvalues and the eigenvectors can be described by functions analytic in the coupling constant $e(\lambda)$. This is described in almost every textbook on quantum mechanics.

In quantum mechanics self-adjoint operators play a prominent role. However, non-self-adjoint operators are also physically relevant. For instance, they are used to describe resonances. In fact, resonances are often defined as complex eigenvalues of analytically deformed Hamiltonians, which are usually non-self-adjoint. The perturbation theory of non-self-adjoint operators is more complicated than that of self-adjoint operators. In the case of non-self-adjoint operators, eigenvalues and eigenvectors are typically described by a multivalued analytic function with a branch point at $\lambda = 0$. This is described, e.g., in Refs. 11 and 16.

The method of analytic functions may be inapplicable if the isolated eigenvalue has infinite degeneracy, because it may then happen that the perturbed operator has continuous spectrum close to the unperturbed eigenvalue. Thus one cannot follow individual eigenvalues.

In practice one is not interested in the full perturbation expansion of eigenvalues or eigenvectors. One usually uses the lowest order approximation. The first order approximation to the eigenvalue is very simple—it is just the appropriate matrix element of the perturbation. More interesting is the second order approximation. Its importance has been noted since the early days of quantum mechanics. Not without a reason the computations based on the second order perturbation theory have been called by Fermi the golden rule of quantum mechanics.

In our paper we describe a method of constructing approximate eigenvalues and approximate eigenprojections that summarizes the usual second order perturbation theory. We do not restrict ourselves to self-adjoint operators. We prove that our construction can be applied without any problem in the case when the eigenvalue has infinite multiplicity. Thus, the formulas that we give are quite robust—they do not need the assumptions typical of the usual approach to perturbation theory through expansion in a power series.

Let us now describe our results a little more closely. Suppose that L_0 is a closed operator having a cluster of isolated eigenvalues Ξ. The spectral projection of L_0 onto Ξ, denoted $\mathbf{1}_\Xi(L_0)$, gives a natural decomposition of the Banach space into a direct sum $\mathcal{H} = \mathcal{H}^v \oplus \mathcal{H}^{\bar{v}}$.

Let Q be a perturbation. Our main object is the perturbed operator

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$$L_{\lambda,\beta} := L_0 + \lambda(Q^{v\bar{v}} + Q^{\bar{v}v}) + \beta Q^{\bar{v}\bar{v}}.$$

Note that we assume that inside $\mathcal{H}^v$ the perturbation is zero. This guarantees that there is no first order shift of the spectrum. $Q^{v\bar{v}}+Q^{\bar{v}v}$ is the "off-diagonal" part of the perturbation—it connects $\mathcal{H}^v$ and $\mathcal{H}^{\bar{v}}$. $Q^{\bar{v}\bar{v}}$ is the "external" part of the perturbation—it acts inside $\mathcal{H}^{\bar{v}}$. We use two perturbation parameters, λ for the off-diagonal and β for the diagonal part. We are interested in what happens for small complex λ and β. We will try to estimate carefully the deviations from our predictions in terms of these two coupling constants.

It is easy to see that for small λ and β, the spectrum of $L_{\lambda,\beta}$ does not differ much from the spectrum L_0. Thus, for small λ and β, for any isolated point e of Ξ, there exists a patch of spectrum of $L_{\lambda,\beta}$ located around e, which we will denote by Θ_e. (We say "a patch," not "a cluster," because the spectrum does not have to be discrete.) We will give the formula for a projection p_e that approximates the spectral projection of $L_{\lambda,\beta}$ onto Θ_e. We will show that this projection approximately diagonalizes $L_{\lambda,\beta}$. By this we mean that $L-p_eL_{\lambda,\beta}p_e-(\mathbf{1}-p_e)L_{\lambda,\beta}(\mathbf{1}-p_e)$ is small.

The above results are contained in Theorem 2.1. They are quite easy. What is more interesting is the study of the splitting of the patch Θ_e, which is the subject of Theorem 2.3—the main result of our paper.

We show that if the eigenvalue e is semisimple then the patch of the spectrum around e naturally splits into subpatches separated by a distance of order $O(|\lambda|^2)$. The subpatches will be parametrized by eigenvalues of the so-called *level shift operator* (LSO). The level shift operator is a certain operator that describes the shift of spectrum under the influence of second order perturbation theory. The subpatch of the spectrum of $L_{\lambda,\beta}$ around $e+\lambda^2m$, where m is an eigenvalue of the LSO, will be denoted by $\Theta_{e,m}$. We will also give a formula for the projection $p_{e,m}$ that approximates the spectral projection of $L_{\lambda,\beta}$ onto $\Theta_{e,m}$. Finally, we will show that $p_{e,m}$ approximately diagonalizes $L_{\lambda,\beta}$.

Clearly, the results that we present are quite general and applicable in many situations. The main motivation for our paper comes, however, from the class of problems first considered by Jaksic and Pillet in Refs. 12, 13, and 15. Using the terminology of Ref. 7 we can say that the results of our paper can be used to describe approximately resonances of Pauli–Fierz Liouvillean. The last section is devoted to a short description of this application.

Let us briefly explain what we mean by resonances of Pauli–Fierz Liouvilleans. We use the name "Pauli–Fierz system" to describe a quantum system consisting of a small system (e.g., an atom) interacting with a bosonic field (e.g., photons or phonons). We are especially interested in the case when the field has a positive density, for instance it is at a positive temperature. The dynamics of this system is generated by a certain self-adjoint operator, which, following Ref. 7, we call the "Pauli–Fierz Liouvillean." Next we apply the so-called Jaksic–Pillet method and we obtain an analytically deformed Pauli–Fierz Liouvillean. Analytically deformed Liouvilleans are non-self-adjoint and have spectrum in the lower half-plane. Moreover, they often have isolated eigenvalues. These eigenvalues are called resonances. They do not depend on the parameter of deformation. They are physically relevant, they are responsible for the decay of certain correlation functions. They can be naturally written as the sum of an explicit operator with discrete eigenvalues and a perturbation. The method of our paper allows us to give approximate predictions about the resonances.

Another class of operators (not discussed in our paper) where our results could be applied are the generators of a Pauli–Fierz dynamics on an operator algebra (called C-Liouvilleans in Ref. 14).

Level shift operators appear in the mathematics and physics literature in various disguises whenever the second order perturbation theory is considered. They are often introduced in the case of embedded eigenvalues. For instance, they appeared implicitly in the work of Ref. 6 devoted to the perturbation theory for embedded eigenvalues of Pauli–Fierz operators. The analysis of the point spectrum given in Ref. 6 is very closely related to the analysis given in our paper. Nevertheless, there are some differences. Reference 6 was devoted to the study of *embedded* eigenvalues, and therefore additional tools were required: the limiting absorption principle and Mourre's positive commutator method. Another difference is the self-adjointness of the operator studied in Ref. 6, whereas in our paper we do not restrict ourselves to self-adjoint operators.

Constructions similar to ours can be found in the papers of Bach–Fröhlich–Sigal.[1,2] The authors study the spectrum of certain operators (similar to the Pauli–Fierz operators considered in Refs. 6 and 7) by an iterative procedure ("renormalization group"). The basic step of this procedure resembles our prescription for locating the spectrum and constructing the approximate spectral projection.

The LSO appears naturally in the so-called weak coupling (van Hove) limit.[4,5,8] In this context it is sometimes called the Davies generator.

II. MAIN RESULTS

A. Notation

If $\Xi \subset \Theta \subset \mathbb{C}$, then we say that Ξ is an isolated subset of Θ if it is closed and open in the relative topology of Θ.

Θ^{cl} denotes the closure of Θ in $\mathbb{C}$.

If L is a linear operator, $\mathrm{sp}\, L$ denotes its spectrum and $\mathrm{Dom}\, L$ its domain. If Ξ is an isolated and bounded subset of $\mathrm{sp}\, L$, then we can define the spectral projection of L onto Ξ by the formula

$$\mathbf{1}_{\Xi}(L) = \frac{1}{2\pi i}\oint_{\gamma} (z-L)^{-1}\,\mathrm{d}z,$$

where γ is a closed path that encircles Ξ counterclockwise.

If e is an isolated point of $\mathrm{sp}\, L$, then we will write $\mathbf{1}_e(L)$ for $\mathbf{1}_{\{e\}}(L)$. For such e set $N_e := (L-e)\mathbf{1}_e(L)$. We say that the degree of nilpotence of e is equal to n iff $N_e^{n-1} \neq 0$ but $N_e^n = 0$. We say that e is semisimple iff $n=1$ (i.e., $N_e = 0$).

If $\Theta \subset \mathbb{C}$ and $\epsilon > 0$, then we set

$$\mathrm{D}(\Theta,\epsilon) := \{z \in \mathbb{C} : \mathrm{dist}(z,\Theta) < \epsilon\}.$$

For $e \in \mathbb{C}$, $\mathrm{D}(e,\epsilon)$ will denote the open disc centered at e with radius ϵ. Moreover, we set $\mathrm{D}(\emptyset,\epsilon) := \emptyset$.

If $A(\lambda,\beta)$ are bounded operators, and $f(\lambda,\beta)$ a positive function, then

$$A(\lambda,\beta) = O(f(\lambda,\beta))$$

means that there exists c such that

$$\|A(\lambda,\beta)\| \leq c f(\lambda,\beta).$$

Moreover,

$$A_1(\lambda,\beta) = A_2(\lambda,\beta) + O(f(\lambda,\beta))$$

or

$$A_1(\lambda,\beta) \overset{O(f(\lambda,\beta))}{=} A_2(\lambda,\beta)$$

means that

$$A_1(\lambda,\beta) - A_2(\lambda,\beta) = O(f(\lambda,\beta)).$$

B. Assumptions

Let L_0 be a closed operator on a Banach space $\mathcal{H}$. Suppose that Ξ is an isolated bounded subset of $\mathrm{sp}\, L_0$.

It will be convenient to denote $\mathbf{1}_{\Xi}(L_0)$ by $\mathbf{1}^{vv}$ and set $\mathbf{1}^{\overline{vv}} := \mathbf{1} - \mathbf{1}^{vv}$. We can also introduce the subspaces

$$\mathcal{H}^{v} := \mathbf{1}^{vv}\mathcal{H}, \quad \mathcal{H}^{\overline{v}} := \mathbf{1}^{\overline{vv}}\mathcal{H},$$

so that $\mathcal{H}$ is decomposed into a direct sum

$$\mathcal{H} = \mathcal{H}^{v} \oplus \mathcal{H}^{\overline{v}}. \tag{2.1}$$

With respect to the decomposition (2.1) any operator B on $\mathcal{H}$ satisfying

$$\mathrm{Dom}(B) = (\mathrm{Dom}(B) \cap \mathcal{H}^{v}) \oplus (\mathrm{Dom}(B) \cap \mathcal{H}^{\overline{v}})$$

can be written as

$$B = \begin{bmatrix} B^{vv} & B^{v\overline{v}} \\ B^{\overline{v}v} & B^{\overline{vv}} \end{bmatrix}. \tag{2.2}$$

In particular, we have

$$L_0 = \begin{bmatrix} L_0^{vv} & 0 \\ 0 & L_0^{\overline{vv}} \end{bmatrix}. \tag{2.3}$$

It will be convenient to write E for L_0^{vv}. Note that E is a bounded operator on $\mathcal{H}^{v}$ and $\mathrm{sp}(E) = \Xi$.

