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Scattering at Zero Energy for Attractive Homogeneous Potentials

Jan Dereziński and Erik Skibsted

Abstract. We compute up to a compact term the zero-energy scattering matrix for a class of potentials asymptotically behaving as $-\gamma|x|^{-\mu}$ with $0 < \mu < 2$ and $\gamma > 0$. It turns out to be the propagator for the wave equation on the sphere at time $\frac{\mu\pi}{2-\mu}$.

1. Introduction and results

The paper is devoted to a study of the zero-energy scattering matrix $S(0)$ for a class of radial potentials on $\mathbb{R}^d$ with $d \geq 2$. This class consists of the potentials of the form $V(x) := -\gamma|x|^{-\mu} + W(|x|)$, where $0 < \mu < 2$ and $\gamma > 0$ and $W(r)$ is a fast decaying perturbation. We will show that the leading term of $S(0)$ can be computed and is an interesting Fourier integral operator.

This paper can be considered as a companion to a series of papers [4–6], where the low-energy scattering theory has been developed for a somewhat more general class of potentials. Note however, that this paper can be read independently of [4–6].

Before stating our main result, which deals with quantum scattering, let us say a few words about its classical analog. Consider the equations of motion in a strictly homogeneous potential $V(r) = -\gamma r^{-\mu}$. It turns out that this problem is exactly solvable at zero energy. The (non-collision) zero-energy orbits are given by the implicit equation (in polar coordinates)

$$\sin\left(1 - \frac{\mu}{2}\right)\theta(t) = \left(\frac{r(t)}{r_{\mathrm{tp}}}\right)^{-1+\frac{\mu}{2}}, \tag{1.1}$$

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see [4, Example 4.3]. Whence the deflection angle of such trajectories equals $-\frac{\mu\pi}{2-\mu}$. In particular, for attractive Coulomb potentials it equals $-\pi$, which corresponds to the well-known fact that in this case zero-energy orbits are parabolas (see [15, p. 126] for example).

One can ask whether a similar behavior can be seen at the quantum level. Our analysis shows that indeed this is the case.

Our main result is stated in terms of the unitary group $\mathrm{e}^{\mathrm{i}\theta\Lambda}$ generated by a certain self-adjoint operator Λ on $L^2(S^{d-1})$. The operator Λ is defined by setting $\Lambda Y = (l + d/2 - 1)Y$ if Y is a spherical harmonic of order l. $\mathrm{e}^{\mathrm{i}\theta\Lambda}$ can be called the *propagator for the wave equation on the sphere*. Note that for any θ, the distributional kernel of $\mathrm{e}^{\mathrm{i}\theta\Lambda}$ can be computed explicitly and its singularities appear at $\omega \cdot \omega' = \cos\theta$. This is expressed in the following fact [16]:

Proposition 1.1. $\mathrm{e}^{\mathrm{i}\theta\Lambda}$ *equals*

1. $c_\theta I$*, where* I *is the identity, if* $\theta \in \pi 2\mathbb{Z}$*;*
2. $c_\theta P$*, where* P *is the parity operator (given by* $\tau(\omega) \mapsto \tau(-\omega)$*), if* $\theta \in \pi(2\mathbb{Z}+1)$*;*
3. *the operator whose Schwartz kernel is of the form* $c_\theta(\omega \cdot \omega' - \cos\theta + \mathrm{i}0)^{-\frac{d}{2}}$ *if* $\theta \in]\pi 2k, \pi(2k+1)[$ *for some* $k \in \mathbb{Z}$*;*
4. *the operator whose Schwartz kernel is of the form* $c_\theta(\omega \cdot \omega' - \cos\theta - \mathrm{i}0)^{-\frac{d}{2}}$ *if* $\theta \in]\pi(2k-1), \pi 2k[$ *for some* $k \in \mathbb{Z}$*.*

We also remark that for all θ, the operator $\mathrm{e}^{\mathrm{i}\theta\Lambda}$ belongs to the class of Fourier integral operators of order 0 in the sense of Hörmander [11, 12].

Let us now briefly recall some points of the time-dependent scattering theory for Schrödinger operators. Set $H_0 := -\frac{1}{2}\Delta$ and $H = H_0 + V(x)$. If the potential $V(x)$ is short-range, following the standard formalism, we can define the usual scattering operator. In the long-range case the usual formalism does not apply. Nevertheless, one can use one of the modified formalisms, which leads to a modified scattering operator S.

Clearly, H_0 commutes with S, and hence S can be written in terms of the direct integral

$$S \simeq \int_{]0,\infty[} \oplus S(\lambda)\mathrm{d}\lambda\,. \tag{1.2}$$

The operators $S(\lambda)$, called scattering matrices, are defined up to a set of measure zero. In the short-range case, one can chose $S(\lambda)$ to be continuous for $\lambda > 0$, which fixes the value of positive energy scattering matrices uniquely. In the long-range case we need to use modified scattering operators, in whose definition there is a freedom of choosing an arbitrary phase factor depending on λ. One can however also choose S so that $S(\lambda)$ is continuous for $\lambda > 0$, which fixes $S(\lambda)$ up to a phase factor.

Actually, we will use the modified formalism also in the short-range case. To us a "scattering matrix" will always mean a "modified scattering matrix", defined up to a phase factor, also in the short-range case.

