General Topology

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Contents

1 General concepts

In this section we discuss the basic axioms of topological spaces. Special classes of topological spaces will be considered in next sections. We assume that the reader is familiar with basic intuitions about topological spaces and will easily fill in missing proofs. We provide only some proofs that are related to the concept of the net and subnet, which are less intuitive.

1.1 Open and closed sets

$(X, \mathcal{T})$ is a topological space iff X is a set and $\mathcal{T} \subset 2^X$ satisfies

(1) $\emptyset, X \in \mathcal{T}$;

(2) $A_i \in \mathcal{T},\ i \in I,\ \Rightarrow \underset{i\in I}{\cup} A_i \in \mathcal{T}$;

(3) $A_1, \ldots, A_n \in \mathcal{T}\ \Rightarrow \overset{n}{\underset{i=1}{\cap}} A_i \in \mathcal{T}$.

Elements of $\mathcal{T}$ are called sets open in X. We will call $\mathcal{T}$ "a topology".
A set $A \subset X$ is called closed in X iff $X \backslash A$ is open.
If $\mathcal{T}$, $\mathcal{S}$ are topologies on X, then we say that $\mathcal{T}$ is weaker than $\mathcal{S}$ iff $\mathcal{T} \subset \mathcal{S}$.
If $\mathcal{T} = 2^X$, then we say that $\mathcal{T}$ is discrete.
If $\mathcal{T} = \{\emptyset, X\}$, then we say that $\mathcal{T}$ is antidiscrete.

1.2 Basis and subbasis of a topology

We say that $\mathcal{B} \subset 2^X$ is a covering of X (or covers X) iff $\underset{A\in\mathcal{B}}{\cup} A = X$

Let $(X, \mathcal{T})$ be a topological space. Let $\mathcal{B} \subset 2^X$. We say that $\mathcal{B}$ is a basis of a topology $\mathcal{T}$ iff $\mathcal{T}$ equals the family of unions of elements of $\mathcal{B}$. Clearly, every basis is a covering of X.

Theorem 1.1 *Let $\mathcal{H} \subset 2^X$. Then the following conditions are equivalent:*

(1) *the family of finite intersections of elements of $\mathcal{H}$ is a basis of $\mathcal{T}$;*

(2) *$\mathcal{T}$ is the least topology containing $\mathcal{H}$ and $\mathcal{H}$ covers X.*

(3) *$\mathcal{T} = \cap\mathcal{T}'$ where $\mathcal{T}'$ runs over all topologies containing $\mathcal{H}$ and $\mathcal{H}$ covers X.*

If $\mathcal{H}$ satisfies one of the above conditions, then it is called a subbasis of $\mathcal{T}$.

1.3 Neighborhoods

Let $x \in X$. We say that $A \subset X$ is a neighborhood of x, if there exists an open U such that $x \in U \subset A$. We say that $\mathcal{V}_x \subset 2^X$ is a basis of neighborhoods of x iff all element of $\mathcal{V}_x$ are neighborhoods of x and for any neighborhood U of x there exists $V \in \mathcal{V}_x$ such that $V \subset U$.

1.4 Convergence

The set with a relation $(I, \leq)$ is called a directed set iff

(1) $i \leq j$, $j \leq k \Rightarrow i \leq k$,

(2) $i \leq j$, $j \leq i \Rightarrow i = j$,

(3) for any i, j there exists k such that $i \leq k$ and $j \leq k$.

(Some authors omit (2)).

A net in X is a directed set $(I, \leq)$ together with a map $I \ni i \mapsto x_i \in X$. We will write $(x_i)_{i\in I}$.

Let $(y_j)_{j\in J}$ be another net.We will say that it is a subnet of $(x_i)_{i\in I}$ iff

(1) There exists a map $J \ni j \mapsto i(j) \in I$ such that for any $i \in I$ there exists $j \in J$ such that for any $j_1 \in J$ if $j_1 \geq j$, then $i(j_1) \geq i$;

(2) $x_{i(j)} = y_j$, $j \in J$.

(It is not necessary to assume that $j \mapsto i(j)$ is increasing).

We say that a net $(x_i)_{i\in I}$ is convergent to $x \in X$, iff for any neighborhood U of x there exists $i_U \in I$ such that if $i \geq i_U$, then $x_i \in U$.

Theorem 1.2 *If $(x_i)_{i\in I}$ is a net convergent to x, then any of its subnets is convergent to x.*

1.5 Interior and closure

Theorem 1.3 *Let $A, C \subset X$. Then the following conditions are equivalent:*

(1) *C is the smallest closed set containing A;*

(2) *$C = \cap C'$ where C' runs over all closed sets containing A;*

(3) *Let $x \in X$. Then $x \in C$ iff for any neighborhood U of x we have $U \cap A \neq \emptyset$.*

(4) *For any $x \in X$ choose a basis of neighborhoods of x, denoted $\mathcal{V}_x$. Then $x \in C$ iff for any $V \in \mathcal{V}_x$ we have $V \cap A \neq \emptyset$.*

(5) *C is the set of all limits of convergent nets in A.*

Proof. (4)$\Rightarrow$(5) Suppose that for any $V \in \mathcal{V}_x$ we can find $x_V \in V \backslash A$. Note that $\mathcal{V}_x$ is a directed set. Hence $(x_V)_{V \in \mathcal{V}_x}$ is a net. Clearly, $x_V \to x$. $\square$

If the above conditions are satisfied then we say that C is the closure of A and we write $C = A^{\mathrm{cl}}$.

Theorem 1.4 *Let $A, B \subset X$. Then the following conditions are equivalent:*

(1) *B is the largest open set contained in A;*

(2) *$B = \cup B'$ where B' runs over all open sets contained in A;*

(3) *Let $x \in X$. Then $x \in B$ iff there exists a neighborhood U of x such that $U \subset A$;*

(4) *For any $x \in X$ choose a basis of neighborhoods of x, denoted $\mathcal{V}_x$. Then $x \in B$ iff there exists a $V \in \mathcal{V}_x$ such that $V \subset A$.*

(5) *Let $x \in X$. Then $x \in B$ iff for any net $(x_i)_{i \in I}$ in X convergent to x there exists $i_0 \in I$ such that for $i \geq i_0$ we have $x_i \in A$;*

If the above conditions are satified, then B is called the interior of A and is denoted A°.
Clearly, $A^\circ = X \backslash (X \backslash A)^{\mathrm{cl}}$

Example 1.5 *Consider the index set I and the spaces $X_i := \{0, 1\}$ with the discrete topology. Consider $X := \underset{i \in I}{\times} X_i$ and let $A \subset X$. Elements of X can be labelled by subsets of I: in fact, if $J \subset I$, then $x_i^J := 1$ for $i \in J$ and $x_i^J = 0$ otherwise. Let X_{fin} be the set of sequences x^J with a finite J. Then the closure of X^{fin} equals X. In fact, if $J \subset I$ then we take the directed set $2^J_{\mathrm{fin}}$of finite subsets of J, then $(x^K)_{K \in 2^J_{\mathrm{fin}}}$ converges to x^J.*

1.6 Dense sets

We say that $Y \subset X$ is dense iff $Y^{\mathrm{cl}} = X$.

Theorem 1.6 *Let Y be dense in X and W open in X. Then $W^{\mathrm{cl}} = (W \cap Y)^{\mathrm{cl}}$.*

Theorem 1.7 *Let $Y \subset X$. TFAE (the following are equivalent):*

(1) *Y is open in Y^{cl}.*

(2) *$Y = A \cap B$ for A open and B closed.*

Proof. (1)$\Rightarrow$(2). Take $B := Y^{\mathrm{cl}}$.
(1)$\Leftarrow$(2). Clearly, $Y \subset Y^{\mathrm{cl}} \subset B$. Hence, $Y = A \cap Y^{\mathrm{cl}}$. Therefore, Y is open in Y^{cl}. $\square$

If the conditions of Theorem 1.7 are satisfied, we say that Y is locally open in X.

1.7 Cluster points of a net

Theorem 1.8 *Let $(x_i)_{\in I}$ be a net and $x \in X$. Then the following conditions are equivalent:*

(1) *For any neighborhood U of x and $i \in I$ there exists $j \geq i$ such that $x_j \in U$;*

(2) *There exists a basis of neighborhoods $\mathcal{V}_x$ such that for any $U \in \mathcal{V}_x$ and $i \in I$ there exists $j \geq i$ such that $x_j \in U$.*

(3) *There exists a subnet $(x_j)_{j \in J}$ convergent to x;*

(4) *x belongs to*

$$\underset{i \in I}{\cap} \{x_j \ : \ j \geq i\}^{\mathrm{cl}}.$$

If x satisfies the above conditions, then we say that it is a cluster point of the net $(x_i)_{i\in I}$.

Proof. (2)⇒(3) Let $J := \{(j,U) \in I \times \mathcal{V}_x \ : \ x_j \in U\}$. We write $(j_1,U_1) \le (j_2,U_2)$ iff $j_1 \le j_2$ and $U_1 \supset U_2$.

Step 1. J is directed.

In fact, let $(j_1,U_1),(j_2,U_2) \in J$. There exists $i \in I$ such that $j_1 \le i$, $j_2 \le i$. There exists $U \in \mathcal{V}_x$ such that $U \subset U_1 \cap U_2$. There exists $j \in I$ such that $i \le j$ and $x_j \in U$. Thus $(j,U) \in J$, $(j_1,U_1) \le (j,U)$, $j_2,U_2) \le (j,U)$.

Step 2. Define $J \ni (j,U) \mapsto j \in I$, $x_{j,U} := x_j$. It is a subnet.

In fact, let $i \in I$, $U \in \mathcal{V}_x$. There exists $j \in I$ with $j \ge i$ and $x_j \in U$. Thus $(j,U) \in J$. Now if $(j,U) \le (j_1,U_1)$, then $j \le j_1$, and hence $i \le j_1$.

Step 3. $x_{j,U}$ is convergent to x.

In fact, let $U \in \mathcal{V}_x$. There exists $j \in I$ with $x_j \in U$. Thus $(j,U) \in J$. Now if $(j,U) \le (j_1,U_1)$, then $x_{(j_1,U_1)} = x_{j_1} \in U_1 \subset U$. □

1.8 Continuity

Let X, Y be topological spaces and $f : X \to Y$.

Theorem 1.9 *Let $x_0 \in X$. TFAE:*

(1) *For any neighborhood V of $f(x_0)$, $f^{-1}(V)$ is a neighborhood of x_0;*

(2) *There exists a basis $\mathcal{V}_{x_0}$ of neighborhoods of x_0 and a basis of neighborhoods $\mathcal{W}_{f(x_0)}$ of $f(x_0)$ such that for any $W \in \mathcal{W}_{f(x_0)}$ there exists $V \in \mathcal{V}_{x_0}$ with $V \subset f^{-1}(W)$.*

(3) *For any net $(x_i)_{i\in I}$ converging to x_0, the net $(f(x_i))_{i\in I}$ converges to $f(x_0)$.*

We say that f is continuous at x_0 iff the conditions of the above theorem hold.

Theorem 1.10 *TFAE:*

(1) *The function f is continuous at every $x \in X$;*

(2) *For any open set $V \in Y$, $f^{-1}(V)$ is open.*

(3) *For any closed set $V \in Y$, $f^{-1}(V)$ is closed.*

(4) *There exists a subbasis $\mathcal{B}$ in Y such that the preimages of elements of $\mathcal{B}$ are open in X.*

We say that f is continuous iff the conditions of the above theorem hold.

The set of continuous functions from X to Y is denoted $C(X,Y)$. We will write

$$C(X) := C(X,\mathbb{C}),\ C_+(X) := C(X,[0,\infty[).$$

Note that $C(X)$ and $C(X,\mathbb{R})$ are commutative complex/real algebras.

Let X, Y be topological spaces. Let $p : X \to Y$ be a continuous map. Then $p^\# : C(Y) \to C(X)$ defined by

$$(p^\# f)(x) := f(p(x)), \quad x \in X \tag{1.1}$$

is a unital homomorphism of commutative algebras.

We can also introduce $C_{\rm bd}(X)$ consisting of bounded elements of $C(X)$. For $f \in C_{\rm bd}(X)$ we set $\|f\|_\infty := \sup\limits_{x\in X} |f(x)|$. It is a norm and $C_{\rm bd}(X)$ becomes a C^*-algebra. Note that if p is as above, then $p^\#$ maps $C_{\rm bd}(Y)$ into $C_{\rm bd}(X)$ and is a continuous $*$-homomorphism.