Let Q be another operator, that we will treat as a perturbation of L_0. More precisely, we make the following assumptions.

Assumption 2.A: $Q^{vv} = 0$.

Assumption 2.B: Off-diagonal elements of Q, *i.e.,* $Q^{v\overline{v}}$ *and* $Q^{\overline{v}v}$, *are bounded.*

We will also use either one of the following two assumptions.

Assumption 2.C: $Q^{\overline{vv}}$ *is an operator bounded perturbation of* $L_0^{\overline{vv}}$ (Ref. 11).

Assumption 2.D: $\mathcal{H}^{\overline{vv}}$ *is a Hilbert space,* $L_0^{\overline{vv}}$ *is self-adjoint, bounded from below and* $Q^{\overline{vv}}$ *is a form bounded perturbation of* $L_0^{\overline{vv}}$ (Ref. 11).

Let $\lambda, \beta \in \mathbb{C}$. Note that under Assumption 2.C or 2.D the operator $L_0^{\overline{vv}} + \beta Q^{\overline{vv}}$ is well defined for small enough β (Ref. 11). Likewise,

$$L_{\lambda,\beta} := L_0 + \lambda Q^{v\overline{v}} + \lambda Q^{\overline{v}v} + \beta Q^{\overline{vv}} = \begin{bmatrix} E & \lambda Q^{v\overline{v}} \\ \lambda Q^{\overline{v}v} & L_0^{\overline{vv}} + \beta Q^{\overline{vv}} \end{bmatrix}$$

is well defined for small enough β. For simplicity we will write L instead of $L_{\lambda,\beta}$.

Fix an open subset $\Omega \subset \mathbb{C}$ such that $\Omega^{\mathrm{cl}} \cap \mathrm{sp}\, L_0 = \Xi$ and $\Xi \subset \Omega$. Note that there exists β_0 such that, for $|\beta| \leqslant \beta_0$, $\mathrm{sp}(L_0^{\overline{vv}} + \beta Q^{\overline{vv}}) \cap \Omega^{\mathrm{cl}} = \emptyset$. We fix β_0 satisfying these conditions.

C. Results

The main results of our paper are stated in the following two theorems. Note that Theorem 2.1 is quite easy and basically describes the well-known stability of spectrum under a perturbation. Theorem 2.3 is more difficult—it describes the splitting of the spectrum according to second order perturbation theory. In that theorem, an important role is played by the level shift operator. Note that we tried to make the two theorems as parallel as possible.

Theorem 2.1: *Suppose that Assumptions 2.A and 2.B hold. We also suppose that either Assumption 2.C or 2.D is satisfied. Then the following is true:*

(1) *There exists a continuous and increasing function*

$$[0,\infty[\, \ni x \mapsto \delta(x) \in [0,\infty],$$

such that $\lim_{x\to 0} \delta(x) = 0$, *and for* $|\beta| < \beta_0$ *and* $|\lambda| < \lambda_0$, *for some* $\lambda_0 > 0$, *we have*

$$\mathrm{sp}(L) \cap \Omega \subset \mathrm{D}(\mathrm{sp}\, E, \delta(|\lambda|))^{\mathrm{cl}}. \tag{2.4}$$

(2) *In what follows we assume that $\mathcal{E}$ is an isolated subset of* sp E. *Clearly,* (1) *implies that there exists* $0 < \lambda_{\mathcal{E}}$ *such that, for* $|\lambda| < \lambda_{\mathcal{E}}$,

$$\Theta_{\mathcal{E}} := \mathrm{D}(\mathcal{E}, \delta(|\lambda|))^{\mathrm{cl}} \cap \mathrm{sp}\, L \tag{2.5}$$

is an isolated subset of sp L *and* $\Theta_{\mathcal{E}} \subset \Omega$.

(3) *For* $|\lambda| < \lambda_{\mathcal{E}}$ *we have*

$$\mathbf{1}_{\Theta_{\mathcal{E}}}(L) - \mathbf{1}_{\mathcal{E}}(L_0) = O(|\lambda|). \tag{2.6}$$

(4) *For* $|\lambda| < \lambda_{\mathcal{E}}$ *we have*

$$\dim \mathbf{1}_{\Theta_{\mathcal{E}}}(L) = \dim \mathbf{1}_{\mathcal{E}}(L_0). \tag{2.7}$$

(5) *In what follows we assume that e is an isolated point of* sp E. *We will write* Θ_e *for* $\Theta_{\{e\}}$. *If the degree of nilpotence of e as an eigenvalue of E is equal to n, then there exists* C_e *such that*

$$\Theta_e \subset \mathrm{D}(e, C_e|\lambda|^{2/n})^{\mathrm{cl}}.$$

(6) *For*

$$|\lambda| < \|\mathbf{1}_e(L_0) Q^{v\bar{v}}(e\mathbf{1}^{\bar{v}\bar{v}} - L_0^{\bar{v}\bar{v}})^{-2} Q^{\bar{v}v} \mathbf{1}_e(L_0)\|^{-1/2} =: \widehat{\lambda_e}, \tag{2.8}$$

we set

$$p_e := (\mathbf{1}_e(L_0) + \lambda(e\mathbf{1}^{\bar{v}\bar{v}} - L_0^{\bar{v}\bar{v}})^{-1} Q^{\bar{v}v} \mathbf{1}_e(L_0))(\mathbf{1}_e(L_0) + \lambda^2 \mathbf{1}_e(L_0) Q^{v\bar{v}}(e\mathbf{1}^{\bar{v}\bar{v}} - L_0^{\bar{v}\bar{v}})^{-2} Q^{\bar{v}v} \mathbf{1}_e(L_0))^{-1}$$
$$\times (\mathbf{1}_e(L_0) + \lambda \mathbf{1}_e(L_0) Q^{v\bar{v}}(e\mathbf{1}^{\bar{v}\bar{v}} - L_0^{\bar{v}\bar{v}})^{-1}). \tag{2.9}$$

Then p_e *is a projection. Moreover,*

(a)

$$\mathbf{1}_{\Theta_e}(L) - p_e = O(|\lambda|); \tag{2.10}$$

(b) *if e is a semisimple eigenvalue of E then*

$$\mathbf{1}_{\Theta_e}(L) - p_e = O(|\lambda|^2 + |\lambda\beta|); \tag{2.11}$$

(c) *if, in addition,* $\mathrm{sp}(E) = \{e\}$, *then*

$$\mathbf{1}_{\Theta_e}(L) - p_e = O(|\lambda|^3 + |\lambda\beta|). \tag{2.12}$$

(7) *For* $|\lambda| < \widehat{\lambda_e}$ *we have*

(a)

$$L - p_e L p_e - (\mathbf{1} - p_e) L (\mathbf{1} - p_e) = O(|\lambda|); \tag{2.13}$$

(b) *if e is a semisimple eigenvalue of E then*

$$L - p_e L p_e - (\mathbf{1} - p_e) L (\mathbf{1} - p_e) = O(|\lambda|^2 + |\lambda\beta|); \tag{2.14}$$

(c) *if, moreover,* $\mathrm{sp}(E) = \{e\}$ *then*

$$L - p_e L p_e - (\mathbf{1} - p_e) L (\mathbf{1} - p_e) = O(|\lambda|^3 + |\lambda\beta|). \tag{2.15}$$

Note that in Eq. (2.9) we use the notation $(e\mathbf{1}^{\bar{v}\bar{v}} - L^{\bar{v}\bar{v}})^{-1}$ for the inverse of the operator $e\mathbf{1}^{\bar{v}\bar{v}} - L^{\bar{v}\bar{v}}$ *restricted to* $\mathcal{H}^{\bar{v}}$. In what follows we will often use similar notation without a comment.

Let us now assume that sp E is a finite set.

Definition 2.2: We define the level shift operator (LSO) as

$$\Gamma \coloneqq \sum_{e \in \mathrm{sp}(E)} \mathbf{1}_e(E) Q^{v\overline{v}} (e \mathbf{1}^{\overline{v v}} - L_0^{\overline{v v}})^{-1} Q^{\overline{v} v} \mathbf{1}_e(E).$$

From now on we will write for shortness $\Gamma^{ee} \coloneqq \mathbf{1}_e(E)\Gamma\mathbf{1}_e(E)$.

Now we are ready to state our main theorem.

Theorem 2.3: Suppose that Assumptions 2.A and 2.B hold. We also assume either Assumption 2.C or 2.D. Assume also that $\mathrm{sp}\, E$ *is a finite set consisting of semisimple eigenvalues. Then the following is true:*

(1) *There exists a continuous and increasing function*

$$[0,\infty[\, \ni x \mapsto \delta(x) \in [0,\infty],$$

such that $\lim_{x\to 0} \delta(x)=0$, *and, for* $|\beta| < \beta_0$, $|\lambda| < \lambda_0$, *for some* $\lambda_0 > 0$, *we have*

$$\mathrm{sp}(L) \cap \Omega \subset \mathrm{D}(\mathrm{sp}(E+\lambda^2\Gamma), |\lambda|^2\delta(|\lambda|^2+|\beta|))^{\mathrm{cl}} = \bigcup_{e\in \mathrm{sp}(E)} \mathrm{D}(e+\lambda^2\,\mathrm{sp}(\Gamma^{ee}), |\lambda|^2\delta(|\lambda|^2+|\beta|))^{\mathrm{cl}}. \tag{2.16}$$

(2) *In what follows we fix* $e \in \mathrm{sp}\, E$, *and* $\mathcal{M}$ *is an isolated subset of* $\mathrm{sp}\,\Gamma^{ee}$. *Clearly, (1) implies that there exists* $0 < \lambda_{e,\mathcal{M}}$ *and* $0 < \beta_{e,\mathcal{M}}$ *such that for* $|\lambda| < \lambda_{e,\mathcal{M}}$ *and* $|\beta| < \beta_{e,\mathcal{M}}$,

$$\Theta_{e,\mathcal{M}} \coloneqq \mathrm{D}(e+\lambda^2\mathcal{M}, |\lambda|^2\delta(|\lambda|^2+|\beta|))^{\mathrm{cl}} \cap \mathrm{sp}\, L$$

is an isolated subset of $\mathrm{sp}\, L$ *and* $\Theta_{e,\mathcal{M}} \subset \Omega$.

(3) *For* $|\lambda| < \lambda_{e,\mathcal{M}}$ *and* $|\beta| < \beta_{e,\mathcal{M}}$ *we have*

$$\mathbf{1}_{\Theta_{e,\mathcal{M}}}(L) - \mathbf{1}_{\mathcal{M}}(\Gamma^{ee}) = O(|\lambda|+|\beta|). \tag{2.17}$$

(4) *For* $|\lambda| < \lambda_{e,\mathcal{M}}$ *and* $|\beta| < \beta_{e,\mathcal{M}}$ *we have*

$$\dim \mathbf{1}_{\Theta_{e,\mathcal{M}}}(L) = \dim \mathbf{1}_{\mathcal{M}}(\Gamma^{ee}). \tag{2.18}$$

(5) *Assume now that m is an isolated point of* $\mathrm{sp}\,\Gamma^{ee}$. *We will write* $\Theta_{e,m}$ *for* $\Theta_{e,\{m\}}$. *Suppose that the degree of nilpotence of m as an eigenvalue of* Γ^{ee} *is equal to n. Then*

$$\Theta_{e,m} \subset \mathrm{D}(e+\lambda^2 m, C_{e,m}|\lambda|^2(|\lambda|^2+|\beta|)^{1/n})^{\mathrm{cl}},$$

for some $C_{e,m} > 0$.

