Scattering in nonrelativistic quantum field theory

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The lecture notes are devoted to some topics in scattering theory for certain models inspired by quantum field theory. As a toy example, we describe scattering theory for van Hove Hamiltonians, where all basic objects can be computed exactly. We also sketch the formalism, basic results and some open problems about the so-called Pauli-Fierz Hamiltonians – a class of models describing a small quantum system interacting with a bosonic field, which have an interesting and nontrivial scattering theory.

Contents

1. Introduction

The main aim of these lectures is to sketch the formalism and basic results of the scattering theory for certain classes of models inspired by quantum field theory (QFT). We hope that we will convince the readers that this subject has both mathematical elegance and physical relevance.

In our lectures, we mostly consider models that are quite simple. In particular, they always have localized, fast decaying interactions. We are not

going to consider relativistic, or even translation invariant models, whose scattering theory is mathematically more dificult, and often problematic.

In Section 2 we describe the standard formalism of scattering theory [24, 28, 32], whose starting point is a pair of operators H and H_0 on a single Hilbert space. Then, in Section 3 we consider scattering theory of Schrödinger operators [28, 10], which, at least in the short range case, is an application of the standard formalism.

Later on we will see that when one wants to study scattering in QFT models, even very simple ones, the standard formalism has to be modified substantially. Therefore, strictly speaking, Sections 2 and 3 do not belong to the main subject of our lectures. Nevertheless, we believe that it is instructive to start with a discussion of these topics, so that the reader can compare them with scattering in QFT.

We use the term "quantum field theory" in a rather broad meaning. Roughly speaking, for the purpose of these lectures, a quantum field theory Hamiltonian is a self-adjoint operator whose definition is based on the formalism of second quantization, involving creation/annihilation operators and Fock spaces. In Section 4 we briefly recall this formalism [5, 7].

In Section 5 we describe in formal terms general principles of scattering in QFT with localized interactions [13, 20, 31]. We explain, in particular, the meaning of renormalization, which in such models is finite and well understood.

In Section 6 we describe scattering theory of a certain exactly solvable class of Hamiltonians – van Hove Hamiltonians [8].

In Section 7 we discuss the so-called representations of the CCR [5, 7]. They arise naturally in the context of scattering theory for bosonic Hamiltonians and allow us to describe some difficult situations typical eg. for the infra-red problem.

Section 8 is devoted to the scattering theory for a class of Hamiltonians describing a small system interacting with bosonic quantum fields. Following our earlier works, we call them Pauli-Fierz Hamiltonians, although other names can be found in the literature as well. We describe some rigorous results about this subject, as well as some intriguing unsolved problems [10, 11, 12, 14, 15].

In our lectures we do not discuss scattering theory for translation invariant QFT models. This subject is more difficult and its rigorous understanding is limited. Let us give a list of what we know rigorously about this subject.

(1) Scattering theory for N-body Schrödinger Hamiltonians is well understood, thanks to the work of Enss, Sigal, Soffer, Graf, the author and others, see [9] and references therein. It can be interpreted as a rather special example of a quantum field theory [6] for a class of Hamiltonians preserving the number of particles.
(2) The Haag-Ruelle theory gives a satsfactory framework for scattering theory in a relativistic quantum field theory satisfying the so-called Haag-Kastler or Wightman axioms in the presence of an isolated shell in the energy-momentum spectrum [23].
(3) Formal perturbative scattering theory for (nonrelativistic) translation invariant QFT models is described in [13, 31].
(4) Compton scattering at weak coupling and small energy has been studied in an interesting paper of Fröhlich, Griesemer and Schlein [15].

2. Basic abstract scattering theory

In this section we recall the standard formalism of scattering theory in an abstract setting. This topic is well known, see e.g. [28, 24, 32]. Later on we will use a different formalism, but we believe that it is instructive to start with the standard approach.

2.1. *Møller and scattering operators*

Suppose that we are given two self-adjoint operators H_0 and $H = H_0 + V$. The *Møller (or wave) operators* (if they exist) are defined as

$$S^{\pm} := \mathrm{s}\!-\lim_{t\to\pm\infty} \mathrm{e}^{\mathrm{i}tH}\,\mathrm{e}^{-\mathrm{i}tH_0}\,.$$

They satisfy $S^{\pm}H_0 = HS^{\pm}$ and are isometric.

The *scattering operator* is introduced as

$$S = S^{+*}S^{-}.$$

It satisfies $H_0S = SH_0$. If $\mathrm{Ran}S^{+} = \mathrm{Ran}S^{-}$, then it is unitary.

Let us note in parenthesis that in the old literature one can sometimes find a scattering operator of a different kind

$$\tilde{S} = S^{+}S^{-*}, \tag{2.1}$$

which satisfies $\tilde{S}H = H\tilde{S}$. Both scattering operators are closely related:

$$\tilde{S} = S^{-}S^{*}S^{-*}.$$

2.2. *Measurement of observables*

In this and the next subsections we try to describe how the scattering operator leads to measurable quantities. We will call them *(abstract) scattering cross-sections*. Let us note that scattering cross-sections in the context of Schrödinger operators or QED are discussed in essentially every textbook on quantum mechanics or quantum field theory. In distinction to those presentations, we will try to do it in an abstract setting, disregarding the concrete form of a quantum system. I find it curious that the abstract formalism of scattering cross-sections is quite complicated and involves a nontrivial condition, which we will call the predictiveness, see (2.6). Note also, that one treats differently the initial time (when the state is prepared) and the final time (when an observable is measured).

We will go back to scattering cross-sections in Subsection 3.2, where we consider them in the context of Schrödinger operators,

Let us start with recalling some of the basic principles of quantum mechanics. Let ρ be the *density matrix* representing a state prepared at time t_-. (Recall that a density matrix is a positive operator of trace 1). Let A be a self-adjoint operator representing an observable measured at time t_+. We learn at basic courses of quantum mechanics that the average outcome of the measurement, which we call the *expectation of the measurement*, equals

$$\operatorname{Tr} A\, \mathrm{e}^{-\mathrm{i}(t_+ - t_-)H}\, \rho\, \mathrm{e}^{\mathrm{i}(t_+ - t_-)H}.$$

In realistic situations, it is often difficult to determine the initial state ρ. Typically, the only thing that the experimenter uses to prepare the initial state can be mathematically described by a certain commuting family of self-adjoint operators, which we will call *the control observable*.

Suppose that the control observable has continuous spectrum (which often happens in practice). Then there does not exist a density matrix, which commutes with the control observable, In fact, this follows from the fact that density matrices have pure point spectrum. Therefore, in such a case it is impossible to prepare a state which has a sharp value of the control observable.

Let us try to describe this situation with a more formal language. The control observable will be represented by a $*$-homomorphism

$$C_\infty(X) \ni f \mapsto \gamma(f) \in B(\mathcal{H}), \tag{2.2}$$

where X is a locally compact Hausdorff space and $C_\infty(X)$ denotes the commutative C^*-algebra of continuous functions on X vanishing at infinity. (For example, we can think of X as $\mathbb{R}^d$ and $\gamma(f)$ as $f(D)$, where D denotes

the momentum). We can assume that the $*$-homomorphism γ is injective. If not, $\mathrm{Ker}\gamma = \{f \in C_\infty(X) \ : \ f = 0 \text{ on } Y\}$ for some closed $Y \subset X$, and we can replace in an obvious way $C_\infty(X)$ with $C_\infty(X\backslash Y)$.

It is convenient to extend the $*$-homomorphism γ to a normal $*$-homomorphism, denoted by the same symbol

$$\mathcal{L}^\infty(X) \ni f \mapsto \gamma(f) \in B(\mathcal{H}), \tag{2.3}$$

where $\mathcal{L}^\infty(X)$ denotes the commutative W^*-algebra of bounded Borel functions on X.

Let U be a Borel set in X and let 1_U denote the characteristic function of U. Let B be a self-adjoint operator. We define two real numbers

$$\sigma_+(U,\gamma,B) := \sup\{\mathrm{Tr}\rho B \ : \ \rho \text{ is a density matrix, } \rho = \gamma(1_U)\rho\gamma(1_U)\},$$
$$\sigma_-(U,\gamma,B) := \inf\{\mathrm{Tr}\rho B \ : \ \rho \text{ is a density matrix, } \rho = \gamma(1_U)\rho\gamma(1_U)\}.$$

Clearly, $\sigma_+(U,\rho,B)$, resp. $\sigma_-(U,\rho,B)$, is an increasing, resp. decreasing function of the set U.

Let $x \in X$. We set

$$\sigma_+(x,\gamma,B) := \inf\{\sigma_+(U,\gamma,B) \ : \ x \in U \text{ open in } X\}, \tag{2.4}$$
$$\sigma_-(x,\gamma,B) := \sup\{\sigma_-(U,\gamma,B) \ : \ x \in U \text{ open in } X\}. \tag{2.5}$$

We will say that x is *predictive for* (γ, B) if

$$\sigma_-(x,\gamma,B) = \sigma_+(x,\gamma,B), \tag{2.6}$$

and then we set $\sigma(x,\gamma,B)$ equal to (2.6).

For instance, if x is closed in X (which, by the injectivity of γ implies $\gamma(1_{\{x\}}) \neq 0$), then x is predictive for (γ, B) iff the value of

$$(\Psi|B\Psi) \tag{2.7}$$

does not depend on a normalized vector $\Psi \in \mathrm{Ran}\gamma(1_{\{x\}})$, and then $\sigma(x,\gamma,B)$ equals (2.7).

Let us go back to the situation where the experiment is prepared at time t_-. and the observable A is measured at time t_+. We assume that the experimenter tries to prepare the initial state so that the initial value of the observable given by γ equals x. We also assume that $x \in X$ is predictive for $(\gamma, \mathrm{e}^{\mathrm{i}(t_+-t_-)H} A \mathrm{e}^{-\mathrm{i}(t_+-t_-)H})$. Then the expectation of the measurement is close to

$$\sigma\left(x, \gamma, \mathrm{e}^{\mathrm{i}(t_+-t_-)H} A \mathrm{e}^{-\mathrm{i}(t_+-t_-)H}\right).$$

2.3. *Physical meaning of the scattering operator*

The physical importance of the scattering theory is based on the fact that in practical situations it takes a long time to prepare states and to measure observables. Scattering theory provides a natural way to take this into account.