The above description of scattering theory applies to a rather general class of potentials. Under more restrictive assumptions described in [5] one can show that the modified scattering operator can be chosen so that $S(\lambda)$ is continuous down to $\lambda = 0$, which allows us to define the zero-energy scattering matrix $S(0)$. This is one of the main results of [5].

One can compute the scattering matrix in terms of asymptotics of generalized eigenfunctions. If the potential is radial this is particularly convenient. Explicitly the asymptotics of the regular solution of the stationary Schrödinger equation for the energy λ and the angular momentum sector l determines the *scattering phase shift*, denoted $\sigma_l(\lambda)$. Again, the usual definition of the scattering phase shift applies only to the short-range case. In the long-range case one needs to introduce a modified scattering phase shift. Our construction differs from these by a trivial term (i.e. an l-independent term), cf. [5, Theorems 7.3 and 7.4].

Suppose that $]0,\infty[\ni r \mapsto V(r)$ is a continuous real function such that for some positive constants ϵ, κ and C

$$\left|V(r) + \gamma r^{-\mu}\right| \le C r^{-1-\frac{\mu}{2}-\epsilon}\,, \quad r > 1\,; \tag{1.3}$$

$$|V(r)| \le C r^{-2+\kappa}\,, \quad r \le 1\,. \tag{1.4}$$

With these assumptions, which will be the main assumptions used in this paper, one can show that our (appropriately modified) phase shift $\sigma_l(\lambda)$ is continuous in λ down to $\lambda = 0$.

We have the following relationship between the phase shift and the scattering matrix:

$$S(\lambda)Y = \mathrm{e}^{\mathrm{i}2\sigma_l(\lambda)}Y\,, \tag{1.5}$$

where Y is any spherical harmonic Y of order l. For positive energies (1.5) is a well-known identity valid under rather general assumptions. For a (partial) justification under the above conditions we refer to [5]. However let us stress that in this paper we can (and will) avoid time-dependent formalisms completely, and in fact we take (1.5) as the definition of $S(\lambda)$, in particular for the limiting case $\lambda = 0$. Whence we define the (modified) scattering matrix through the (modified) phase shift.

Here is the main result of our paper:

Theorem 1.2. *Assume the conditions* (1.3) *and* (1.4) *on the potential* $V(r)$*. Then, for some* $c \in \mathbb{R}$ *and a compact operator* K *on* $L^2(S^{d-1})$*, we have*

$$S(0) = \mathrm{e}^{\mathrm{i}c}\mathrm{e}^{-\mathrm{i}\frac{\mu\pi}{2-\mu}\Lambda} + K\,.$$

We prove Theorem 1.2 by a careful one-dimensional WKB-analysis, *simultaneously in each angular momentum sector.* Therefore our results do not follow easily from the literature on 1-dimensional Schrödinger operators that we know.

Consider the potential $V(r)$ equal *exactly* to $-\gamma|x|^{-\mu}$, and the corresponding Hamiltonian $H_\mu := H_0 - \gamma r^{-\mu}$. It is not difficult to show that H_μ is an analytic family of operators for $\operatorname{Re}\mu \in]0,2[$. In the preprint version of our paper we formulated a conjecture that in the case of H_μ

$$S(0) = \mathrm{e}^{\mathrm{i}c}\mathrm{e}^{-\mathrm{i}\frac{\mu\pi}{2-\mu}\Lambda}\,,$$

without the compact error term, or alternatively, that the terms $o(l^0)$ in Proposition 3.2 vanish identically. A special case of this conjecture is the formula $S(0) = \mathrm{e}^{\mathrm{i}c}P$ in the attractive Coulomb case, which has been known for a long time, see [20]. (Here, $(P\tau)(\omega) = \tau(-\omega)$). Recently, the above conjecture has been proven by R. Frank, see [7].

2. Propagator of the wave equation on the sphere

2.1. Distributional kernel of the propagator

For any $1 \le i < j \le d$, define the corresponding angular momentum operator

$$L_{ij} := -\mathrm{i}(x_i\partial_{x_j} - x_j\partial_{x_i})\,.$$

Set

$$L^2 := \sum_{1\le i<j\le d} L_{ij}^2\,, \quad \Lambda := \sqrt{L^2 + (d/2-1)^2}\,.$$

Note that Λ is a self-adjoint operator on $L^2(S^{d-1})$ and its eigenfunctions with eigenvalue $l + d/2 - 1$ are the lth order spherical harmonics for $l = 0, 1, \dots$.

For any θ one can compute exactly the integral kernel of $\mathrm{e}^{\mathrm{i}\theta\Lambda}$. Although the result already appears in the literature, see [16, Chapter 4, (2.13)], we shall for the readers convenience give its complete derivation (this proof is different from Taylor's). Note that the operator appears naturally when we solve the wave equation on the sphere, therefore we call it the propagator of the wave equation on the sphere.