(6) *For*

$$|\lambda| < \|\mathbf{1}_m(\Gamma^{ee}) Q^{v\overline{v}} (e\mathbf{1}^{\overline{v v}} - L_0^{\overline{v v}})^{-2} Q^{\overline{v} v}\mathbf{1}_m(\Gamma^{ee})\|^{-1/2} \eqqcolon \widehat{\lambda_{e,m}}, \tag{2.19}$$

we set

$$\begin{aligned} p_{e,m} \coloneqq & (\mathbf{1}_m(\Gamma^{ee}) + \lambda(e\mathbf{1}^{\overline{v v}} - L_0^{\overline{v v}})^{-1} Q^{\overline{v} v}\mathbf{1}_m(\Gamma^{ee}))(\mathbf{1}_m(\Gamma^{ee}) + \lambda^2 \mathbf{1}_m(\Gamma^{ee}) Q^{v\overline{v}}(e\mathbf{1}^{\overline{v v}} - L_0^{\overline{v v}})^{-2} \\ & \times Q^{\overline{v} v}\mathbf{1}_m(\Gamma^{ee}))^{-1}(\mathbf{1}_m(\Gamma^{ee}) + \lambda \mathbf{1}_m(\Gamma^{ee}) Q^{v\overline{v}}(e\mathbf{1}^{\overline{v v}} - L_0^{\overline{v v}})^{-1}). \end{aligned} \tag{2.20}$$

Then $p_{e,m}$ *is a projection. Moreover,*

(a)

$$\mathbf{1}_{\Theta_{e,m}}(L) - p_{e,m} = O(|\lambda|+|\beta|); \tag{2.21}$$

(b) *if m is a semisimple eigenvalue of* Γ^{ee} *then*

$$\mathbf{1}_{\Theta_{e,m}}(L) - p_{e,m} = O(|\lambda|^2+|\beta|). \tag{2.22}$$

(7) *For* $|\lambda| < \widehat{\lambda_{e,m}}$ *we have*

(a)

$$L - p_{e,m}Lp_{e,m} - (\mathbf{1} - p_{e,m})L(\mathbf{1} - p_{e,m}) = O(|\lambda|^2 + |\lambda\beta|);$$

(b) *if* $\mathbf{1}^{vv} = \mathbf{1}_m(\Gamma^{ee})$ *then*

$$L - p_{e,m}Lp_{e,m} - (\mathbf{1} - p_{e,m})L(\mathbf{1} - p_{e,m}) = O(|\lambda|^3 + |\lambda\beta|).$$

Remark 2.4: Note that in both theorems (6) describes how close the projections p_e *and* $p_{e,m}$ *are to the corresponding spectral projections of* L *and (7) describes how well they diagonalize* L.

In Theorem 2.1, we have the same order of smallness in (6) and (7). We will see from the proof, that (7) essentially follows from (6).

On the other hand, in Theorem 2.3, the order of smallness of (7) is much better than that of (6). Thus (7) requires a separate proof.

III. PROOFS

Let us begin with a general fact about the stability of the spectrum of bounded operators (Ref. 11).

Theorem 3.1: *Let* $A \in B(\mathcal{H})$. *Then there exists an increasing and continuous function,*

$$[0,\infty[\ \ni x \mapsto \mu_A(x) \in [0,\infty],$$

such that $\lim_{x\to 0} \mu_A(x) = 0$ *and for any* $B \in B(\mathcal{H})$ *we have* $\mathrm{sp}(A+B) \subset \mathrm{D}(\mathrm{sp}(A), \mu_A(\|B\|))^{\mathrm{cl}}$.

If $a \in \mathrm{sp}(A)$ *is an isolated eigenvalue with the degree of nilpotence equal to n, then there exists* $\epsilon > 0$ *such that for* $z \in \mathrm{D}(a,\epsilon)\backslash\{a\}$ *we have*

$$\|(z-A)^{-1}\| \leqslant C|z-a|^{-n}, \tag{3.1}$$

for some $C > 0$. *Moreover, for* $|\lambda| < \Lambda_a$ *for some* $\Lambda_a > 0$ *we have*

$$\mathrm{sp}(A + \lambda B) \cap \mathrm{D}(a,\epsilon) \subset \mathrm{D}(a, c|\lambda|^{1/n})^{\mathrm{cl}}, \tag{3.2}$$

where $c = (C\|B\|)^{1/n}$.

Proof: We prove only the last statement. Let $|\lambda| < \epsilon^n (C\|B\|)^{-1}$. If $z \in \mathrm{D}(a,\epsilon)\backslash \mathrm{D}(a,(C\|\lambda B\|)^{1/n})^{\mathrm{cl}}$ then

$$|z-a| > (C\|\lambda B\|)^{1/n},$$

so by the inequality (3.1),

$$1 > C\|\lambda B\|\,|z-a|^{-n} \geqslant \|\lambda B\|\,\|(z-A)^{-1}\| \geqslant \|\lambda B(z-A)^{-1}\|.$$

This shows that $z - A - \lambda B$ is invertible and hence $z \notin \mathrm{sp}(A + \lambda B)$, so we get (3.2). ∎

Let us comment on some additional notation that we will use. For $\mathcal{E}$ an isolated subset of $\mathrm{sp}(E)$ we will write

$$\mathbf{1}^{\mathcal{E}\mathcal{E}} := \mathbf{1}_{\mathcal{E}}(E) = \mathbf{1}_{\mathcal{E}}(L_0), \quad \mathbf{1}^{\overline{\mathcal{E}\mathcal{E}}} := \mathbf{1} - \mathbf{1}^{\mathcal{E}\mathcal{E}}, \quad \mathbf{1}^{\underline{\mathcal{E}\mathcal{E}}} := \mathbf{1}^{vv} - \mathbf{1}^{\mathcal{E}\mathcal{E}},$$

$$\mathcal{H}^{\mathcal{E}} := \mathrm{Ran}\,\mathbf{1}_{\mathcal{E}}(E), \quad \mathcal{H}^{\overline{\mathcal{E}}} := \mathrm{Ran}\,\mathbf{1}^{\overline{\mathcal{E}\mathcal{E}}}, \quad \mathcal{H}^{\underline{\mathcal{E}}} := \mathrm{Ran}\,\mathbf{1}^{\underline{\mathcal{E}\mathcal{E}}}.$$

Now the Banach space $\mathcal{H}$ can be decomposed in the following way:

$$\mathcal{H} = \mathcal{H}^{\mathcal{E}} \oplus \mathcal{H}^{\overline{\mathcal{E}}} = \mathcal{H}^{\mathcal{E}} \oplus \mathcal{H}^{\underline{\mathcal{E}}} \oplus \mathcal{H}^{\overline{v}},$$

and operator L can be written as

$$L = \begin{bmatrix} E^{\mathcal{E}\mathcal{E}} & \lambda Q^{\mathcal{E}\bar{\mathcal{E}}} \\ \lambda Q^{\bar{\mathcal{E}}\mathcal{E}} & L^{\bar{\mathcal{E}}\bar{\mathcal{E}}} \end{bmatrix} = \begin{bmatrix} E^{\mathcal{E}\mathcal{E}} & 0 & \lambda Q^{\mathcal{E}\bar{v}} \\ 0 & E^{\underline{\mathcal{E}}\underline{\mathcal{E}}} & \lambda Q^{\underline{\mathcal{E}}\bar{v}} \\ \lambda Q^{\bar{v}\mathcal{E}} & \lambda Q^{\bar{v}\underline{\mathcal{E}}} & L_0^{\bar{v}\bar{v}} + \beta Q^{\bar{v}\bar{v}} \end{bmatrix}. \tag{3.3}$$

If e is an isolated point of $\mathrm{sp}(E)$ then we write $\mathbf{1}^{ee}$ for $\mathbf{1}^{\{e\}\{e\}}$, $\mathcal{H}^{e}$ for $\mathcal{H}^{\{e\}}$, etc. Note that $E^{\bar{\mathcal{E}}\bar{\mathcal{E}}} = E^{\underline{\mathcal{E}}\underline{\mathcal{E}}}$.

We will use the following theorem for several operators and for various decompositions of the space $\mathcal{H}$.

Theorem 3.2: *Let H be a closed operator on a Banach space $\mathcal{H} = \mathcal{H}^{v} \oplus \mathcal{H}^{\bar{v}}$. Assume that off-diagonal elements of H,* i.e., $H^{v\bar{v}}$ and $H^{\bar{v}v}$ *are bounded. For $z \in \mathbb{C}\backslash \mathrm{sp}(H^{\bar{v}\bar{v}})$ define*

$$G_v(z) \coloneqq z\mathbf{1}^{vv} - H^{vv} - H^{v\bar{v}}(z\mathbf{1}^{\bar{v}\bar{v}} - H^{\bar{v}\bar{v}})^{-1}H^{\bar{v}v}.$$

Then for $z \notin \mathrm{sp}(H^{\bar{v}\bar{v}})$ we have

(1) $z \in \mathrm{sp}(H)$ *iff* $0 \in \mathrm{sp}(G_v(z))$,
(2) *if* $0 \notin \mathrm{sp}(G_v(z))$ *then*

$$(z-H)^{-1} = (z\mathbf{1}^{\bar{v}\bar{v}} - H^{\bar{v}\bar{v}})^{-1} + (\mathbf{1}^{vv} + (z\mathbf{1}^{\bar{v}\bar{v}} - H^{\bar{v}\bar{v}})^{-1}H^{\bar{v}v})G_v^{-1}(z)(\mathbf{1}^{vv} + H^{v\bar{v}}(z\mathbf{1}^{\bar{v}\bar{v}} - H^{\bar{v}\bar{v}})^{-1}).$$

The last equation is often called the Feshbach formula. We will keep this name. For more information about the above theorem the reader is referred to Refs. 6 and 10.