Suppose that H_0 is an operator, which is "easy to control" by the experimentalist. Let ρ be a density matrix and A a self-adjoint operator. We assume for the moment that the experimentalist is able to prepare the state $\mathrm{e}^{-\mathrm{i}t_- H_0}\,\rho\,\mathrm{e}^{\mathrm{i}t_- H_0}$ at time t_- , and to measure the observable $\mathrm{e}^{\mathrm{i}t_+ H_0}\,A\,\mathrm{e}^{-\mathrm{i}t_+ H_0}$ at time t_+. Suppose also that the standard Møller operators exist, and hence the scattering operator S is well defined. Then it is easy to see that, for $t_- \to -\infty$, $t_+ \to \infty$, the expectation of the measurement converges to

$$\mathrm{Tr}\, AS\rho S^*. \tag{2.8}$$

Thus, in principle, we can determine the full information about the operator S, up to a phase factor, from experiments.

One can argue that the experiment described above is rather difficult to perform for arbitrary A and ρ. Let us modify it to make it more realistic.

Assume that the observable A commutes with H_0. Then

$$\mathrm{e}^{\mathrm{i}t_+ H_0}\,A\,\mathrm{e}^{-\mathrm{i}t_+ H_0} = A$$

does not depend on the time of measurement t_+, and thus should be easy to measure.

ρ is a trace class operator, hence there exists an orthonormal basis consisting of its eigenvectors. If $[H_0, \rho] = 0$, then H_0 has pure point spectrum. But in typical situations H_0 has continuous spectrum. Therefore, there are no density matrices commuting with H_0. It is therefore natural to apply the formalism described in the previous subsection.

First we need to choose a control observable. A possible choice would be the free Hamiltonian H_0, or in the language of the previous subsection, the $*$-homomorphism

$$C_\infty(\mathrm{sp}H_0) \ni f \mapsto f(H_0) \in B(\mathcal{H}),$$

given by the functional calculus. Physically, it means the only observable that we control when preparing the initial state is the energy.

In practice, the experimentalist, when preparing the initial state, controls other observables as well (eg. the momentum). Assume that they can be described by a $*$-homomorphism γ defined on a C^*-algebra $C_\infty(X)$, see

(2.2). It is natural to assume that $f(H_0)$ for $f \in C_\infty(\mathbb{R})$ belongs to the range of γ (which means that the free Hamiltonian is one of control observables).

Suppose that the experimentalist prepares the state at time t_- with the control observable γ arbitrarily close to $x \in X$. Then he performs the measurement of the observable A at time t_+. It follows from the definitions (2.4) and (2.5) that the expectation of such a measurement lies inside or very close to the interval

$$\left[\sigma_-\left(x,\gamma,\mathrm{e}^{\mathrm{i}(t_+-t_-)H}\,A\,\mathrm{e}^{-\mathrm{i}(t_+-t_-)H}\right),\sigma_+\left(x,\gamma,\mathrm{e}^{\mathrm{i}(t_+-t_-)H}\,A\,\mathrm{e}^{-\mathrm{i}(t_+-t_-)H}\right)\right].$$

Let us now take the limits $t_- \to -\infty$, $t_+ \to \infty$. Assume that we are allowed to change the order of relevant limits. Then, for any $\epsilon > 0$, there exists T such that, for $t_- \leq -T, \quad T \leq t_+$, the expectation of the measurement lies in

$$\left[\sigma_-\left(x,\gamma,S^*AS\right)-\epsilon,\quad \sigma_+\left(x,\gamma,S^*AS\right)+\epsilon\right].$$

In particular, let us assume also that x is predictive for (γ, S^*AS). Then, as $t_- \to -\infty$ and $t_+ \to \infty$, the expectation of the experiment becomes close to

$$\sigma(x,\gamma,S^*AS). \tag{2.9}$$

(2.9) can be called the *scattering cross-section at $x \in X$ for the observable A*.

2.4. *Problem with eigenvalues*

As before, H and H_0 is a pair of self-adjoint operators. It is easy to see that if the standard Møller operators exist and $H_0\Psi = E\Psi$, then $H\Psi = E\Psi$. Thus, on the subspace spanned by eigenvectors of H_0, the Møller and scattering operators are equal to the identity. Because of that, in practice the standard formalism of scattering theory is usually applied to Hamiltonians H_0 without point spectrum.

In models inspired by QFT, typically, both H_0 and H have ground states, and these ground states are different. Thus, standard scattering theory is not applicable. Instead, one can sometimes try other approaches.

2.5. *Alternative kinds of Møller operators*

There are various possible alternative kinds of Møller operators, which can be used instead of standard ones. Let us describe two of them.

The *strong Abelian Møller operators* are defined as

$$S^{\pm}_{\mathrm{Ab}} := \mathrm{s}\text{-}\lim_{\epsilon\searrow 0} \epsilon \int_0^{\infty} \mathrm{e}^{-\epsilon t}\, \mathrm{e}^{\pm \mathrm{i}tH}\, \mathrm{e}^{\mp \mathrm{i}tH_0}\, \mathrm{d}t.$$

They satisfy $S^{\pm}_{\mathrm{Ab}} H_0 = H S^{\pm}_{\mathrm{Ab}}$, but do not have to be isometric. If the standard Møller operator exists, then so do the Abelian Møller operators, and they coincide.

Another type of Møller operators that can be found in the literature are *adiabatic Møller operators*. To define them we first introduce the *dynamics with an adiabatically switched on interaction*

$$U_\epsilon(0) = 1, \quad \frac{\mathrm{d}}{\mathrm{d}t} U_\epsilon(t) = \mathrm{i} U_\epsilon(t)(H_0 + \mathrm{e}^{-\epsilon|t|}\, V).$$

Then one sets

$$S^{\pm}_{\mathrm{ad}} := \mathrm{w}\text{-}\lim_{\epsilon\searrow 0} \lim_{t\to\pm\infty} U_\epsilon(t)\, \mathrm{e}^{-\mathrm{i}tH_0}\,.$$

One expects that under quite general assumptions $S^{\pm}_{\mathrm{Ab}}$ coincides with $S^{\pm}_{\mathrm{ad}}$. In such a case, we will denote them by $S^{\pm}_{\mathrm{ur}}$. (The subscript "ur" stands for unrenormalized)

Suppose that the *vacuum amplitude operators* $Z^{\pm} := S^{\pm *}_{\mathrm{ur}} S^{\pm}_{\mathrm{ur}}$ have trivial kernels. Then we can define the renormalized Møller operators

$$S^{\pm}_{\mathrm{rn}} := S^{\pm}_{\mathrm{ur}} (Z^{\pm})^{-1/2}.$$

They also satisfy $S^{\pm}_{\mathrm{rn}} H_0 = H S^{\pm}_{\mathrm{rn}}$ and are isometric.

If $\mathrm{Ran} S^{+}_{\mathrm{rn}} = \mathrm{Ran} S^{-}_{\mathrm{rn}}$, then the renormalized scattering operator

$$S_{\mathrm{rn}} = S^{+*}_{\mathrm{rn}} S^{-}_{\mathrm{rn}}$$

is unitary and $H_0 S_{\mathrm{rn}} = S_{\mathrm{rn}} H_0$.

2.6. *Dyson series for Møller and scattering operators*

Set $V(t) = \mathrm{e}^{\mathrm{i}tH_0}\, V\, \mathrm{e}^{-\mathrm{i}tH_0}$. Expanding in formal power series we obtain

$$\begin{aligned} S^{+}_{\mathrm{Ab}} &= \lim_{\epsilon\searrow 0} \sum_{n=0}^{\infty} \int_{\infty > t_n > \cdots > t_1 > 0} \\ &\qquad \mathrm{i}^n\, \mathrm{e}^{-\epsilon t_n}\, V(t_n) \cdots V(t_1) \mathrm{d}t_n \cdots \mathrm{d}t_1, \\ S^{+}_{\mathrm{ad}} &= \lim_{\epsilon\searrow 0} \sum_{n=0}^{\infty} \int_{\infty > t_n > \cdots > t_1 > 0} \\ &\qquad \mathrm{i}^n\, \mathrm{e}^{-\epsilon(t_n + \cdots + t_1)}\, V(t_n) \cdots V(t_1) \mathrm{d}t_n \cdots \mathrm{d}t_1. \end{aligned}$$

For $S_{\rm ur} := S^{+*}_{\rm ur} S^{-}_{\rm ur}$, after performing the $\epsilon \searrow 0$ limit we get

$$S_{\rm ur} = \sum_{n=0}^{\infty} \int_{\infty > t_n > \cdots > t_1 > -\infty} \mathrm{i}^n V(t_n) \cdots V(t_1) \mathrm{d}t_n \cdots \mathrm{d}t_1.$$

After expanding each term in Feynman diagrams, this formal expansion is the usual starting point for analysis of scattering amplitudes in quantum field theory.

2.7. *Other formalisms of scattering theory*

The formalism of scattering theory that we described in this section started from a pair of operators H_0 and H acting on the same Hilbert space. Note that this formalism does not apply to all situations of physical interest, including many QFT models.

Usually, the main aim of scattering theory is to describe a certain *single* self-adjoint Hamiltonian H acting on a Hilbert space $\mathcal{H}$. We will call H and $\mathcal{H}$ the *physical Hamiltonian* and the *physical Hilbert space* respectively. The "free Hamiltonian", or better to say, the "asymptotic Hamiltonian" is not a priori given. It is even not clear that it should act on the same Hilbert space and that it should be the same for the past and future. In fact, part of our job is to guess the *asymptotic Hilbert spaces* $\mathcal{H}^{\pm \rm as}$ as well as the *asymptotic Hamiltonians* $H^{\pm \rm as}$ together with a construction of the Møller operators $S^{\pm} : \mathcal{H}^{\pm \rm as} \to \mathcal{H}$, which should be isometric (preferably unitary), and intertwine the asymptotic and physical Hamiltonians, i.e. $HS^{\pm} = S^{\pm} H^{\pm \rm as}$. I do not know a single formalism that gives a universal recipe how to do this. For various situations one often needs to find it separately. An example of such a formalism is given in Section 8 where we describe scattering theory for Pauli-Fierz Hamiltonians.

Let us mention that a common way to define Møller operators is to introduce appropriate identification operators $J^{\pm} : \mathcal{H}^{\pm \rm as} \to \mathcal{H}$ such that

$$S^{\pm} := \mathrm{s}\!-\!\lim_{t \to \infty} \mathrm{e}^{\mathrm{i}tH} J^{\pm} \mathrm{e}^{-\mathrm{i}tH^{\pm \rm as}}. \tag{2.10}$$

Note that the usual scattering operator $S = S^{+*} S^{-}$ maps $\mathcal{H}^{-\rm as}$ into $\mathcal{H}^{+\rm as}$. The alternative scattering operator $\tilde{S} = S^{+} S^{-*}$, introduced in (2.1), acts on the physical space $\mathcal{H}$.