Theorem 1.11 *$p^\#$ is injective ⇐ p has a dense image.*

If $Y \subset X$, Then we define

$$C_Y(X) := \{f \in C(X) \ : \ f = 0 \text{ on } Y\}.$$

Clearly, $C_Y(X) = C_{Y^{\rm cl}}(X)$, so it is enough to consider closed subsets Y. $C_Y(X)$ is an ideal in $C(X)$.

1.9 Semicontinuity

Theorem 1.12 *Let X be a topological space and $f : X \to [-\infty, \infty]$. TFAE:*

(1) *For any $t \in \mathbb{R}$, $f^{-1}(]t, \infty])$ is open.*

(2) *For any $t \in \mathbb{R}$, $f^{-1}([-\infty, t])$ is closed.*

(3) *If (x_i) is a net in X convergent to x, then $f(x) \leq \liminf_i f(x_i)$.*

If the above conditions are satisfied, then we say that f is lower semicontinuous. We say that it is upper semicontinuous iff $-f$ is lower semicontinuous. Let $C^{\mathrm{lsc}}(X, [-\infty, \infty])$, $C^{\mathrm{usc}}(X, [-\infty, \infty])$ denote the spaces of lower and upper semicontinuous function on X.

Theorem 1.13 (1) *$C^{\mathrm{lsc}}(X, [-\infty, \infty])$ is stable under addition and multiplication by positive numbers.*

(2) *The supremum of any family of lower semicontinuous functions is lower semicontinuous.*

(3) *The infimum of a finite family of lower semicontinuous functions is lower semicontinuous.*

(4) *The uniform limit of a sequence of lower semicontinuous functions is lower semicontinuous.*

1.10 Basic constructions

If $Y \subset X$, and $(X, \mathcal{T})$ is a topological space, then

$$\{A \cap Y \ : \ A \in \mathcal{T}\}.$$

is a topology on Y called the relative topology on Y.

If $(X_i, \mathcal{T}_i)_{i \in I}$ is a family of disjoint topological spaces, then $\underset{i \in I}{\cup} X_i$ is equipped with a natural topology

$$\{\underset{i \in I}{\cup} A_i \ : \ A_i \in \mathcal{T}_i\}.$$

If $(X_i, \mathcal{T}_i)_{i \in I}$ is a family of topological spaces, then we define the product topology on the Cartesian product $\underset{i \in I}{\Pi} X_i$ as the topology with the subbasis

$$\{\overset{n}{\underset{j=1}{\cap}} \pi_i^{-1}(U_i) \ : \ U_i \in \mathcal{T}_i, \ i \in I\},$$

where $\pi_i : X \to X_i$ are the canonical projections.

Let (x_j) be a net in $\underset{i \in I}{\Pi} X_i$. Then $x_j \to x$ iff $\pi_i(x_j) \to \pi_i(x)$ for any $i \in I$.

Let $\mathcal{V}_{i,x_i}$ be a basis of neighborhoods of $x_i \in X_i$. Let $x = (x_i) \in \underset{i \in I}{\Pi} X_i$ has basis of neighborhoods consisting of $\underset{i \in I}{\Pi} A_i$, where all $A_i = X_i$ except for a finite number satisfying $A_i \in \mathcal{V}_{i,x_i}$

2 Compactness

2.1 Compact spaces

Let X be a topological space and $Y \subset X$. $\mathcal{S} \subset 2^X$ is called a covering of Y if $Y \subset \bigcup_{U \in \mathcal{S}} U$. $\mathcal{S}_0$ is called a subcovering of $\mathcal{S}$ if it is a covering contained in $\mathcal{S}$.

A family $\mathcal{R} \subset 2^X$ is called centered iff any finite subfamily of $\mathcal{R}$ has a non-empty intersection.

Theorem 2.1 *TFAE*

(1) *Every covering $\mathcal{S}$ of X by open sets contains a finite subcovering;*

(2) *Every centered family $\mathcal{R} \subset 2^X$ of closed set has a non-empty intersection;*

(3) *Every net in X has a cluster point.*

Proof. 1.⇔2. The condition 1. can be reformulated as follows. If $\mathcal{S}$ is a family of open sets in X such that for any finite subfamily $\mathcal{S}_0$ we have $\underset{U \in \mathcal{S}_0}{\cup} U \neq X$, then $\underset{U \in \mathcal{S}}{\cup} U \neq X$. Since the complements of open sets are closed, we immediately get 2.

2.⇒3. Set

$$F_i := \{x_j \ : \ i \leq j\}^{\mathrm{cl}}, \quad i \in I.$$

The family $\{F_i \ : \ i \in I\}$ is a centered family of closed sets. Thus $\underset{i \in I}{\cap} F_i$ is nonempty. But this is the set of cluster points of $(x_i)_{i \in I}$.

2.⇐3. Let $\mathcal{R}$ be a centered family of closed sets. Consider the index set Σ consisting of finite subsets of $\mathcal{R}$ ordered by inclusion. For any $\sigma := \{F_1, \ldots, F_n\} \in \Sigma$ we choose $x_\sigma \in \overset{n}{\underset{j=1}{\cap}} F_j$. Then, by (3), $(x_\sigma)_{\sigma \in \Sigma}$ has a cluster point $x \in X$.

Let $F \in \mathcal{R}$. Let U be a neighborhood of x. Then there exists $\sigma \in \Sigma$ such that $\sigma = \{F_1, \ldots, F_n\}$, $F \leq \{F_1, \ldots, F_n\}$ and $x_\sigma \in U$. One of $F_1, \ldots, F_n$ equals F, hence $x_\sigma \in \overset{n}{\underset{j=1}{\cap}} F_j \subset F$. Hence $F \cap U \neq \emptyset$. Thus, $x \in F^{\mathrm{cl}}$. But F is closed. Thus $x \in F$. Consequently, $x \in \underset{F \in \mathcal{R}}{\cap} F$. □

A space satisfying the above conditions is called compact.

A subset of a topological space is called precompact if its closure is compact.

Theorem 2.2 *If $f : X \to Y$ is a continuous function and X is compact, then $f(X)$ is compact.*

Proof. If $\mathcal{S}$ is a covering of $f(X)$ by open sets, then $\{f^{-1}(A) \ : \ A \in \mathcal{S}\}$ is a covering of X by open sets. We can choose a finite $\mathcal{S}_1 \subset \mathcal{S}$, such that $\{f^{-1}(A) \ : \ A \in \mathcal{S}_1\}$ is a covering of X. Using the fact that $ff^{-1}(A) = A$, we see that $\mathcal{S}_1$ is a covering of $f(X)$ □

Theorem 2.3 *Every closed set of a compact space is compact.*

Proof. Let A be a closed set in a compact space X. If $(x_i)_{i \in I}$ is a net in A then we can choose a subnet $(x_i(j))_{j \in J}$ convergent in X. Since A is closed, its limit belongs to A. □

Theorem 2.4 *If $(X_i)_{i \in I}$ is a family of disjoint compact spaces, then $\underset{i \in I}{\cup} X_i$ is compact iff I is finite.*

Theorem 2.5 (Tikhonov) *If $(X_i)_{i \in I}$ is a family of topological spaces, then $\underset{i \in I}{\Pi} X_i$ is compact iff, for every $i \in I$, X_i is compact.*

Theorem 2.6 (Dini) *If X is compact, $f_n, f \in C(X, \mathbb{R})$, $f_{n+1} \geq f_n$, $f = \sup f_n$, then $f_n \to f$ uniformly.*

Clearly, every finite topological space is compact.

2.2 Proof of Tikhonov's theorem

Let $\mathcal{W} \subset 2^X$ we say that $\mathcal{W}$ is of a finite type iff

(1) $\emptyset \in \mathcal{W}$;

(2) $A \in \mathcal{W} \Leftrightarrow$ for any finite $B \subset A$, $B \in \mathcal{W}$.

The following lemma follows from the axiom of choice:

Lemma 2.7 (Teichmüller-Tukey lemma) *Let $\mathcal{W}$ be of a finite type and $A \in \mathcal{W}$. Then there exists $M \in \mathcal{W}$ that contains A and is maximal wrt $\subset$.*

Lemma 2.8 *Let $\mathcal{R}_0 \subset 2^X$ be a centered family. Then there exists a centered family containing $\mathcal{R}_0$ which is maximal wrt $\subset$.*

Proof. We note that the class of centered families in 2^X is of the finite type and then apply Lemma 2.7. □

Lemma 2.9 *Let $\mathcal{R} \subset 2^X$ be a centered family maximal wrt $\subset$. Then*

(1) $A_1, \dots, A_k \in \mathcal{R} \;\Rightarrow\; A_1 \cap \dots \cap A_k \in \mathcal{R}$;

(2) $A_0 \cap A \neq \emptyset$, $A \in \mathcal{R} \Rightarrow A_0 \in \mathcal{R}$

Proof. 1. We easily see that if $\mathcal{R}$ is centered, $A_1, \dots, A_k \in \mathcal{R}$, then $\mathcal{R} \cup \{A_1 \cap \dots \cap A_k\}$ is centered.

2. If $\mathcal{R}$ is centered, satisfies 1. and $A_0 \cap A \neq \emptyset$, $A \in \mathcal{R}$, then we easily see that $\mathcal{R} \cup \{A_0\}$ is centered. □

Proof of Theorem 2.5 Let $\mathcal{R}_0$ be a centered family of closed subsets of $X := \underset{i \in I}{\times} X_i$. Let $\pi_i : X \to X_i$ be the coordinate projections. Let $\mathcal{R}$ be a maximal centered family of subsets of X containing $\mathcal{R}_0$. For any $i \in I$

$$\{\pi_i(A)^{\mathrm{cl}} \;:\; A \in \mathcal{R}\}.$$

is a centered family of closed subsets of X_i. Hence there exists $x_i \in \underset{A \in \mathcal{R}}{\cap} \pi_i(A)^{\mathrm{cl}}$. We will show that

$$x := (x_i)_{i \in I} \in \underset{A \in \mathcal{R}_0}{\cap} A.$$

Let $A \in \mathcal{R}$. Let W_i be a neighborhood of x_i in X_i. We have $\pi_i(A) \cap W_i \neq \emptyset$. Hence $\pi_i^{-1}(W_i) \cap A \neq \emptyset$. By Lemma 2.9 2., $\pi_i^{-1}(W_i) \in \mathcal{R}$. By Lemma 2.9 1., if $i_1, \dots, i_n$ is a finite subset of I, then $\overset{n}{\underset{j=1}{\cap}} \pi_{i_j}^{-1}(W_{i_j})$ belongs to $\mathcal{R}$. Thus

$$A \cap \overset{n}{\underset{j=1}{\cap}} \pi_{i_j}^{-1}(W_{i_j}) \neq \emptyset.$$

Hence $x \in A^{\mathrm{cl}}$. In particular, if $A \in \mathcal{R}_0$, then $x \in A$ □

2.3 One-point compactificaton

Definition 2.10 *Let X be a topological space. Then $X^{\mathrm{Al}} := X \cup \{\infty\}$ is called the one-point or Alexandrov compactification of X if it is equipped with the following topology: $A \subset X^{\mathrm{Al}}$ is open in X^{Al}, iff $A \cap X$ is open in X and if $\infty \in A$, then there exists a compact $K \subset X$ such that $A = (X \backslash K) \cup \{\infty\}$.*

Theorem 2.11 *For any space X, its one-point compactification X^{Al} is compact. X is dense in X^{Al} iff X is not compact.*

Proof. Let $\mathcal{S}$ be an open covering of X^{Al}. Then there exists $A_0 \in \mathcal{S}$ such that $\infty \in A_0$. Since A_0 is open, there exists a compact $K \subset X$ such that $A \supset X \backslash K$. Let

$$\mathcal{S}' := \big\{A \cap X \;:\; A \in \mathcal{S}\big\}.$$

Then $\mathcal{S}'$ covers X and hence also K. We can choose a finite subcovering $A_1 \cap X, \dots, A_n \cap X$ of K. Then $A_0, A_1, \dots, A_n$ is finite subcovering of X^{Al}. □

Definition 2.12 *Let X be a topological space. $C_{\rm c}(X)$ is the set of continuous functions with a compact support. $C_\infty(X)$ is the set of of continuous functions on X such that for any $\epsilon > 0$ there exists a compact $K \subset X$ such that $|f(x)| \leq \epsilon$ on $X \backslash K$.*

$C_\infty(X)$ can be identified with

$$\{f \in C(X^{\rm Al}) \ : \ f(\infty) = 0\}.$$

Thus $C_\infty(X)$ is a maximal ideal in $C(X^{\rm Al})$.

