Quadratic Hamiltonians and their renormalization

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1. Introduction

In this article I would like to review the results of two recent papers [D, BD] on quadratic bosonic Hamiltonians with an infinite number of degrees of freedom. I would like to convince the reader that their theory is surprisingly rich.

Let a_ξ^*/a_ξ denote *creation/annihilation operators* satisfying

$$[a_\xi, a_{\xi'}] = [a_\xi^*, a_{\xi'}^*] = 0, \quad [a_\xi, a_{\xi'}^*] = \delta_{\xi,\xi'}, \tag{1}$$

and acting on a *bosonic Fock space*. (Above, $\delta_{\xi,\xi'}$ denotes the delta function. Strictly speaking, a_ξ^*/a_ξ are operator valued measures and they acquire the meaning of an operator only after smearing out with appropriate test functions).

The first class of Hamiltonians that I would like to discuss was studied in [D] and is given by a formal expression of the form

$$H = \int h(\xi) a_\xi^* a_\xi \mathrm{d}\xi + \int \overline{z}(\xi) a_\xi \mathrm{d}\xi + \int z(\xi) a_\xi^* \mathrm{d}\xi + c, \tag{2}$$

where $h(\xi) \geq 0$. Note that in the above expression the constant c can be infinite. Following [Sch], operators of the form (2) will be called *van Hove Hamiltonians*.

The second class of operators was recently studied by L. Bruneau together with myself in [BD]. Operators from this class are given by a formal expression of the form

$$\begin{aligned} H = & \int h(\xi) a_\xi^* a_\xi \mathrm{d}\xi \\ & + \frac{1}{2} \int g(\xi,\xi') a_\xi^* a_{\xi'}^* \mathrm{d}\xi + \frac{1}{2} \int \overline{g}(\xi,\xi') a_\xi a_{\xi'} \mathrm{d}\xi + c. \end{aligned} \tag{3}$$

We will call them *Bogoliubov Hamiltonians*. This name is justified by the famous application of such Hamiltonians in the study of the Bose gas due to Bogoliubov [Bog].

There are several questions that one can pose about these operators. They include: When the above formal expressions defines a self-adjoint operator? When they are bounded from below? When they have a ground state? What is their scattering theory? Rather complete answers to these questions exist in the case of van Hove Hamiltonians. For Bogoliubov Hamiltonians, the answers are not so complete, but still we have a number of interesting results about them.

Van Hove and Bogoliubov Hamiltonians are used in quantum physics very often. A lot of interesting physical phenomena can be explained just with help of quadratic Hamiltonians.

In my paper I would like to convince the reader that also from the mathematical point of view they are interesting objects and illustrate various curious properties of unbounded operators.

Quadratic Hamiltonians are also useful, because they help to understand properties of more complicated Hamiltonians used in quantum theory such as those studied in [Fr, DG1, BFS, DJ].

2. Notation

Let us briefly review the notation for bosonic Fock spaces that we will use in our paper [Be, RS2, BR, GJ, DG1, D1]. Suppose that $\mathcal{Z}$ is a Hilbert space. The bosonic Fock space over the one-particle space $\mathcal{Z}$ is defined as

$$\Gamma_{\rm s}(\mathcal{Z}) := \mathop{\oplus}_{n=0}^{\infty} \otimes_{\rm s}^n \mathcal{Z}.$$

It has a distinguished vector called the vacuum vector $\Omega = 1 \in \otimes_{\rm s}^0 \mathcal{Z} = \mathbb{C}$.

The bosonic Fock space can be viewed as a commutative algebra with the product defined as follows: if $\Psi \in \otimes_{\rm s}^n \mathcal{Z}$, $\Phi \in \otimes_{\rm s}^m \mathcal{Z}$, then

$$\Psi \otimes_{\rm s} \Phi := \Theta_{\rm s}\ \Psi \otimes \Phi \in \otimes_{\rm s}^{n+m} \mathcal{Z},$$

where $\Theta_{\rm s}$ is the symmetrizing operator.

For $z \in \mathcal{Z}$ we define the creation operator

$$a^*(z)\Psi := \sqrt{n+1} z \otimes_{\rm s} \Psi, \quad \Psi \in \otimes_{\rm s}^n \mathcal{Z},$$

and the annihilation operator $a(z) := (a^*(z))^*$.