Let us mention some interesting set-ups of scattering theory, which we will not discuss in these notes:

(1) Many-body Schrödinger operators, see eg. [10]
(2) Local relativistic QFT, the Haag-Ruelle theory, see eg. [23]
(3) Obstacle scattering for classical waves.

3. Scattering theory for 2-body Schrödinger operators

In this section we describe basic elements of scattering theory for Schrödinger operators [28, 10]. In the short-range case they follow the rules of the standard formalism, outlined in the previous section. In the long-range case a modification is needed.

3.1. *Short-range case*

Consider the Hilbert space $L^2(\mathbb{R}^d)$ and set

$$H_0 = -\frac{1}{2}\Delta, \quad H = -\frac{1}{2}\Delta + V(x).$$

We say that the potential $V(x)$ is short range if

$$|V(x)| \leq C(1+|x|)^{-1-\mu}, \quad \mu > 0. \tag{3.1}$$

Under this assumption one can show that the standard Møller operators $S^{\pm} := \mathrm{s}-\lim_{t\to\pm\infty} \mathrm{e}^{\mathrm{i}tH}\,\mathrm{e}^{-\mathrm{i}tH_0}$ exist and their ranges equal the absolute continuous spectral subspace of the operator H, denoted $\mathrm{Ran1_c}(H)$. The last statement is called the asymptotic completeness.

We define as usual the scattering operator S and we introduce the T-operator:

$$S = 1 + \mathrm{i}T.$$

3.2. *Physical meaning of scattering cross-sections*

Let ξ be the momentum variable. Let $\hat{\xi} = \xi|\xi|^{-1}$ denote the angular variable. Recall that T commutes with H_0. Therefore, the T-operator has the distributional kernel in the momentum representation:

$$T(\xi_+, \xi_-) = \delta(|\xi_+| - |\xi_-|)T(|\xi_+|, \hat{\xi}_+, \hat{\xi}_-).$$

The scattering cross-section at the energy $\lambda^2/2$, incoming angle $\hat{\xi}_-$ and outgoing angle $\hat{\xi}_+$ is defined as

$$\sigma(\lambda, \hat{\xi}_+, \hat{\xi}_-) := |T(\lambda, \hat{\xi}_+, \hat{\xi}_-)|^2. \tag{3.2}$$

It is commonly accepted that the scattering cross-sections are physically the most relevant quantities that are contained in the scattering operator.

Let us try to explain their physical meaning, following the idea sketched in Subsection 2.3

The rough idea of the scattering cross-section is as follows. Suppose that we prepare a state concentrated around the momentum ξ_- and measure the probability of finding the particle of momentum around ξ_+. Assume that the energies are the same: $|\xi_-|^2/2 = |\xi_+|^2/2$. Then the probability of the measurement is proportional to $\sigma(|\xi_+|, \hat{\xi}_+, \hat{\xi}_-)$, at least if the scattering amplitude is well behaved (sufficiently continuous).

Let us make it more precise. Let $D = -\mathrm{i}\nabla_x$ denote the momentum operator. Suppose that we want to measure the observable $a(D)$ at time t_+. At time t_- prepare the state $\mathrm{e}^{-\mathrm{i}t_- H_0}\,\rho\,\mathrm{e}^{\mathrm{i}t_- H_0}$, where for simplicity we assume that the density matrix factorizes in the energy and momenta:

$$\rho(\xi_-, \xi'_-) = \rho_{\mathrm{en}}(|\xi_-|, |\xi'_-|)\rho_{\mathrm{an}}(\hat{\xi}_-, \hat{\xi}'_-).$$

We also assume that $a(D)\rho = 0$ (so that we measure only scattered states). By (2.8), the expectation of the measurement converges to

$$\int\int\int \overline{T(|\xi_+|, \hat{\xi}_+, \hat{\xi}_-)}a(\xi_+)T(|\xi_+|, \hat{\xi}_+, \hat{\xi}'_-) \times\rho_{\mathrm{en}}(|\xi_+|, |\xi_+|)\rho_{\mathrm{an}}(\hat{\xi}_-, \hat{\xi}'_-)|\xi_+|^{d-1}\mathrm{d}\xi_+\mathrm{d}\hat{\xi}_-\mathrm{d}\hat{\xi}'_-. \tag{3.3}$$

Let us make some additional assumptions. Fix the incoming angle $\eta_- \in S^{d-1}$. Let us assume that $\hat{\xi}_- \mapsto T(|\xi_+|, \hat{\xi}_+, \hat{\xi}_-)$ is continuous at $\hat{\xi}_- = \hat{\eta}_-$, uniformly for $\xi_+ \in \operatorname{supp} a$. Then it is easy to see that, for any $\epsilon > 0$, there exists $\delta > 0$ such that if $\rho_{\mathrm{an}}(\hat{\xi}_-, \hat{\xi}'_-)$ is supported in the set

$$|\hat{\xi}_- - \hat{\eta}_-| \leq \delta, \quad |\hat{\xi}'_- - \hat{\eta}_-| \leq \delta,$$

then the expectation value of the measurement (3.3) differs from

$$\int a(\xi_+)\sigma(|\xi_+|, \hat{\xi}_+, \hat{\eta}_-)\rho_{\mathrm{en}}(|\xi_+|, |\xi_+|)|\xi_+|^{d-1}\mathrm{d}\xi_+ \times \int \rho_{\mathrm{an}}(\hat{\xi}_-, \hat{\xi}'_-)\mathrm{d}\hat{\xi}_-\mathrm{d}\hat{\xi}'_-. \tag{3.4}$$

by at most ϵ.

Note that the operator T enters (3.4) only through the scattering cross-section. Therefore, scattering cross-sections are sufficient to describe experiments with a well collimated incident beam.

3.3. *Long-range case*

Suppose that the potential satisfies $V = V_{\rm l} + V_{\rm s}$ where $V_{\rm s}$ is short-range (satisfies (3.1)) and

$$|\partial_x^\alpha V_{\rm l}| \leq C_\alpha (1+|x|)^{-|\alpha|-\mu}, \quad \mu > 0,\ |\alpha| = 0, 1, \dots. \tag{3.5}$$

We then say that the potential is long range.

It includes the physically relevant Coulomb potential $V(x) = z|x|^{-1}$, where z is the charge.

One can show that for such potentials standard Møller operators in general do not exist. This is one of manifestations of the infra-red problem in quantum physics. Nevertheless, it is possible to compute scattering cross-sections for long range potentials.

There are several methods to do this. The method presented in many quantum mechanics textbooks goes as follows. First one approximates a given long-range potential by a sequence of short-range potentials. Eg. the Coulomb potential is approximated by the Yukawa potentials $V_\mu = z\,\mathrm{e}^{-\mu|x|}\,|x|^{-1}$. For short-range potentials one can construct Møller and scattering operators, and hence the scattering cross-sections

$$\sigma_\mu(\lambda, \hat\xi_1, \hat\xi_2)$$

are well defined. Then one shows that there exists

$$\lim_{\mu \searrow 0} \sigma_\mu(\lambda, \hat\xi_1, \hat\xi_2),$$

which is interpreted as the scattering cross-section for V.

There exist better approaches to the long-range scattering. Instead of the standard Møller operators, one defines the so-called modified Møller operators for long-range potentials, see eg. [9]. One way to do it, which works for $\mu > \frac{1}{2}$ in (3.5), is as follows. One introduces the function

$$S(t,\xi) = \frac{t\xi^2}{2} + \int_0^t V_{\rm l}(s\xi)\mathrm{d}s.$$

Then one can show that there exists

$$S^\pm_{\rm lr} := \mathrm{s}\!-\!\lim_{t\to\pm\infty} \mathrm{e}^{\mathrm{i}tH}\,\mathrm{e}^{-\mathrm{i}S(t,D)}\,. \tag{3.6}$$

(3.6) are called modified Møller operators. They are isometric, intertwine the free and full Hamiltonian, that is $S^\pm_{\rm lr} H_0 = H S^\pm_{\rm lr}$. They also satisfy asymptotic completeness, in other words $\operatorname{Ran} S^\pm_{\rm lr} = \operatorname{Ran} 1_{\rm c}(H)$.

We introduce the modified scattering operator by setting $S_{\mathrm{lr}} := S_{\mathrm{lr}}^{+*} S_{\mathrm{lr}}^{-}$ and the T-operator by $S_{\mathrm{lr}} = 1 + \mathrm{i}T_{\mathrm{lr}}$. We can write the distributional kernel as

$$T_{\mathrm{lr}}(\xi_+, \xi_-) = \delta(|\xi_+| - |\xi_-|) T_{\mathrm{lr}}(|\xi_+|, \hat{\xi}_+, \hat{\xi}_-).$$

cattering cross-section are defined as

$$\sigma(\lambda, \hat{\xi}_+, \hat{\xi}_-) := |T_{\mathrm{lr}}(\lambda, \hat{\xi}_+, \hat{\xi}_-)|^2. \tag{3.7}$$

3.4. *Freedom of the choice of modified Møller operators*

The main disadvantage of the formalism described above is the fact that in general there is no canonical choice of $S_{\mathrm{lr}}^{\pm}$. Nevertheless, this arbitrariness is quite limited. If we have two modified Møller operators $S_{\mathrm{lr},1}^{\pm}$ and $S_{\mathrm{lr},2}^{\pm}$, then there exists a phase function $\psi^{\pm}$ such that

$$S_{\mathrm{lr},1}^{\pm} = S_{\mathrm{lr},2}^{\pm}\, \mathrm{e}^{\mathrm{i}\psi^{\pm}(D)},$$

where recall that $D = -\mathrm{i}\nabla_x$. This arbitrariness disappears in scattering cross-sections, which are canonically defined.

There is, however, another construction, which is unique and canonical. For long-range potentials, there exists self-adjoint operators $D^{\pm}$ such that, for any $g \in C_{\mathrm{c}}(\mathbb{R}^d)$,

$$g(D^{\pm}) = \mathrm{s}\!-\!\lim_{t \to \pm\infty} \mathrm{e}^{\mathrm{i}tH}\, g(D)\, \mathrm{e}^{-\mathrm{i}tH}\, 1_{\mathrm{c}}(H).$$

Unlike modified Møller operators, asymptotic momenta are canonically defined. Following [10], one can define canonically the whole class of modified Møller operators as isometric operators $S_{\mathrm{lr}}^{\pm}$ satisfying

$$g(D^{\pm}) = S_{\mathrm{lr}}^{\pm} g(D) S_{\mathrm{lr}}^{\pm *}.$$

4. Second quantization

In this section we will fix our notation for operators on Fock spaces, which will be the main language in the sequel.