$C_{\rm c}(X)$ is dense in $C_\infty(X)$.

Let X, Y be topological spaces and $p : X \to Y$. We say that p is proper if the preimage of a compact set is compact.

Note that the composition of proper maps is proper.

Theorem 2.13 *Let X, Y be topological spaces and $p : X \to Y$ be continuous. TFAE:*

(1) *p is proper.*

(2) *p can be extended to a continuous map $p : X^{\rm Al} \to Y^{\rm Al}$ by setting $p(\infty) = \infty$.*

(3) *$p^\#$ maps $C_\infty(Y)$ into $C_\infty(X)$.*

(4) *$p^\#$ maps $C_{\rm c}(Y)$ into $C_{\rm c}(X)$.*

Example 2.14 *Let X be a discrete space. Consider the space $X^{\rm Al}$. Let $A \subset X^{\rm Al}$. Then $A^{\rm cl} = A$ if A is finite and $A^{\rm cl} = A \cup \{\infty\}$ otherwise.*

3 Separation axioms

One can argue that the general definition of a topological space considered in the previous section is too general and admits many pathological examples of no practical value. To my experience, in applications to functional analysis all topological spaces are the so-called Tikhonov or $T_{3\frac{1}{2}}$ spaces. Especially important in applications are normal or T_4 spaces, which is a smaller class. These two concepts belong to a chain of axioms that go under the name of separation axioms. In this section we discuss various separation axioms. One can argue that for practical purposes we could limit ourselves just to the concept of a Tikhonov and normal space. Nevertheless, for aesthetical, historical and pedagogical reasons we describe the whole chain of axioms from T_0 to T_4. We find that a discussion of these axioms is quite a charming intellectual exercise, even if it is essentially useless (except for $T_{3\frac{1}{2}}$ and T_4 spaces).

The separation axioms are ordered from the weakest T_0 to the strongest T_4, with the exception of the pair $T_{2\frac{1}{2}}$ and T_3, which are not comparable with one another: not every T_3 space is $T_{2\frac{1}{2}}$, and, to my knowledge, not every $T_{2\frac{1}{2}}$ space is T_3. The name $T_{2\frac{1}{2}}$-space has been introduced by us and we did not see this class of spaces discussed in the literature.

3.1 T_0 spaces

We say that a space X is T_0 iff for any distinct $x, y \in X$ there exist an open U such that $x \in U$ and $y \notin U$, or $y \in U$ and $x \notin U$.

3.2 T_1 spaces

We say that a space X is T_1 iff for any distinct $x, y \in X$ there exist an open U such that $x \in U$ and $y \notin U$.

Theorem 3.1 *X is T_1 iff all 1-element subsets of X are closed..*

Theorem 3.2 *If X is a T_1 space $C \subset X$ is compact and $y \in X \backslash C$, then there exists an open sets $U \subset X$ such that*

$$C \subset U, \quad y \notin U.$$

Clearly, every T_1 space is T_0.

3.3 T_2 or Hausdorff spaces

X is a T_2 or Hausdorff space iff for any distinct $x_1, x_2 \in X$ there exist disjoint open sets A_1, A_2 such that $x_1 \in A_1$ and $x_2 \in A_2$;

Theorem 3.3 *TFAE*

(1) *X is T_2.*

(2) *Every net converges to at most one point.*

(3) *The diagonal $\{(x,x) \;:\; x \in X\}$ is closed in $X \times X$*

Clearly, every T_2 space is T_1.

Theorem 3.4 *If X is Hausdorff and $Y \subset X$ is compact, then Y is closed.*

Proof. Let $(x_i)_{i\in I}$ be a net in Y convergent to $x \in X$. Y is compact, hence it possesses a cluster point in Y. Since X is Hausdorff, the cluster point is only one and equals x. Hence $x \in Y$. □

Theorem 3.5 *If X is Hausdorff and $C, D \subset X$ are compact and disjoint, then there exist disjoint open sets $U, V \subset X$ such that*

$$C \subset U, \quad D \subset V.$$

Proof. **Step 1** We will show that if C is compact and $x \in X \backslash C$, then we will find disjoint open U, V such that $x \in U$ and $C \subset V$.

For any $y \in C$ we will find open disjoint sets U_y, V_y such that $x \in U_y$, $y \in V_y$. Then $\{V_y \;:\; y \in C\}$ is an open covering of C. Let $\{V_{y_1}, \ldots, V_{y_n}\}$ be its finite subcovering. Then $U := \bigcap_{i=1}^{n} U_{y_i}$, $V := \bigcup_{i=1}^{n} V_{y_i}$ has the property we are looking for.

Step 2 Let C, D be compact. For any $x \in D$ we will find disjoint open U_x, V_x such that $x \in U_x$, $C \subset V_x$. Now $\{U_x \;:\; x \in D\}$ is an open covering of D. Let $\{U_{x_1}, \ldots, U_{x_n}\}$ be its finite subcovering $U := \bigcup_{i=1}^{n} U_{x_i}$, $V := \bigcap_{i=1}^{n} V_{x_i}$ has the property described in the theorem. □

3.4 $T_{2\frac{1}{2}}$ spaces

Let X be a T_1 space. We wil call it a $T_{2\frac{1}{2}}$ space iff for any distinct $x_1, x_2 \in X$ there exists $f \in C(X)$ such that $f(x_1) \neq f(x_2)$.

Clearly, every $T_{2\frac{1}{2}}$ space is T_2.

Theorem 3.6 *Let X be a $T_{2\frac{1}{2}}$ space and $C, D \subset X$ are compact and disjoint. Then there exists $f \in C(X, [0,1])$ such that $f = 0$ on C and $F = 1$ on D.*

Theorem 3.7 *Let X, Y be $T_{2\frac{1}{2}}$ space and $p : X \to Y$ be continuous. Then*

(1) $p^\#$ *is surjective* $\Rightarrow$ p *is injective.*

(2) $p^\#$ *is injective* $\Leftarrow$ *the range of* p *is dense.*

Proof. (1) $\Rightarrow$ Let p be not injective. Let $x_1, x_2 \in Y$ with $p(x_1) = p(x_2)$. Then $(p^\# f)(x_1) = f(p(x_1)) = f(p(x_2)) = (p^\# f)(x_2)$. But there exist $g \in C(X)$ such that $g(x_1) \neq g(x_2)$. Hence $g \notin p^\#(C(Y))$ and thus $p^\#$ is not surjective.

(2) $\Leftarrow$ follows from Theorem 1.11. $\square$

Theorem 3.8 *Let* X *be a topological space. Define*

$$Z := \prod_{f \in C(X,[0,1])} [0,1].$$

Define $J : X \to Z$ *by* $J(x) := \big(f(x) \ : \ f \in C(X,[0,1])\big)$. *Clearly,* J *is continuous. Moreover,* J *is injective iff* X *is* $T_{2\frac{1}{2}}$.

3.5 T_3 or regular spaces

Let X be a T_1 space. It is called a T_3 or regular space iff for any closed $C \subset X$ and $x \in X \backslash C$ there exist disjoint open sets U, W such that $x \in U$ and $C \subset W$.

Clearly, every T_3 space is T_2.

Theorem 3.9 X *is a* T_3 *space iff for every* $x \in X$ *the family of closed neighborhoods of* x *is a basis of neighborhoods of* x.

Theorem 3.10 *If* X *is Hausdorff and* $C, D \subset X$ *are disjoint,* C *is compact and* D *is closed, then there exist disjoint open sets* $U, V \subset X$ *such that*

$$C \subset U, \quad D \subset V.$$

Theorem 3.11 *Let* $A \subset X$ *be dense and* Y *be a* T_3 *space. Let* $f : A \to Y$. *Then there exists a continuous map* $\tilde{f} : X \to Y$ *extending* f *iff for any net* (x_i) *in* A *convergent to* $x \in X$ *there exists* $\lim_i f(x_i)$. $\tilde{f}$ *is then uniquely defined.*

Proof. For $x \in X$ set $\tilde{f}(x) := \lim_i f(x_i)$ where (x_i) is a net convergent to x. It is easy to see that the definition is correct.

Let us show that $\tilde{f}$ is continuous. Let $x \in X$ and let W be a closed neighborhood of $\tilde{f}(x)$ in Y. There exists an open neighborhood U of x in X such that $f(U \cap A) \subset W$. For any $z \in U$, Let (z_i) be a net in $V \cap A$ convergent to z.

$$\tilde{f}(z) = \lim f(z_i) \subset f(V \cap A)^{\mathrm{cl}} \subset W^{\mathrm{cl}} \subset W.$$

But closed neighborhoods form a basis of neighborhoods of $\tilde{f}(x)$. $\square$

3.6 $T_{3\frac{1}{2}}$, Tikhonov or completely regular spaces

Let X be a T_1 space. It is called a $T_{3\frac{1}{2}}$, Tikhonov or completely regular space iff for any closed $C \subset X$ and $x \in X \backslash C$ there exists $f \in C(X)$ such that $f(x) = 0$ and $f = 1$ on C.

Clearly, every $T_{3\frac{1}{2}}$ space is T_3 and $T_{2\frac{1}{2}}$.

Theorem 3.12 *Let X be a $T_{3\frac{1}{2}}$ space and $C, D \subset X$ are disjoint, C is compact and D is closed. Then there exists $f \in C(X,[0,1])$ such that $f = 0$ on C and $F = 1$ on D.*

Theorem 3.13 *Let X be T_n for $0 \le n \le 3\frac{1}{2}$ and $Y \subset X$. Y is T_n for $0 \le n \le 3\frac{1}{2}$.*

Theorem 3.14 *Let $(X_i)_{i\in I}$ be a family of disjoint topological spaces. Then, for $0 \le n \le 3\frac{1}{2}$, $\bigcup\limits_{i\in I} X_i$ is T_n iff all X_i are T_n.*

Theorem 3.15 *Let $(X_i)_{i\in I}$ be a family of topological spaces. Then, for $0 \le n \le 3\frac{1}{2}$, $\prod\limits_{i\in I} X_i$ is T_n iff all X_i are T_n.*

Theorem 3.16 *Let X, $J : X \to Z$ be as in Theorem 3.8. Then J is a homeomorphism onto $J(X)$ iff X is Tikhonov.*

Theorem 3.17 *Let $(X, \mathcal{T})$ be a topologigal space. TFAE:*

(1) *X is Tikhonov.*

(2) *X is homeomorphic to a subset of $\prod\limits_{i\in I}[0,1]$.*

(3) *If $\mathcal{S}$ is the weakest topology such that elements of $C(X)$ are continuous, then $\mathcal{S} = \mathcal{T}$.*

Theorem 3.18 *Let X, Y be Tikhonov spaces and $p : X \to Y$ continuous. Then*

(1) *$p^{\#}$ is surjective $\Rightarrow$ p is injective.*

(2) *$p^{\#}$ is injective iff the range of p is dense.*

Proof. (1) follows from Theorem 3.7.

(2) $\Rightarrow$ Let $p(X)$ be not dense. Let $y_0 \in Y \backslash p(X)^{\mathrm{cl}}$. Then there exists $f \in C(Y)$ such that $f = 0$ on $p(X)^{\mathrm{cl}}$ and $f(y_0) \neq 0$. Thus $p^{\#} f = 0$. Hence $p^{\#}$ is not injective.