In a large part of the literature one assumes that $\mathcal{Z}$ equals $L^2(\Xi)$ for some measure space $(\Xi, {\rm d}\xi)$. One introduces "operator valued measures" a_ξ / a_ξ^* satisfying (1). If $\xi \mapsto z(\xi)$ is a square integrable function then

$$a^*(z) = \int z(\xi) a_\xi^* {\rm d}\xi, \quad a(z) = \int \overline{z}(\xi) a_\xi {\rm d}\xi.$$

We will use both notations. The notation involving the operator valued measures will be called "traditional" – it is lengthy and depends on an arbitrary identification $\mathcal{Z} = L^2(\Xi)$, but is perhaps more familiar to some readers and often convenient.

For an operator q on $\mathcal{Z}$ we define the operator $\Gamma(q)$ on $\Gamma_{\mathrm{s}}(\mathcal{Z})$ by

$$\Gamma(q)\Big|_{\otimes_{\mathrm{s}}^n\mathcal{Z}} = q\otimes\cdots\otimes q.$$

For an operator h on $\mathcal{Z}$ we define the operator $\mathrm{d}\Gamma(h)$ on $\Gamma_{\mathrm{s}}(\mathcal{Z})$ by

$$\mathrm{d}\Gamma(h)\Big|_{\otimes_{\mathrm{s}}^n\mathcal{Z}} = h\otimes 1^{(n-1)\otimes}+\cdots 1^{(n-1)\otimes}\otimes h.$$

If h is the multiplication operator by $h(\xi)$, then in the traditional notation we have

$$\mathrm{d}\Gamma(h) = \int h(\xi)a_\xi^* a_\xi \mathrm{d}\xi.$$

Note the identity $\Gamma(\mathrm{e}^{\mathrm{i}th}) = \mathrm{e}^{\mathrm{i}t\mathrm{d}\Gamma(h)}$.

For $g\in\otimes_{\mathrm{s}}^2\mathcal{Z}$ we define the 2-particle creation operator

$$a^*(g)\Psi := \sqrt{(n+2)(n+1)}\,g\otimes_{\mathrm{s}}\Psi,\quad \Psi\in\otimes_{\mathrm{s}}^n\mathcal{Z},$$

and the annihilation operator $a(g)=a^*(g)^*$.
In the traditional notation, if g equals the function $g(\xi,\xi')$, then we have

$$a^*(g) = \int g(\xi,\xi')a_\xi^* a_{\xi'}^*\mathrm{d}\xi\mathrm{d}\xi',\quad a(g) = \int \overline{g}(\xi,\xi')a_\xi a_{\xi'}\mathrm{d}\xi\mathrm{d}\xi'.$$

3. Van Hove Hamiltonians

In this section we summarize properties of van Hove Hamiltonians, following [D].

Let $\mathcal{Z}=L^2(\Xi)$. Let $\Xi\ni\xi\mapsto h(\xi)$ be a positive function. Let $\xi\mapsto z(\xi)$ be a function on Ξ such that

$$\int_{h<1}|z(\xi)|^2\mathrm{d}\xi + \int_{h\geq 1}\frac{|z(\xi)|^2}{h(\xi)^2}\mathrm{d}\xi<\infty.$$

Then we can define a family of unitary operators on $\Gamma_{\mathrm{s}}(\mathcal{Z})$

$$V(t) := \Gamma(\mathrm{e}^{\mathrm{i}th})\exp\left(a^*((1-\mathrm{e}^{-\mathrm{i}th})h^{-1}z)-\mathrm{hc}\right).$$

One can easily check that

$$V(t_1)V(t_2) = c(t_1,t_2)V(t_1+t_2)$$

for some complex numbers $c(t_1,t_2)$.

For an operator $B\in B(\Gamma_{\mathrm{s}}(\mathcal{Z}))$ we define

$$\beta_t(B) := V(t)BV(t)^*.$$

Then β is a 1-parameter group of $*$-automorphisms of the algebra of bounded operators on the Fock space, pointwise continuous in the strong operator topology.

By a general theorem [BR], there exists a self-adjoint operator H such that

$$\beta_t(B) = \mathrm{e}^{\mathrm{i}tH}\,B\,\mathrm{e}^{-\mathrm{i}tH}.$$

H is defined uniquely up to an additive constant. We call it a *van Hove Hamiltonian*. It is easy to see that formally it is given by (2), which contains an arbitrary constant c. One can ask if there is a natural choice of c. It turns out that there exist two such natural choices. To describe them it is convenient (especially, if we want to be rigorous) to use the unitary groups generated by van Hove Hamiltonians.