4.1. *Fock spaces*

Let $\mathcal{Z}$ be a Hilbert space. Physically, it will have the meaning of a 1-particle space. On $\otimes^n \mathcal{Z}$ we have the obvious natural action of the permutation group, denoted

$$S_n \ni \sigma \mapsto \Theta(\sigma) \in U(\otimes^n \mathcal{Z}).$$

Let us introduce the orthogonal projections onto symmetric/antisymmetric tensors:

$$\Theta^n_{\rm s} := \frac{1}{n!}\sum_{\sigma\in S_n}\Theta(\sigma),$$
$$\Theta^n_{\rm a} := \frac{1}{n!}\sum_{\sigma\in S_n}{\rm sgn}\sigma\Theta(\sigma).$$

Many concepts are paralel for the symmetric (bosonic) and antisymmetric (fermionic) case. The former will be often denoted by the subscript "s" and the latter by the subscript "a". We will write "s/a" to denote "either s or a".

The *n-particle bosonic/fermionic space* is defined as $\otimes^n_{\rm s/a}\mathcal{Z} := \Theta^n_{\rm s/a}\otimes^n\mathcal{Z}$. The *bosonic/fermionic Fock space* is $\Gamma_{\rm s/a}(\mathcal{Z}) := \bigoplus_{n=0}^{\infty}\otimes^n_{\rm s/a}\mathcal{Z}$. The vector $\Omega = 1 \in \otimes^0_{\rm s/a}\mathcal{Z} = \mathbb{C}$ is called the *vacuum*.

4.2. *Creation and annihilation operators*

For $f \in \mathcal{Z}$ we define the *creation operator*

$$a^*(f)\Psi := \sqrt{n+1}\,\Theta^{n+1}_{\rm s/a} f\otimes\Psi, \quad \Psi\in\otimes^n_{\rm s/a}\mathcal{Z},$$

and the *annihilation operator* $a(f) := (a^*(f))^*$.

Note that traditionally, in most physics textbooks, one uses a somewhat different notation for creation and annihilation operators. One identifies $\mathcal{Z}$ with $L^2(\Xi)$ for some measure space $(\Xi, {\rm d}\xi)$. If f equals a function $\Xi\ni\xi\mapsto f(\xi)$, then one writes

$$a^*(f) = \int f(\xi)a^*(\xi){\rm d}\xi, \quad a(f) = \int \overline{f}(\xi)a(\xi){\rm d}\xi. \tag{4.1}$$

4.3. *Field and Weyl operators*

In the bosonic case, for $f\in\mathcal{Z}$ we introduce the *field operators*

$$\phi(f) := \frac{1}{\sqrt{2}}(a^*(f)+a(f)).$$

and the *Weyl operators*

$$W(f) := {\rm e}^{{\rm i}\phi(f)}.$$

For later reference note that

$$(\Omega|W(f)\Omega) = {\rm e}^{-\|f\|^2/4}.$$

4.4. *Wick quantization*

Let $b \in B\left(\otimes^n_{\mathrm{s/a}}\mathcal{Z}, \otimes^m_{\mathrm{s/a}}\mathcal{Z}\right)$. We would like to define its *Wick quantization*. To this end, it will be convenient to use the traditional notation, which involves an identification of $\mathcal{Z}$ with $L^2(\Xi)$. This identification allows us (at least formally) to represent the operator b by its integral kernel of b, which is a function $b(\xi_1, \cdots \xi_m, \xi'_n, \cdots, \xi'_1)$ symmetric/antisymmetric in its first and last coordinates. The Wick quantization of the polynomial b will be denoted by

$$B = \int b(\xi_1, \cdots \xi_m, \xi'_n, \cdots, \xi'_1) \\ a^*(\xi_1)\cdots a^*(\xi_m)a(\xi'_n)\cdots a(\xi'_1)\mathrm{d}\xi_1, \cdots \xi_n \mathrm{d}\xi'_m \cdots \mathrm{d}\xi'_1. \tag{4.2}$$

It is the operator whose only nonzero matrix elements are between $k+m$ and $k+n$ particle vectors. For $\Phi \in \otimes^{k+m}_{\mathrm{s/a}}\mathcal{Z}$, $\Psi \in \otimes^{k+n}_{\mathrm{s/a}}\mathcal{Z}$, the corresponding matrix element equals

$$(\Phi|B\Psi) = \frac{\sqrt{(n+k)!(m+k)!}}{k!}(\Phi|b \otimes 1^{\otimes k}_{\mathcal{Z}}\Psi).$$

Let us remark that the operator (4.2) does not depend on the the choice of the identification of $\mathcal{Z}$ with $L^2(\Xi)$. Moreover, (4.2) is consistent with the usual traditional notation, in particular with (4.1).

4.5. *Second quantization of operators*

For an operator q on $\mathcal{Z}$ we define the operator $\Gamma(q)$ on $\Gamma_{\mathrm{s/a}}(\mathcal{Z})$ by

$$\Gamma(q)\Big|_{\otimes^n_{\mathrm{s/a}}\mathcal{Z}} := q \otimes \cdots \otimes q\Big|_{\otimes^n_{\mathrm{s/a}}\mathcal{Z}}.$$

Similarly, for an operator h we define the operator $\mathrm{d}\Gamma(h)$ by

$$\mathrm{d}\Gamma(h)\Big|_{\otimes^n_{\mathrm{s/a}}\mathcal{Z}} := \Big(h \otimes 1^{(n-1)\otimes} + \cdots + 1^{(n-1)\otimes} \otimes h\Big)\Big|_{\otimes^n_{\mathrm{s/a}}\mathcal{Z}}.$$

In the traditional notation, if h is the multiplication operator by $h(\xi)$, then $\mathrm{d}\Gamma(h) = \int h(\xi)a^*_\xi a_\xi \mathrm{d}\xi$.

Note the identity $\Gamma(\mathrm{e}^{\mathrm{i}th}) = \mathrm{e}^{\mathrm{i}t\mathrm{d}\Gamma(h)}$.

5. Scattering for Hamiltonians of quantum field theory

In this section we describe the basics of scattering theory of QFT Hamiltonians with localized interaction and without the "small system" (see Section

8). Unfortunately, in many cases one has to work with formal power series (see however [11]). Most of the general references on the subject are quite old [13, 20, 31].

5.1. *QFT Hamiltonians*

Typical Hamiltonians of QFT have (at least formally) the form

$$H_\lambda := H_0 + \lambda V, \tag{5.1}$$

where

$$H_0 := \int h(\xi) a^*(\xi) a(\xi) \mathrm{d}\xi, \tag{5.2}$$

$$V := \int \sum_{n,m} v_{n,m}(\xi_1, \cdots \xi_m, \xi'_n, \cdots, \xi'_1)$$
$$a^*(\xi_1) \cdots a^*(\xi_m) a(\xi'_n) \cdots a(\xi'_1) \mathrm{d}\xi_1, \cdots \xi_m \mathrm{d}\xi'_1 \cdots \mathrm{d}\xi'_n.$$

The polynomials $v_{n,m}$ should be even in fermionic variables. We will assume that the one-particle energy is $h(\xi) = \sqrt{\xi^2 + m^2}$.

The variable ξ has the interpretation of a 1-particle momentum. Clearly, H_0 is translation invariant. The perturbation V is translation invariant iff it has the form

$$v_{n,m}(\xi_1, \cdots \xi_m, \xi'_n, \cdots, \xi'_1)$$
$$= \tilde{v}_{n,m}(\xi_1, \cdots \xi_m, \xi'_n, \cdots, \xi'_1)$$
$$\delta(\xi_1 + \cdots + \xi_m - \xi'_n - \cdots - \xi'_1).$$

In our notes we will not consider translation invariant interactions. We will always assume that $v_{n,m}(\xi_1, \cdots \xi_m, \xi'_n, \cdots, \xi'_1)$ are smooth and decay fast in all directions. This simplifying assumption expresses in particular the fact that the interaction is well localized. The scattering theory for such interactions is much easier to study and better understood than that for translation invariant interactions.

We will not worry too much about the self-adjointness of H_λ. If we encounter problems, we will work with formal power series.

Actually, in the case of fermions one can define (5.1) as a self-adjoint operator, since the perturbation is bounded. In the case of bosons, the self-adjointness holds if the perturbation is of degree 1. It is also true for 2nd order perturbation that is sufficiently small. Otherwise it can be proven only under special assumptions (e.g. for spacially cut-off $P(\phi)_2$ interactions [18]).

5.2. *QFT Hamiltonians that do not polarize vacuum*

Suppose that

$$v_{n,0} = v_{0,n} = 0. \tag{5.3}$$

Then Ω is an eigenvector of both H_0 and H, and the standard wave operators exist, at least formally, see eg. [31].

Unfortunately, physically realistic Hamiltonians often polarize the vacuum, and the standard formalism of scattering theory is inapplicable in these cases.

5.3. *Ground state*

In general, at least formally, H_λ possesses a *ground state* Ω_λ with the *ground state energy* E_λ. They depend on λ in terms of a formal perturbation expansion:

$$\Omega_\lambda = \sum_{n=0}^{\infty} \lambda^n \Omega_n, \qquad E_\lambda = \sum_{n=0}^{\infty} \lambda^n E_n.$$

5.4. *Feynman-Dyson approach*

There exist two basic formalisms for scattering theory of QFT Hamiltonians with localized interaction. The first approach can be traced back to the early works on QED. We will call it the *Feynman-Dyson approach*. It starts with introducing the unrenormalized Møller operators. One can prove their existence, at least as formal power series

$$\begin{aligned} S^{\pm}_{\mathrm{ur}} &= \mathrm{s}\!-\!\lim_{\epsilon \searrow 0} \epsilon \int_0^\infty \mathrm{e}^{-\epsilon t}\, \mathrm{e}^{\pm \mathrm{i}tH}\, \mathrm{e}^{\mp \mathrm{i}t(H_0 - E)}\, \mathrm{d}t \\ &= \sum_{n=0}^{\infty} \lambda^n S^{\pm}_{\mathrm{ur},n}. \end{aligned}$$

One can also show that the *vacuum amplitude operator* $Z = S^{-*}_{\mathrm{ur}} S^{-}_{\mathrm{ur}} = S^{+*}_{\mathrm{ur}} S^{+}_{\mathrm{ur}}$ is proportional to identity and equals $Z = |(\Omega_\lambda|\Omega)|^2$. The *renormalized Møller operators* $S^{\pm}_{\mathrm{rn}} := S^{\pm}_{\mathrm{ur}} Z^{-1/2}$ are formally unitary and so is the *renormalized scattering operator* $S_{\mathrm{rn}} := S^{+*}_{\mathrm{rn}} S^{-}_{\mathrm{rn}}$.

