Introduction to Quantization–Problems about Pseudodifferential Calculus

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Problem 0.1. *Let $a(x,p) \neq 0$ everywhere. Then there exist $b_0, b_{-2}, b_{-4}, \dots$ such that*

$$a \star \sum_{j=0}^{\infty} \hbar^{2j} b_{-2j} = 1 + O(\hbar^{\infty}), \tag{0.1}$$

$$a \star \sum_{j=0}^{n} \hbar^{2j} b_{-2j} = 1 + O(\hbar^{2n+2}). \tag{0.2}$$

Moreover,

$$b_0 = \frac{1}{a}, \tag{0.3}$$

$$b_{-2} = \frac{\partial_p^2 a \partial_x^2 a - (\partial_x \partial_p a)^2}{4a^3} + \frac{2\partial_x \partial_p a \partial_x a \partial_p a - \partial_x^2 a (\partial_p a)^2 - \partial_p^2 a (\partial_x a)^2}{4a^4}. \tag{0.4}$$

Proof. It is clear that

$$a \star a^{-1} = 1 + O(\hbar). \tag{0.5}$$

Hence $b_0 = a^{-1}$. Let us compute up to $O(\hbar^2)$:

$$a \star a^{-1}(x,p) \;=\; 1 + \hbar r_{-1} + \hbar^2 r_{-2} + O(\hbar^3) \tag{0.6}$$

$$= \Big(1 - \mathrm{i}\frac{\hbar}{2}(\partial_{p_1}\partial_{x_2} - \partial_{x_1}\partial_{p_2}) + \frac{\hbar^2}{8}(\partial_{p_1}\partial_{x_2} - \partial_{x_1}\partial_{p_2})^2\Big) a(x_1,p_1) a(x_2,p_2)^{-1} \Big|_{\substack{x = x_1 = x_2 \\ p = p_1 = p_2}} + O(\hbar^3).$$

It is easy to see that $r_{-1} = 0$.

$$r_{-2} = \frac{1}{8}\partial_p^2 a \partial_x^2 a^{-1} + \frac{1}{8}\partial_x^2 a \partial_p^2 a^{-1} - \frac{1}{4}\partial_p \partial_x a \partial_p \partial_x a^{-1} \tag{0.7}$$

$$= \frac{1}{4}\frac{\big(-\partial_p^2 a \partial_x^2 a + (\partial_p \partial_x a)^2\big)}{a^2} + \frac{1}{4}\frac{\big(\partial_p^2 a (\partial_x a)^2 + \partial_x^2 a (\partial_p a)^2 - 2\partial_x \partial_p a \partial_x a \partial_p a\big)}{a^3} \tag{0.8}$$

Now let us find b_{-2}.

$$a \star (a^{-1} + \hbar^2 b_{-2}) = 1 + \hbar r_{-2} + \hbar^2 a b_{-2} + O(\hbar^3). \tag{0.9}$$

Therefore, if we set

$$b_{-2} = -a^{-1} r_{-2}, \tag{0.10}$$

then

$$a \star (a^{-1} + \hbar^2 b_{-2}) = 1 + O(\hbar^3). \tag{0.11}$$

It is easy to see that making an ansatz

$$b := \sum_{n=0}^{\infty} \hbar^n b_{-n}, \quad c := \sum_{n=0}^{\infty} \hbar^n c_{-n}, \tag{0.12}$$

we find unique formal power series satisfying

$$a \star b = c \star a = 1. \tag{0.13}$$

Clearly,

$$a \star b(x,p) = \mathrm{e}^{\mathrm{i}\frac{\hbar}{2}(\partial_{p_1}\partial_{x_2} - \partial_{x_1}\partial_{p_2})} a(x_1,p_1) b(x_2,p_2) \Big|_{\substack{x = x_1 = x_2 \\ p = p_1 = p_2}}, \tag{0.14}$$

$$c \star a(x,p) = \mathrm{e}^{-\mathrm{i}\frac{\hbar}{2}(\partial_{p_1}\partial_{x_2} - \partial_{x_1}\partial_{p_2})} a(x_1,p_1) c(x_2,p_2) \Big|_{\substack{x = x_1 = x_2 \\ p = p_1 = p_2}}. \tag{0.15}$$

Thus we obtain the recursion for b from the recursion for a by switching the sign of $\hbar$. But by the associativity of the starproduct

$$c = c \star a \star b = b. \tag{0.16}$$

Hence all the terms with odd powers of $b = c$ are zero.