The following theorem describes the unitary group generated by van Hove Hamiltonians of the first kind:

Theorem 1. *Let*

$$\int_{h(\xi)<1}|z(\xi)|^2\mathrm{d}\xi+\int_{h(\xi)\geq 1}\frac{|z(\xi)|^2}{h(\xi)}\mathrm{d}\xi<\infty.$$

Then

$$U_{\mathrm{I}}(t):=\exp\left(\mathrm{i}\int|z(\xi)|^2\frac{\sin th(\xi)-th(\xi)}{h^2(\xi)}\mathrm{d}\xi\right)V(t) \tag{4}$$

is a strongly continuous unitary group.

We define the *type I van Hove Hamiltonian* H_{I} to be the self-adjoint generator of (4), that is $U_{\mathrm{I}}(t)=\mathrm{e}^{\mathrm{i}tH_{\mathrm{I}}}$. Formally,

$$H_{\mathrm{I}} \ = \ \int h(\xi)a_\xi^*a_\xi\mathrm{d}\xi+\int\overline{z}(\xi)a_\xi\mathrm{d}\xi+\int z(\xi)a_\xi^*\mathrm{d}\xi.$$

It satisfies $\Omega\in\mathrm{Dom}H_{\mathrm{I}}$, $(\Omega|H_{\mathrm{I}}\Omega)=0$.

Note that

$$\inf\mathrm{sp}H_{\mathrm{I}}=-\int\frac{|z(\xi)|^2}{h(\xi)}\mathrm{d}\xi,$$

(which can be $-\infty$). The linear perturbation contained in (2) is an operator iff $\int|z(\xi)|^2\mathrm{d}\xi<\infty$, otherwise it is a quadratic form.

Another natural class of van Hove Hamiltonians is described in the following theorem:

Theorem 2. *Let*

$$\int_{h(\xi)<1}\frac{|z(\xi)|^2}{h(\xi)}\mathrm{d}\xi+\int_{h(\xi)\geq 1}\frac{|z(\xi)|^2}{h^2(\xi)}\mathrm{d}\xi<\infty. \tag{5}$$

Then

$$U_{\mathrm{II}}(t):=\exp\left(\mathrm{i}\int|z(\xi)|^2\frac{\sin th(\xi)}{h^2(\xi)}\mathrm{d}\xi\right)V(t) \tag{6}$$

is a strongly continuous unitary group.

We define the *type II van Hove Hamiltonian* H_{II} to be the self-adjoint generator of (6), that is $U_{\mathrm{II}}(t)=\mathrm{e}^{\mathrm{i}tH_{\mathrm{II}}}$. Formally,

$$H_{\mathrm{II}} \ = \ \int h(\xi)\left(a_\xi^*+\frac{\overline{z}(\xi)}{h(\xi)}\right)\left(a_\xi+\frac{z(\xi)}{h(\xi)}\right)\mathrm{d}\xi.$$

It satisfies $\inf\mathrm{sp}H_{\mathrm{II}}=0$.

Let us introduce the following unitary operator called sometimes the *dressing operator*:

$$U := \exp\left(-a^*(\frac{z}{h}) + a(\frac{z}{h})\right). \tag{7}$$

Note that (7) is well defined iff

$$\int \frac{|z(\xi)|^2}{h^2(\xi)} \mathrm{d}\xi < \infty.$$

It intertwines H_{II} and the free van Hove Hamiltonian:

$$H_{\mathrm{II}} = U \int h(\xi) a_\xi^* a_\xi \mathrm{d}\xi \; U^*.$$

Hence, in this case H_{II} has a ground state. Otherwise H_{II} has no ground state.

Both H_{I} and H_{II} are well defined iff

$$\int \frac{|z(\xi)|^2}{h(\xi)} \mathrm{d}\xi < \infty,$$

and then

$$H_{\mathrm{II}} = H_{\mathrm{I}} + \int \frac{|z(\xi)|^2}{h(\xi)} \mathrm{d}\xi < \infty.$$

If

$$\int_{h(\xi)<1} \frac{|z(\xi)|^2}{h(\xi)} \mathrm{d}\xi = \int_{h(\xi)\geq 1} \frac{|z(\xi)|^2}{h(\xi)} \mathrm{d}\xi = \infty,$$

then neither H_{I} nor H_{II} is well defined.