5.5. *The LSZ formalism*

Instead of the scattering theory based on Møller operators, one can proceed differently. Following Lehman-Symanzik-Zimmermann, one can start by introducing the so-called *asymptotic creation/annihilation operators* defined

as the limits

$$a^{\pm}_{\lambda}(f) := \lim_{t\to\pm\infty} \mathrm{e}^{\mathrm{i}tH}\, a(\mathrm{e}^{-\mathrm{i}th}\, f)\, \mathrm{e}^{-\mathrm{i}tH},$$
$$a^{*\pm}_{\lambda}(f) := \lim_{t\to\pm\infty} \mathrm{e}^{\mathrm{i}tH}\, a^{*}(\mathrm{e}^{-\mathrm{i}th}\, f)\, \mathrm{e}^{-\mathrm{i}tH}\,.$$

One can show their existence at least as formal power series. They satisfy the usual *canonical commutation/anticommutation relations (CCR/CAR)*. Moreover, asymptotic annihilation operators kill the perturbed ground state:

$$a^{\pm}_{\lambda}(f)\Omega_{\lambda} = 0.$$

The renormalized Møller operators can be defined with help of asymptotic operators

$$S^{\pm}_{\mathrm{rn},\lambda} a^{*}(f_1)\cdots a^{*}(f_n)\Omega = a^{*\pm}_{\lambda}(f_1)\cdots a^{*\pm}_{\lambda}(f_n)\Omega_{\lambda}.$$

They are formally unitary and intertwine the CCR/CAR:

$$S^{\pm}_{\mathrm{rn},\lambda} a^{*}(f) = a^{*\pm}_{\lambda}(f) S^{\pm}_{\mathrm{rn},\lambda},$$
$$S^{\pm}_{\mathrm{rn},\lambda} a(f) = a^{\pm}_{\lambda}(f) S^{\pm}_{\mathrm{rn},\lambda}.$$

Note that there is no need for renormalization.

One can construct the alternative renormalized scattering operator $\tilde{S}$ with help of asymptotic operators, even skipping the Møller operators, as the unique (up to a phase factor) unitary operator satisfying

$$\tilde{S}_{\mathrm{rn},\lambda} a^{*-}_{\lambda}(f) = a^{*+}_{\lambda}(f) \tilde{S}_{\mathrm{rn},\lambda},$$
$$\tilde{S}_{\mathrm{rn},\lambda} a^{-}_{\lambda}(f) = a^{+}_{\lambda}(f) \tilde{S}_{\mathrm{rn},\lambda}.$$

6. Scattering theory of Van Hove Hamiltonians

A *van Hove Hamiltonian* is a self-adjoint operator formally defined as

$$H = \int h(\xi) a^{*}(\xi) a(\xi) \mathrm{d}\xi + \int \overline{z}(\xi) a(\xi) \mathrm{d}\xi + \int z(\xi) a^{*}(\xi) \mathrm{d}\xi,$$

where $\xi \mapsto h(\xi) \in [0, \infty[$ describes the energy and $\xi \mapsto z(\xi)$ the interaction. Van Hove Hamiltonians form a very instructive class of operators, whose properties, and in particular the scattering theory, are very well understood [8]. They can also serve as a simple illustration of the infra-red and ultraviolet problem. In our lectures we will not discuss the ultraviolet problem

and we will always assume that at high energies the coupling function is sufficiently regular, which is expressed by the condition

$$\int_{h\geq 1} |z(\xi)|^2 \mathrm{d}\xi < \infty.$$

Following [8], we will however discuss the infra-red behavior of van Hove Hamiltonians, which is relevant for their scattering theory. One can distinguish 3 cases of the infra-red behavior of the coupling function. In the order of an increasing singularity, we call them A, B and C.

6.1. *Infra-red case A*

We say that the coupling function belongs to Case A if

$$\int_{h<1} \frac{|z(\xi)|^2}{h(\xi)^2} \mathrm{d}\xi < \infty. \tag{6.1}$$

(The integral (6.1) is restricted to ξ with $h(\xi) < 1$). Van Hove Hamiltonians with the coupling function satisfying this condition are the most regular. It is easy to see that they are bounded from below self-adjoint operators with the ground state energy

$$E := -\int \frac{|z(\xi)|^2}{h(\xi)} \mathrm{d}\xi, \tag{6.2}$$

and the spectrum $[E, \infty[$. Besides, the coherent vector

$$\Psi = \exp\left(-\int \frac{|z(\xi)|^2}{2h(\xi)^2} \mathrm{d}\xi\right) \exp\left(\int a^*(\xi) \frac{z(\xi)}{h(\xi)} \mathrm{d}\xi\right) \Omega.$$

is its unique ground state.

To see this it is enough to introduce the so-called *dressing operator*

$$U := \exp\left(-a^*(\frac{z}{h}) + a(\frac{z}{h})\right). \tag{6.3}$$

If we set

$$H_0 = \int h(\xi) a^*_\xi a_\xi \mathrm{d}\xi,$$

then the operator H is up to a constant unitarily equivalent to H_0:

$$H - E = U H_0 U^*. \tag{6.4}$$

6.2. *Infra-red case B*

Let

$$\int_{h<1} \frac{|z(\xi)|^2}{h(\xi)} \mathrm{d}\xi < \infty,$$
$$\int_{h<1} \frac{|z(\xi)|^2}{h(\xi)^2} \mathrm{d}\xi = \infty.$$

In this case H can be still defined as a self-adjoint operator and is bounded from below. The equation (6.2) defines a finite number E, which is the infimum of the spectrum of H. However, H has no eigenvalues. This is related to the fact that the dressing operator (6.3) is ill defined, and hence we cannot write (6.4).

6.3. *Infra-red case C*

Let

$$\int_{h<1} |z(\xi)|^2 \mathrm{d}\xi < \infty,$$
$$\int_{h<1} \frac{|z(\xi)|^2}{h(\xi)} \mathrm{d}\xi = \infty.$$

H can be still defined as a self-adjoint operator. However, H has no eigenvectors and its spectrum covers the whole real line.

For coupling functions satisfying

$$\int_{h<1} |z(\xi)|^2 \mathrm{d}\xi = \infty$$

one cannot define a van Hove Hamiltonian at all.

6.4. *Feynman-Dyson scattering theory for van Hove Hamiltonians*

Assume that h has an absolutely continuous spectrum (as an operator on $L^2(\Xi)$) and Case A or B:

$$\int \frac{|z(\xi)|^2}{h(\xi)} \mathrm{d}\xi < \infty.$$

Then it is easy to show that there exists the strong Abelian Møller operator

$$S^{\pm}_{\mathrm{ur}} := \mathrm{s}\text{-}\lim_{\epsilon \searrow 0} \epsilon \int_0^{\infty} \mathrm{e}^{-\epsilon t}\, \mathrm{e}^{\mathrm{i}tH}\, \mathrm{e}^{-\mathrm{i}t(H_0+E)}\, \mathrm{d}\xi.$$

We have $S^{\pm}_{\rm ur} = UZ$, where

$$Z = \exp\left(-\int \frac{|z(\xi)|^2}{h^2(\xi)}\mathrm{d}\xi\right).$$

In Case A, the vacuum amplitude constant is nonzero and we can renormalize $S^{\pm}_{\rm ur}$, obtaining the dressing operator

$$S^{\pm}_{\rm rn} := S^{\pm}_{\rm ur} Z^{-1/2} = U.$$

The scattering operator is (unfortunately) trivial:

$$S = S^{+*}_{\rm rn} S^{-}_{\rm rn} = 1.$$

In case B, the vacuum amplitude constant is zero. The Møller operators are not defined. However, if we are willing to introduce and then remove a cut-off, then we can informally conclude that the scattering operator is again equal to identity.

6.5. *The LSZ formalism for van Hove Hamiltonians*

It is easy to see that in Case A, B and C, for $f \in \mathrm{Dom} h^{-1}$, there exist asymptotic fields:

$$\begin{aligned} a^{\pm}(f) &:= \lim_{t\to\pm\infty} \mathrm{e}^{\mathrm{i}tH}\, a(\mathrm{e}^{-\mathrm{i}th} f)\, \mathrm{e}^{-\mathrm{i}tH} = a(f) + (f|h^{-1}z), \\ a^{*\pm}(f) &:= \lim_{t\to\pm\infty} \mathrm{e}^{\mathrm{i}tH}\, a^*(\mathrm{e}^{-\mathrm{i}th} f)\, \mathrm{e}^{-\mathrm{i}tH} = a^*(f) + (z|h^{-1}f). \end{aligned}$$

This allows us to compute the scattering operator $\tilde{S}$ even in Case B and C. It is trivial – proportional to the identity.

From the point of view of asymptotic fields, the difference between Case A and Cases B and C consists in the type of representations of the CCR: in Case A it is Fock, but in Case B and C it is not. (Here we use the terminology that we will develop in the next section).

7. Representations of the CCR

We have seen that the LSZ formalism leads to asymptotic operators satisfying the usual canonical commutation/anticommutation relations (CCR/CAR). These operators can have unusual properties, different from the properties of the usual creation/annihilation operators on a Fock space, as we saw for van Hove Hamiltonians in Case B and C. Therefore, it is useful to develop a theory of representations of the CCR/CAR in an abstract form. In these lectures we will restrict ourselves to the case of the CCR. We will follow [5, 7].

7.1. *Definition of a representation of the CCR*

Let $\mathcal{Y}$ be a real vector space equipped with an antisymmetric form ω. (Usually we assume that ω is symplectic, i.e. nondegenerate). Let $U(\mathcal{H})$ denote the set of unitary operators on a Hilbert space $\mathcal{H}$. We say that

$$\mathcal{Y} \ni y \mapsto W^\pi(y) \in U(\mathcal{H})$$

is a *representation of the CCR* over $\mathcal{Y}$ in $\mathcal{H}$ if

$$W^\pi(y_1)W^\pi(y_2) = \mathrm{e}^{-\frac{\mathrm{i}}{2}y_1\omega y_2}\, W^\pi(y_1+y_2), \qquad y_1, y_2 \in \mathcal{Y}.$$

7.2. *Regular representations of the CCR*

Let $\mathcal{Y} \ni y \mapsto W^\pi(y)$ be a representation of the CCR. Clearly,

$$\mathbb{R} \ni t \mapsto W^\pi(ty) \in U(\mathcal{H})$$

is a 1-parameter group. We say that a representation of the CCR is *regular* if this group is strongly continuous for each $y \in \mathcal{Y}$.