(2) $\Leftarrow$ follows from Theorem 1.11. □

Theorem 3.19 *Let X be Tikhonov and $f \in C_+^{\mathrm{lsc}}(X)$. Then*

$$f = \sup\{g \in C(X) \ : \ g \le f\}.$$

3.7 T_4 or normal spaces

Let X be a T_1 space. We say that X is a T_4 or normal space iff for any disjoint closed sets $C_1, C_2 \subset X$ there exist disjoint open $U_1, U_2 \subset X$ such that

$$C_1 \subset U_1,\ C_2 \subset U_2;$$

Theorem 3.20 *Let X be a T_1 space. TFAE:*

(1) *X is normal.*

(2) *For any closed set C and open set W containing C, there exists an open set U such that $C \subset U$ and $U^{\mathrm{cl}} \subset W$.*

(3) *For any closed set C and open set W containing C, there exist open sets $U_1, U_2, \dots$ such that $C \subset \bigcup\limits_{i=1}^{\infty} U_i$ and $U_i^{\mathrm{cl}} \subset W$.*

(4) *Let $C_1, \dots, C_n \subset X$ be disjoint and closed. Then there exist open disjoint $U_1, \dots, U_n$ such that $C_j \subset U_j$, $j = 1, \dots, n$;*

(5) *Let $C, D \subset X$ closed disjoint. Then there exists $f \in C_+(X)$ such that $f = 0$ on C and $f = 1$ on D.*

(6) *Let $A_1, \dots, A_n \subset X$ be open and $C \subset \bigcup_{j=1}^n A_j$ closed. Then there exist $h_j \in C_+(X)$, $j = 1, \dots, n$ such that* $\operatorname{supp} h_j \subset A_j$ *and*

$$\sum_{j=1}^n h_j = 1 \text{ on } C.$$

Remark 3.21 *The implication (1)⇒(5) is called* **Urysohn's lemma**.

Proof. (1)⇔(2)⇒(3) is obvious.

(3)⇒(1). Let C, D be closed. Applying (3) to C, $X\backslash D$ we obtain sets $U_1, U_2 \dots$ satisfying

$$C \subset \bigcup_{i=1}^{\infty} U_i, \quad U_i^{\mathrm{cl}} \cap D = \emptyset.$$

Applying (3) to D, $X\backslash C$ we obtain sets $V_1, V_2, \dots$ satisfying

$$D \subset \bigcup_{i=1}^{\infty} V_i, \quad V_i^{\mathrm{cl}} \cap C = \emptyset.$$

Set

$$G_i := U_i \backslash \bigcup_{j=1}^{i} V_j^{\mathrm{cl}}, \quad H_i := V_i \backslash \bigcup_{j=1}^{i} U_j^{\mathrm{cl}}.$$

They are open and satisfy $G_i \cap H_j = \emptyset$ for all i, j. Hence $U := \bigcup_{i=1}^{\infty} G_i$ and $V := \bigcup_{i=1}^{\infty} H_i$ are disjoint. Clearly, they are open and $C \subset U$, $D \subset V$.

(1)⇒(5) We first prove by induction that for all $\theta = p2^{-n}$ with $0 \leq p \leq 2^n$ we can find an open set U_θ such that $C \subset U_\theta$, $U^{\mathrm{cl}} \subset X\backslash D$ and $\theta_1 \leq \theta_2$ implies $U_{\theta_1}^{\mathrm{cl}} \subset U_{\theta_2}$. For $n = 1, 2 \dots$ we set

$$f_n := \sum_{p=1}^{2^n} 2^{-n} 1_{U_{p2^{-n}}}.$$

The limit $f := \lim_{n\to\infty} f_n$ exists and defines a continuous function satisfying the required properties.

(5)⇒ (6) We construct the sequence $k_j \in C(X)$ such that

$$\begin{aligned}
&k_1 = 1 \text{ on } C, \ \operatorname{supp} k_1 \subset \textstyle\bigcup_{j=1}^n A_j,\\
&\dots,\\
&k_{j+1} = 1 \text{ on } \operatorname{supp} k_j \backslash A_j, \ \operatorname{supp} k_{j+1} \subset \textstyle\bigcup_{i=j+1}^n A_i,\\
&\dots,\\
&k_{n+1} = 0.
\end{aligned}$$

We put

$$h_j = k_j(1 - k_{j+1}).$$

We see that

$$\operatorname{supp} h_j \subset A_j.$$

We prove by induction that

$$h_1 + \dots + h_j + k_{j+1} = 1 \text{ on } C.$$

□

Theorem 3.22 (Tietze) *If X is normal, $C \subset X$ is a closed subset and $f \in C(C, \mathbb{R})$, then there exists $\tilde{f} \in C(X, \mathbb{R})$, which is an extension of f and $\sup|f| = \sup \tilde{f}$.*

Proof. It suffices to assume that $f(C) \subset [-1, 1]$. The sets $A_1 = f^{-1}([-1, -1/3])$ and $B_1 = f^{-1}([1/3, 1])$ are disjoint and closed in X. Hence, there exists $g_1 \in C(X, [-1/3, 1/3])$, such that $g_1 = -1/3$ on A_1 and $g_1 = 1/3$ on B_1. Note that $\|g_1\| \leq 1/3$ and $\|g_1|_C - f\| \leq 2/3$.

We repeat the same construction to $f - g_1$ obtaining a function g_2. We iterate this. Thus we obtain a sequence of functions $g_1, g_2, \dots$ such that $\|g_n\| \leq 2^{n-1}3^{-n}$ and $\| \sum_{j=1}^{n} g_j|_C - f\| \leq 2^n 3^{-n}$. Then we set $\tilde{f} := \sum_{j=1}^{\infty} g_j$. □

4 Compact and locally compact Hausdorff spaces

From the point of view of C^*-algebras, the most important classes of topological spaces are compact Hausdorff and locally compact Hausdorff spaces.

4.1 Compact Hausdorff spaces

Theorem 4.1 *If $f : X \to Y$ is continuous and injective, X is compact Hausdorff and Y is Hausdorff, then*

$$f^{-1} : f(X) \to X$$

is continuous.

Proof. Let F be closed in X. Then it is compact. So $f(F)$ is compact. Since Y is Hausdorff, $f(F)$ is closed in Y. So $f(F)$ is closed in $f(X)$.

Since f is injective, $f(F) = (f^{-1})^{-1}(F)$. Thus the preimages of closed sets wrt f^{-1} are closed. So f^{-1} is continuous. □

Theorem 4.2 *Let $(X, \mathcal{T})$ be a compact Hausdorff space.*

(1) *If $\mathcal{S}$ is a topology weaker than $\mathcal{T}$ and Hausdorff, then $\mathcal{S} = \mathcal{T}$;*

(2) *If $\mathcal{S}$ is a topology stronger than $\mathcal{T}$ and compact, then $\mathcal{S} = \mathcal{T}$.*

Proof. Let id denote the identity map from $(X, \mathcal{T})$ to $(X, \mathcal{S})$.

(1) Let $\mathcal{S}$ be Hausdorff and weaker than $\mathcal{T}$. Then id is continuous. By Theorem 4.1, id^{-1} is continuous as well. Hence $\mathcal{S} = \mathcal{T}$.

(2) Let $\mathcal{S}$ be compact and stronger than $\mathcal{T}$. Then it is necessarily Hausdorff. Besides, id^{-1} is continuous. By Theorem 4.1, so is id. Hence, $\mathcal{S} = \mathcal{T}$. □

Theorem 4.3 *Every compact Hausdorff space is normal.*

Proof. Let C, D be disjoint closed sets in X. Then they are compact. Hence we can apply Theorem 3.5. □

4.2 Real continuous function on a compact Hausdorff space

Theorem 4.4 *Let X, Y be compact Hausdorff spaces and let $p : X \to Y$ be continuous. Then*

(1) *$p^{\#}$ is surjective iff p is injective.*

(2) *$p^{\#}$ is injective iff p is surjective.*

(3) *$p^{\#}$ is bijective iff p is bijective (in this case it is homeomorphic).*

Proof. (1)$\Rightarrow$. follows by theorem 3.7.

(1)$\Leftarrow$. Let p be injective. p is a bijection from X to $p(X)$. Hence it has a continuous inverse. Let $g \in C(Y)$. We can define f_0 on $p(X)$ by $f_0(y) := g(p^{-1}y)$. f_0 is continuous on. By Tietze's theorem, we can extend it to a continuous function f on Y. Then $p^{\#}f = g$. Hence $p^{\#}$ is surjective.

(2)$\Rightarrow$. follows by Theorem 3.18

(2)$\Leftarrow$. follows by Theorem 1.11. $\square$

Theorem 4.5 *Let X be a compact Hausdorff space.*

(1) *Let Y be a closed subset of X. Then $C_Y(X)$ is a closed ideal of $C(X)$.*

(2) *Let $\mathfrak{N}$ be a closed ideal of $C(X)$. Set*

$$Y := \bigcap_{F \in \mathfrak{N}} F^{-1}(0).$$

Then Y is closed and $\mathfrak{N} = C_Y(X)$.

(3)

$$C(X)/C_Y(X) \ni F + C_Y(X) \mapsto F\Big|_Y \in C(Y)$$

is an isometric $*$-homomorphism.

Proof. (1) is obvious.

(2) Let $\mathfrak{N}$ be a closed ideal.

Let $F \in \mathfrak{N}$. Then $\{F \neq 0\} \subset X \backslash Y$. Therefore, Y is closed and

$$\mathfrak{N} \subset C_Y(X).$$

Let $F \in C_Y(X)$. For any $x \in X$, we can find $G_x \in C(X)$ with $G_x \geq 0$, $G_x(x) > 0$ and

(i) if $x \in Y$, then $\operatorname{supp} G_x \subset \{|F| < \epsilon\}$,

(ii) if $x \notin Y$, then $G_x \in \mathfrak{N}$.

Clearly, $\{G_x > 0\}$ is an open cover of X. We can find a finite subcobver indexed by $x_1, \dots, x_n$. Set

$$G := \sum_{j=1}^{n} G_{x_j}.$$

Then $G(x) > 0$ $x \in X$. Set

$$H := G^{-1} \sum_{x_j \in Y}^{n} G_{x_j}.$$

Then $0 \leq H \leq 1$ and

$$1 - H = G^{-1} \sum_{x_j \in X \backslash Y}^{n} G_{x_j} \in \mathfrak{N}.$$

Now

$$F = (1 - H)F + HF,$$

where $(1 - H)F \in \mathfrak{N}$ and $\|HF\|_\infty < \epsilon$. Using the fact that $\mathfrak{N}$ is closed and $\epsilon > 0$ arbitrary, we see that $F \in \mathfrak{N}$. Thus

$$\mathfrak{N} \supset C_Y(X),$$

which ends the proof of (2).

Let $F_0 \in C(X)$ and $f := F_0\Big|_Y$. Clearly,

$$\|F_0 + C_Y(X)\| = \inf\Big\{\|F\|_\infty \ : \ F \in C(X) \ : \ f = F\Big|_Y\Big\} \geq \|f\|_\infty.$$

Let $f \in C(Y)$. By the Tietze theorem, there exists $F \in C(X)$ such that $f := F\Big|_Y$. Let $\epsilon > 0$ and $U := \{|F| < \|f\|_\infty + \epsilon\}$. Then U is open and contains Y. We will find $G \in C(X)$ such that $0 \leq G \leq 1$, $G = 1$ on Y and $\{G \neq 0\} \subset U$. Then $\|FG\|_\infty \leq \|f\|_\infty + \epsilon$ and $FG\Big|_Y = f$. Hence

$$\inf\Big\{\|F\|_\infty \ : \ F \in C(X) \ : \ f = F\Big|_Y\Big\} \leq \|f\|_\infty.$$

□

4.3 Locally compact spaces

Definition 4.6 *X is locally compact iff every point in X has a compact neighborhood.*

Clearly, every compact space is locally compact.

Theorem 4.7 (1) *Every closed subset of a locally compact space is locally compact;*
(2) *Every open subset of a locally compact Hausdorff space is locally compact Hasdorff.*

Proof. (1) Let F be a closed set of a locally compact space X. Let $x \in F$ and let K be a compact neighborhood of x in X. Then $K \cap F$ is a compact neighborhood of x in F.