Altogether we have 3 kinds of situations that lead to different infrared behaviors of the van Hove hamiltonians. Likewise, we have 3 possible ultraviolet behaviors. Thus, altogether we have $3 \times 3 = 9$ situations that lead to van Hove Hamiltonians with distinct properties. They are summarized in the following table:

	$\int_{h>1} \|z\|^2 < \infty$	$\int_{h>1} \|z\|^2 = \infty$ $\int_{h>1} \frac{\|z\|^2}{h} < \infty$	$\int_{h>1} \frac{\|z\|^2}{h} = \infty$ $\int_{h>1} \frac{\|z\|^2}{h^2} < \infty$	
$\int_{h<1} \frac{\|z\|^2}{h^2} < \infty$				H_{II} defined gr. st. exists
$\int_{h<1} \frac{\|z\|^2}{h^2} = \infty$ $\int_{h<1} \frac{\|z\|^2}{h} < \infty$				H_{II} defined no gr. st.
$\int_{h<1} \frac{\|z\|^2}{h} = \infty$ $\int_{h<1} \|z\|^2 < \infty$				unbounded from below
	H_{I} defined pert. is an operator	H_{I} defined pert. is not an operator	infinite renormalization	

In the literature, the analysis of the ultraviolet problem of van hove Hamiltonians can be found in [Be, Sch], following earlier treatments [vH, EP, To, GS]. The understanding of the infrared problem can be traced back to [BN], and then was discussed in a series of papers [Ki]. Closely related problems of coherent representations was discussed already in [Frie]. Nevertheless, it seems that [D] gives the first complete treatment of this subject in the literature.

4. Scattering theory of Van Hove Hamiltonians

The main goal of this section is a description of scattering theory for van Hove Hamiltonians. It is based on [D]

Let us start with some remarks about scattering theory in an abstract setting (see e.g. [Ya, Kato, RS3]). Suppose we are given two self-adjoint operators: H_0 and H.

In the standard approach to scattering theory, which works e.g. for 2-body Schrödinger operators, the *wave operators* are defined by

$$\Omega^{\pm} := \mathrm{s-}\lim_{t\to\pm\infty} \mathrm{e}^{\mathrm{i}tH}\,\mathrm{e}^{-\mathrm{i}tH_0}\,. \tag{8}$$

They satisfy $\Omega^{\pm}H_0 = H\Omega^{\pm}$ and are isometric. If $\mathrm{Ran}\Omega^{+} = \mathrm{Ran}\Omega^{-}$, then the *scattering operator*

$$S = \Omega^{+*}\Omega^{-}$$

is unitary and $H_0S = SH_0$.

Unfortunately, this approach does not work for van Hove Hamiltonians. Let us describe an alternative, less known approach to scattering theory.

Again, we start from two self-adjoint operators: H_0 and H. We introduce the *unrenormalized Abelian wave operators*:

$$\Omega_{\mathrm{ur}}^{\pm} := \mathrm{s-}\lim_{\epsilon\searrow 0} 2\epsilon\int_0^\infty \mathrm{e}^{-2\epsilon t}\,\mathrm{e}^{\pm\mathrm{i}tH}\,\mathrm{e}^{\mp\mathrm{i}tH_0}\,\mathrm{d}t. \tag{9}$$

They satisfy $\Omega_{\mathrm{ur}}^{\pm}H_0 = H\Omega_{\mathrm{ur}}^{\pm}$ but do not have to be isometric. Note that if the usual wave operators $\Omega^{\pm}$ defined by (8) exist, then so do the unrenormalized Abelian ones, and they coincide. However, (9) may exist even if the usual wave operators do not.

Assume that the operators $Z^{\pm} := \Omega_{\mathrm{ur}}^{\pm *}\Omega_{\mathrm{ur}}^{\pm}$ have a zero kernel. Then we can define the *renormalized Abelian wave operators*

$$\Omega_{\mathrm{rn}}^{\pm} := \Omega_{\mathrm{ur}}^{\pm}(Z^{\pm})^{-1/2}.$$

They also satisfy $\Omega_{\mathrm{rn}}^{\pm}H_0 = H\Omega_{\mathrm{rn}}^{\pm}$ and are isometric.