First we need to introduce some notation about distributions. For any $\epsilon > 0$ and $s \in \mathbb{R}$, the expression

$$\mathbb{R} \ni y \mapsto (y \pm \mathrm{i}\epsilon)^{-\frac{s}{2}}$$

defines uniquely a function on a real line, which can be viewed as a distribution in $\mathcal{S}'(\mathbb{R})$. It is well-known that for any $\phi \in \mathcal{S}(\mathbb{R})$ there exists a limit

$$\lim_{\epsilon\searrow 0}\int (y \pm \mathrm{i}\epsilon)^{-\frac{s}{2}}\phi(y)\mathrm{d}y =: \int (y \pm \mathrm{i}0)^{-\frac{s}{2}}\phi(y)\mathrm{d}y\,,$$

which defines a distribution in $\mathcal{S}'(\mathbb{R})$. In the sequel we will treat this distribution as if it were a function, denoting it by $(y \pm \mathrm{i}0)^{-\frac{s}{2}}$. Note that for $s, \epsilon > 0$ we have the identity

$$(y \pm \mathrm{i}\epsilon)^{-\frac{s}{2}} = \frac{\mathrm{e}^{\mp\mathrm{i}\pi\frac{s}{4}}}{\Gamma(s/2)}\int_0^\infty \mathrm{e}^{\mathrm{i}t(\pm y+\mathrm{i}\epsilon)}t^{\frac{s-2}{2}}\,\mathrm{d}t\,. \tag{2.1}$$

We shall in this section show the following result:

Proposition 2.1.

1. *If $\theta = \pi 2k$, $k \in \mathbb{Z}$, then $\mathrm{e}^{\mathrm{i}\theta\Lambda} = (-1)^{kd}$ times the identity.*
2. *If $\theta = \pi(2k+1)$, $k \in \mathbb{Z}$, then $\mathrm{e}^{\mathrm{i}\theta\Lambda} = \mathrm{e}^{\mathrm{i}\pi(2k+1)(d/2-1)}P$, where P is the parity operator.*

3. *If* $\theta \in]\pi 2k, \pi(2k+1)[$, $k \in \mathbb{Z}$, *then* $\mathrm{e}^{\mathrm{i}\theta\Lambda}$ *has the distributional kernel*

$$\mathrm{e}^{\mathrm{i}\theta\Lambda}(\omega,\omega') = (2\pi)^{-d/2}\sin\theta\,\Gamma(d/2)\mathrm{e}^{-\mathrm{i}\pi/2}(-\omega\cdot\omega' + \cos\theta - \mathrm{i}0)^{-d/2}.$$

4. *If* $\theta \in]\pi(2k-1), \pi 2k[$, $k \in \mathbb{Z}$, *then* $\mathrm{e}^{\mathrm{i}\theta\Lambda}$ *has the distributional kernel*

$$\mathrm{e}^{\mathrm{i}\theta\Lambda}(\omega,\omega') = (2\pi)^{-d/2}\sin\theta\,\Gamma(d/2)\mathrm{e}^{-\mathrm{i}\pi/2}(-\omega\cdot\omega' + \cos\theta + \mathrm{i}0)^{-d/2}\,.$$

2.2. Tchebyshev and Gegenbauer polynomials

Recall that the Tchebyshev polynomials (of the first kind) are defined by the identity

$$T_n(\cos\phi) := \cos n\phi\,, \quad n = 0, 1, \dots\,.$$

Let $|t| < 1$.

An elementary calculation yields the following generating function of Tchebyshev polynomials:

$$-\ln(1 - 2wt + t^2) = \sum_{l=1}^{\infty}\frac{2t^l}{l}T_l(w)\,. \tag{2.2}$$

Gegenbauer polynomials are defined by the generating function [1, 14]

$$\frac{1}{(1-2wt+t^2)^{(d-2)/2}} = \sum_{l=0}^{\infty} t^l C_l^{(d-2)/2}(w)\,. \tag{2.3}$$

The left hand sides of (2.2) and (2.3) look different. But after simple manipulations (involving differentiation of both sides) they become quite similar:

$$\frac{-t+t^{-1}}{(t-2w+t^{-1})^{\frac{d}{2}}} = \begin{cases} T_0(w) + \sum_{l=1}^{\infty} t^l 2T_l(w)\,, & d = 2\,; \\ \sum_{l=0}^{\infty} t^{l+\frac{d}{2}-1}\frac{2l+d-2}{d-2}C_l^{(d-2)/2}(w)\,, & d \geq 3\,. \end{cases} \tag{2.4}$$

By substituting $t = \mathrm{e}^{\mathrm{i}\theta}$ for $\operatorname{Im}\theta > 0$, we rewrite this as

$$\frac{-\mathrm{i}2\sin\theta}{2^{d/2}(\cos\theta - w)^{\frac{d}{2}}} = \begin{cases} T_0(w) + \sum_{l=1}^{\infty} \mathrm{e}^{\mathrm{i}l\theta} 2T_l(w)\,, & d = 2\,; \\ \sum_{l=0}^{\infty} \mathrm{e}^{\mathrm{i}(l+\frac{d}{2}-1)\theta}\frac{2l+d-2}{d-2}C_l^{(d-2)/2}(w)\,, & d \geq 3\,. \end{cases} \tag{2.5}$$

2.3. Projection onto lth sector of spherical harmonics

It is well-known that the integral kernel of the projection onto lth sector of spherical harmonics in $L^2(S^{d-1})$ can be computed explicitly. This fact is usually presented in the literature as the addition theorem for spherical harmonics, see e.g. Theorem 2, Sect. 2 of [14]. In the case $d = 3$ it can also be found in [17].