Lemma 3.3: Suppose that Assumptions 2.A and 2.B hold. We also assume either Assumption 2.C or 2.D. Let $\mathcal{E}$ be an isolated subset of $\mathrm{sp}(E)$ *and fix $r>0$. Then there exists $0<\Lambda_{\mathcal{E}}$ such that for $|\lambda|<\Lambda_{\mathcal{E}}$, $|\beta|<\beta_0$ we have* $(\Omega\backslash \mathrm{D}(\mathrm{sp}(E^{\bar{\mathcal{E}}\bar{\mathcal{E}}}),r)) \cap \mathrm{sp}(L^{\bar{\mathcal{E}}\bar{\mathcal{E}}}) = \emptyset$ *and*

$$\sup_{\substack{z\in\Omega\backslash \mathrm{D}(\mathrm{sp}(E^{\bar{\mathcal{E}}\bar{\mathcal{E}}}),r) \\ |\lambda|<\Lambda_{\mathcal{E}},\ |\beta|<\beta_0}} \|(z\mathbf{1}^{\bar{\mathcal{E}}\bar{\mathcal{E}}} - L^{\bar{\mathcal{E}}\bar{\mathcal{E}}})^{-1}\| < \infty. \tag{3.4}$$

Proof: If $z \in \Omega$ and $|\beta|<\beta_0$ then $z \notin \mathrm{sp}(L^{\bar{v}\bar{v}})$ and hence we can use the Theorem 3.2 for the operator $L^{\bar{\mathcal{E}}\bar{\mathcal{E}}}$ and for decomposition $\mathcal{H}^{\bar{\mathcal{E}}} \coloneqq \mathcal{H}^{\underline{\mathcal{E}}} \oplus \mathcal{H}^{\bar{v}}$. We obtain that, for some $\Lambda_{\mathcal{E}}>0$, and $|\lambda| < \Lambda_{\mathcal{E}}$ and for $z\in\Omega\backslash \mathrm{D}(\mathrm{sp}(E^{\underline{\mathcal{E}}\underline{\mathcal{E}}}),r)$,

$$G_{\underline{\mathcal{E}}}(z) = z\mathbf{1}^{\underline{\mathcal{E}}\underline{\mathcal{E}}} - E^{\underline{\mathcal{E}}\underline{\mathcal{E}}} - \lambda^2 Q^{\underline{\mathcal{E}}\bar{v}}(z\mathbf{1}^{\bar{v}\bar{v}} - L^{\bar{v}\bar{v}})^{-1}Q^{\bar{v}\underline{\mathcal{E}}},$$

is invertible and hence $z \notin \mathrm{sp}(L^{\bar{\mathcal{E}}\bar{\mathcal{E}}})$. Moreover, for such z, $G_{\underline{\mathcal{E}}}(z)$ has a uniformly bounded inverse. Therefore, the Feshbach formula implies (3.4). ■

Proof of the Theorem 2.1: (1) By Theorem 3.2, $z \in \mathrm{sp}(L)\cap\Omega$ iff $z \in \mathrm{sp}(E + \lambda^2 Q^{v\bar{v}}(z1^{\bar{v}\bar{v}} - L^{\bar{v}\bar{v}})^{-1}Q^{\bar{v}v})\cap\Omega$. By Theorem 3.1,

$$\mathrm{sp}(E + \lambda^2 Q^{v\bar{v}}(z\mathbf{1}^{\bar{v}\bar{v}} - L^{\bar{v}\bar{v}})^{-1}Q^{\bar{v}v}) \subset \mathrm{D}(\mathrm{sp}(E), \mu_E(|\lambda|^2 c))^{\mathrm{cl}},$$

where

$$c = \sup_{z\in\Omega, |\beta|<\beta_0} \|Q^{v\bar{v}}(z\mathbf{1}^{\bar{v}\bar{v}} - L^{\bar{v}\bar{v}})^{-1}Q^{\bar{v}v}\|,$$

(which as we know is finite), and $\mu_E:[0,\infty[\to[0,\infty[$ is a continuous increasing function with $\lim_{x\to 0}\mu_E(x)=0$. Thus $\mathrm{sp}(L)\cap\Omega \subset \mathrm{D}(\mathrm{sp}(E), \delta(|\lambda|))^{\mathrm{cl}}$, where $\delta(x) = \mu_E(x^2 c)$.

(2) A simple consequence of (1).

(3) For some $0<\lambda_{\mathcal{E}}$, $|\lambda|<\lambda_{\mathcal{E}}$, and $|\beta|<\beta_0$, there exists a closed path $\gamma\subset\Omega$ that encircles $\Theta_{\mathcal{E}}$ counterclockwise, but no other parts of $\mathrm{sp}(L)$. We have

$$\sup_{\substack{z\in\gamma,\ |\beta|<\beta_0\\ |\lambda|<\lambda_{\mathcal{E}}}} \|(z-L)^{-1}\| < \infty.$$

Besides,

$$(2\pi i)^{-1}\oint_\gamma (z\mathbf{1}^{\overline{vv}} - L_0^{\overline{vv}})^{-1}\,\mathrm{dz} = (2\pi i)^{-1}\oint_\gamma (z\mathbf{1}^{\overline{vv}} - L^{\overline{vv}})^{-1}\,\mathrm{dz} = 0.$$

Therefore,

$$\begin{aligned}\mathbf{1}_{\Theta_{\mathcal{E}}}(L) - \mathbf{1}_{\mathcal{E}}(L_0) &= (2\pi i)^{-1}\oint_\gamma ((z\mathbf{1}-L)^{-1} - (z\mathbf{1}^{vv}-E)^{-1} - (z\mathbf{1}^{\overline{vv}} - L^{\overline{vv}})^{-1})\mathrm{dz}\\ &= \lambda(2\pi i)^{-1}\oint_\gamma (z\mathbf{1}-L)^{-1}(Q^{v\overline{v}} + Q^{\overline{v}v})((z\mathbf{1}^{vv}-E)^{-1} + (z\mathbf{1}^{\overline{vv}} - L^{\overline{vv}})^{-1})\mathrm{dz} = O(|\lambda|).\end{aligned}$$

(4) Equation (2.6) implies that for $|\lambda|$ sufficiently small we have $\|\mathbf{1}_{\Theta_{\mathcal{E}}}(L) - \mathbf{1}_{\mathcal{E}}(L_0)\| < 1$ so by a well-known theorem (Ref. 11) Eq. (2.7) holds for $|\lambda|$ small. But $\lambda \mapsto \dim \mathbf{1}_{\Theta_{\mathcal{E}}}(L) \in \mathbb{N}$ is a continuous function so (2.7) holds for all $|\lambda| < \lambda_{\mathcal{E}}$.

(5) Let $\mathcal{E}$, r, and $\Lambda_{\mathcal{E}}$ be the same as in the Lemma 3.3. For $|\lambda| < \Lambda_{\mathcal{E}}$, $|\beta| < \beta_0$, we can use Theorem 3.2, which implies that for $z \in \Omega \setminus \mathrm{D}(\mathrm{sp}(E^{\overline{\mathcal{E}\mathcal{E}}}), r)$ we have $z \in \mathrm{sp}(L)$ iff

$$z \in \mathrm{sp}(E^{\mathcal{E}\mathcal{E}} + \lambda^2 Q^{\mathcal{E}\overline{\mathcal{E}}}(z\mathbf{1}^{\overline{\mathcal{E}\mathcal{E}}} - L^{\overline{\mathcal{E}\mathcal{E}}})^{-1} Q^{\overline{\mathcal{E}}\mathcal{E}}).$$

By Theorem 3.1 we get

$$\mathrm{sp}(E^{\mathcal{E}\mathcal{E}} + \lambda^2 Q^{\mathcal{E}\overline{\mathcal{E}}}(z\mathbf{1}^{\overline{\mathcal{E}\mathcal{E}}} - L^{\overline{\mathcal{E}\mathcal{E}}})^{-1} Q^{\overline{\mathcal{E}}\mathcal{E}}) \subset \mathrm{D}(\mathcal{E}, \mu_{E^{\mathcal{E}\mathcal{E}}}(|\lambda|^2 c))^{\mathrm{cl}}, \tag{3.5}$$

where

$$c := \sup_{\substack{z\in\Omega\setminus\mathrm{D}(\mathrm{sp}(E^{\overline{\mathcal{E}\mathcal{E}}}),r)\\ |\beta|<\beta_0,\ |\lambda|<\Lambda_{\mathcal{E}}}} \|Q^{\mathcal{E}\overline{\mathcal{E}}}(z\mathbf{1}^{\overline{\mathcal{E}\mathcal{E}}} - L^{\overline{\mathcal{E}\mathcal{E}}})^{-1} Q^{\overline{\mathcal{E}}\mathcal{E}}\|$$

is finite by Lemma 3.3.

Now set $\mathcal{E} = \{e\}$ and assume that e has a degree of nilpotence equal to n. Then by Theorem 3.1 we can take $\mu_{E^{ee}}(x) := c_1 x^{1/n}$.

(6) For $|\lambda| < \widehat{\lambda_e}$,

$$\mathbf{1}^{ee} + \lambda^2 Q^{e\overline{v}}(e\mathbf{1}^{\overline{vv}} - L_0^{\overline{vv}})^{-2} Q^{\overline{v}e},$$

is an invertible operator so the expression for p_e makes sense. Direct computations show that $p_e^2 = p_e$. Note that

$$(\mathbf{1}^{ee} + \lambda^2 Q^{e\overline{v}}(e\mathbf{1}^{\overline{vv}} - L_0^{\overline{vv}})^{-2} Q^{\overline{v}e})^{-1} = \mathbf{1}^{ee} - \lambda^2 Q^{e\overline{v}}(e\mathbf{1}^{\overline{vv}} - L_0^{\overline{vv}})^{-2} Q^{\overline{v}e} + O(\lambda^4)$$

so

$$\begin{aligned}p_e = \mathbf{1}^{ee} &+ \lambda(Q^{e\overline{v}}(e\mathbf{1}^{\overline{vv}} - L_0^{\overline{vv}})^{-1} + (e\mathbf{1}^{\overline{vv}} - L_0^{\overline{vv}})^{-1} Q^{\overline{v}e}) + \lambda^2((e\mathbf{1}^{\overline{vv}} - L_0^{\overline{vv}})^{-1} Q^{\overline{v}e} Q^{e\overline{v}}(e\mathbf{1}^{\overline{vv}} - L_0^{\overline{vv}})^{-1}\\ &- Q^{e\overline{v}}(e\mathbf{1}^{\overline{vv}} - L_0^{\overline{vv}})^{-2} Q^{\overline{v}e}) + O(|\lambda|^3).\end{aligned} \tag{3.6}$$

We have

$$\mathbf{1}_{\Theta_e}(L) = \mathbf{1}_e(L_0) + \lambda(2\pi i)^{-1}\oint_\gamma ((z\mathbf{1}^{vv} - E)^{-1}Q^{v\bar{v}}(z\mathbf{1}^{\bar{v}\bar{v}} - L_0^{\bar{v}\bar{v}})^{-1} + (z\mathbf{1}^{\bar{v}\bar{v}} - L_0^{\bar{v}\bar{v}})^{-1}Q^{\bar{v}v}(z\mathbf{1}^{vv} - E)^{-1})\mathrm{d}z$$
$$+ \lambda^2(2\pi i)^{-1}\oint_\gamma ((z\mathbf{1}^{vv} - E)^{-1}Q^{v\bar{v}}(z\mathbf{1}^{\bar{v}\bar{v}} - L_0^{\bar{v}\bar{v}})^{-1}Q^{\bar{v}v}(z\mathbf{1}^{vv} - E)^{-1}$$
$$+ (z\mathbf{1}^{\bar{v}\bar{v}} - L_0^{\bar{v}\bar{v}})^{-1}Q^{\bar{v}v}(z\mathbf{1}^{vv} - E)^{-1}Q^{v\bar{v}}(z\mathbf{1}^{\bar{v}\bar{v}} - L_0^{\bar{v}\bar{v}})^{-1})\mathrm{d}z + O(|\lambda|^3 + |\lambda\beta|).$$

e is the only one eigenvalue of E inside γ so $(z\mathbf{1}^{vv}-E)^{-1}$ has only one pole inside γ. All points on and inside γ are not in $\mathrm{sp}(L_0^{\bar{e}\bar{e}})$ so $(z\mathbf{1}^{\bar{v}\bar{v}}-L_0^{\bar{v}\bar{v}})^{-1}$ is analytic inside and continuous on γ. If e is semisimple then $(z\mathbf{1}^{vv}-E)^{-1}=(z-e)^{-1}\mathbf{1}^{ee}+$*analytic part* and hence

$$\mathbf{1}_{\Theta_e}(L) = \mathbf{1}_e(L_0) + \lambda(Q^{e\bar{v}}(e\mathbf{1}^{\bar{v}\bar{v}} - L_0^{\bar{v}\bar{v}})^{-1} + (e\mathbf{1}^{\bar{v}\bar{v}} - L_0^{\bar{v}\bar{v}})^{-1}Q^{\bar{v}e}) + \lambda^2\Bigg((2\pi i)^{-1}\oint_\gamma (z\mathbf{1}^{vv} - E)^{-1}Q^{v\bar{v}}$$
$$\times(z\mathbf{1}^{\bar{v}\bar{v}} - L_0^{\bar{v}\bar{v}})^{-1}Q^{\bar{v}v}(z\mathbf{1}^{vv} - E)^{-1}\,\mathrm{d}z + (e\mathbf{1}^{\bar{v}\bar{v}} - L_0^{\bar{v}\bar{v}})^{-1}Q^{\bar{v}e}Q^{e\bar{v}}(e\mathbf{1}^{\bar{v}\bar{v}} - L_0^{\bar{v}\bar{v}})^{-1}\Bigg) + O(|\lambda|^3$$
$$+ |\lambda\beta|). \tag{3.7}$$

Now part (b) [Eq. (2.11)] is a simple consequence of (3.7) and (3.6). In general, when e is not semisimple, terms of order $O(|\lambda|)$ will not cancel so part (a) [Eq. (2.10)] cannot be improved.