(2) Let U be an open subset of a locally compact Hausdorff space X. Let $x \in U$ and let K be a compact neighborhood of x in X. Then $K \backslash U$ is closed in K, hence compact. Applying Theorem 3.5 to disjoint compact sets $\{x\}$ and $K \backslash U$ we will find disjoint sets A, B open in X such that $x \in A$ and $K \backslash U \subset B$. Then $K \backslash B$ is a compact neighborhood of x in U. □

Theorem 4.8 (1) *Let X be Hausdorff and $Y \subset X$ be locally compact. Then Y is open in Y^{cl}.*
(2) *Let X be locally compact Hausdorff. Then $Y \subset X$ is locally compact iff Y is dense in Y^{cl}.*

Proof. (1) It suffices to assume that Y is dense in X. Let $x \in X$. There exists $U \subset Y$ open in Y such that $x \in U$ and $U^{\mathrm{cl}} \cap Y$ is compact ($U^{\mathrm{cl}} \cap Y$ is the closure of U in Y). Now $U \subset U^{\mathrm{cl}} \cap Y$ and $U^{\mathrm{cl}} \cap Y$ is closed in X (because it is compact). Hence $U^{\mathrm{cl}} \subset U^{\mathrm{cl}} \cap Y$, or $U^{\mathrm{cl}} \subset Y$.

There exists $W \subset X$ open in X such that $U = Y \cap W$. By Theorem 1.6,

$$W^{\mathrm{cl}} = (W \cap Y)^{\mathrm{cl}} = U^{\mathrm{cl}} \subset Y.$$

Hence, x belongs to the interior of Y.

(2) follows from (1) and Theorem 4.7 □

Theorem 4.9 *Let X be a topological space. Its one-point compactification $X^{\rm Al}$ is Hausdorff iff X is a Hausdorff locally compact space.*

Proof. $\Leftarrow$ Let $x \in X$. Let K be a compact neighborhood of x. Then there exists a set $U \subset K$ open in X such that $x \in U$. Then U and $V := X^{\rm Al}\backslash K$ are disjoint sets open in $X^{\rm Al}$ with $x \in U$ and $\infty \in V$.

$\Rightarrow$. Suppose that $x \in X$. Let U, V be disjoint open subsets of $X^{\rm Al}$ such that $x \in U$ and $\infty \in V$. Then there exists a compact $K \subset X$ such that $K \supset X\backslash V$. Thus K is a compact neighborhood of x. $\Box$

Theorem 4.10 (Tietze for locally compact Hausdorff spaces) *If X is locally compact Hausdorff, $C \subset X$ is closed and $f \in C_\infty(C)$, then there exists $\tilde{f} \in C_\infty(X)$, which is an extension of f and $\|f\| = \|\tilde{f}\|$.*

Proof. Consider the one-point compactification $X^{\rm Al}$ of X. We have two cases:

If C is compact, then we argue as follows. $X^{\rm Al}$ is normal, and C is closed in X. Hence there exists a function $g \in C(X, [0,1])$ such that $g = 1$ on C and 0 on ∞. By the Tietze Theorem for normal spaces, there exists $f_1 \in C(X^{\rm Al}, \mathbb{R})$ extending f. Then we set $\tilde{f} := f_1 g$.

If C is not compact, then we extend f to $F \in C(C \cup \{\infty\}, \mathbb{R})$ by setting $F(\infty) = 0$. Clearly, $C \cup \{\infty\}$ is compact, hence closed in $X^{\rm Al}$. By the Tietze theorem, we can extend F to a function $\tilde{F} \in C(X^{\rm Al})$. Then we restrict $\tilde{F}$ to X. $\Box$

Theorem 4.11 *Every locally compact Hausdorff space is Tikhonov.*

Proof. Let X be a locally compact Hausdorff space. Then $X^{\rm Al}$ is compact Hausdorff, hence Tikhonov. But a subset of a Tikhonov space is Tikhonov. $\Box$

Theorem 4.12 *Consider the discrete spaces X and Y, where X is infinite countable and Y is infinite uncountable. Then $X^{\rm Al} \times Y^{\rm Al}$ is Hausdorff compact, hence $Z := X^{\rm Al} \times Y^{\rm Al}\backslash\{(\infty, \infty)\}$ is locally compact Hausdorff. Consider the closed subsets $A := X \times \{\infty\}$ and $B := \{\infty\} \times Y$. Then there do not exist disjoint open sets U and V such that $A \subset U$ and $B \subset V$. Thus Z is not normal*

Proof. Let U contain $X \times \{\infty\}$ and be open in Z. For $x \in X$, let π_x be the injection of $Y^{\rm Al}$ onto $X^{\rm Al} \times Y^{\rm Al}$ sending y to (x, y). Then $\pi_x^{-1}(U) =: U_x$ is open in $Y^{\rm Al}$. We have $\infty \in U_x$, hence $Y\backslash U_x$ is finite. Therefore, $\underset{x\in X}{\cup}(Y\backslash U_x)$ is countable. So, $W := \underset{x\in X}{\cap} U_x = Y\backslash \underset{x\in X}{\cup}(Y\backslash U_x)$ is uncountable. Now, $X \times W \subset U$. Clearly, $\{\infty\} \times W \subset (X \times W)^{\rm cl}$. Therefore, $\emptyset \neq \{\infty\} \times W \subset \{\infty\} \times Y \cap U^{\rm cl}$. $\Box$

4.4 Algebra of continuous functions on a locally compact Hausdorff space

Theorem 4.13 *Let X be a locally compact Hausdorff space. Let $\rho : C_\infty(X) \to C_\infty(X)$ be a linear map. Then TFAE:*

(1) *There exists $f \in C_{\rm bd}(X)$ such that*

$$\rho(g) = fg, \quad g \in C_\infty(X).$$

(2) *For any $g_1, g_2 \in C_\infty(X)$*

$$\rho(g_1 g_2) = g_1 \rho(g_2).$$

Proof. (1)$\Rightarrow$(2) is obvious.

(2)$\Leftarrow$(1). First note that if $g(x) = 0$, then $\rho(g)(x) = 0$. In fact, we can write $g = g_1 g_2$ with $g_2(x) = 0$ and $\rho(g)(x) = \rho(g_1)(x) g_2(x) = 0$. Therefore, $f(x) := \rho(g)(x)$ with $g \in C_\infty(X)$, $g(x) = 0$, defines a unique function on X

If U is a precompact open set, we easily check that f is continuous on U. Hence, it is continuous on X.

If f is not bounded, then we can find a sequence of disjoint open sets $U_1, U_2, \dots$ and points $x_i \in U_i$ with $f(x_i) \to \infty$. Then we find $g_i \in C(X, [0,1])$ with $\mathrm{supp} g_i \subset U_i$ and $g_i(x_i) = 1$. Set $g := \sum_{j=1}^{\infty} (f(x_j))^{-1} g_j$. Then $g \in C_\infty(X)$ and $\rho(g) \not\in\in C_\infty(X)$, which is a contradiction. $\Box$

Remark 4.14 *In the algebraic language, $C_{\mathrm{bd}}(X)$ is the multiplier algebra of $C_\infty(X)$.*

Let (f_i) be a net in $C_{\mathrm{bd}}(X)$ and $f \in C_{\mathrm{bd}}(X)$. We say that f_i converges to f in the strict topology if for any $g \in C_\infty(X)$, gf_i converges to gf uniformly. In other words, $C_{\mathrm{bd}}(X)$ is equipped in the locally convex topology given by the family of seminorms $p_g(f) := \|fg\|$ for $g \in C_\infty(X)$.

Theorem 4.15 *Let X be locally compact Hausdorff. Then $C_\infty(X)$ is dense in $C_{\mathrm{bd}}(X)$ in the strict topology.*

Proof. Let $f \in C_{\mathrm{bd}}(X)$. Let $\mathcal{K}$ be the family of compact subsets of X ordered by inclusion. For any $K \in \mathcal{K}$, we can find $g_K \in C_\infty(X)$ such that $g_K = 1$ on K and $0 \le g_K \le 1$. We claim that the net $(fg_K)_{K\in\mathcal{K}}$ converges to f. In fact, let $h \in C_\infty(X)$. For any $\epsilon > 0$, let $K_\epsilon := \{|h| > \epsilon\}$. Then K_ϵ is compact and for $K \in \mathcal{K}$, $K \supset K_\epsilon$ we have

$$\|(fg_K - f)h\|_\infty \le \epsilon \|f\|_\infty.$$

$\Box$

Theorem 4.16 *Let X, Y be locally compact Hausdorff spaces and $p : X \to Y$ be continuous.*

(1) *$p^\#$ is strictly continuous from $C_{\mathrm{bd}}(Y)$ to $C_{\mathrm{bd}}(X)$.*

(2) *$p^\#(C_\infty(Y))C_\infty(X)$ is strictly dense in $p^\#(C_\infty(Y))$.*

(3) *p is injective $\Leftrightarrow$ $p^\#(C_\infty(Y))$ is strictly dense in $C_{\mathrm{bd}}(X)$.*

Proof. (1) Let $g \in C_\infty(X)$, $0 \le g \le 1$ and $K_n := \{|g| \ge 2^{-n}\}$. Clearly, K_n is compact, and hence so is $p(K_n)$. Therefore, there exists $h_n \in C_\infty(Y)$ such that $0 \le h_n \le 1$, $h_n = 1$ on $p(K_n)$. Set

$$h(y) := \sum_{n=1}^{\infty} 2^{-n} h_n(y).$$

We will show that

$$g(x) \le h(p(x)), \quad x \in X. \tag{4.2}$$

If $x \in X \backslash K_{n-1}$, then $g(x) \le 2^{-n+1}$. If $x \in K_n$, then $2^{-n+1} = \sum_{j=n}^{\infty} 2^{-j} \le h(p(x))$. Hence for $x \in K_n \backslash K_{n-1}$,

$$g(x) \le 2^{-n+1} \le h(p(x)).$$

Now $\{x \ : \ g(x) \neq 0\} = \bigcup_{n=1}^{\infty} K_n \backslash K_{n-1}$, (where $K_0 = \emptyset$). Hence (4.2) is true. This shows that for any $g \in C_\infty(X)$ satisfying $0 \le g \le 1$ we can find $h \in C_\infty(Y)$ such that

$$\|gp^\#(f)\|_\infty \le \|hf\|_\infty,$$

which implies the strict continuity of $p^\#$.

(2) Let $f \in C_\infty(Y)$, Let $(g_K)_{K\in\mathcal{K}}$ be the net in $C_\infty(X)$ defined in the proof of Theorem 4.15. We will show that $p^\#(f)g_K$ converges strictly to $p^\#(f)$.

Let $h \in C_\infty(X)$ and $K_\epsilon := \{|h| \geq \epsilon\}$. Then for $K_\epsilon \subset K$,

$$\|(g_K f \circ p - f \circ p)h\|_\infty \leq \epsilon \|f\|_\infty.$$

(3)$\Leftarrow$. Suppose that p is not injective. Then for some distinct $x_1, x_2 \in X$, we have $p(x_1) = p(x_2)$. There exists $h \in C_\infty(X)$ such that $h(x_1) = 0$ and $h(x_2) = 1$. There exists also $g \in C_\infty(X)$ such that $g(x_1) = g(x_2) = 1$, because $\{x_1, x_2\}$ is a compact set. Thus

$$gh(x_1) = 0, \quad gh(x_2) = 0.$$

Now if (f_i) is a net in $C_\infty(Y)$, then

$$g(x_1) f_i \circ p(x_1) = g(x_2) f_i \circ p(x_2).$$

Thus $gf_i \circ p$ does not converge to gh.

(3)$\Rightarrow$. Assume that p is injective. Let $h \in C_{\rm bd}(X)$. Let K be compact. Then p has a continuous inverse from $p(K)$ to K. So, $h \circ p^{-1}\Big|_{p(K)}$ is continuous. So it can be extended to a function $f_K \in C_\infty(Y)$ with

$$\|f_K\|_\infty = \left\| h \circ p^{-1}\Big|_{p(K)} \right\|_\infty \leq \|h\|_\infty.$$

Now let $g \in C_\infty(X)$ and $K_\epsilon := \{|g| \geq \epsilon\}$. Then K_ϵ is compact. If $K_\epsilon \subset K$, then $\|(f_K \circ p - h)g\|_\infty \leq \epsilon 2 \|h\|_\infty$. $\square$

Theorem 4.16 (3) is due to D. Ellwood.

4.5 Commutative C^*-algebras

Let us recall the famous Gelfand theory.