In what follows $\mathrm{d}\hat{y}$ will denote the natural measure on the unit sphere S^{d-1}. Note that for this measure the area of S^{d-1} equals $s_{d-1} = \frac{2\pi^{d/2}}{\Gamma(d/2)}$.

Proposition 2.2. *Let* Y *be an* l*th order spherical harmonic in* $L^2(S^{d-1})$.

1. *In the case* $d = 2$,

$$\int_{S^1} \frac{1}{2\pi} T_0(\hat{x} \cdot \hat{y}) Y(\hat{y}) \mathrm{d}\hat{y} = \delta_{l0} Y(\hat{x}) \,; \tag{2.6}$$
$$\int_{S^1} \frac{1}{\pi} T_n(\hat{x} \cdot \hat{y}) Y(\hat{y}) \mathrm{d}\hat{y} = \delta_{ln} Y(\hat{x}) \,, \quad n = 1, 2, \dots \,.$$

2. *In the case* $d \geq 3$,

$$\int_{S^{d-1}} \frac{(d-2+2l)\Gamma(d/2-1)}{4\pi^{d/2}} C_n^{(d-2)/2}(\hat{x} \cdot \hat{y}) Y(\hat{y}) \mathrm{d}\hat{y} = \delta_{ln} Y(\hat{x}) \,. \tag{2.7}$$

Proof. The case (2.6) is elementary. In the proof below we restrict ourselves to $d \geq 3$.

Let us first recall the formula for the Green's function in $\mathbb{R}^d$ for $d \geq 3$:

$$G_d(x) = -\frac{\Gamma(d/2-1)}{4\pi^{d/2}|x|^{d-2}} = -\frac{1}{s_{d-1}(d-2)|x|^{d-2}} \,, \tag{2.8}$$

It satisfies

$$\Delta G_d = \delta_0 \,,$$

where δ_0 is Dirac's delta at zero. Recall also the 3rd Green's identity: if $\Delta g = 0$ and Ω is a sufficiently regular domain containing x, then

$$g(x) = \int_{\partial\Omega} g(y) \nabla_y G_d(x-y) \mathrm{d}\vec{s}(y) - \int_{\partial\Omega} (\nabla g)(y) G_d(x-y) \mathrm{d}\vec{s}(y) \,. \tag{2.9}$$

We extend Y to $\mathbb{R}^d$ by setting $g(x) = |x|^l Y(\hat{x})$. Note that

$$\Delta g(x) = 0 \,, \quad \hat{x} \nabla_x g(x) = l g(x) \,.$$

By (2.3), for $|x| < |y|$,

$$G_d(x-y) = -\frac{\Gamma(d/2-1)}{4\pi^{d/2}} \sum_{n=0}^{\infty} C_n^{(d-2)/2}(\hat{x}\hat{y}) |x|^n |y|^{-d+2-n} \,,$$
$$\hat{y} \cdot \nabla_y G_d(x-y) = \frac{\Gamma(d/2-1)}{4\pi^{d/2}} \sum_{n=0}^{\infty} (d-2+n) C_n^{(d-2)/2}(\hat{x}\hat{y}) |x|^n |y|^{-d+1-n} \,.$$

We apply (2.9) to the unit ball, so that $|y| = 1$ and $|x| < 1$:

$$\begin{aligned} |x|^l Y(\hat{x}) &= \int_{S^{d-1}} g(\hat{y}) \hat{y} \cdot \nabla G_d(x-\hat{y}) \mathrm{d}\hat{y} - \int_{S^{d-1}} (\hat{y} \cdot \nabla g)(\hat{y}) G_d(x-\hat{y}) \mathrm{d}\hat{y} \\ &= \frac{\Gamma(d/2-1)}{4\pi^{d/2}} \sum_{n=0}^{\infty} (d-2+n+l) \int_{S^{d-1}} Y(\hat{y}) C_n^{(d-2)/2}(\hat{x}\hat{y}) |x|^n \mathrm{d}\hat{y} \,. \end{aligned} \tag{2.10}$$

Comparing the powers of $|x|$ on both sides of (2.10), we obtain (2.7). □

2.4. Proof of Proposition 2.1

Let Q_l^{d-1} be the orthogonal projection onto lth order spherical harmonics on S^{d-1}. We multiply (2.5) by $\Gamma(d/2)2^{-1}\pi^{-d/2}$, set $w = \omega\cdot\omega'$ and use Proposition 2.2. We obtain

$$\begin{aligned}\frac{-\mathrm{i}\sin\theta\,\Gamma(d/2)}{(2\pi)^{d/2}(\cos\theta-\omega\cdot\omega')^{d/2}} &= \sum_{l=0}^{\infty} Q_l^{d-1}(\omega,\omega')\mathrm{e}^{\mathrm{i}(l+d/2-1)\theta}\\ &= \mathrm{e}^{\mathrm{i}\theta\Lambda}(\omega,\omega')\,.\end{aligned}$$

Replace θ with $\theta+\mathrm{i}\epsilon$, where θ is real and ϵ positive. For small ϵ we have

$$\cos(\theta+\mathrm{i}\epsilon)\approx\cos\theta-\mathrm{i}\sin\theta\epsilon\,.$$

Now $\sin\theta>0$ for $\theta\in]\pi 2k,\pi(2k+1)[$ and $\sin\theta<0$ for $\theta\in]\pi(2k-1),\pi 2k[$, which ends the proof for the case $\theta\in\mathbb{R}\setminus\pi\mathbb{Z}$.