If $\mathrm{sp}(E)=\{e\}$ and e is semisimple then $(z\mathbf{1}^{vv}-E)^{-1}=(z-e)^{-1}\mathbf{1}^{ee}=(z-e)^{-1}\mathbf{1}^{vv}$. Now

$$(2\pi i)^{-1}\oint_\gamma (z\mathbf{1}^{vv} - E)^{-1}Q^{v\bar{v}}(z\mathbf{1}^{\bar{v}\bar{v}} - L_0^{\bar{v}\bar{v}})^{-1}Q^{\bar{v}v}(z\mathbf{1}^{vv} - E)^{-1}\,\mathrm{d}z = -Q^{e\bar{v}}(e\mathbf{1}^{\bar{v}\bar{v}} - L_0^{\bar{v}\bar{v}})^{-2}Q^{\bar{v}e}.$$

Now part (c) [Eq. (2.12)] is a simple consequence of (3.7) and (3.6).

(7) The proof of (6) in the cases (a), (b), and (c) shows actually slightly improved results,

$$(\mathbf{1}_{\Theta_e}(L) - p_e)L = O(|\lambda|), \quad O(|\lambda|^2 + |\lambda\beta|), \quad \text{and } O(|\lambda|^3 + |\lambda\beta|),$$

$$L(\mathbf{1}_{\Theta_e}(L) - p_e) = O(|\lambda|), \quad O(|\lambda|^2 + |\lambda\beta|), \quad \text{and } O(|\lambda|^3 + |\lambda\beta|).$$

To obtain (7) we use

$$L - p_eLp_e - (\mathbf{1} - p_e)L(\mathbf{1} - p_e) = -[p_e,[p_e,L]] = -[p_e,[p_e - \mathbf{1}_e(L),L]].$$

■

Proof of the Theorem 2.3: (1) Let $e \in \mathrm{sp}(E)$. Let $\mathcal{E}=\{e\}$, $\Lambda_e=\Lambda_{\mathcal{E}}$ and r be the same as in the Lemma 3.3 and in the proof of (5) of the previous theorem. For $|\lambda|<\Lambda_e$, $|\beta|<\beta_0$ and for $z \in \Omega\setminus\mathrm{D}(\mathrm{sp}(E^{\bar{e}\bar{e}}),r)$ we can use Theorem 3.2 for the operator L and for decomposition $\mathcal{H}:=\mathcal{H}^e \oplus \mathcal{H}^{\bar{e}}$. If $z \in \mathrm{sp}(L)\cap\Omega\setminus\mathrm{D}(\mathrm{sp}(E^{\bar{e}\bar{e}}),r)$ then

$$0 \in \mathrm{sp}(z\mathbf{1}^{ee} - E^{ee} - \lambda^2Q^{e\bar{e}}(z\mathbf{1}^{\bar{e}\bar{e}} - L^{\bar{e}\bar{e}})^{-1}Q^{\bar{e}e}). \tag{3.8}$$

Note that $e \in \mathrm{sp}(E)$ is semisimple so $E^{ee}=e\mathbf{1}^{ee}$ and moreover, we have $Q^{\bar{e}e}=Q^{\bar{v}e}$ and $Q^{e\bar{e}}=Q^{e\bar{v}}$. Now (3.8) can be written as

$$\frac{z-e}{\lambda^2} \in \mathrm{sp}(Q^{e\bar{v}}(z\mathbf{1}^{\bar{e}\bar{e}} - L^{\bar{e}\bar{e}})^{-1}Q^{\bar{v}e}). \tag{3.9}$$

Note that

$$
\begin{aligned}
(z\mathbf{1}^{\bar{e}\bar{e}} - L^{\bar{e}\bar{e}})^{-1} &= (e\mathbf{1}^{\bar{e}\bar{e}} - L^{\bar{e}\bar{e}})^{-1} + (e-z)(z\mathbf{1}^{\bar{e}\bar{e}} - L^{\bar{e}\bar{e}})^{-1}(e\mathbf{1}^{\bar{e}\bar{e}} - L^{\bar{e}\bar{e}})^{-1} \\
&= (e\mathbf{1}^{\bar{e}\bar{e}} - L_0^{\bar{e}\bar{e}})^{-1} + (e\mathbf{1}^{\bar{e}\bar{e}} - L^{\bar{e}\bar{e}})^{-1}(\lambda(Q^{\underline{e}\overline{v}} + Q^{\overline{v}\underline{e}}) + \beta Q^{\overline{vv}})(e\mathbf{1}^{\bar{e}\bar{e}} - L_0^{\bar{e}\bar{e}})^{-1} \\
&\quad + (e-z)(z\mathbf{1}^{\bar{e}\bar{e}} - L^{\bar{e}\bar{e}})^{-1}(e\mathbf{1}^{\bar{e}\bar{e}} - L^{\bar{e}\bar{e}})^{-1},
\end{aligned}
\tag{3.10}
$$

and $(e\mathbf{1}^{\bar{e}\bar{e}} - L_0^{\bar{e}\bar{e}})^{-1} = (e\mathbf{1}^{\underline{ee}} - E^{\underline{ee}})^{-1} + (e\mathbf{1}^{\overline{vv}} - L_0^{\overline{vv}})^{-1}$. Now we can write

$$
Q^{e\overline{v}}(z\mathbf{1}^{\bar{e}\bar{e}} - L^{\bar{e}\bar{e}})^{-1}Q^{\overline{v}e} = \Gamma^{ee} + \mathbf{I} + \mathbf{II} + \mathbf{III},
\tag{3.11}
$$

where

$$
\begin{aligned}
\mathbf{I} &= \beta Q^{e\overline{v}}(e\mathbf{1}^{\bar{e}\bar{e}} - L^{\bar{e}\bar{e}})^{-1}Q^{\overline{vv}}(e\mathbf{1}^{\bar{e}\bar{e}} - L_0^{\bar{e}\bar{e}})^{-1}Q^{\overline{v}e}, \\
\mathbf{II} &= \lambda Q^{e\overline{v}}(e\mathbf{1}^{\bar{e}\bar{e}} - L^{\bar{e}\bar{e}})^{-1}(Q^{\overline{v}\underline{e}} + Q^{\underline{e}\overline{v}})(e\mathbf{1}^{\bar{e}\bar{e}} - L_0^{\bar{e}\bar{e}})^{-1}Q^{\overline{v}e}, \\
\mathbf{III} &= (e-z)Q^{e\overline{v}}(z\mathbf{1}^{\bar{e}\bar{e}} - L^{\bar{e}\bar{e}})^{-1}(e\mathbf{1}^{\bar{e}\bar{e}} - L^{\bar{e}\bar{e}})^{-1}Q^{\overline{v}e}.
\end{aligned}
\tag{3.12}
$$

Clearly, $\|\mathbf{I}\| \leqslant C_{\mathrm{I}}|\beta|$. If we note that

$$
\mathbf{1}^{\overline{vv}}(e\mathbf{1}^{\bar{e}\bar{e}} - L^{\bar{e}\bar{e}})^{-1}\mathbf{1}^{\underline{ee}} = \lambda(e\mathbf{1}^{\overline{vv}} - L^{\overline{vv}})^{-1}Q^{\overline{v}\underline{e}}G_{\underline{e}}^{-1}(e) = O(\lambda)
$$

and similarly $\mathbf{1}^{\underline{ee}}(e\mathbf{1}^{\bar{e}\bar{e}} - L^{\bar{e}\bar{e}})^{-1}\mathbf{1}^{\overline{vv}} = O(\lambda)$ then we get $\|\mathbf{II}\| \leqslant C_{\mathrm{II}}|\lambda|^2$. Moreover, Theorem 2.1 implies that $|z-e| < C\lambda^2$ and hence by the Lemma 3.3 [Eq. (3.4)] we get $\|\mathbf{III}\| \leqslant C_{\mathrm{III}}\lambda^2$. So for $|\lambda| < \Lambda_e$ and $|\beta| < \beta_0$ we have

$$
\|\mathbf{I} + \mathbf{II} + \mathbf{III}\| < C_e(|\lambda|^2 + |\beta|)
$$

for some $C_e > 0$. Now we can apply the Theorem 3.1 to the expression (3.9) and get for $|\lambda| < \Lambda_e$ and $|\beta| < \beta_0$,

$$
\frac{z-e}{\lambda^2} \in \mathrm{D}(\mathrm{sp}(\Gamma^{ee}), \mu_{\Gamma^{ee}}(C_e(|\lambda|^2 + |\beta|)))^{\mathrm{cl}},
\tag{3.13}
$$

where functions $\mu_{\Gamma^{ee}} : [0,\infty[\to [0,\infty]$ are continuous, increasing and $\lim_{x\to 0}\mu_{\Gamma^{ee}}(x) = 0$. This implies

$$
\mathrm{sp}(L) \cap \Omega \setminus \mathrm{D}(\mathrm{sp}(E^{\bar{e}\bar{e}}), r) \subset \mathrm{D}(e + \lambda^2\,\mathrm{sp}(\Gamma^{ee}), |\lambda|^2\mu_{\Gamma^{ee}}(C_e(|\lambda|^2 + |\beta|)))^{\mathrm{cl}},
$$

and hence for $|\lambda| < \lambda_0 := \min_{e\in\mathrm{sp}(E)}\Lambda_e$ and $|\beta| < \beta_0$ we have

$$
\mathrm{sp}(L) \cap \Omega \subset \bigcup_{e\in\mathrm{sp}(E)} \mathrm{D}(e + \lambda^2\,\mathrm{sp}(\Gamma^{ee}), |\lambda|^2\delta(|\lambda|^2 + |\beta|))^{\mathrm{cl}},
$$

where we denoted $\delta(x) := \max_{e\in\mathrm{sp}(E)}(\mu_{\Gamma^{ee}}(C_e x))$.

(2) A simple consequence of (1).