Let us go back to (0.11). We know that we can find b_{-4} such that

$$a \star (a^{-1} + \hbar^2 b_{-2} + \hbar^4 b_{-4}) = 1 + O(\hbar^4). \tag{0.17}$$

Therefore, we can replace $O(\hbar^3)$ in (0.11) with $O(\hbar^4)$. $\square$

Problem 0.2. *Let $a(x,p) \neq 0$ everywhere. Then there exist $d_0, d_{-2}, d_{-4}, \ldots$ such that*

$$(\hbar^2 + a) \star \sum_{j=0}^{\infty} \hbar^{2j} d_{-2j} = 1 + O(\hbar^\infty), \tag{0.18}$$

$$(\hbar^2 + a) \star \sum_{j=0}^{n} \hbar^{2j} d_{-2j} = 1 + O(\hbar^{2n+2}). \tag{0.19}$$

Moreover,

$$d_0 = \frac{1}{a}, \tag{0.20}$$

$$d_{-2} = -\frac{1}{a^2} + \frac{\partial_p^2 a \partial_x^2 a - (\partial_x \partial_p a)^2}{4a^3} + \frac{2\partial_x \partial_p a \partial_x a \partial_p a - \partial_x^2 a (\partial_p a)^2 - \partial_p^2 a (\partial_x a)^2}{4a^4}. \tag{0.21}$$

Proof. We have

$$(a+\hbar^2)^{-1\star} = a^{-1\star} \star (1+\hbar^2 a^{-1\star})^{-1\star} \tag{0.22}$$

$$= \sum_{j=0}^{\infty} \hbar^{2j} (a^{-1\star})^{j\star} \tag{0.23}$$

$$= a^{-1\star} - \hbar^2 a^{-1\star} \star a^{-1\star} + O(\hbar^4).. \tag{0.24}$$

Then we use

$$a^{-1\star} = \sum_{j=0}^{\infty} \hbar^{2j} b_{-2j} \tag{0.25}$$

calculated in Problem 1. □

Problem 0.3. *Let $a(x,\xi) \neq 0$ be homogeneous in ξ of degree 2 away from $|\xi| < 1$ and elliptic. Then there exist b_{-2j} homogeneous in ξ of degree $-2j$, $j = 1, 2, 3, \ldots$, such that*

$$a \star \sum_{j=1}^{\infty} b_{-2j} = 1 + S^{-\infty}, \tag{0.26}$$

$$a \star \sum_{j=1}^{n} b_{-2j} = 1 + S^{-2n-2}. \tag{0.27}$$

Moreover,

$$b_{-2} = \frac{1}{a}, \tag{0.28}$$

$$b_{-4} = \frac{\partial_\xi^2 a \partial_x^2 a - (\partial_x \partial_\xi a)^2}{4a^3} + \frac{2\partial_x \partial_\xi a \partial_x a \partial_\xi a - \partial_x^2 a (\partial_\xi a)^2 - \partial_\xi^2 a (\partial_x a)^2}{4a^4}. \tag{0.29}$$

Problem 0.4. *Let $a(x,\xi) \neq 0$ be homogeneous in ξ of degree 2 away from $|\xi| < 1$ and elliptic. Then there exist d_{-2j} homogeneous in ξ of degree $-2j$, $j = 1, 2, \ldots$, such that*

$$(1+a) \star \sum_{j=1}^{\infty} d_{-2j} = 1 + S^{-\infty}, \tag{0.30}$$

$$(1+a) \star \sum_{j=1}^{n} d_{-2j} = 1 + S^{-2n-2}. \tag{0.31}$$

Moreover,

$$d_{-2} = \frac{1}{a}, \tag{0.32}$$

$$d_{-4} = -\frac{1}{a^2} + \frac{\partial_\xi^2 a \partial_x^2 a - (\partial_x \partial_\xi a)^2}{4a^3} + \frac{2\partial_x \partial_\xi a \partial_x a \partial_\xi a - \partial_x^2 a (\partial_\xi a)^2 - \partial_\xi^2 a (\partial_x a)^2}{4a^4}. \tag{0.33}$$