The case $\theta\in\pi\mathbb{Z}$ is obvious.

2.5. Propagator as a FIO

The operator $\mathrm{e}^{\mathrm{i}\theta\Lambda}$ is an interesting explicit example of a Fourier integral operator (whenceforth abbreviated FIO) in the sense of Hörmander [11, 12]. As a side remark, let us check this directly. (The material of this subsection will not be used in what follows.)

Let X be a smooth compact manifold of dimension n. Let us recall some basic definitions related to Fourier integral operators on X, cf. [12].

We say that $X\times X\times\mathbb{R}^k\ni(x,x',\theta)\mapsto\phi(x,x',\theta)\in\mathbb{R}$ is a non-degenerate phase function if it is a function homogeneous of degree 1 in θ, smooth and satisfying $\nabla\phi\neq 0$ away from $\theta=0$, and such that

$$\big\{(x,x',\theta)\in X\times X\times\mathbb{R}^k\mid\nabla_\theta\phi(x,x',\theta)=0\big\}$$

is a smooth manifold on which $\nabla\nabla_{\theta_1}\phi,\dots,\nabla\nabla_{\theta_k}\phi$ are linearly independent.

Let χ be a smooth and homogeneous transformation on $\mathbb{T}^*X\setminus X{\times}\{0\}$. We say that it is associated to a non-degenerate phase function ϕ iff two pairs $(x,\xi),(x',\xi')\in\mathbb{T}^*X\setminus X{\times}\{0\}$ satisfy $\chi(x',\xi'):=(x,\xi)$ exactly when

$$\begin{aligned}\xi &= \nabla_x\phi(x,x',\theta)\ ,\\ \xi' &= -\nabla_{x'}\phi(x,x',\theta)\ ,\\ 0 &= \nabla_\theta\phi(x,x',\theta)\ .\end{aligned}\tag{2.11}$$

The transformation χ is automatically canonical, that is, it preserves the symplectic form of $\mathbb{T}^*X$.

We say that a smooth function $X\times X\times\mathbb{R}^k\ni(x,x',\theta)\mapsto u(x,x',\theta)$ is an amplitude of order m iff

$$\partial_x^\alpha\partial_{x'}^{\alpha'}\partial_\theta^\beta u = O\big(\langle\theta\rangle^{m-|\beta|}\big)\,.$$

Recall from [12] that an operator U from $C^\infty(X)$ to $\mathcal{D}'(X)$ is called a Fourier integral operator of order

$$m - \frac{n}{2} + \frac{k}{2}$$

iff in local coordinate patches its distributional kernel can be written as

$$U(x,x') = \int \mathrm{e}^{\mathrm{i}\phi(x,x'\theta)} u(x,x',\theta)\,\mathrm{d}\theta\,, \tag{2.12}$$

where $\theta \in \mathbb{R}^k$ are auxiliary variables, the function ϕ is a non-degenerate phase function, and u is an amplitude of order m.

If the phase of U is associated to a canonical transformation χ, we say that U itself is associated to χ.

Let $WF(v)$ denote the wave front set of a distribution v, as defined in [12, Section 2.5]). Let us remark that under appropriate additional assumptions on a FIO U, for all $v \in \mathcal{D}'(X)$ we have

$$WF(Uv) \subseteq \chi\big(WF(v)\big)\,;$$

see [12, Proposition 2.5.7 and Theorem 2.5.14]. (Note that these additional assumptions are fulfilled for the example $U = U_\theta$ given below.)

Theorem 2.3. *The operator* $U_\theta := \mathrm{e}^{\mathrm{i}\theta\Lambda}$ *is a FIO of order* 0.

Proof. If $\theta \in \pi\mathbb{Z}$, then $\mathrm{e}^{\mathrm{i}\theta\Lambda}$ is a so-called point transformation. But point transformations given by diffeomorphisms of the underlying manifold are always FIO of order zero.

Assume that $\theta \notin \pi\mathbb{Z}$. Consider e.g. the case $\theta \in]\pi 2k, \pi(2k+1)[$. By (2.1) and Proposition 2.1 the kernel of U_θ can then be written as

$$U_\theta(\omega,\omega') = C\int_0^\infty \mathrm{e}^{\mathrm{i}t(\omega\,\cdot\,\omega' - \cos\theta)} t^{\frac{d-2}{2}}\,\mathrm{d}t\,. \tag{2.13}$$

If we compare (2.13) with the definition of a FIO given above, we see that $t(\omega\cdot\omega' - \cos\theta)$ is a non-degenerate phase function. We also have $n = d-1$, $m = \frac{d-2}{2}$ and $k = 1$. Thus U_θ is a FIO of order

$$\frac{d-2}{2} - \frac{d-1}{2} + \frac{1}{2} = 0\,. \qquad \square$$

Let us describe the canonical transformation associated to the FIO U_θ. Let $(\omega,\xi) \in \mathbb{T}^*(S^{d-1})$. It is enough to assume that $|\xi| = 1$. Then the canonical transformation χ_θ associated to U_θ is given by $\chi_\theta(\omega',\xi') = (\omega,\xi)$, where

$$\begin{aligned} \omega &= \omega'\cos\theta - \xi'\sin\theta\,, \\ \xi &= \omega'\sin\theta + \xi'\cos\theta\,. \end{aligned}$$

3. Main result

3.1. Scattering matrix at positive energies

Throughout the paper we fix $\mu \in]0,2[$ and $\gamma > 0$, and impose the conditions (1.3) and (1.4) on the potential $V(r)$. It will be convenient to fix $R_0 > 0$ such that $V(r) < 0$ for $r > R_0$.