Assume that $y \mapsto W^\pi(y)$ is a regular representation of the CCR. The *field operator* corresponding to $y \in \mathcal{Y}$ is defined as

$$\phi^\pi(y) := -\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}W^\pi(ty)\Big|_{t=0}.$$

We have the *Heisenberg canonical commutation relations*

$$[\phi^\pi(y_1), \phi^\pi(y_2)] = \mathrm{i}y_1\omega y_2.$$

7.3. *Creation/annihilation operators associated with a representation of the CCR*

Let $\mathcal{Z}$ be a complex vector space with a scalar product $(\cdot|\cdot)$. It is a symplectic space with the form $\mathrm{Im}(\cdot|\cdot)$. Suppose that

$$\mathcal{Z} \ni f \mapsto W^\pi(f) \in U(\mathcal{H}) \tag{7.1}$$

is a regular representation of the CCR. For $f \in \mathcal{Z}$ we introduce the *creation/annihilation operators* corresponding to (7.1)

$$a^{\pi *}(f) := \frac{1}{\sqrt{2}}(\phi^\pi(f) + \mathrm{i}\phi^\pi(\mathrm{i}f)), \;\; a^\pi(f) := \frac{1}{\sqrt{2}}(\phi^\pi(f) - \mathrm{i}\phi^\pi(\mathrm{i}f)).$$

They satisfy the usual relations

$$[a^\pi(f_1), a^\pi(f_2)] = 0, \;\; [a^{\pi *}(f_1), a^{\pi *}(f_2)] = 0,$$
$$[a^\pi(f_1), a^{\pi *}(f_2)] = (f_1|f_2).$$

7.4. *The Fock representation*

We still consider a complex vector space $\mathcal{Z}$ with a scalar product. Let $\mathcal{Z}^{\mathrm{cpl}}$ denote its completion. Consider the creation/annihilation operators acting on the Fock space $\Gamma_{\mathrm{s}}(\mathcal{Z}^{\mathrm{cpl}})$. Then $\phi(f) := \frac{1}{\sqrt{2}}\,(a^*(f) + a(f))$ are self-adjoint operators and

$$\mathcal{Z} \ni f \mapsto \exp \mathrm{i}\phi(f) \in U\left(\Gamma_{\mathrm{s}}(\mathcal{Z}^{\mathrm{cpl}})\right)$$

is a regular representation of the CCR called the *Fock representation*. The vacuum Ω is characterized by either of the following equivalent equations:

$$\begin{aligned} a(f)\Omega &= 0, \qquad f \in \mathcal{Z};\\ (\Omega|\,\mathrm{e}^{\mathrm{i}\phi(f)}\,\Omega) &= \mathrm{e}^{-\frac{1}{4}(f|f)}, \quad f \in \mathcal{Z}. \end{aligned}$$

7.5. *Coherent representations*

In this subsection, following [12], we describe an important class of representations of the CCR on a Fock space – coherent representations.

Let g be an antilinear functional on $\mathcal{Z}$ (not necessarily bounded). Then

$$\mathcal{Z} \ni f \mapsto W_g(f) := W(f)\,\mathrm{e}^{\mathrm{iRe}(g|f)} \in U(\Gamma_{\mathrm{s}}(\mathcal{Z}^{\mathrm{cpl}})) \tag{7.2}$$

is a regular representation of the CCR. It will be called *the g-coherent representation*. The corresponding creation/annihilation operators are

$$\begin{aligned} a_g(f) &= a(f) + \frac{1}{\sqrt{2}}(f|g),\\ a_g^*(f) &= a^*(f) + \frac{1}{\sqrt{2}}(g|f). \end{aligned}$$

The vector Ω is characterized by either of the following equations:

$$\begin{aligned} a_g(f)\Omega &= \frac{1}{\sqrt{2}}(f|g)\Omega,\\ (\Omega|W_g(f)\Omega) &= \mathrm{e}^{-\frac{1}{4}(f|f)+\mathrm{iRe}(f|g)}\,. \end{aligned}$$

It is easy to show that the representation $f \mapsto W_g(f)$ is unitarily equivalent to the Fock representation iff g is a bounded functional, equivalently, $g \in \mathcal{Z}^{\mathrm{cpl}}$. More generally, W_{g_1} is equivalent to W_{g_2} iff $g_1 - g_2 \in \mathcal{Z}^{\mathrm{cpl}}$. This gives an obvious equivalence relation on the dual of $\mathcal{Z}$. The equivalence class of g with respect to this relation will be denoted $[g]$.

7.6. *Coherent sectors*

Suppose that

$$\mathcal{Z} \ni f \mapsto W^\pi(f) \in U(\mathcal{H}) \tag{7.3}$$

is a representation of the CCR (e.g. obtained by asymptotic limits, so that $\pi = \pm$). Let g be be an antilinear functional on $\mathcal{Z}$. In this subsection we describe a method that allows us to determine the largest subrepresentation of W^π equivalent to a multiple of the g-coherent representation.

Let $\mathrm{Span}^{\mathrm{cl}}(K)$ denote the closure of the linear span of K. Define

$$\begin{aligned} \mathcal{K}_g^\pi &:= \{\Psi \in \mathcal{H} \ : \ a^\pi(f)\Psi = \sqrt{2}(g|f)\Psi\} \\ &= \{\Psi \in \mathcal{H} \ : \ (\Psi|W^\pi(f)\Psi) = \|\Psi\|^2 \, \mathrm{e}^{-\frac{1}{4}(f|f)+\mathrm{iRe}(f|g)}\}, \end{aligned} \tag{7.4}$$

$$\begin{aligned} \mathcal{H}_{[g]}^\pi &:= \mathrm{Span}^{\mathrm{cl}} \left\{a^{\pi *}(f_1)\cdots a^{\pi *}(f_1)\Psi \ : \ \Psi \in \mathcal{K}_g^\pi, \ f_i \in \mathcal{Z}\right\} \\ &= \mathrm{Span}^{\mathrm{cl}} \left\{W^\pi(f)\Psi \ : \ \Psi \in \mathcal{K}_g^\pi, \ f \in \mathcal{Z}\right\}. \end{aligned} \tag{7.5}$$

$\mathcal{K}_g^\pi$ is called the *space of g-coherent vectors* and $\mathcal{H}_{[g]}^\pi$ is called the $[g]$-*coherent sector* of W^π. In the case $g = 0$, we have a somewhat different terminology: $\mathcal{K}_0^\pi$ is called the *space of Fock vacua* and $\mathcal{H}_{[0]}^\pi$ is called the *Fock sector* of W^π.

We also define an isometric operator $S_g^\pi : \mathcal{K}_g^\pi \otimes \Gamma_{\mathrm{s}}(\mathcal{Z}^{\mathrm{cpl}}) \to \mathcal{H}$ by

$$\begin{aligned} & S_g^\pi \ \Psi \otimes a_g^*(f_1)\cdots a_g^*(f_n)\Omega \\ = & \qquad a^{\pi *}(f_1)\cdots a^{\pi *}(f_n)\Psi, \end{aligned} \tag{7.6}$$

$$\begin{aligned} & S_g^\pi \ \Psi \otimes W_g(f)\Omega \\ = & \qquad W^\pi(f)\Psi. \end{aligned}$$

(In (7.4), (7.5) and (7.6) we give two alternative equivalent definitions. One of them involves creation/annihilation operators and ther other one involves Weyl operators).

Theorem 7.1: *The following statements are true:*

(1) $\mathcal{H}_{[g]}^\pi$ is an invariant subspace for W^π.
(2) $S_g^\pi : \mathcal{K}_g^\pi \otimes \Gamma_{\mathrm{s}}(\mathcal{Z}^{\mathrm{cpl}}) \to \mathcal{H}_{[g]}^\pi$ is unitary.
(3) $S_g^\pi \ 1 \otimes W_g(f) = W^\pi(f) \ S_g^\pi$, $f \in \mathcal{Z}$.
(4) If U is isometric such that $U \ 1 \otimes W_g(f) = W^\pi(f) \ U$, $f \in \mathcal{Z}$, then $\mathrm{Ran}U \subset \mathcal{H}_{[g]}^\pi$.

Thus $\mathcal{H}_{[g]}^\pi$ is the biggest subspace of $\mathcal{H}$, on which W^π is unitarily equivalent to W_g.

7.7. *Covariant representations*

We still consider a representation of the CCR (7.3). Let h be a self-adjoint operator on $\mathcal{Z}^{\mathrm{cpl}}$ and H a self-adjoint operator on $\mathcal{H}$. We say that (W^π, h, H) is a *covariant representation of the CCR* iff

$$\mathrm{e}^{\mathrm{i}tH}\, W^\pi(f)\, \mathrm{e}^{-\mathrm{i}tH} = W^\pi(\mathrm{e}^{\mathrm{i}th}\, f),\ \ f \in \mathcal{Z}.$$

The most obvious example of a covariant representation is $(W, h, \mathrm{d}\Gamma(h))$, where W is the Fock representation. This follows from the identity

$$\mathrm{e}^{\mathrm{i}t\mathrm{d}\Gamma(h)}\, W(f)\, \mathrm{e}^{-\mathrm{i}t\mathrm{d}\Gamma(h)} = W(\mathrm{e}^{\mathrm{i}th}\, f).$$

Let us now describe a somewhat more complicated example of a covariant representation. Let $g \in h^{-1}\mathcal{Z}^{\mathrm{cpl}}$. Set $z = \frac{1}{\sqrt{2}} hg$. Introduce the van Hove Hamiltonian

$$\mathrm{d}\Gamma_g(h) := \mathrm{d}\Gamma(h) + a^*(z) + a(z) + (z|h^{-1}z).$$

Let W^g by the g-coherent representation. Then $(W_g, h, \mathrm{d}\Gamma_g(h))$ is a covariant representation of the CCR, that is

$$\mathrm{e}^{\mathrm{i}t\mathrm{d}\Gamma_g(h)}\, W_g(f)\, \mathrm{e}^{-\mathrm{i}t\mathrm{d}\Gamma_g(h)} = W_g(\mathrm{e}^{\mathrm{i}th}\, f). \tag{7.7}$$

Note that (7.7) is obvious for $g \in \mathcal{Z}^{\mathrm{cpl}}$, because then

$$\mathrm{d}\Gamma_g(h) = W(\mathrm{i}g)\mathrm{d}\Gamma(h)W(-\mathrm{i}g),$$
$$W_g(f) = W(\mathrm{i}g)W(f)W(-\mathrm{i}g).$$

7.8. *Coherent sectors of a covariant representation*

The following theorem [12] shows that in some cases subrepresentations of a covariant representation of the CCR are also covariant.