If $\mathrm{Ran}\Omega_{\mathrm{rn}}^{+} = \mathrm{Ran}\Omega_{\mathrm{rn}}^{-}$, then the *renormalized scattering operator*

$$S_{\mathrm{rn}} = \Omega_{\mathrm{rn}}^{+*}\Omega_{\mathrm{rn}}^{-}$$

is unitary and $H_0S_{\mathrm{rn}} = S_{\mathrm{rn}}H_0$.

Note that the alternative approach is more suitable for quantum field theory than the standard. In particular, as we will see, it works well in the case of van Hove Hamiltonians.

Let

$$\begin{aligned} H_0 &= \int h(\xi) a_\xi^* a_\xi \mathrm{d}\xi, \\ H &= \int h(\xi) a_\xi^* a_\xi \mathrm{d}\xi + \int \overline{z}(\xi) a_\xi \mathrm{d}\xi + \int z(\xi) a_\xi^* \mathrm{d}\xi + \int \frac{|\boldsymbol{z}(\xi)|^2}{h(\xi)} \mathrm{d}\xi. \end{aligned}$$

(In other words, H is a van Hove Hamiltonian of type II). Suppose that h has an absolutely continuous spectrum and the assumption (5) is satisfied. Then it is not difficult to show that the unrenormalized Abelian wave operators exist. One can compute explicitely the wave and scattering operators:

$$\Omega_{\mathrm{ur}}^{\pm} = Z^{1/2} U, \quad \Omega_{\mathrm{rn}}^{\pm} = U, \quad S_{\mathrm{rn}} = 1.$$

where U is the dressing operator and

$$Z = \exp \int \frac{|\boldsymbol{z}(\xi)|^2}{h^2(\xi)} \mathrm{d}\xi.$$

Unfortunately, the scattering operator is trivial.

Note in parenthesis that scattering theory for operators similar but more complicated than van Hove Hamiltonians can be quite interesting [DG1, DG2].

5. Bogoliubov Hamiltonians

In this section we describe mathematical theory of Bogoliubov Hamiltonians following [BD]. Again, it is not obvious how to define those Hamiltonians. The expression (3) is not very convenient for their rigorous definition. In order to fomulate a definition that is natural and as general as possible, it is convenient to think in terms of the *classical phase space* underlying the given bosonic Fock space. To this end we need to recall some notions from linear algebra and the formalism of second quantization.

Let $\mathcal{Z}$ be a complex Hilbert space. We will write $\overline{\mathcal{Z}}$ for the space complex conjugate to $\mathcal{Z}$. The real vector space

$$\mathcal{Y} := \{(z, \overline{z}) \ : \ z \in \mathcal{Z}\} \subset \mathcal{Z} \oplus \overline{\mathcal{Z}}$$

equipped with a natural symplectic form

$$(z_1, \overline{z}_1) \omega (z_2, \overline{z}_2) := \mathrm{Im}(z_1 | z_2).$$

has the meaning of the *dual of the classical phase space* of the quantum system described by the bosonic Fock space $\Gamma_{\mathrm{s}}(\mathcal{Z})$.

For $y = (z, \overline{z}) \in \mathcal{Y}$ we define the corresponding *Weyl operator*

$$W(y) := \mathrm{e}^{\mathrm{i} a^*(z) + \mathrm{i} a(z)}.$$

Note that $W(y_1) W(y_2) = \mathrm{e}^{-\frac{\mathrm{i}}{2} y_1 \omega y_2} W(y_1 + y_2)$.

A map r on $\mathcal{Y}$ is called *symplectic* if

$$(ry_1)\omega(ry_2) = y_1\omega y_2.$$

For such r,

$$W(ry_1)W(ry_2) = \mathrm{e}^{-\frac{\mathrm{i}}{2}y_1\omega y_2}\, W(r(y_1+y_2)),$$

and thus the commutation relations of Weyl operators are preserved.