Since our potential is radial, to define the scattering operator we can use the scattering phase shift formalism. This formalism is, at least under some mild additional conditions on the potential, equivalent to the usual time-dependent formalism of scattering theory, see [5] for an elaboration.

In the paper we will need just the scattering matrix at zero energy. Let us however start with defining the scattering matrix at a positive energy.

Let $l \in \mathbb{N} \cup \{0\}$ have the meaning of a total angular momentum and $\lambda > 0$ be the energy. Introduce the notation

$$V_l(r) = 2V(r) + \frac{(l + \frac{d}{2} - 1)^2 - 4^{-1}}{r^2}\,.$$

Consider the reduced Schrödinger equation on the half-line $]0, \infty[$ for energy λ:

$$-u'' + V_l u = 2\lambda u\,. \tag{3.1}$$

One can show that all real solutions of (3.1) satisfy

$$\lim_{r\to\infty} \left((\lambda + \gamma r^{-\mu})^{\frac{1}{4}} u(r) - C \sin\left(\int_{R_0}^{r} (2\lambda + 2\gamma \tilde{r}^{-\mu})^{\frac{1}{2}} \mathrm{d}\tilde{r} + D \right) \right) = 0 \tag{3.2}$$

for some $C > 0$ and $D \in \mathbb{R}$. The regular solution is the solution satisfying

$$\lim_{r\to 0} r^{-l-\frac{d-1}{2}} u(r) = 1\ . \tag{3.3}$$

(The existence and uniqueness of the regular solution is usually proven by studying an integral equation of Volterra type, cf. [15].) Now the phase shift at energy λ is defined in terms of the constant D for the regular solution by

$$\begin{aligned} &\sigma_l(\lambda) \\ &= D + \sqrt{2} \int_{R_0}^{\infty} \left(\sqrt{\lambda + \gamma r^{-\mu}} - \sqrt{\lambda - V(r)} \right) \mathrm{d}r - \sqrt{2\lambda R_0} + \frac{d - 3 + 2l}{4}\pi\,. \end{aligned} \tag{3.4}$$

We define the (modified) scattering matrix at energy λ as the unitary operator on $L^2(S^{d-1})$ that on lth order spherical harmonics Y acts as

$$S(\lambda)Y = \mathrm{e}^{\mathrm{i}2\sigma_l(\lambda)} Y\,.$$

Note that the above definition is adapted to the long-range case. However, we use it also in the short-range case, because it makes possible to take the limit as $\lambda \searrow 0$. $\sigma_l(\lambda)$ defined above is also consistent with the convention adopted in [5].

For comparison, let us mention the standard definition of the phase shift in the short-range case. (3.2) needs to be replaced with

$$\lim_{r\to\infty} \Big(u(r) - C \sin\big((2\lambda)^{\frac{1}{2}} r + D_{\mathrm{sr}} \big) \Big) = 0\,, \tag{3.5}$$

and (3.4) with

$$\sigma_l^{\rm sr}(\lambda) = D_{\rm sr} + \frac{d-3+2l}{4}\pi \,. \tag{3.6}$$

In particular under (1.3) and (1.4) with $\mu \in]1,2[$

$$\sigma_l^{\rm sr}(\lambda) = \sigma_l(\lambda) + \sqrt{2}\int_{R_0}^{\infty} \Big(\sqrt{\lambda} - \sqrt{\lambda - V(r)}\Big)\,{\rm d}r \,. \tag{3.7}$$

The integral to the right in (3.7) does not have a (finite) limit as $\lambda \searrow 0$ in this case.

3.2. Scattering matrix at zero energy

It turns out that under the conditions (1.3) and (1.4) the definition of the scattering matrix can be extended to zero energy.

The definitions of $\sigma_l(0)$ and $S(0)$ are special cases of the definitions of $\sigma(\lambda)$ and $S(\lambda)$ for $\lambda > 0$ described in Subsection 3.1.

Explicitly, consider the zero-energy case of (3.1)

$$-u'' + V_l u = 0 \,. \tag{3.8}$$

It follows from the WKB-analysis given in the bulk of Subsection 3.3 that all real solutions of (3.8) satisfy

$$\lim_{r\to\infty} \left((\gamma r^{-\mu})^{\frac14} u(r) - C \sin\left(\int_{R_0}^{r} (2\gamma \tilde r^{-\mu})^{\frac12}\,{\rm d}\tilde r + D \right)\right) = 0 \tag{3.9}$$

for some $C > 0$ and $D \in \mathbb{R}$. Consider D corresponding to the regular solution which is fixed by the requirement (3.3). Now we define the (modified) zero-energy phase shift as

$$\sigma_l(0) = D + \int_{R_0}^{\infty} \Big(\sqrt{2\gamma r^{-\mu}} - \sqrt{-2V(r)}\Big)\,{\rm d}r + \frac{d-3+2l}{4}\pi \,. \tag{3.10}$$

We define the (modified) scattering matrix at energy 0 by

$$S(0)Y = {\rm e}^{{\rm i}2\sigma_l(0)}Y \,,$$

where Y is any lth order spherical harmonic.