(3) Let γ be a closed path such that γ encircles $\mathcal{M}$ but no other parts of $\mathrm{sp}(\Gamma^{ee})$. By (1) and (2), for small enough λ and β, the translated and rescaled path $e + \lambda^2\gamma$ encircles only $\Theta_{e,\mathcal{M}}$ but no other parts of $\mathrm{sp}(L)$. Now

$$
\mathbf{1}_{\Theta_{e,\mathcal{M}}}(L) = (2\pi i)^{-1}\oint_{e+\lambda^2\gamma} \frac{1}{\eta\mathbf{1} - L}\,\mathrm{d}\eta.
$$

For all $\eta \in e + \lambda^2\gamma$ we can use Feshbach formula for the operator L and for the decomposition $\mathcal{H} = \mathcal{H}^e \oplus \mathcal{H}^{\bar{e}}$. We get

$$\oint_{e+\lambda^2\gamma}(\eta\mathbf{1}-L)^{-1}\,\mathrm{d}\eta=\oint_{e+\lambda^2\gamma}(\mathbf{1}^{ee}+(\eta\mathbf{1}^{\overline{ee}}-L^{\overline{ee}})^{-1}\lambda Q^{\overline{ee}e})G_e^{-1}(\eta)(\mathbf{1}^{ee}+\lambda Q^{e\overline{ee}}(\eta\mathbf{1}^{\overline{ee}}-L^{\overline{ee}})^{-1})\mathrm{d}\eta,$$

where

$$G_e(\eta)=\eta\mathbf{1}^{ee}-e\mathbf{1}^{ee}-\lambda^2Q^{e\overline{e}}(\eta\mathbf{1}^{\overline{ee}}-L^{\overline{ee}})^{-1}Q^{\overline{e}e}.$$

Note that

$$((\eta-e)\mathbf{1}^{ee}-\lambda^2\Gamma^{ee})^{-1}-G_e^{-1}(\eta)=\lambda^2((\eta-e)\mathbf{1}^{ee}-\lambda^2\Gamma^{ee})^{-1}Q^{e\overline{v}}((e\mathbf{1}^{\overline{vv}}-L_0^{\overline{vv}})^{-1}$$
$$-\mathbf{1}^{\overline{vv}}(\eta\mathbf{1}^{\overline{ee}}-L^{\overline{ee}})^{-1}\mathbf{1}^{\overline{vv}})Q^{\overline{v}e}G_e^{-1}(\eta), \tag{3.14}$$

where we used $Q^{\overline{v}e}=Q^{\overline{e}e}$ and $Q^{e\overline{v}}=Q^{e\overline{e}}$. Moreover,

$$(e\mathbf{1}^{\overline{vv}}-L_0^{\overline{vv}})^{-1}-\mathbf{1}^{\overline{vv}}(\eta\mathbf{1}^{\overline{ee}}-L^{\overline{ee}})^{-1}\mathbf{1}^{\overline{vv}}$$
$$=(e\mathbf{1}^{\overline{vv}}-L_0^{\overline{vv}})^{-1}-(\eta\mathbf{1}^{\overline{vv}}-L_0^{\overline{vv}}-\beta Q^{\overline{vv}})^{-1}-\lambda^2(\eta\mathbf{1}^{\overline{vv}}-L_0^{\overline{vv}}-\beta Q^{\overline{vv}})^{-1}Q^{\overline{v}\underline{e}}G_{\underline{e}}^{-1}(\eta)$$
$$\times Q^{\underline{e}\overline{v}}(\eta\mathbf{1}^{\overline{vv}}-L_0^{\overline{vv}}-\beta Q^{\overline{vv}})^{-1}$$
$$=(e\mathbf{1}^{\overline{vv}}-L_0^{\overline{vv}})^{-1}((\eta-e)\mathbf{1}^{\overline{vv}}+\beta Q^{\overline{vv}})(\eta\mathbf{1}^{\overline{vv}}-L_0^{\overline{vv}}-\beta Q^{\overline{vv}})^{-1}-\lambda^2(\eta\mathbf{1}^{\overline{vv}}-L_0^{\overline{vv}}-\beta Q^{\overline{vv}})^{-1}$$
$$\times Q^{\overline{v}\underline{e}}G_{\underline{e}}^{-1}(\eta)Q^{\underline{e}\overline{v}}(\eta\mathbf{1}^{\overline{vv}}-L_0^{\overline{vv}}-\beta Q^{\overline{vv}})^{-1}, \tag{3.15}$$

where

$$G_{\underline{e}}(\eta)=\eta\mathbf{1}^{\underline{ee}}-E^{\underline{ee}}-\lambda^2Q^{\underline{e}\overline{v}}(\eta\mathbf{1}^{\overline{vv}}-L^{\overline{vv}})^{-1}Q^{\overline{v}\underline{e}}.$$

If we change the variable $\eta=e+\lambda^2z$ and use the equations (3.14) and (3.15) we get

$$\oint_{e+\lambda^2\gamma}(\eta\mathbf{1}-L)^{-1}\,\mathrm{d}\eta=\oint_{\gamma}(\mathbf{1}^{ee}+((e+\lambda^2z)\mathbf{1}^{\overline{ee}}-L^{\overline{ee}})^{-1}\lambda Q^{\overline{ee}e})(z\mathbf{1}^{ee}-\Gamma^{ee})^{-1}$$
$$\times(\mathbf{1}^{ee}+\lambda Q^{e\overline{ee}}((e+\lambda^2z)\mathbf{1}^{\overline{ee}}-L^{\overline{ee}})^{-1})\mathrm{d}z+O(|\lambda|^2+|\beta|). \tag{3.16}$$

Since

$$(2\pi i)^{-1}\oint_{\gamma}(z\mathbf{1}^{ee}-\Gamma^{ee})^{-1}\,\mathrm{d}z=\mathbf{1}_{\mathcal{M}}(\Gamma^{ee}),$$

we get $\mathbf{1}_{\Theta_{e,\mathcal{M}}}(L)-\mathbf{1}_{\mathcal{M}}(\Gamma^{ee})=O(|\lambda|+|\beta|)$.

(4) Equation (2.17) implies that for $|\lambda|$ and $|\beta|$ sufficiently small we have $\|\mathbf{1}_{\Theta_{e,\mathcal{M}}}(L)-\mathbf{1}_{\mathcal{M}}(\Gamma^{ee})\|<1$ so by a well-known theorem (Ref. 11) equality (2.18) holds. But dim $\mathbf{1}_{\Theta_{\mathcal{E}}}(L)\in\mathbb{N}$ is a continuous function of λ and β so (2.18) holds for all $|\lambda|<\lambda_{e,\mathcal{M}}$ and $|\beta|<\beta_{e,\mathcal{M}}$.

(5) If the degree of nilpotence of m as an eigenvalue of Γ^{ee} is n then due to the Theorem 3.1, Eq. (3.13) can be written as

$$\frac{z-e}{\lambda^2}\in \mathrm{D}(m,C_{e,m}(|\lambda|^2+|\beta|)^{1/n})^{\mathrm{cl}}\cup \mathrm{D}(\mathrm{sp}(\Gamma^{ee})\setminus\{m\},\mu_{\Gamma^{ee}}(C_e(|\lambda|^2+|\beta|)))^{\mathrm{cl}}.$$

(6) For $|\lambda|<\widehat{\lambda_{e,m}}$,

$$\mathbf{1}_m(\Gamma^{ee})+\lambda^2\mathbf{1}_m(\Gamma^{ee})Q^{v\overline{v}}(e\mathbf{1}^{\overline{vv}}-L_0^{\overline{vv}})^{-2}Q^{\overline{v}v}\mathbf{1}_m(\Gamma^{ee})$$

is an invertible operator so the expression for $p_{e,m}$ makes sense. Direct computations show that $p_{e,m}^2=p_{e,m}$.

In the proof of the part (3) we showed that $\mathbf{1}_{\Theta_{e,m}}(L)=\mathbf{1}_m(\Gamma^{ee})+O(|\lambda|+|\beta|)$ so (a) is already done. To show (b), we use the approximation for $\mathbf{1}_{\Theta_{e,m}}(L)$ given by (3.16);

$$\mathbf{1}_{\Theta_{e,m}}(L) \overset{O(|\lambda^2|+|\beta|)}{=} \frac{1}{2\pi i}\oint_\gamma (\lambda((e+\lambda^2 z)\mathbf{1}^{\overline{ee}}-L^{\overline{ee}})^{-1}Q^{\overline{ee}}+\mathbf{1}^{ee})(z\mathbf{1}^{ee}-\Gamma^{ee})^{-1}$$

$$\times(\lambda Q^{e\overline{e}}((e+\lambda^2 z)\mathbf{1}^{\overline{ee}}-L^{\overline{ee}})^{-1}+\mathbf{1}^{ee})\mathrm{d}z \overset{O(|\lambda^2|+|\lambda\beta|)}{=} \frac{1}{2\pi i}\oint_\gamma (\lambda(e+\lambda^2 z)\ \mathbf{1}^{\overline{ee}}-L_0^{\overline{ee}})^{-1}$$

$$\times Q^{\overline{ee}}+\mathbf{1}^{ee})(z\mathbf{1}^{ee}-\Gamma^{ee})^{-1}(\lambda Q^{e\overline{e}}((e+\lambda^2 z)\mathbf{1}^{\overline{ee}}-L_0^{\overline{ee}})^{-1}+\mathbf{1}^{ee})\mathrm{d}z$$

$$=(\lambda((e+\lambda^2 m)\mathbf{1}^{\overline{ee}}-L_0^{\overline{ee}})^{-1}Q^{\overline{ee}}+\mathbf{1}^{ee})\mathbf{1}_m(\Gamma^{ee})(\lambda Q^{e\overline{e}}((e+\lambda^2 m)\mathbf{1}^{\overline{ee}}-L_0^{\overline{ee}})^{-1}+\mathbf{1}^{ee}) \overset{O(|\lambda|^3)}{=} p_{e,m},$$

where we used

$$((e+\lambda^2 z)\mathbf{1}^{\overline{ee}}-L_0^{\overline{ee}})^{-1}-((e+\lambda^2 z)\mathbf{1}^{\overline{ee}}-L^{\overline{ee}})^{-1}=O(|\lambda|+|\beta|).$$

(7) Let us denote $\mathbf{1}^{mm}:=\mathbf{1}_m(\Gamma^{ee})$, $Q^{m\overline{v}}:=\mathbf{1}^{mm}Q^{v\overline{v}}$ and $Q^{\overline{v}m}:=Q^{\overline{v}v}\mathbf{1}^{mm}$, so that

$$p_{e,m}=(\mathbf{1}^{mm}+\lambda(e\mathbf{1}^{\overline{vv}}-L_0^{\overline{vv}})^{-1}Q^{\overline{v}m})(\mathbf{1}^{mm}+\lambda^2 Q^{m\overline{v}}(e\mathbf{1}^{\overline{vv}}-L_0^{\overline{vv}})^{-2}Q^{\overline{v}m})^{-1}$$
$$\times(\mathbf{1}^{mm}+\lambda Q^{m\overline{v}}(e\mathbf{1}^{\overline{vv}}-L_0^{\overline{vv}})^{-1}).$$