Suppose that $\mathcal{Z} \ni f \mapsto W^\pi(f) \in U(\mathcal{H})$ is a representation of the CCR. We will use the notation $\mathcal{K}_g^\pi$, $\mathcal{H}_{[g]}^\pi$ and S_g^π introduced in (7.4), (7.5) and (7.6).

Theorem 7.2: *Let (W^π, h, H) be covariant. Then the following is true:*

(1) $\mathcal{K}_0^\pi$ and $\mathcal{H}_{[0]}^\pi$ are $\mathrm{e}^{\mathrm{i}tH}$-invariant. Let $K_0^\pi := H\Big|_{\mathcal{K}_0^\pi}$ and set

$$H_0^\pi = K_0^\pi \otimes 1 + 1 \otimes \mathrm{d}\Gamma(h).$$

Then $HS_0^\pi = S_0^\pi H_0^\pi$.

(2) Let $g \in h^{-1/2}\mathcal{Z}$. Then $\mathcal{H}^\pi_{[g]}$ is $\mathrm{e}^{\mathrm{i}tH}$-invariant. Moreover, for some operator K^π_g on $\mathcal{K}^\pi_g$, if we set

$$H^\pi_g := K^\pi_g \otimes 1 + 1 \otimes \mathrm{d}\Gamma_g(h),$$

then we have $HS^\pi_g = S^\pi_g H^\pi_g$.

(1) of the above theorem shows that one can always restrict a covariant representation to its Fock sector, obtaining a covariant representation. This covariant representation is very easy – the Hamiltonian restricted to this sector decouples into a sum of non-interacting simple-minded terms.

(2) says that, under some conditions on g, the representation W^π restricted to the $[g]$-coherent sector is still covariant. Moreover, it is unitarily equivalent to

$$\left(1 \otimes W^g, 1 \otimes h, K^\pi_g \otimes 1 + 1 \otimes \mathrm{d}\Gamma_g(h)\right).$$

This fact can be used to analyze dynamics that are seemingly difficult, eg. such as those typical for the infra-red problem [4, 30, 33, 25, 26]. In fact, if $g \notin \mathcal{Z}^{\mathrm{cpl}}$, then the Hamiltonian H restricted to the $[g]$-coherent sector has no eigenvectors, and in spite of that it is under control – its main part is a well understood van Hove Hamiltonian.

8. Pauli-Fierz Hamiltonians

Many physical situations are well described in terms of a "small quantum system" interacting with quantized fields. The small quantum system can be an atom, a molecule, a "quantum dot", etc. One often assumes that it is finite dimensional, or at least that its Hamiltonian has a discrete spectrum. The quantized fields can describe electromagnetic radiation (photons), crystal vibrations (phonons), etc. One often assumes that they are described by a simple free dynamics.

The Hamiltonian of a composite system typically consists of three terms: the Hamiltonian of the small system, the Hamiltonian of the quantum field, and the interaction that couples them.

8.1. *Definition of Pauli-Fierz Hamiltonians*

We will restrict ourselves to the case of bosonic fields and we will assume that the interaction is linear in the fields.

More explicitly, suppose that $\mathcal{K}$ be a Hilbert space with a self-adjoint operator K describing the small system. For instance, we can consider the

space $L^2(\mathbb{R}^d)$ with a Schrödinger operator $K = -\Delta + V(x)$. Usually, we will assume that K has discrete eigenvalues, which is the case if $\lim_{|x|\to\infty} V(x) = \infty$.

We assume that the bosons are described by the Fock space $\Gamma_{\rm s}(\mathcal{Z})$, where, for concreteness, the one-particle space is $\mathcal{Z} = L^2(\mathbb{R}^d)$. As usual, the dispersion relation of the bosons is assumed to be $h(\xi) := \sqrt{\xi^2 + m^2}$, $m \geq 0$. The parameter m will be called "the mass".

The full Hilbert space is $\mathcal{K} \otimes \Gamma_{\rm s}(\mathcal{Z})$. We fix a coupling function

$$\xi \mapsto v(\xi) \in B(\mathcal{K}).$$

An operator of the form

$$H := H_0 + V, \tag{8.1}$$

where

$$H_0 := K \otimes 1 + 1 \otimes \int h(\xi) a^*(\xi) a(\xi) \mathrm{d}\xi, \tag{8.2}$$

$$V := \int v(\xi) \otimes a^*(\xi) \mathrm{d}\xi + \mathrm{hc},$$

will be called a *Pauli-Fierz Hamiltonian.* Note in parenthesis that the terminology in this area is not settled and other names are used in this context as well, such as a *generalized spin-boson Hamiltonian.*

8.2. *Spectral properties of Pauli-Fierz Hamiltonians*

Let us start with some results about the spectral properties of Pauli-Fierz Hamiltonians.

Theorem 8.1:

(1) [10] *Assume that* $(K + \mathrm{i})^{-1}$ *is compact and*

$$\int (1 + h(\xi)^{-1}) \|v(\xi)\|^2 \mathrm{d}\xi < \infty.$$

Then H *is self-adjoint and bounded from below. If* $E := \inf \mathrm{sp} H$*, then*

$$\mathrm{sp}_{\rm ess} H = [E + m, \infty[. \tag{8.3}$$

(2) [16]*, see also* [1, 2, 19]*. If in addition*

$$\int (1 + h(\xi)^{-2}) \|v(\xi)\|^2 \mathrm{d}\xi < \infty,$$

then H *has a ground state (the infimum of its spectrum is an eigenvalue).*

(1) can be called an HVZ-type theorem for Pauli-Fierz Hamiltonians (after a well known Hunziker-van Winter-Zhislin Theorem about N-body Schrödinger Hamiltonians [29]). It implies that if m is positive, then H necessarily has a ground state. By (2), if the interaction is sufficiently regular in the infrared region, this ground state survives even if $m = 0$.

In typical situations one expects that H has no eigenvalues embedded in its continuous spectrum. This expecation is often confirmed by rigorous results. In fact, for a small non-zero coupling constant and some generic assumptions on the interaction, one can show that the spectrum of $H_\lambda := H + \lambda V$ in $]E + m, \infty[$ is purely absolutely continuous, eg. [2, 3].

In particular, if $m = 0$, this means that the only eigenvalue of H_λ is at the bottom of its spectrum. One can often prove that it is nondegenerate.

8.3. *Scattering theory of Pauli-Fierz Hamiltonians*

In the case of Pauli-Fierz Hamiltonians, the formalism of scattering theory based on Abelian Møller operators (which in Section 5 we called the Feynman-Dyson formalism) does not apply. Note that (8.1) is not an operator of the form (5.1), because of the presence of the small system.

It turns out, however, that a certain version of the LSZ formalism works well for Pauli-Fierz Hamiltonians. This formalism will be described below, following its version described by Gérard and the author in [10, 11, 12]. (Fröhlich-Griesemer-Schlein use a slightly different setup in [14]).

Theorem 8.2: [10] *Suppose that for f from a dense subspace we have*

$$\int_0^\infty \left\| \int \mathrm{e}^{\mathrm{i}th(\xi)}\, f(\xi) v(\xi) \mathrm{d}\xi + \mathrm{hc} \right\| \mathrm{d}t < \infty. \tag{8.4}$$

Define $\mathcal{Z}_1 := \mathrm{Dom} h^{-1/2} \subset L^2(\mathbb{R}^d)$. Then the following holds:

(1) For $f \in \mathcal{Z}_1$, there exists

$$W^\pm(f) := \mathrm{s}\text{-}\lim_{t \to \pm\infty} \mathrm{e}^{\mathrm{i}tH}\, 1 \otimes W(\mathrm{e}^{-\mathrm{i}th} f)\, \mathrm{e}^{-\mathrm{i}tH}; \tag{8.5}$$

(2) $W^\pm(f_1) W^\pm(f_2) = \mathrm{e}^{-\mathrm{i}\mathrm{Im}(f_1|f_2)}\, W^\pm(f_1 + f_2)$, $f_1, f_2 \in \mathcal{Z}_1$;
(3) $\mathbb{R} \ni t \mapsto W^\pm(tf)$ is strongly continuous;
(4) $\mathrm{e}^{\mathrm{i}tH}\, W^\pm(f)\, \mathrm{e}^{-\mathrm{i}tH} = W^\pm(\mathrm{e}^{\mathrm{i}th} f)$;
(5) If $H\Psi = E\Psi$, then $(\Psi | W^\pm(f) \Psi) = \mathrm{e}^{-\|f\|^2/4}\, \|\Psi\|^2$.

Note that the assumption (8.4) is very weak and it allows for $m = 0$.

Now we can follow the strategy developed in in Section 7. Using *asymptotic Weyl operators* $W^{\pm}(f)$ we introduce *asymptotic fields*

$$\phi^{\pm}(f) := \frac{\mathrm{d}}{\mathrm{i}\mathrm{d}t} W^{\pm}(tf)\Big|_{t=0}$$

and *asymptotic creation/annihilation operators*

$$a^{*\pm}(f) := \frac{1}{\sqrt{2}}(\phi(f) + \mathrm{i}\phi(\mathrm{i}f)),$$
$$a^{\pm}(f) := \frac{1}{\sqrt{2}}(\phi(f) - \mathrm{i}\phi(\mathrm{i}f)).$$

We also define the space of *asymptotic Fock vacua*:

$$\mathcal{K}_0^{\pm} := \left\{\Psi \ : \ (\Psi|W^{\pm}(f)\Psi) = \mathrm{e}^{-\|f\|^2/4}\,\|\Psi\|^2\right\} \tag{8.6}$$
$$= \left\{\Psi \ : \ a^{\pm}(f)\Psi = 0\right\}. \tag{8.7}$$

(Remember that (8.6) and (8.7) are equal to one another).