Every linear map r on $\mathcal{Y}$ can be uniquely extended to a complex linear map on $\mathcal{Z}\oplus\overline{\mathcal{Z}}$ and written as

$$r = \left[\begin{array}{cc} p & q \\ \overline{q} & \overline{p} \end{array}\right].$$

r is symplectic iff

$$\begin{array}{ll} p^*p - \overline{q}^*\overline{q} = 1, & -\overline{p}^*\overline{q} + q^*p = 0, \\ pp^* - qq^* = 1, & \overline{q}p^* - \overline{p}q^* = 0. \end{array}$$

We have the decomposition

$$r = \left[\begin{array}{cc} 1 & 0 \\ d^* & 1 \end{array}\right]\left[\begin{array}{cc} p & 0 \\ 0 & \overline{p}^{*-1} \end{array}\right]\left[\begin{array}{cc} 1 & c \\ 0 & 1 \end{array}\right],$$

with symmetric operators $d := q\overline{p}^{-1}$, $c := p^{-1}q$. (We say that d is a *symmetric* operator iff $d = \overline{d}^*$).

Theorem 3 (Shale Theorem)**.** [Sh] *Let r be symplectic. There exists a unitary U, which we call a* Bogoliubov implementer*, such that*

$$UW(y)U^* = W(ry), \quad y \in \mathcal{Y},$$

iff $\mathrm{Tr} q^*q < \infty$.

The map $B(\Gamma_{\mathrm{s}}(\mathcal{Z})) \ni A \mapsto UAU^*$, where U is a Bogoliubov implementer, will be called a *Bogoliubov automorphism*. For a given r, a Bogoliubov implementer is determined up to a phase. There exists a distinguished choice, denoted U_{nat}, satisfying $(\Omega|U_{\mathrm{nat}}\Omega) > 0$, given by

$$U_{\mathrm{nat}} := |\det pp^*|^{-\frac{1}{4}}\, \mathrm{e}^{-\frac{1}{2}a^*(d)}\, \Gamma(p^{*-1})\, \mathrm{e}^{\frac{1}{2}a(c)}\,.$$

An important role in our considerations will be played by *strongly continuous 1-parameter groups of symplectic transformations.* If $\mathbb{R}\ni t\mapsto r(t)$ is such a group, we introduce the maps $t\mapsto p(t), q(t)$ defined by

$$r(t) = \left[\begin{array}{cc} p(t) & q(t) \\ \overline{q}(t) & \overline{p}(t) \end{array}\right]. \tag{10}$$

If h is a *self-adjoint* operator on $\mathcal{Z}$ and g is a bounded symmetric operator from $\overline{\mathcal{Z}}$ to $\mathcal{Z}$ then

$$r(t) = \exp \mathrm{i}t \left[\begin{array}{cc} h & g \\ \overline{g} & \overline{h} \end{array}\right] \tag{11}$$

is a 1-parameter symplectic group.

Let us consider for a while the case of finite dimensional $\mathcal{Z}$. In this case theory of Bogoliubov Hamiltonians, while not quite trivial, is well understood. (We still keep the notation $\mathcal{Z} = L^2(\Xi)$, but now Ξ has to be a finite set and the integration over Ξ is just summation)

Clearly, in finite dimension every continuous 1-parameter symplectic group is of the form (11). Consider a classical quadratic Hamiltonian

$$\begin{aligned} H(\overline{z}, z) &= \int h(\xi)\overline{z}_\xi z_\xi \mathrm{d}\xi \\ &+\frac{1}{2}\int g(\xi,\xi')\overline{z}_\xi\overline{z}_\xi \mathrm{d}\xi\mathrm{d}\xi' + \frac{1}{2}\int \overline{g}(\xi,\xi')z_\xi z_{\xi'}\mathrm{d}\xi\mathrm{d}\xi'. \end{aligned}$$

It is a function on the *classical phase space*

$$\overline{\mathcal{Y}} := \{(\overline{z}, z) \ : \ z\in\mathcal{Z}\} \subset \overline{\mathcal{Z}}\oplus\mathcal{Z}.$$

The *Weyl quantization* of $H(\overline{z}, z)$ equals

$$\begin{aligned} H &= \frac{1}{2}\int h(\xi)a_\xi^* a_\xi \mathrm{d}\xi + \frac{1}{2}\int h(\xi)a_\xi a_\xi^* \mathrm{d}\xi \\ &+\frac{1}{2}\int g(\xi,\xi')a_\xi^* a_{\xi'}^* \mathrm{d}\xi\mathrm{d}\xi' + \frac{1}{2}\int \overline{g}(\xi,\xi')a_\xi a_{\xi'}\mathrm{d}\xi\mathrm{d}\xi' \end{aligned}$$

and corresponds to the choice of c in (3) given by

$$c = \frac{1}{2}\int h(\xi,\xi)\mathrm{d}\xi = \frac{1}{2}\mathrm{Tr}h,$$

H is essentially self-adjoint on finite particle vectors. We have

$$\mathrm{e}^{\mathrm{i}tH} = (\det p(t))^{-\frac{1}{2}}\,\mathrm{e}^{-\frac{1}{2}a^*(d(t))}\,\Gamma(p(t)^{*-1})\,\mathrm{e}^{\frac{1}{2}a(c(t))}\,.$$