We compute

$$p_{e,m}(L-e)=(\mathbf{1}^{mm}+\lambda(e\mathbf{1}^{\overline{vv}}-L_0^{\overline{vv}})^{-1}Q^{\overline{v}m})(\mathbf{1}^{mm}+\lambda^2 Q^{m\overline{v}}(e\mathbf{1}^{\overline{vv}}-L_0^{\overline{vv}})^{-2}Q^{\overline{v}m})^{-1}$$
$$\times(\lambda^2 Q^{m\overline{v}}(e\mathbf{1}^{\overline{vv}}-L^{\overline{vv}})^{-1}Q^{\overline{v}v}+\lambda\beta Q^{m\overline{v}}(e\mathbf{1}^{\overline{vv}}-L^{\overline{vv}})^{-1}Q^{\overline{vv}})=O(|\lambda|^2+|\lambda\beta|),$$

$$p_{e,m}Lp_{e,m}-ep_{e,m}=(\mathbf{1}^{mm}+\lambda(e\mathbf{1}^{\overline{vv}}-L_0^{\overline{vv}})^{-1}Q^{\overline{v}m})(\mathbf{1}^{mm}+\lambda^2 Q^{m\overline{v}}(e\mathbf{1}^{\overline{vv}}-L_0^{\overline{vv}})^{-2}Q^{\overline{v}m})^{-1}$$
$$\times(\lambda^2 Q^{m\overline{v}}(e\mathbf{1}^{\overline{vv}}-L^{\overline{vv}})^{-1}Q^{\overline{v}m}+\lambda^2\beta Q^{m\overline{v}}(e\mathbf{1}^{\overline{vv}}-L^{\overline{vv}})^{-1}Q^{\overline{vv}}(e\mathbf{1}^{\overline{vv}}-L^{\overline{vv}})^{-1}Q^{\overline{v}m})$$
$$\times(\mathbf{1}^{mm}+\lambda^2 Q^{m\overline{v}}(e\mathbf{1}^{\overline{vv}}-L_0^{\overline{vv}})^{-2}Q^{\overline{v}m})^{-1}(\mathbf{1}^{mm}+\lambda Q^{m\overline{v}}(e\mathbf{1}^{\overline{vv}}-L_0^{\overline{vv}})^{-1})$$
$$=O(|\lambda|^2+|\lambda\beta|).$$

Thus

$$p_{e,m}(L-e)(\mathbf{1}-p_{e,m})=O(|\lambda|^2+|\lambda\beta|).$$

Similarly,

$$(\mathbf{1}-p_{e,m})(L-e)p_{e,m}=O(|\lambda|^2+|\lambda\beta|).$$

Finally, we use

$$L-p_{e,m}Lp_{e,m}-(\mathbf{1}-p_{e,m})L(\mathbf{1}-p_{e,m})=p_{e,m}(L-e)(\mathbf{1}-p_{e,m})+(\mathbf{1}-p_{e,m})(L-e)p_{e,m}.$$

This proves (a).

Assume now that $\mathrm{sp}(E)=\{e\}$. Then

$$Q^{m\overline{v}}(e\mathbf{1}^{\overline{vv}}-L^{\overline{vv}})^{-1}Q^{\overline{v}v}=Q^{m\overline{v}}(e\mathbf{1}^{\overline{vv}}-L^{\overline{vv}})^{-1}Q^{\overline{v}e}=Q^{m\overline{v}}(e\mathbf{1}^{\overline{vv}}-L^{\overline{vv}})^{-1}Q^{\overline{v}m}.$$

[The first identity follows from $\mathrm{sp}(E)=\{e\}$, the second is a consequence of the definition of $\mathbf{1}^{mm}$.] Using this we get

$$p_{e,m}(L-e)=(\mathbf{1}^{mm}+\lambda(e\mathbf{1}^{\overline{vv}}-L_0^{\overline{vv}})^{-1}Q^{\overline{v}m})(\mathbf{1}^{mm}+\lambda^2Q^{m\overline{v}}(e\mathbf{1}^{\overline{vv}}-L_0^{\overline{vv}})^{-2}Q^{\overline{v}m})^{-1}\lambda^2Q^{m\overline{v}}(e\mathbf{1}^{\overline{vv}}-L^{\overline{vv}})^{-1}$$
$$\times Q^{\overline{v}m}(\mathbf{1}^{mm}+\lambda^2Q^{m\overline{v}}(e\mathbf{1}^{\overline{vv}}-L_0^{\overline{vv}})^{-2}Q^{\overline{v}m})^{-1}(\mathbf{1}^{mm}+\lambda Q^{m\overline{v}}(e\mathbf{1}^{\overline{vv}}-L_0^{\overline{vv}})^{-1})+O(|\lambda|^3).$$

This proves (b). ■

IV. APPLICATION: ANALYTICALLY DEFORMED PAULI–FIERZ LIOUVILLEANS

In this section we describe a class of operators to which the results of our paper can be applied. These operators arise naturally as models used in quantum physics. They provided for us a part of motivation to write this paper.

In order to introduce these operators we have to introduce a number of concepts taken from operator algebra and mathematical physics. Our presentation is based on Refs. 6, 7, 12, 13, and 9.

A. W^*-dynamical systems and Liouvilleans

Let us start with a brief description of some elements of theory of operator algebras, that we will use.[3,9]

A pair $(\mathfrak{M},\tau^t)$, where $\mathfrak{M}$ is a W^*-algebra and τ^t is a σ-weakly continuous group of automorphisms of $\mathfrak{M}$, is called a W^*-dynamical system. In many circumstances it is convenient to describe a quantum system by a W^*-dynamical system. One of important results of theory of W^*-algebras says that there exists a distinguished representation, unique up to the unitary equivalence, called the standard representation.[3,9] It is a quadruple $(\pi,\mathcal{W},J,\mathcal{W}_+)$, where $\mathcal{W}$ is a Hilbert space, $\pi:=\mathfrak{M}\to B(\mathcal{W})$ is a *-representation, J an antiunitary involution, called the modular conjugation, and $\mathcal{W}_+$ is a self-dual cone, called the positive cone, in $\mathcal{W}$ satisfying certain axioms. In this representation there exists a unique self-adjoint operator L, called the Liouvillean, that implements the dynamics

$$\pi(\tau^t(A)):=e^{itL}\pi(A)e^{-itL}$$

and leaves invariant the positive cone, $e^{itL}\mathcal{W}_+=\mathcal{W}_+$.

The properties of the W^*-dynamics τ^t are encoded in a simple way in the Liouvillean. For instance, the dynamics τ^t has no stationary states iff L has no point spectrum; it has a single stationary state iff L has a simple eigenvalue at zero.

One can argue that the resonances of L correspond to metastable states of the system $(\mathfrak{M},\tau)$.

B. Massless bosons at zero density interacting with a small quantum system

Our main object of interest will be Pauli–Fierz systems at a positive density. They will be introduced in the next section. In order, however, to understand their physical content it is appropriate to describe first Pauli–Fierz systems at a zero density (in other words, at zero temperature), which we will do in this section.

Let $\mathcal{K}$ be a Hilbert space associated with quantum mechanical system and let K be a self-adjoint Hamiltonian for this system.

Let $L^2(\mathbb{R}^d)$ be the one particle bosonic space and let h be the one particle energy operator given by the multiplication by $|\xi|$ where $\xi\in\mathbb{R}^d$. The Hamiltonian $\mathrm{d}\Gamma(h)$ of the Bose gas acts on the symmetric (bosonic) Fock space $\Gamma_s(L^2(\mathbb{R}^d))$.

Let the interaction between systems be given by a measurable operator valued function $\mathbb{R}^d\ni\xi\mapsto v(\xi)\in B(\mathcal{K})$. The following sections are based on Ref. 7 (see also Ref. 6).

The Hilbert space of the system at zero density (zero temperature) is $\mathcal{H}=\mathcal{K}\otimes\Gamma_s(L^2(\mathbb{R}^d))$ and the free Hamiltonian is

$$H_{\mathrm{fr}} := K \otimes \mathbf{1}_{\Gamma_s(L^2(\mathbb{R}^d))} + \mathbf{1}_{\mathcal{K}} \otimes \int |\xi| a^*(\xi) a(\xi) \mathrm{d}\xi,$$

where $a^*(\xi)/a(\xi)$ are the usual creation/annihilation operators of the boson of momentum ξ. The interaction is given by the operator

$$V := \int (v(\xi) \otimes a^*(\xi) + v^*(\xi) \otimes a(\xi)) \mathrm{d}\xi.$$

The full Pauli–Fierz Hamiltonian equals

$$H := H_{\mathrm{fr}} + \lambda V,$$

where $\lambda \in \mathbb{R}$. To guarantee the self-adjointness of H we can assume that $\int (1+|\xi|^{-1}) \|v(\xi)\|^2 \, \mathrm{d}\xi < \infty$.

C. Massless bosons at density ρ interacting with a small quantum system

In this section we explain the notion of a Pauli–Fierz system at density ρ.

Suppose that we are given a measurable function

$$\mathbb{R}^d \ni \xi \mapsto \rho(\xi) \in [0,\infty[.$$

Let us consider the "doubled" Fock space $\Gamma_s(L^2(\mathbb{R}^d) \oplus L^2(\mathbb{R}))$. The creation/annihilation operators corresponding to the left/right $L^2(\mathbb{R}^d)$ will be denoted by $a_1^*(\xi)/a_1(\xi)$ and $a_r^*(\xi)/a_r(\xi)$, respectively. Let us introduce the left and right Araki–Woods creation and annihilation operators

$$a_{\rho,1}^*(\xi) := \sqrt{1+\rho(\xi)} a_1^*(\xi) + \sqrt{\rho(\xi)} a_r(\xi),$$

$$a_{\rho,1}(\xi) := \sqrt{1+\rho(\xi)} a_1(\xi) + \sqrt{\rho(\xi)} a_r^*(\xi),$$

$$a_{\rho,r}^*(\xi) := \sqrt{\rho(\xi)} a_1(\xi) + \sqrt{1+\rho(\xi)} a_r^*(\xi),$$

$$a_{\rho,r}(\xi) := \sqrt{\rho(\xi)} a_1^*(\xi) + \sqrt{1+\rho(\xi)} a_r(\xi).$$

The sub-W^*-algebra of $\mathcal{B}(\mathcal{K} \otimes \overline{\mathcal{K}} \otimes \Gamma_s(L^2(\mathbb{R}^d) \oplus L^2(\mathbb{R}^d))$ generated by operators of the form

$$A \otimes \mathbf{1}_{\overline{\mathcal{K}}} \otimes \exp\left(i \int f(\xi) a_{\rho,1}^*(\xi) \mathrm{d}\xi + i \int \overline{f}(\xi) a_{\rho,1}(\xi) \mathrm{d}\xi \right),$$

where $A \in \mathcal{B}(\mathcal{K})$ and $\int |f(\xi)|^2 \delta(\xi) d\xi < \infty$, will be called the Pauli–Fierz W^*-algebra. It is in a standard representation.

Note that the Pauli–Fierz algebra is isomorphic to the tensor product of the algebra of the small system $\mathcal{B}(\mathcal{K})$ and the algebra of Araki–Woods canonical commutation relations at density ρ.

The free Liouvillean is given by

$$L_{\mathrm{fr}} := K \otimes \mathbf{1} \otimes \mathbf{1} - \mathbf{1} \otimes \overline{K} \otimes \mathbf{1} + \mathbf{1} \otimes \mathbf{1} \otimes \int (|\xi| a_1^*(\xi) a_1(\xi) - |\xi| a_r^*(\xi) a_r(\xi)) \mathrm{d}\xi;$$

the perturbation is

$$Q_\rho := \int v(\xi) \otimes 1 \otimes a_{\rho,1}^*(\xi) \mathrm{d}\xi + hc - \int (\mathbf{1} \otimes \overline{v}(\xi)) \otimes a_{\rho,r}^*(\xi) \mathrm{d}\xi + hc.$$

Assumption 4.A: If $\int (1+|\xi|^2)(1+\rho(\xi)) \|v(\xi)\|^2 \, \mathrm{d}\xi < \infty$ *holds then*

$$L_\rho := L_{\rm fr} + \lambda Q_\rho$$

essentially self-adjoint on the intersection of the domains of $L_{\rm fr}$ *and* Q_ρ.