Here is a reformulation of Theorem 8.2, where we use the terminology introduced in Section 7:

Theorem 8.3: *Under the assumptions of Theorem 8.2 the following is true:*

(1) For $f \in \mathcal{Z}_1$ the limit (8.5) exists. Denote it by $W^{\pm}(f)$.
(2) $\mathcal{Z}_1 \ni f \mapsto W^{\pm}(f)$ are representations of the CCR.
(3) These representations are regular.
(4) $(W^{\pm}, h, H)$ are covariant.
(5) $\mathcal{H}_{\mathrm{p}}(H) \subset \mathcal{K}_0^{\pm}$, where $\mathcal{H}_{\mathrm{p}}(H)$ denotes the span of eigenvectors of H.

8.4. *Asymptotic dynamics*

Let us stress that so far in our scattering theory for Pauli Fierz amiltonians, the starting point was a single Hamiltonian H, and not a pair of Hamiltonians (H, H_0). In fact, a priori it is not clear which operator should play the role of the "free Hamiltonian", or better to say, the "asymptotic Hamiltonian". The operator H_0 of (8.2), obtained by dropping the interaction term, is in general not the right choice. In fact, typically, it even has a completely different spectrum than H. In this subsection we will describe how to introduce natural asymptotic Hamiltonians and to construct Møller operators.

First let us introduce the operator

$$K_0^{\pm} := H\Big|_{\mathcal{K}_0^{\pm}}.$$

It describes the energies of asymptotic vacua. (Under the assumptions of Theorem 8.4 below we can prove, and under more general conditions we expect, that the spectrum of $K_0^\pm$ coincides with the point spectrum of H).

Define

$$\mathcal{H}_{[0]}^\pm := \mathrm{Span}^{\mathrm{cl}} \left\{ W^\pm(f)\Psi \ : \ \Psi \in \mathcal{K}_0^\pm, \ f \in \mathcal{Z}_1 \right\}.$$

Clearly, $\mathcal{H}_{[0]}^\pm$ is the smallest space containing the asymptotic vacua and invariant wrt asymptotic creation operators. It is the largest space on which the asymptotic representations are Fock.

Define the *asymptotic Fock Hilbert space* $\mathcal{H}_0^{\pm\mathrm{as}} := \mathcal{K}_0^\pm \otimes \Gamma_{\mathrm{s}}(L^2(\mathbb{R}^d))$ and the *asymptotic Hamiltonian for the Fock sector*

$$H_0^{\pm\mathrm{as}} := K_0^\pm \otimes 1 + 1 \otimes \int h(\xi) a^*(\xi) a(\xi) \mathrm{d}\xi.$$

Note that there exist unitary operators

$$S_0^\pm : \mathcal{H}_0^{\pm\mathrm{as}} \to \mathcal{H}_{[0]}^\pm \subset \mathcal{H},$$

which we will call the *Møller operators for the Fock sector*, such that

$$\begin{aligned} &S_0^\pm \Psi \otimes a^*(f_1) \cdots a^*(f_n) \ \Omega \\ &= a^{*\pm}(f_1) \cdots a^{*\pm}(f_n) \ \Psi, \quad \Psi \in \mathcal{K}_0^\pm. \end{aligned}$$

The Møller operators intertwine the creation/annihilation operators and the Hamiltonian on the asymptotic space, and those on the physical space:

$$\begin{aligned} S_0^\pm 1 \otimes a^*(f) &= a^{*\pm}(f) S_0^\pm, \\ S_0^\pm 1 \otimes a(f) &= a^\pm(f) S_0^\pm, \\ S_0^\pm H_0^{\pm\mathrm{as}} &= H S_0^\pm. \end{aligned}$$

The *scattering operators for the Fock sector* is defined as

$$S_{00} = S_0^{+*} S_0^-.$$

It satisfies $S_{00} H_0^{-\mathrm{as}} = H_0^{+\mathrm{as}} S_{00}$. If $\mathcal{H}_{[0]}^+ = \mathcal{H}_{[0]}^-$, then S_{00} is unitary on $\mathcal{H}_0^{+\mathrm{as}} = \mathcal{H}_0^{-\mathrm{as}}$.

The operator S_{00} can be used to compute various physically interesting scattering cross-sections.

8.5. *Asymptotic completeness*

Theorem 8.2 is not difficult to prove. The following theorem is deeper, especially its second part.

Theorem 8.4: ***Asymptotic completeness for massive Pauli-Fierz Hamiltonians.*** *Assume that $m > 0$. Then*

(1) [21, 10, 11] $\mathcal{H}^{\pm}_{[0]} = \mathcal{H}$, *in other words, the asymptotic representations of the CCR are Fock.*
(2) [10] $\mathcal{K}^{\pm}_{0} = \mathcal{H}_{\mathrm{p}}(H)$, *in other words, all the asymptotic vacua are linear combinations of eigenvectors.*

In the proof of Theorem 8.4 an important role is played by the methods developed in the study of N-body scattering theory [9]. It is a rather satisfactory result except for one aspect: it assumes the positivity of the mass, which is not very physical. It would be very interesting to extend it to the case $m = 0$. Here is a possible conjecture [12]:
Conjecture: Asymptotic completeness for massless Pauli-Fierz Hamiltonians. *Assume that $h(\xi) = |\xi|$ and*

$$\int (1 + h(\xi)^{-2}) \|v(\xi)\|^2 \mathrm{d}\xi < \infty.$$

Then

(1) $\mathcal{H}^{\pm}_{[0]} = \mathcal{H}$,
(2) $\mathcal{K}^{\pm}_{0} = \mathcal{H}_{\mathrm{p}}(H)$.

Note that the above conjecture is true if $\dim \mathcal{K} = 1$ (i.e. for van Hove Hamiltonians). It is also true if $v(\xi) = 0$ for $|\xi| < \epsilon$, $\epsilon > 0$, (as remarked in [14]).

8.6. *Relaxation to the ground state*

Common wisdom says that a typical small system interacting with a reservoir at zero temperature will relax to its ground state. For a wide and generic class of Pauli-Fierz Hamiltonians this idea can be rigorously expressed and proven, and is essentially an easy corollary of their asymptotic completeness and spectral properties.

As we remarked before, one can often prove that Pauli-Fierz Hamiltonians have only absolutely continuous spectrum except for a unique ground

state $\Psi_{\rm gr}$ [2, 3]. If in addition asymptotic completeness holds [10, 14], then the asymptotic space is $\mathcal{H}_0^{\pm{\rm as}} = \Gamma_{\rm s}(\mathcal{Z})$.

Introduce the C^*-algebra

$$\mathfrak{A} := B(\mathcal{K}) \otimes {\rm CCR}(\mathcal{Z}) \subset B\left(\mathcal{K} \otimes \Gamma(\Gamma_{\rm s}(\mathcal{Z}))\right),$$

where ${\rm CCR}(\mathcal{Z}) = {\rm Span}^{\rm cl}\{W(f) \ : \ f \in \mathcal{Z}\}$. The following theorem comes from [22, 14]:

Theorem 8.5: ***Relaxation to the ground state.*** *Assume that H is a Pauli-Fierz Hamiltonian for which asymptotic completeness holds, and there are no eigenvectors except for a unique ground state $\Psi_{\rm gr}$. Let $A \in \mathfrak{A}$. Then*

$$\mathrm{w}\!-\lim_{|t|\to\infty} \mathrm{e}^{\mathrm{i}tH} A \, \mathrm{e}^{-\mathrm{i}tH} = (\Psi_{\rm gr}|A\Psi_{\rm gr}) \ \ 1_{\mathcal{H}}.$$

8.7. *Coherent asymptotic representations*

In the massless case asymptotic completeness does not always hold. In particular, the Fock property of asymptotic fields may be not true. To see this it is enough to consider the case of van Hove Hamiltonians; more complicated examples can be found in [12]. Nevertheless, following the formalism of Subsection 7.6 and [12], one can try to look for coherent asymptotic representations. This will allow us to study scattering amplitudes also in the case where the Fock property breaks down.

In fact, assume that g belongs to the dual of $\mathcal{Z}_1$. Then one can define the *subspaces of asymptotic g-coherent vectors*

$$\mathcal{K}_g^{\pm} := \{\Psi \in \mathcal{H} \ : \ (\Psi|W^{\pm}(f)\Psi) = \|\Psi\|^2 \, \mathrm{e}^{-\frac{1}{4}(f|f)+\mathrm{i}{\rm Re}(f|g)}\},$$

the *$[g]$-coherent sector*

$$\mathcal{H}_{[g]}^{\pm} := {\rm Span}^{\rm cl}\left\{W^{\pm}(f)\Psi \ : \ \Psi \in \mathcal{K}_g^{\pi}, \ f \in \mathcal{Z}\right\},$$

the *g-coherent asymptotic Hilbert space*

$$\mathcal{H}_g^{\pm{\rm as}} := \mathcal{K}_g^{\pm} \otimes \Gamma_{\rm s}(\mathcal{Z}^{\rm cpl}),$$

and the *g-coherent asymptotic Hamiltonians*

$$H_g^{\pm{\rm as}} := K_g^{\pm} \otimes 1 + 1 \otimes {\rm d}\Gamma(h).$$

The *Møller operators for the g-coherent sectors* $S_g^{\pm} : \mathcal{H}_g^{\pm \mathrm{as}} \to \mathcal{H}_{[g]}^{\pm} \subset \mathcal{H}$ intertwine creation/annihilation operators and the Hamiltonians:

$$S_g^{\pm} 1 \otimes a_g^*(f) = a^{*\pm}(f) S_g^{\pm},$$
$$S_g^{\pm} 1 \otimes a_g(f) = a^{\pm}(f) S_g^{\pm},$$
$$S_g^{\pm} H_g^{\pm \mathrm{as}} = H S_g^{\pm}.$$

There exists an alternative time-dependent definition of the Møller operator, which follows the pattern (2.10). Define the *g-coherent identifier* $J_g^{\pm} : \mathcal{H}_g^{\pm \mathrm{as}} \to \mathcal{H}$ by

$$J_g^{\pm} \ \Psi \otimes W_g(f)\Omega = 1 \otimes W(f) \ \Psi.$$

Then we can introduce the Møller operators using this identifier:

$$S_g^{\pm} = \mathrm{s}- \lim_{t \to \pm\infty} \mathrm{e}^{\mathrm{i}tH} \, J_g^{\pm} \, \mathrm{e}^{-\mathrm{i}tH_g^{\pm \mathrm{as}}}.$$

Let g_1, g_2 belong to the dual of $\mathcal{Z}_1$. Then one can define the *scattering operator between the sectors corresponding to* g_1 *and* g_2:

$$S_{g_2,g_1} := S_{g_2}^{+*} S_{g_1}^{-}.$$

This operator can be used to define and compute scattering crossections even if asymptotic fields have no Fock vacua.

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