Note that the set of operators of the form

$$(\det p)^{-\frac{1}{2}}\,\mathrm{e}^{-\frac{1}{2}a^*(d)}\,\Gamma(p^{*-1})\,\mathrm{e}^{\frac{1}{2}a(c)} \tag{12}$$

is closed wrt the multiplication. It is called the *metaplectic group* $Mp(\mathcal{Y})$.

Let us now relax the condition $\dim\mathcal{Z} < \infty$ and ask about possible genralizations of the above costruction to the case of an arbitrary number of degrees of freedom. Clearly, (12) is well defined provided that $p-1$ is trace class, or equivalently, $r-1$ is trace class. The set of operators of this form is also closed wrt multiplication. Thus, as noticed by Lundberg, the metaplectic group can be defined also in the case of an infinite number of degrees of freedom.

We say that a strongly continuous 1-parameter group of symplectic transformations $t\mapsto r(t)$ is *implementable* iff there exists a strongly continuous 1-parameter unitary group $t\mapsto U(t)$, called the *implementing unitary group*, such that

$$U(t)W(y)U^*(t) = W(r(t)y), \quad y\in\mathcal{Y}. \tag{13}$$

Only now, after so much preparation, we introduce the rigorous definition of a Bogoliubov Hamiltonian: A self-adjoint operator H is called a *Bogoliubov Hamiltonian* if there exists a 1-parameter strongly-continuous implementable symplectic group $t \mapsto r(t)$ such that $H := -\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}U(t)\Big|_{t=0}$, where $t \mapsto U(t)$ is its implementing unitary group.

The following theorem is proven in [BD]:

Theorem 4. *$t \mapsto r(t)$ is implementable iff* $\mathrm{Tr}q^*(t)q(t) < \infty$ *and*

$$\lim_{t\to 0} \mathrm{Tr}q^*(t)q(t) = 0.$$

Once again, given an implementable Bogoliubov dynamics we have a 1-parameter family of Hamiltonians, formally differing by the constant c. We would like to discuss some of their natural choices.

Let us describe our first choice. Let $t \mapsto r(t)$ be an implementable symplectic group. Let $p(t)$ be defined by (10). We say that $t \mapsto r(t)$ is of *type I* iff $\frac{\mathrm{d}}{\mathrm{d}t}p(t)\Big|_{t=0} = \mathrm{i}h$, $p(t)\,\mathrm{e}^{-\mathrm{i}th} - 1$ is trace class and $\|p(t)\,\mathrm{e}^{-\mathrm{i}th} - 1\|_1 \to 0$.

Theorem 5. *In the type I case*

$$U_{\mathrm{I}}(t) := \det(p(t)\,\mathrm{e}^{-\mathrm{i}th})^{-\frac{1}{2}}\,\mathrm{e}^{-\frac{1}{2}a^*(d(t))}\,\Gamma(p(t)^{*-1})\,\mathrm{e}^{\frac{1}{2}a(c(t))}$$

is a strongly continuous 1-parameter unitary group.

A *type I Bogoliubov Hamiltonian* is defined as

$$H_{\mathrm{I}} := -\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}U_{\mathrm{I}}(t)\Big|_{t=0}.$$

Let $t \mapsto r(t)$ be implementable. We say that it is of *type II* iff the implementing 1-parameter group has a generator, which is bounded from below. In this case we define the *type II Hamiltonian* to be

$$H_{\mathrm{II}} := -\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}U_{\mathrm{II}}(t)\Big|_{t=0}.$$

such that $\inf \mathrm{sp}H_{\mathrm{II}} = 0$ and $U_{\mathrm{II}}(t)$ implements $r(t)$.