Note that it follows from the proof given in the bulk of this section (as well as from [5]) that

$$\sigma_l(0) = \lim_{\lambda\searrow 0} \sigma_l(\lambda)\,,$$
$$S(0) = {\rm s-}\lim_{\lambda\searrow 0} S(\lambda)\,.$$

The following theorem is the main result of the paper:

Theorem 3.1. *For a certain compact operator K on $L^2(S^{d-1})$, we have*

$$S(0) = {\rm e}^{{\rm i}c_0}{\rm e}^{-{\rm i}\frac{\mu\pi}{2-\mu}\Lambda} + K \,,$$

where

$$c_0 = \frac{4\sqrt{2\gamma}}{2-\mu} R_0^{1-\frac{\mu}{2}} + 2\int_{R_0}^{\infty} \Big(\sqrt{2\gamma r^{-\mu}} - \sqrt{-2V(r)}\Big)\,{\rm d}r \,.$$

3.3. One-dimensional WKB-analysis

We shall show the following asymptotics:

Proposition 3.2. *The phase shift obeys*

$$\sigma_l(0) = -\frac{\mu\pi}{2(2-\mu)}l + \frac{c}{2} + o(l^0)\,, \tag{3.11}$$
$$\frac{c}{2} = -\frac{\pi\mu(d-2)}{4(2-\mu)} + \frac{2\sqrt{2\gamma}}{2-\mu}R_0^{1-\frac{\mu}{2}} + \int_{R_0}^{\infty}\left(\sqrt{2\gamma r^{-\mu}} - \sqrt{-2V(r)}\right)\mathrm{d}r\,.$$

Clearly Theorem 3.1 is a consequence of Proposition 3.2.

This subsection is devoted to the main part of the proof of Proposition 3.2. It is based on detailed 1-dimensional analysis.

For convenience, let us note that the effective potential V_l of (3.8) for the case $V(r) = -\gamma r^{-\mu}$ is given by

$$V_l(r) = -2\gamma r^{-\mu} + \frac{k(k+1)}{r^2}\,, \quad k := l + \frac{d-3}{2}\,.$$

Abusing slightly notation, we shall henceforth denote this expression by V_k, and similarly $\sigma_k(0) := \sigma_l(0)$. Note that now we have $V_0(r) = -2\gamma r^{-\mu}$.

In the case $V = -\gamma r^{-\mu}$, there is for $k > 0$ a unique zero, say, denoted r_0, of the effective potential V_k. Explicitly,

$$V_k(r_0) = 0 \quad \text{for} \quad r_0 = \left(\frac{k(k+1)}{2\gamma}\right)^{\frac{1}{2-\mu}}\,. \tag{3.12}$$

For later applications, let us notice that

$$V_k'(r_0) = -(2-\mu)\frac{k(k+1)}{r_0^3}\,. \tag{3.13}$$

Clearly V_k is positive to the left of r_0 and negative to the right of r_0.

Proposition 3.3. *The regular solution satisfies* (*up to multiplication by a positive constant*)

$$u(r) = (-V_k)^{-\frac{1}{4}}(r)\left(\sin\left(\int_{r_0}^{r}\sqrt{-V_k(\tilde r)}\,\mathrm{d}\tilde r + \frac{\pi}{4} + o(k^0)\right) + O(r^{-\epsilon_k})\right), \tag{3.14}$$

where $o(k^0)$ signifies a vanishing term that is independent of r and $\epsilon_k > 0$.

3.3.1. Scheme of proof of Proposition 3.3. We shall first concentrate on the case where $V = -\gamma r^{-\mu}$; the general case will be treated by the same scheme (to be discussed later).

We introduce a partition of $]0,\infty[$ into four subintervals given as follows in terms of $\epsilon_1, \epsilon_2, \epsilon_3 \in (0,1]$ to be fixed later:

1. $I_1 =]0, r_1],\ r_1 = r_0 k^{-\frac{\epsilon_1}{2-\mu}}$.
2. $I_2 =]r_1, r_2],\ r_2 = r_0(1 - k^{-\epsilon_2})$.
3. $I_3 =]r_2, r_3],\ r_3 = r_0(1 + k^{-\epsilon_3})$.
4. $I_4 =]r_3, \infty[$.

In each of the intervals I_j where $j = 2, 3$ or 4, we shall specify a certain model Schrödinger equation together with its two linearly independent solutions $\phi_j^{\pm}$. In terms of these, we can construct exact solutions to the reduced equation

$$-u'' + V_k u = 0 \tag{3.15}$$

by the method of variation of parameters, cf. for example [9]. Our subject of study is formulas for the regular solution $u = u_k$. Specifically, in the interval I_1 we shall use a comparison argument to get estimates of the regular solution at $r = r_1$. Then we shall use a connection formula to get estimates of the "coefficients" a_2^+ and a_2^- of the ansatz

$$u = a_j^+ \phi_j^+ + a_j^- \phi_j^- \tag{3.16}$$

with $j = 2$ at the same point $r = r_1$. Next, using the differential equation for a_2^+ and a_2^- we shall derive estimates of these quantities at $r = r_2$. Proceeding similarly we shall consecutively represent u by (3.16) on I_3 and I_4 using connection formulas at r_2 and r_3, and eventually get estimates in the interval I_4, and whence derive the relevant asymptotics of u.