Let $\mathfrak{A}$ be a commutative C^*-algebra. A linear map $\phi : \mathfrak{A} \to \mathbb{C}$ is called a character iff $\phi \neq 0$, $\phi(AB) = \phi(A)\phi(B)$. Let $\mathrm{Char}(\mathfrak{A})$ denote the set of characters of $\mathfrak{A}$. For any $A \in \mathfrak{A}$, let $\hat{A} : \mathrm{Char}(\mathfrak{A}) \to \mathbb{C}$ be defined by

$$\hat{A}(\phi) := \phi(A)$$

We equip $\mathrm{Char}(\mathfrak{A})$ with the weakest topology such that $\hat{A}$ are continuous.

Theorem 4.17 *Let $\mathfrak{A}$ be a commutative C^*-algebra. Then*

(1) $\mathrm{Char}(\mathfrak{A})$ *is locally compact Hausdorff.*

(2) $\mathfrak{A} \ni A \mapsto \hat{A} \in C_\infty(\mathrm{Char}(\mathfrak{A}))$ *is a $*$-isomorphism.*

(3) $\mathrm{Char}(\mathfrak{A})$ *is compact Hausdorff iff $\mathfrak{A}$ is unital iff* $C_\infty(\mathrm{Char}(\mathfrak{A})) = C(\mathrm{Char}(\mathfrak{A}))$.

Let X be an arbitrary topological space. Then $C_{\rm bd}(X)$ and $C_\infty(X)$ are commutative C^*-algebras. If $x \in X$, then we set $\phi_x(f) := f(x)$. Clearly, ϕ_x is a character both on $C_{\rm bd}(X)$ and $C_\infty(X)$.

Theorem 4.18 *$X \ni x \mapsto \phi_x \in \mathrm{Char}(C_{\rm bd}(X))$ is a continuous map. It is a homeomorphism iff X is compact Hausdorff (and then $C_{\rm bd}(X) = C(X)$).*

Theorem 4.19 *$X \ni x \mapsto \phi_x \in \mathrm{Char}(C_\infty(X))$ is a continuous map. It is a homeomorphism iff X is locally compact Hausdorff.*

4.6 Morphisms of commutative C^*-algebras

Let X, Y be compact Hausdorff spaces. Let $\pi : C(Y) \to C(X)$ be a unital $*$-homomorphism. $x \in X$ will be identified with the character $\phi_x \in \mathrm{Char}(X)$, thus we will write $x(g)$ for $g(x)$, where $g \in C(Y)$. Set

$$p(x)(f) := x \circ \pi(f), \quad f \in C(Y).$$

Then $p(x)$ is a character on $C(Y)$, and thus an element of Y. (It is nonzero, because $p(x)(1) = 1$).

Theorem 4.20 *The map $p : X \to Y$ is continuous and, in the notation of (1.1), $\pi = p^{\#}$.*

Remark 4.21 *By the above theorem the category of compact spaces with continuous maps as morphisms is isomorphic to the category of commutative C^*-algebras with $*$-homomorphisms if we reverse the direction of arrows.*

Now let X, Y be locally compact Hausdorff spaces. Recall that $C_\infty(Y)_{\mathrm{un}}$ denotes the algebra $C_\infty(Y)$ with the adjoined unit, which can be identified with $C(Y \cup \{\infty\})$.

Let $\pi : C_\infty(Y) \to C_{\mathrm{bd}}(X)$ be a $*$-homomorphism. We can extend π to a unital homomorphism $\pi_{\mathrm{un}} : C_\infty(Y)_{\mathrm{un}} \to C_{\mathrm{bd}}(X)$ by setting

$$\pi_{\mathrm{un}}(f + \lambda) := \pi(f) + \lambda, \quad f \in C_\infty(Y), \quad \lambda \in \mathbb{C}.$$

For any $x \in X$ we set

$$p_{\mathrm{un}}(x)(g) := x(\pi \circ g), \quad g \in C_\infty(Y).$$

Theorem 4.22 *$p_{\mathrm{un}} : X \to Y \cup \{\infty\}$ is continuous and $\pi_{\mathrm{un}} = p_{\mathrm{un}}^{\#}$. Moreover, TFAE:*

(1) *$p_{\mathrm{un}}(X) \subset Y$, so that if we set $p := p_{\mathrm{un}}\Big|_X$, then $\pi = p^{\#}\Big|_{C_\infty(Y)}$.*

(2) *$\pi(C_\infty(Y))C_\infty(X)$ is dense in $C_\infty(X)$.*

Proof. (1)$\Leftarrow$(2). Suppose that $p_{\mathrm{un}}(x_0) = \infty$ for some $x_0 \in X$. Let $g \in C_\infty(Y)$. Then $\pi^{\#}(g)(x_0) = g \circ p_{\mathrm{un}}(x_0) = 0$. Hence for any $h \in C_\infty(X)$, $\pi^{\#}(g)(x_0)h(x_0) = 0$.

Thus if $f \in C_\infty(X)$ with $f(x_0) \neq 0$, then f does not belong to the closure of $\pi(C_\infty(Y))C_\infty(X)$. $\square$

Remark 4.23 *The above theorem suggests a natural category of C^*-algebras advocated by S. L. Woronowicz.*

4.7 Stone-Čech compactification

Let X be a topological space. We denote by $X^{\mathrm{St\check{C}}}$ the set of all characters of $C_{\mathrm{bd}}(X)$. By the Gelfand Theorem, $X^{\mathrm{St\check{C}}}$ is a compact Hausdorff space. We have the continuous map

$$X \ni x \mapsto \pi_x \in X^{\mathrm{St\check{C}}},$$

where $\pi_x(f) := f(x)$.

Theorem 4.24 *Let X be Tikhonov.*

(1) *$x \to \pi_x$ is a homeomorphism onto its image and the image is dense in $X^{\mathrm{St\check{C}}}$.*

(2) *If Y is a compact Hausdorff space and $\rho : X \to Y$ is a continuous map, then there exists a unique continuous extension of ρ to a map $\rho^{\mathrm{St\check{C}}} : X^{\mathrm{St\check{C}}} \to Y$.*

4.8 The Stone-Weierstrass Theorem

Let $\mathfrak{A}$ be a family of functions on X. We say that it separates points of X iff for any distinct $x_1, x_2 \in X$ there exists $f \in \mathfrak{A}$ such that $f(x_1) \neq f(x_2)$.

If $\mathfrak{A}$ is a family of complex valued functions on X, we say that $\mathfrak{A}$ is self-adjoint if $f \in \mathfrak{A}$ implies $\overline{f} \in \mathfrak{A}$.

Theorem 4.25 (Stone-Weierstrass) *Let X be a compact Hausdorff space*

(1) *Let $\mathfrak{A} \subset C(X, \mathbb{R})$ be a closed algebra that contains constants and separates points. Then it is uniformly dense in $C(X, \mathbb{R})$.*

(2) *Let $\mathfrak{A} \subset C(X)$ be a closed complex self-adjoint algebra that contains constants and separates points. Then it is uniformly dense in $C(X)$.*

Define $t_+ := \frac{1}{2}(t + |t|)$.

Lemma 4.26 *There exists a sequence of polynomials p_n such that $p_n(t) \to t_+$ uniformly on $[-1, 1]$ and $p_n(0) = 0$.*

Proof. For any $\epsilon > 0$, $s \mapsto (\epsilon^2 + s)^{1/2}$ is analytic in the disc $|s - \frac{1}{2}| < \frac{1}{2} + \epsilon^2$. Hence, there exists a sequence q_n of polynomials converging to $(\epsilon^2 + s)^{1/2}$ uniformly on $[0, 1]$. Hence $q_n(t^2)$ converges to $(\epsilon^2 + t^2)^{1/2}$ uniformly on $[-1, 1]$. But $(\epsilon^2 + t^2)^{1/2} - \epsilon$ converges to $|t|$ uniformly on $[-1, 1]$ as $\epsilon \to 0$. □

Lemma 4.27 *Let X be a topological space and let $\mathfrak{A}$ be a closed subalgebra of $C(X)$. Then $f, g \in \mathfrak{A}$ implies* $\min(f, g), \max(f, g) \in \mathfrak{A}$.

Proof. It suffices to assume that $|f|, |g| \leq 1/2$. We have $\max(f, g) = f + (g - f)_+$ and $|g - f| \leq 1$ Let p_n be a sequence of the previous lemma. Then $p_n(g - f) \to (g - f)_+$ uniformly. □

Proof of Theorem 4.25. (1) Let $f \in C(X, \mathbb{R})$ and $\epsilon > 0$. Let $x, y \in X$. Using the fact that $\mathfrak{A}$ separates points x and y and constants belong to $\mathfrak{A}$ we see that there exists $g_{xy} \in \mathfrak{A}$ such that

$$g_{xy}(x) = f(x), \quad g_{xy}(y) = f(y).$$

Define the sets

$$U_{xy} := \{z \in X \ : \ f(z) < g_{xy}(z) + \epsilon\}, \quad V_{xy} := \{z \in X \ : \ g_{xy}(z) < f(z) + \epsilon\}.$$

Note that $x, y \in U_{x,y} \cap V_{x,y}$.

For fixed x and variable y, the open sets U_{xy} cover X. We can find $y_1, \dots y_n$ such that $X = \bigcup\limits_{k=1}^{n} U_{xy_k}$. By Lemma 4.27, $g_x := \max\limits_{k=1}^{n} g_{xy_k} \in \mathfrak{A}^{\mathrm{cl}}$. Clearly, $f < g_x + \epsilon$. Moreover, $g_x(z) < f(z) + \epsilon$ for $z \in W_x := \bigcap\limits_{k=1}^{n} V_{xy_k}$, which is an open neighborhood of x. Varying x we obtain a covering of X by open sets W_x. We take a finite subcover labelled by $x_1, \dots, x_m$. Now $g := \min\limits_{j=1}^{m} g_{x_j}$ satisfies $g - \epsilon < f < g + \epsilon$. Therefore, $g \in \mathfrak{A}^{\mathrm{cl}}$.

(2) Let $\mathrm{Re}\mathfrak{A} := \{\mathrm{Re} f \ : \ f \in \mathfrak{A}\}$. Using $\mathrm{Re} f = \frac{1}{2}(f + \overline{f})$ we see that $\mathrm{Re}\mathfrak{A} \subset \mathfrak{A}$. Using the fact that $\mathfrak{A}$ is stable wrt multiplication by complex numbers we see that $\mathfrak{A} = \mathrm{Re}\mathfrak{A} + \mathrm{i}\mathrm{Re}\mathfrak{A}$. We check that $\mathrm{Re}\mathfrak{A}$ satisfies the assumptions of (1). Thus $\mathrm{Re}\mathfrak{A} = C(X, \mathbb{R})$. □

Theorem 4.28 (Stone-Weierstrass for locally compact Hausdorff spaces) *Let X be a locally compact Hausdorff space.*

(1) *Let $\mathfrak{A} \subset C_\infty(X, \mathbb{R})$ be a real closed algebra that separates points of X and for every $x \in X$ there exists $f \in \mathfrak{A}$ with $f(x) \neq 0$. Then $\mathfrak{A}$ is uniformly dense in $C_\infty(X, \mathbb{R})$.*

(2) *Let $\mathfrak{A} \subset C_\infty(X)$ be a complex closed self-adjoint algebra that separates points of X and for every $x \in X$ there exists $f \in \mathfrak{A}$ with $f(x) \neq 0$. Then $\mathfrak{A}$ is uniformly dense in $C_\infty(X)$.*

5 Connectedness

5.1 Arcwise connected spaces

We say that X is arcwise connected iff for any $x_0, x_1 \in X$ there exists a continuous map $\gamma : [0,1] \to X$ with $\gamma(0) = x_0$, $\gamma(1) = x_1$.

Theorem 5.1 *Let $f : X \to Y$ be continuous and X arcwise connected. Then so is $f(X)$.*

Theorem 5.2 *Let Y_1, Y_2 be arcwise connected subsets of X and $Y_1 \cap Y_2 \neq \emptyset$. Then $Y_1 \cup Y_2$ is arcwise connected.*

Theorem 5.3 *Let $(X_i)_{i \in I}$ be a family of arcwise connected spaces Then $\prod\limits_{i \in I} X_i$ is arcwise connected.*

5.2 Connected spaces

Let X be a topological space. We say that X is connected iff the only closed open subsets of X are $\emptyset$ and X.