For a finite number of degrees of freedom it is easy to see that we have a complete characterization of type I and II Bogoliubov Hamiltonians:

Theorem 6. *Let $\mathcal{Z}$ be finite dimensional. Then*

(1) *$r(t)$ is always type I and*

$$H_{\mathrm{I}} = \mathrm{d}\Gamma(h) + \frac{1}{2}a^*(g) + \frac{1}{2}a(g).$$

(2) *$r(t)$ is type II iff its classical Hamiltonian is positive definite*

$$\overline{z}hz + \frac{1}{2}\overline{z}g\overline{z} + \frac{1}{2}z\overline{g}z \geq 0,$$

and then

$$H_{\mathrm{II}} = H_{\mathrm{I}} - \frac{1}{4}\mathrm{Tr}\left[\left(\begin{array}{cc} \overline{h}^2 - \overline{g}g & \overline{h}\overline{g} - \overline{g}h \\ hg - g\overline{h} & h^2 - g\overline{g} \end{array}\right)^{1/2} - \left(\begin{array}{cc} \overline{h} & 0 \\ 0 & h \end{array}\right)\right].$$

In the case of an infinite number of degrees of freedom, our results about Bogoliubov Hamiltonians are only partial. Let us give some examples taken from [BD]:

Theorem 7. *Let g be Hilbert-Schmidt. Then*

$$H_{\mathrm{I}} = \mathrm{d}\Gamma(h) + \frac{1}{2}a^*(g) + \frac{1}{2}a(g)$$

is essentially self-adjoint on the algebraic Fock space over $\mathrm{Dom}(h)$ *and* $\mathrm{e}^{\mathrm{i}tH_{\mathrm{I}}}$ *implements $r(t)$ given by (11).*

Theorem 8. *Let h be positive,*

$$\|h^{-1/2}\otimes h^{-1/2}\ g\|_{\Gamma_{\mathrm{s}}^2(\mathcal{Z})} < 1,$$
$$\|h^{-1/2}g\|_{B(\overline{\mathcal{Z}},\mathcal{Z})} < \infty.$$

Then $\frac{1}{2}a^*(g)+\frac{1}{2}a(g)$ is relatively $\mathrm{d}\Gamma(h)$*-bounded with the bound less than* 1*. Therefore, in this case, both the type I and type II Bogoliubov Hamiltonians are well defined.*

There is one class of Bogoliubov Hamiltonians, that we were able to analyze rather completely: those satisfying the condition $g\overline{h} = hg$. In this case, without loss of generality we can assume that they are diagonal in a common orthonormal basis $e_1, e_2, \dots$:

$$he_n = h_n e_n,\ h_n \in \mathbb{R};\quad g\bar{e}_n = g_n e_n,\quad g_n \in \mathbb{C}.$$

We will say that such Hamiltonians are *diagonalizable.*

Theorem 9. [BD] *Suppose h, g are diagonalizable in the above sense.*
(1) *$r(t)$ is well defined iff for some b, $a<1$, $|g_n| \le a|h_n| + b$.*
(2) *$r(t)$ is implementable iff* $\sum\limits_n \frac{|g_n|^2}{1+h_n^2} < \infty$.
(3) *$r(t)$ is type I iff* $\sum\limits_n \frac{|g_n|^2}{1+|h_n|} < \infty$.
(4) *$r(t)$ is type II iff $|g_n| \le h_n$ and* $\sum\limits_n \frac{|g_n|^2}{h_n+h_n^2} < \infty$.

Theorem 9 shows that there exist implementable 1-parameter symplectic groups, which are not type II, even though their classical Hamiltonian is positive definite. Thus there exist Bogoliubov Hamiltonians unbounded from below with positive classical symbols. This is an example of an interesting infrared behavior of Bogoliubov Hamiltonians.

Theorem 9 shows also that there exist implementable 1-parameter symplectic groups, which are not type I. This means that, in order to express them in terms of creation and annihilation operators, one needs to add an infinite constant – perform an appropriate renormalization. This is an example of an interesting ultraviolet behavior of Bogoliubov Hamiltonians.

There remain various open questions concerning Bogoliubov Hamiltonians. For instance, it would be interesting to give sufficient and necessary conditions for symplectic group $r(t)$ to be of type II in terms of its generator.

Note that Bogoliubov Hamiltonians were studied by various authors, among them Friedrichs [Frie], Berezin [Be], Ruijsenaars [Ru1, Ru2], Araki and his collaborators [A, AY], Matsui and Shimada [MS], Ito and Hiroshima [IH]. Nevertheless, the approach contained [BD], briefly described above, seems to be the most general and flexible.

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