The most important class of densities is that given by the Planck law at the inverse temperature β,

$$\rho_\beta{}^{(\xi)} := (e^{\beta|\xi|} - 1)^{-1}.$$

In particular, $\beta=\infty$ corresponds to the temperature zero (and density zero), and the corresponding Liouvillean is unitarily equivalent to

$$H \otimes \mathbf{1} - \mathbf{1} \otimes \overline{H}. \tag{4.1}$$

Thus in this case all the information is encoded in the Pauli–Fierz Hamiltonian described in the preceding section. One can argue that for a general ρ, L_ρ is a kind of a thermodynamical limit (4.1).

D. Analytically deformed Pauli–Fierz Liouvilleans

Pauli–Fierz Liouvilleans have continuous spectrum that covers the whole real line. They may also have some embedded eigenvalues. In particular, a thermal Pauli–Fierz Liouvillean (i.e., whose density is given by the Planck law) always has a zero eigenvalue corresponding to a KMS state. In general, eigenvalues of a Liouvillean are related to stationary states, therefore their study is very important from the physical point of view.

Another physically relevant question about Pauli–Fierz Liouvilleans is whether they have resonances and if so what is their location. They may manifest themselves as poles of an S-matrix or decay rates of certain correlation functions.

In order to define resonances we use the approach of Jaksic–Pillet. The first step of this approach consists of "gluing" the "left" and "right" one-particle subspaces. This is done as follows. We use the spherical coordinates in $\mathbb{R}^d$ and we introduce the Jaksic–Pillet gluing map defined as

$$L^2(\mathbb{R}^d) \oplus \overline{L^2(\mathbb{R}^d)} \ni (f_+, \overline{f_-}) \mapsto f \in L^2(\mathbb{R}) \otimes L^2(S^{d-1}), \tag{4.2}$$

$$f(p,\omega) := \begin{cases} p^{(d-1)/2} f_+(p\omega), & p > 0, \\ (-p)^{(d-1)/2}\overline{f_-}(-p\omega), & p \leq 0. \end{cases}$$

Here, $(p,\omega) \in \mathbb{R}\times S^{d-1}$ and S^{d-1} denotes $(d-1)$ dimensional sphere. The canonical conjugation in $L^2(\mathbb{R})\otimes L^2(S^{d-1})$ is given by the complex conjugation.

If we assume that

$$v^*(\xi) = v(-\xi), \quad \rho(\xi) = \rho(-\xi)$$

and introduce

$$v_\rho(p,\omega) := \begin{cases} p^{(d-1)/2}(1+\rho(p\omega))^{1/2} v(p\omega), & p > 0, \\ (-p)^{(d-1)/2}\rho(p\omega)^{1/2} v(p\omega), & p \leq 0. \end{cases}$$

In the new representation, the free Liouvillean and its perturbation can be written as

$$L_{\rm fr} := K \otimes \mathbf{1} \otimes \mathbf{1} - \mathbf{1} \otimes \overline{K} \otimes \mathbf{1} + \mathbf{1} \otimes \mathbf{1} \otimes \int p a^*(p,\omega) a(p,\omega) \mathrm{d}p\, \mathrm{d}\omega,$$

$$Q_\rho = \int (v_\rho(p,\omega) \otimes \mathbf{1} \otimes a^*(p,\omega) + v_\rho^*(p,\omega) \otimes \mathbf{1} \otimes a(p,\omega))\mathrm{d}p\,\mathrm{d}\omega$$
$$+ \int (\mathbf{1} \otimes \overline{v_\rho}(p,\omega) \otimes a^*(-p,\omega) + \mathbf{1} \otimes \overline{v_\rho^*}(p,\omega) \otimes a(-p,\omega))\mathrm{d}p\,\mathrm{d}\omega$$

as an operator on $\mathcal{K}\otimes\overline{\mathcal{K}}\otimes\Gamma_s(L^2(\mathbb{R})\otimes L^2(S^{d-1}))$.

Let us make the following assumption.

Assumption 4.B: The function

$$\mathbb{R} \ni p \mapsto v_\rho(p,\cdot) \in \mathcal{B}(\mathcal{K},\mathcal{K}\otimes L^2(S^{d-1}))$$

extends to an analytic function in a strip $|\mathrm{Im}\, p| < \eta_0$ *and*

$$\sup_{|\mathrm{Im} p|<\eta_0} \int \|v_\rho(p,\cdot)\|^2 \mathrm{d}(\mathrm{Re}\, p) < \infty.$$

Let $i^{-1}\nabla_p$ be the generator of translations on $L^2(\mathbb{R})$ in the spectral parameter p. Let $S := \mathrm{d}\Gamma(i^{-1}\nabla_p)$ be its second quantization. Note that for any complex η,

$$L_{\mathrm{fr}}(\eta) := e^{i\eta S}L_{\mathrm{fr}}e^{-i\eta S} = L_{\mathrm{fr}} + \eta\mathbf{1}^{\otimes 1}\otimes N,$$

where $N=\mathrm{d}\Gamma(1)$ is the number operator. Moreover, for $|\mathrm{Im}\ \eta| < \eta_0$,

$$\begin{aligned} Q_\rho(\eta) &:= e^{i\eta S}Q_\rho e^{-i\eta S} \\ &= \int v_\rho((p+\eta),\omega) \otimes \mathbf{1} \otimes a^*(p,\omega)\mathrm{d}p\,\mathrm{d}\omega + \int v_\rho^*((p+\overline{\eta}),\omega) \otimes \mathbf{1} \otimes a(p,\omega)\mathrm{d}p\,\mathrm{d}\omega \\ &\quad + \int \mathbf{1} \otimes \overline{v_\rho}((p+\overline{\eta}),\omega) \otimes a^*(-p,\omega)\mathrm{d}p\,\mathrm{d}\omega + \int \mathbf{1} \otimes \overline{v_\rho^*}((p+\eta),\omega) \otimes a(-p,\omega)\mathrm{d}p\,\mathrm{d}\omega. \end{aligned}$$

Theorem 4.1: *Assume that 4.A, and 4.B hold. Then we have the following.*

(1) *There exists a unique operator-valued function* $\eta \mapsto L_\rho(\eta)$ *defined for* $0 \leqslant -\mathrm{Im}\,\eta < \eta_0$ *such that*

 (a) $L_\rho(\eta) = e^{i\eta S}L_\rho e^{-i\eta S}$ *for* $\eta \in \mathbb{R}$.
 (b) *For* $0 < \mathrm{Im}\,\eta < \eta_0$, $\eta \mapsto L_\rho(\eta)$ *is an analytic family.*
 (c) *For* $\mathrm{Im} z > 0$, $(z - L_\rho(\eta))^{-1}$ *is strongly continuous up to* $\mathrm{Im}\,\eta = 0$.

(2) *For and open* $U \subset \mathbb{C}$, $U \cap \mathrm{sp}_{\mathrm{disc}}(L_\rho(\eta))$ *is locally independent of* η, *as long as* $U \cap \mathrm{sp}_{\mathrm{ess}}(L_\rho(\eta)) = \emptyset$.

If we assume that $\dim K < \infty$, *then for* $0 \leqslant -\mathrm{Im}\,\eta < \eta_0$ *there exists* $\lambda_0 > 0$ *such that for* $|\lambda| < \lambda_0$ *the following statements hold:*

(3) $\mathrm{sp} L_\rho(\eta) \subset \{z \in \mathbb{C} : \mathrm{Im} z \leqslant 0\}$,
(4) *There exists* $c > 0$ *such that*

$$\mathrm{sp}_{\mathrm{ess}}\, L_\rho(\eta) \subset \{z \in \mathbb{C} : \mathrm{Im} z < -|\mathrm{Im}\,\eta|(1 - c|\lambda|)\}.$$

(5) *Real eigenvalues of* $L_\rho(\eta)$ *are semisimple and*

$$\mathrm{sp}_{\mathrm{pp}}\, L_\rho = \mathrm{sp}\, L_\rho(\eta) \cap \mathbb{R}.$$

So real discrete eigenvalues of $L_\rho(\eta)$ are semisimple, independent of η and coincide with the embedded eigenvalues of L_ρ. The nonreal discrete eigenvalues of $L_\rho(\eta)$, which are called resonances or metastable states, are also independent of η but they do not have to be semisimple.

E. LSO for Pauli–Fierz Liouvilleans

In this section we indicate how one can apply the method described in our paper to an analytically deformed Pauli–Fierz Liouvillean. We will see that many objects, including the LSO, do not depend on the parameter of deformation η, or depend rather mildly.

It is easy to see that $\mathbb{R}\cap \mathrm{sp}\, L_{\mathrm{fr}}(\eta)$ is an isolated subset of $\mathrm{sp}\, L_{\mathrm{fr}}(\eta)$ equal to

$$\mathrm{sp}(K\otimes \mathbf{1}-\mathbf{1}\otimes \overline{K}) = \{k_1-k_2 : k_1,k_2\in \ \mathrm{sp}\, K\}.$$

The corresponding spectral projection equals the orthogonal projection onto $\mathcal{K}\otimes\overline{\mathcal{K}}\otimes\Omega$, where Ω is the Fock vacuum. Note that it does not depend on η. Denote this projection by $\mathbf{1}^{v\overline{v}}$. Clearly, $E=\mathbf{1}^{vv}L_{\mathrm{fr}}(\eta)$ does not depend on η either and can be identified with $K\otimes\mathbf{1}-\mathbf{1}\otimes\overline{K}$.

We can apply the method developed in this paper to the operator $L_\rho(\eta)=L_{\mathrm{fr}}(\eta)+\lambda Q_\rho(\eta)$ obtaining the LSO, which again does not depend on η,

$$\Gamma_\rho \coloneqq \sum_{e\in\mathrm{sp}(E)} \mathbf{1}_e(E)Q_\rho^{v\overline{v}}(\eta)(e\mathbf{1}^{\overline{vv}}-L_{\mathrm{fr}}^{\overline{vv}}(\eta))^{-1}Q_\rho^{\overline{v}v}(\eta)\mathbf{1}_e(E).$$

One can compute Γ_ρ from the undeformed Liouvillean as well,

$$\Gamma_\rho = \lim_{\epsilon\searrow 0}\sum_{e\in\mathrm{sp}(E)} \mathbf{1}_e(E)Q_\rho^{v\overline{v}}((e+i\in)\mathbf{1}^{\overline{vv}}-\ L_{\mathrm{fr}}^{\overline{vv}}))^{-1}Q_\rho^{\overline{v}v}\mathbf{1}_e(E). \tag{4.3}$$

Note that (4.3) coincides with the definition of LSO contained in Ref. 7.

One can also compute the projectors $p_e(\eta)$ and $p_{e,m}(\eta)$. They depend on η, but in a rather controlled way, they are analytic functions of η for satisfying $s\in\mathbb{R}$,

$$p_e(\eta)=e^{isS}p_e(\eta+s)e^{-isS},\quad p_{e,m}(\eta)=e^{isS}p_{e,m}(\eta+s)e^{-isS}.$$

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