Theorem 5.4 *Every arcwise connected space is connected.*

Theorem 5.5 *Let $f : X \to Y$ be continuous and X connected. Then so is $f(X)$.*

Theorem 5.6 *Let $(Y_i)_{i \in I}$ be connected subsets of X. Suppose that for any partition $I = I_1 \cup I_2$ we have $\bigcup\limits_{i \in I_1} Y_i \cap \bigcup\limits_{i \in I_2} Y_i \neq \emptyset$. Then $\bigcup\limits_{i \in I} Y_i$ is connected.*

Theorem 5.7 *Let $(X_i)_{i \in I}$ be a family of connected spaces Then $\prod\limits_{i \in I} X_i$ is connected.*

Theorem 5.8 *Let $Y \subset X$ be connected. Then any set A such that $Y \subset A \subset Y^{\mathrm{cl}}$ is connected.*

5.3 Components

Let X be a topological space. Let $x_0, x_1 \in X$.

(1) We will write $x_0 \sim_{\mathrm{arc}} x_1$ iff there exists a continuous map $\gamma : [0,1] \to X$ such that $\gamma(0) = x_0$ and $\gamma(1) = x_1$.

(2) We will write $x_0 \sim x_1$ iff there exists a connected subset $Y \subset X$ such that $x_0, x_1 \in Y$.

(3) We will write $x_0 \sim_{\mathrm{q}} x_1$ iff for any $Y \subset X$ closed open in X, $x_1 \in Y$ implies $x_2 \in Y$.

We easily see that $\sim$, $\sim_{\mathrm{q}}$ and $\sim_{\mathrm{arc}}$ are equivalence relations.

We introduce the following definition:

(1) An equivalence class of $\sim_{\mathrm{arc}}$ is called an arcwise connected component of X. It is a connected set.

(2) An equivalence class of $\sim$ is called a connected component of X.

(3) An equivalence class of $\sim_{\mathrm{q}}$ is called a quasicomponent of X. It is also a closed connected set.

Clearly, every arcwise connected component is contained in a connected component, which is contained in a connected quasicomponent.

Theorem 5.9 *Let $x \in X$.*

(1) *The arcwise connected component of x is the union of arcwise connected subsets of X containing x.*

(2) *The connected component of x is equal to the union of connected subsets of X containing x.*

(3) *The quasicomponent of x is the intersection of subsets closed open in X containing x.*

Theorem 5.10 (1) *An arcwise connected component is arcwise connected.*

(2) *A connected component is closed connected.*

(3) *A quasicomponent is closed.*

(4) *If X is compact Hausdorff, then connected components and quasicomponents coincide.*

Proof. (4) Let X be compact and let Y be a quasicomponent. Y is closed in X, hence compact Hausdorff. Let $Y = A_1 \cup A_2$ be a partition into subsets closed open in Y. A_1, A_2 are closed in Y, hence compact Hausdorff. Hence, they are closed in X. By the normality of X, there exist disjoint sets U_1, U_2 open in X such that $A_1 \subset U_1$, $A_2 \subset U_2$.

Let $(F_i)_{i\in I}$ be the family of sets closed open in X such that $Y \subset F_i$. Now $(X\backslash F_i)_{i\in I}$, together with the open set $U_1 \cup U_2$ is an open covering of X. Hence we can choose a finite family $F_1, \dots F_n$ such that $X\backslash F_1, \dots, X\backslash F_n$ and $U_1 \cup U_2$ cover X. This means that $F := \bigcap_{i=1}^{n} F_i \subset U_1 \cup U_2$. But $Y \subset F$ and F is closed open. Now $F \cap U_1$ and $F \cap U_2$ are closed open and disjoint.

By the definition of the quasicomponent, any set closed open in X and intersecting Y has to contain Y. We have two closed open disjoint sets $F \cap U_1$ and $F \cap U_2$ whose union contains Y. Thus, one of them, say, the latter, has an empty intersection with Y. Thus, $Y \subset F \subset U_1$. Hence $Y \subset U_1$. This means that $Y = A_1$. Hence Y is connected. □

6 Countability axioms and metrizability

6.1 Countability axioms

Theorem 6.1 *Let X be a topological space. Then TFAE*

(1) *X has a countable basis.*

(2) *X has a countable subbasis.*

If the above condions are satisfied, then we say that X satisfies the second axiom of countability or is 2nd countable.

X satisfies the first axiom of countability or is 1st countable if every point has a countable basis of neighborhoods.

We say that X is separable iff there exists in X a countable dense subset.

Theorem 6.2 *If X is 2nd countable, then it is separable and 1st countable.*

Theorem 6.3 (1) *Every subset of a separable space is separable.*

(2) *Every open subset of a 1st countable space is 1st countable.*

(3) *Every open subset of a 2nd countable space is 2nd countable.*

Theorem 6.4 (1) *If $(X_n)_{n\in\mathbb{N}}$ is a sequence of disjoint 2nd countable spaces, then $\bigcup_{n\in\mathbb{N}} X_n$ is 2nd countable.*

(2) *If $(X_n)_{n\in\mathbb{N}}$ is a sequence of disjoint separable spaces, then $\bigcup_{n\in\mathbb{N}} X_n$ is separable.*

(3) *If $(X_i)_{i\in I}$ be a sequence of disjoint 1st countable spaces, then $\bigcup_{i\in I} X_i$ is 1st countable .*

Theorem 6.5 (1) *If $(X_n)_{\in\mathbb{N}}$ is a sequence of nonempty topological spaces, then $\prod\limits_{n\in\mathbb{N}} X_n$ is 1st countable iff, for every $n \in \mathbb{N}$, X_n is 1st countable*

(2) *If $(X_n)_{\in\mathbb{N}}$ is a sequence of nonempty topological spaces, then $\prod\limits_{n\in\mathbb{N}} X_n$ is 2nd countable iff, for every $n \in \mathbb{N}$, X_n is 2nd countable*

(3) *If $(X_n)_{\in\mathbb{N}}$ is a sequence of nonempty topological spaces, then $\prod\limits_{n\in\mathbb{N}} X_n$ is separable iff, for every $n \in \mathbb{N}$, X_n is separable.*

6.2 σ-compact spaces

We say that X is σ-compact if it is a countable union of compact spaces.

A subset of a topological space is called σ-precompact if its closure is σ-compact.

Theorem 6.6 *If X is σ-compact, then:*

(1) *From every open covering of X we can choose a countable subcovering.*

(2) *Every closed subset of X is σ-compact*

Theorem 6.7 *Every locally compact, σ-compact Hausdorff space is normal.*

Proof. Let E, F be disjoint closed subsets of X.

Step 1 There exists a countable open covering $\{A_1, A_2, \dots\}$ of E such that $A_i^{\mathrm{cl}} \cap F = \emptyset$. In fact, for any $x \in E$ we will find an open A_x such that $x \in A_x$ and $A_x^{\mathrm{cl}} \cap F = \emptyset$. Since E is σ-compact, we can choose a countable subcovering.

Step 2 There exists a countable open covering $\{B_1, B_2, \dots\}$ of F such that $B_i^{\mathrm{cl}} \cap E = \emptyset$.

Step 3 We set

$$\tilde{A}_n := A_n \backslash \bigcup_{k\leq n} B_k^{\mathrm{cl}}, \qquad \tilde{B}_n := B_n \backslash \bigcup_{k\leq n} A_k^{\mathrm{cl}}.$$

Then $\tilde{A}_n$ and $\tilde{B}_m$ are disjoint for any n, m.

Step 4 We set

$$A := \bigcup_{n=0}^{\infty} \tilde{A}_n, \quad B := \bigcup_{n=0}^{\infty} \tilde{B}_n.$$

Then A, B are disjoint open subsets of X satisfing $E \subset A$ and $F \subset B$. □

Theorem 6.8 *Let X be a locally compact Hausdorff space.*

(1) *$C_\infty(X)$ is separable $\Leftrightarrow$ X is 2nd countable.*

(2) *If X is 2nd countable, then X is σ-compact.*

6.3 Sequences

A sequence is a net with the index set equal to $\mathbb{N}$. If (x_i) is a sequence, then (y_j) is a subsequence of (x_i) iff there exists an increasing map $\mathbb{N} \ni j \mapsto i(j) \in \mathbb{N}$ such that $x_{i(j)} = y_j$. Note that not every subnet of a sequence is a subsequence.

Let X be a topological space and $A \subset X$. The sequential closure of A is denoted as A^{scl} and is defined as the set of limits of all convergent sequences contained in A. Clearly, $A \subset A^{\mathrm{scl}} \subset (A^{\mathrm{scl}})^{\mathrm{scl}} \subset A^{\mathrm{cl}}$, but we may have proper inclusions.

We say that A is sequentially closed iff $A = A^{\mathrm{scl}}$.

We say that a topological space X is Fréchet iff for any $A \subset X$ we have $A^{\mathrm{cl}} = A^{\mathrm{scl}}$.

We say that the space X is sequential iff any $A \subset X$ is closed iff it is sequentially closed.

Theorem 6.9 *Consider the conditions*

(1) *X is 1st countable;*

(2) *X is Fréchet;*

(3) *X is sequential.*

Then we have (1)⇒(2)⇒(3).

Let X, Y be topological spaces. We say that $f : X \to Y$ is sequentially continuous iff for any convergent sequence (x_n) in X we have $\lim_{n\to\infty} f(x_n) = f(x)$, where $\lim\limits_{n\to\infty} x_n = x$.

Theorem 6.10 *If X is sequential, then $f : X \to Y$ is continuous iff it is sequentially continuous.*

6.4 Metric and metrizable spaces

Let (X, d) be a metric space. The topology whose basis equals

$$\{K(x, r) \ : \ x \in X, \ r > 0\}$$

is called the topology generated by the metric. A topology $\mathcal{T}$ is metrizable iff there exists a metric such that $\mathcal{T}$ is generated by the metric.

Theorem 6.11 *Let X be a metric space.*

(1) *$\{K(x, 1/n) \ : \ n \in \mathbb{N}\}$ is a basis of neighborhoods of x. Thus metric spaces are 1st countable.*

(2) *If Y is a dense subset in X, then*

$$\{K(y, 1/n) \ : \ y \in Y, \ n \in \mathbb{N}\}$$

is a basis of the topology. Thus a metric space is separable iff it is 2nd countable.

Theorem 6.12 *Every metrizable space is normal.*

Proof. Let $C, D \subset X$ be closed and disjoint. For any $x \in C$ there exists $r_x > 0$ such that $K(x, r) \cap D = \emptyset$. Set

$$U := \bigcup_{x \in C} K(x, r_x/3).$$

Similarly we construct V. Clearly, U and V are open and disjoint. □

Theorem 6.13 *Let X be a metrizable space. Then every closed set is δ-open.*

Proof. Let C be closed. Define

$$C_n := \{x \in X \ : \ \mathrm{d}(x, C) < 1/n\}.$$

Then C_n are open and

$$C = \bigcap_{n=1}^{\infty} C_n.$$

Hence C is a δ-open set. □

6.5 Basic constructions in metric spaces

If (X, d) is a metric space and $\tilde{\mathrm{d}} := \mathrm{d}(\mathrm{d}+1)$, then $(X, \tilde{\mathrm{d}})$ is a metric space with a metric bounded by 1.

If Y is a subset of a metric space, then clearly Y is a metric space.

If (X_i, d_i) is a family of disjoint metric spaces, then we can equip $\underset{i\in I}{\cup} X_i$ with the metric

$$\mathrm{d}(x_i, y_j) := \begin{cases} \mathrm{d}_i(x_i, y_j)(\mathrm{d}(x_i, x_j)+1)^{-1}, & i = j, \\ , & i \neq j. \end{cases}$$

If $(X, \mathrm{d}_X), (Y, \mathrm{d}_Y)$ are metric spaces $\mathrm{d}((x_1, y_1)(x_2, y_2)) := \mathrm{d}_X(x_1, x_2) + \mathrm{d}_Y(y_1, y_2)$ defines a metric on $X \times Y$.

If (X_n, d_n) is a countable family of metric spaces, then $X := \underset{n\in\mathbb{N}}{\Pi} X_n$ is a metrizable space where the metric can be chosen as

$$\mathrm{d}(x, y) := \sum_{n=1} 2^{-n} \frac{\mathrm{d}_n(x_n, y_n)}{\mathrm{d}_n(x_n, y_n)+1}.$$

Theorem 6.14 *Let X be a topological space. TFAE:*

(1) *X can be embedded in $\underset{n\in\mathbb{N}}{\Pi}[0, 1]$.*

(2) *X is metrizable and 2nd countable.*

(3) *X is normal and 2nd countable.*

(4) *X is regular and 2nd countable.*

Proof. (1)⇒(2)⇒(3)⇒(4) is clear.

Let us prove (4)⇒(3). Let X be regular and 2nd countable. Let us check Theorem 3.20 (3). Let $C \subset X$ be closed, $W \subset X$ open and $C \subset W$. For every $x \in C$ there exists open U_x such that $x \in U_x$ and $U_x^{\mathrm{cl}} \subset W$. We can find V_x in the basis such that $x \in V_x \subset U_x$. Since the basis is countable, the family of sets $\{V_x \ : \ x \in X\}$ is countable. It clearly covers C.

It remains to prove (3)⇒(1). □

7 Completeness and complete boundedness

7.1 Completeness

Let X be a metric space. We say that a sequence (x_n) in X is a Cauchy sequence iff for any $\epsilon > 0$ there exists N such that for $n, m > N$ we have $\mathrm{d}(x_n, x_m) < \epsilon$. In other words, $\lim_{n,m\to\infty} \mathrm{d}(x_n, x_m) = 0$.

Theorem 7.1 (1) *Every convergent sequence is a Cauchy sequence.*

(2) *Every Cauchy sequence is bounded.*

(3) *If (x_n) is a Cauchy sequence and it has a subsequence convergent to y, then (x_n) itself is convergent to y.*

(4) *Every uniformly continuous function transforms a Cauchy sequence into a Cauchy sequence.*

We say that X is complete iff every Cauchy sequence is convergent.

Theorem 7.2 (1) *Let X be complete and let A be a closed subset of X. Then A is complete.*

(2) *Let A be a complete subset of a metric space X. Then A is closed in X.*

(3) *If X, Y are complete, then so is $X \cup Y$.*

(4) *If X, Y are complete, then so is $X \times Y$.*

(5) *If X_n are complete for $n \in \mathbb{N}$, then so is $\prod\limits_{n\in\mathbb{N}} X_n$.*

Let X, Y be metric spaces. A function $p : X \to Y$ is called uniformly continuous if for any $\epsilon > 0$ there exists $\delta > 0$ such that if $\mathrm{d}(x_1, x_2) < \delta$, then $\mathrm{d}(p(x_1), p(x_2)) < \epsilon$. Clearly, every uniformly continuous function is continuous.

$p : X \to Y$ is called an isometry iff $\mathrm{d}(p(x_1), p(x_2)) = \mathrm{d}(x_1, x_2)$. Clearly, every isometry is uniformly continuous.

Theorem 7.3 *Let X, Y be complete spaces and X_0 a dense subset of X. Let $F_0 : X_0 \to Y$ be a uniformly continuous map, then there exists a unique continuous function $F : X \to Y$ such that $F\big|_{X_0} = F_0$. F is uniformly continuous. If F_0 is isometric, then so is F.*

Proof. If $x \in X$, then there exists a sequence (x_n) in X_0 converging to x. (x_n) is a Cauchy sequence and F is uniformly continuous, hence $(F(x_n))$ is Cauchy. Because of the completeness of Y, $(F(x_n))$ is convergent to $y \in Y$. We set $F(x) := y$. Then we check that the definition does not depend on the choice of a sequence. □

Theorem 7.4 *Let X be a metric space Then*

(1) *There exists a complete metric space X^{cpl} and an isometric map $\pi : X \to X^{\mathrm{cpl}}$ such that $\pi(X)$ is dense in X^{cpl}.*

(2) *If Y is a complete space and $F : X \to Y$ a uniformly continuous map, then there exists a unique continuous function $F^{\mathrm{cpl}} : X^{\mathrm{cpl}} \to Y$ such that $F^{\mathrm{cpl}} \circ \pi = F$. F^{cpl} is uniformly continuous. If F is isometric, then so is F^{cpl}.*

(3) *If X is a subset of a complete space Z, then there exists a unique bijective isometry $\rho : X^{\mathrm{cl}} \to X^{\mathrm{cpl}}$ such that $\rho\big|_X = \pi$.*

Proof. (1) Let $\mathcal{M}$ be the set of Cauchy sequences in X. We introduce in $\mathcal{M}$ the relation

$$(x_n) \sim (y_n) \;\Leftrightarrow\; \lim_{n\to\infty} \mathrm{d}(x_n, y_n) = 0.$$

We check that $\sim$ is an equivalence relation. We define

$$X^{\mathrm{cpl}} := \mathcal{M}/\sim.$$

We define in $\mathcal{M}$

$$\mathrm{d}_{\mathcal{M}}((x_n), (y_n)) := \lim_{n\to\infty} \mathrm{d}(x_n, y_n).$$

We check that $\mathrm{d}_{\mathcal{M}}$ is well defined and is compatible with the relation $\sim$. We define

$$\mathrm{d}^{\mathrm{cpl}}([(x_n)], (y_n)]) := \mathrm{d}_{\mathcal{M}}((x_n), (y_n)).$$

We define

$$\pi(x) := [(x)],$$

where (x) denotes the constant sequence equal to x. Clearly, π is an isometry.

If $(x_n) \in \mathcal{M}$, then $\lim_{n\to\infty} \pi(x_n) = [(x_n)]$. Hence, $\pi(X)$ is dense in X^{cpl}.

(2) is a reformulation of Theorem 7.3. □

In what follows we will treat X as a subset of X^{cpl}. If X is embedded in a complete space, we will identify X^{cpl} with X^{cl}.

7.2 Uniform convergence

Theorem 7.5 *Let X be a topological space and Y be a metric space. Then*

(1) *if $f_n \in C(X,Y)$ and $f_n \to f$ uniformly, then $f \in C(X,Y)$.*

(2) *equipped with the metric* $\min(\sup(\mathrm{d}(f(x),g(x)) \ : \ x \in X), 1)$, *$C(X,Y)$ becomes a metric space.*

(3) *If Y is complete, then $C(X,Y)$ is complete.*

7.3 Sequential compactness

We say that a set X is sequentially compact if from every sequence in X we can choose a convergent subsequence.

Theorem 7.6 *Every closed set of a sequentially compact space is sequentially compact.*

Theorem 7.7 *If $(X_i)_{i\in I}$ is a family of disjoint sequentially compact spaces, then $\bigcup_{i\in I} X_i$ is sequentially compact iff I is finite.*

Theorem 7.8 *If $(X_n)_{\in\mathbb{N}}$ is a (countable) family of nonempty topological spaces, then $\prod_{n\in\mathbb{N}} X_n$ is sequentially compact iff, for every $n \in \mathbb{N}$, X_n is sequentially compact.*

7.4 Compactness in metric spaces

Theorem 7.9 *If X is metrizable, then it is compact iff it is sequentially compact.*

Theorem 7.10 *If X is metric and compact then it is bounded and complete.*

Theorem 7.11 *$Y \subset \mathbb{R}^n$ is compact iff it is closed and bounded.*

Let X be a metric space. We say that X is completely bounded iff for any $\epsilon > 0$ there exist $x_1, \ldots, x_n \in X$ such that $X \subset \bigcup_{i=1}^{n} K(x_i, \epsilon)$

Theorem 7.12 *TFAE:*

(1) *X is completely bounded.*

(2) *X^{cpl} is compact.*

7.5 Equicontinuity

Let X be a topological space and (Y,d) a metric space. Let $\mathfrak{A} \subset C(X,Y)$. We say that $\mathfrak{A}$ is equicontinuous iff for every $\epsilon > 0$ and $x \in X$, there exists a neighborhood $V_{x,\epsilon}$ of x such that for every $f \in \mathfrak{A}$

$$y \in V_{x,\epsilon} \Rightarrow \mathrm{d}(f(x), f(y)) < \epsilon.$$

Theorem 7.13 *Let X be a topological space and (Y,d) a metric space. Let (f_n) be an equicontinuous sequence in $C(X,Y)$.*

(1) *Suppose that f_n is pointwise convergent to f. Then f is continuous.*

(2) *Suppose that f_n is pointwise convergent on a dense set and Y is complete. Then f_n is convergent on the whole X (and by (1) its limit is continous).*

Proof. (1) Let $\epsilon > 0$ and $x \in X$. We can find a neighborhood $V_{x,\epsilon}$ of x such that $x' \in V_{x,\epsilon}$ implies $\mathrm{d}(f_n(x), f_n(x')) < \epsilon$. We have $\lim_{n\to\infty} \mathrm{d}(f_n(x), f_n(x')) = \mathrm{d}(f(x), f(x')) \le \epsilon$.

(2) Let $x \in X$. For any $\epsilon > 0$ we can fnd $x' \in V_{x,\epsilon/3}$ such that $\lim\limits_{n\to\infty} f_n(x')$ exists. Thus there exists N such that for any $n, m > N$ we have $\mathrm{d}(f_n(x'), f_m(x')) < \epsilon/3$ Then

$$\mathrm{d}(f_n(x), f_m(x)) \le \mathrm{d}(f_n(x), f_n(x')) + \mathrm{d}(f_n(x'), f_m(x')) + \mathrm{d}(f_m(x'), f_m(x)) < \epsilon.$$

Thus $f_n(x)$ satisfies the Cauchy condition. □

Theorem 7.14 (Ascoli) *Suppose that X is a compact space. Let $\mathfrak{A} \subset C(X)$ satisfy the following conditions*

(1) *for any $x \in X$, $\sup\{|f(x)| \ : \ f \in \mathfrak{A}\} < \infty$.*

(2) *$\mathfrak{A}$ is equicontinuous*

Then $\mathfrak{A}$ is completely bounded in $C(X)$.

Proof. Let $\epsilon > 0$. The family $V_{x,\epsilon}$ covers X hence we can choose a finite subcover labelled by $x_1, \dots x_n$. We define $p : \mathfrak{A} \to \mathbb{C}^n$ by $p(f) := (f(x_1), \dots, f(x_n))$. Then the image of p is bounded, hence completely bounded. This means that there exist $f_1, \dots, f_m \in \mathfrak{A}$ such that every $p(f)$ is at a distance less than ϵ from $p(f_k)$.

Now let $f \in \mathfrak{A}$. There exists f_k with $|f(x_i) - f_k(x_i)| < \epsilon$. There exists i such that $x \in V_{x_i,\epsilon}$. Hence

$$|f(x) - f(x_i)| < \epsilon, \quad |f_k(x) - f_k(x_i)| < \epsilon.$$

Therefore, $|f(x) - f_k(x)| < 3\epsilon$ for all $x \in X$. Thus the balls with centers at $f_1, \dots, , f_m$ and radii 3ϵ cover $\mathfrak{A}$. □

7.6 Uniform eqicontinuity

Let (X, D), (Y, d) be metric spaces. We say hat $\mathfrak{A} \subset C(X, Y)$ is uniformly equicontinuous iff for every $\epsilon > 0$, there exists $\delta > 0$ such that for every $x \in X$ and $f \in \mathfrak{A}$

$$\mathrm{D}(x, y) < \delta \Rightarrow \mathrm{d}(f(x), f(y) < \epsilon.$$

Theorem 7.15 *Let X completely bounded and (f_n) a uniformly equicontinuous sequence of functions $X \to Y$ convergent pointwise to f. Then f_n is convergent uniformly to f.*

Proof. Let $\epsilon > 0$. We can find $\delta > 0$ such that $\mathrm{D}(x, x') < \delta$ implies $\mathrm{d}(f_n(x), f_n(x')) < \epsilon/3$.

Let $x_1, \dots, x_k$ be a family in X such that $X \subset \bigcup\limits_{i=1}^{k} K(x_i, \delta)$. We can find N such that for $i = 1, \dots, k$ and $n, m > N$ $\mathrm{d}(f_n(x_i), f_m(x_i)) < \epsilon/3$. Now if $x \in K(x_i, \delta)$, then

$$\mathrm{d}(f_n(x), f_m(x) \le \mathrm{d}(f_n(x), f_n(x_i)) + \mathrm{d}(f_n(x_i), f_m(x_i)) + \mathrm{d}(f_m(x_i), f_m(x)) < \epsilon.$$

Thus f_n is Cauchy sequence in the uniform metric. We know that $f_n \to f$ pointwise. Hence, $f_n \to f$ uniformly. □

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