Suppose ϕ^- and ϕ^+ solve the same one-dimensional Schrödinger equation, say,

$$-\phi'' + A\phi = 0\,.$$

The variation of parameter method for the equations (3.15) and (3.16) yields

$$\begin{bmatrix} \phi^+ & \phi^- \\ \frac{\mathrm{d}}{\mathrm{d}\tau}\phi^+ & \frac{\mathrm{d}}{\mathrm{d}\tau}\phi^- \end{bmatrix} \frac{\mathrm{d}}{\mathrm{d}\tau} \begin{bmatrix} a^+ \\ a^- \end{bmatrix} = (V_k - A) \begin{bmatrix} 0 & 0 \\ \phi^+ & \phi^- \end{bmatrix} \begin{bmatrix} a^+ \\ a^- \end{bmatrix}. \tag{3.17}$$

(We have omitted the subscript j). We introduce the notation $W(\phi^-, \phi^+)$ for the Wronskian $W(\phi^-, \phi^+) = \phi^- \frac{\mathrm{d}}{\mathrm{d}r}\phi^+ - \phi^+ \frac{\mathrm{d}}{\mathrm{d}r}\phi^-$. Then we write $B = V_k - A$ and transform (3.17) into

$$\frac{\mathrm{d}}{\mathrm{d}r}\begin{pmatrix} a^+ \\ a^- \end{pmatrix} = N \begin{pmatrix} a^+ \\ a^- \end{pmatrix},$$

where

$$N = \frac{B}{W(\phi^-, \phi^+)} \begin{pmatrix} \phi^-\phi^+ & (\phi^-)^2 \\ -(\phi^+)^2 & -\phi^-\phi^+ \end{pmatrix}.$$

For a positive increasing continuous function f on I (to be specified), we introduce the matrix $T = \mathrm{diag}(1, f^{-1})$. We compute

$$TNT^{-1} = \frac{B}{W(\phi^-, \phi^+)} \begin{pmatrix} \phi^-\phi^+ & f(\phi^-)^2 \\ -f^{-1}(\phi^+)^2 & -\phi^-\phi^+ \end{pmatrix}.$$

Introducing the operator $(M_j z)(r) = \int_{r_{j-1}}^{r} N_j(r')z(r')\,\mathrm{d}r'$, $j \geq 2$, acting on continuous functions $z(\,\cdot\,) : I_j \to \mathbb{R}^2$, the above differential equation is solved by

$$\begin{pmatrix} a_j^+ \\ a_j^- \end{pmatrix}(r) - z_j = \sum_{m=1}^{\infty} M_j^m z_j\,; \quad z_j = \begin{pmatrix} a_j^+ \\ a_j^- \end{pmatrix}(r_{j-1})\,.$$

Whence we have the bound

$$\left\| T_j(r) \left\{ \begin{pmatrix} a_j^+ \\ a_j^- \end{pmatrix} (r) - z_j \right\} \right\| \le \sum_{m=1}^{\infty} \left\| \Big(\big(T_j M_j T_j^{-1}\big)^m T_j z_j \Big)(r) \right\| ; \tag{3.18}$$

to the right T_j is considered as an operator acting as $(T_j z)(r') = (T_j)(r') z(r')$. Using that f_j is increasing, we can estimate

$$\|(T_j M_j T_j^{-1} z)(r)\| \le \int_{r_{j-1}}^{r} \|(T_j N_j T_j^{-1})(r')\| \, \|z(r')\| \, \mathrm{d}r' ,$$

which applied repeatedly in (3.18) yields the following bound for $r \in I_j$:

$$\begin{aligned} &\left\| T_j(r) \left\{ \begin{pmatrix} a^+ \\ a^- \end{pmatrix} (r) - z_j \right\} \right\| \\ &\qquad \le \left\{ \left(\exp \int_{r_{j-1}}^{r} \|(T_j N_j T_j^{-1})(r')\| \, \mathrm{d}r' \right) - 1 \right\} \sup_{\tilde{r} \in I_j} \|T_j(\tilde{r}) z_j\| \\ &\qquad = \left\{ \left(\exp \int_{r_{j-1}}^{r} \|(T_j N_j T_j^{-1})(r')\| \, \mathrm{d}r' \right) - 1 \right\} \|T_j(r_{j-1}) z_j\| . \end{aligned} \tag{3.19}$$

We specify in the following $\phi_j^{\pm}$, A_j, B_j and f_j for $j = 2, 3$ and 4; in all cases $W(\phi_j^-, \phi_j^+) = 1$:

Re interval I_2. We define

$$\phi_2^{\pm}(r) = 2^{-\frac{1}{2}} V_k^{-\frac{1}{4}} \mathrm{e}^{\pm \int_{r_1}^{r} \sqrt{V_k} \, \mathrm{d}r'} , \tag{3.20a}$$

compute

$$\begin{aligned} B_2 &= -\big(V_k^{-\frac{1}{4}}\big)'' V_k^{\frac{1}{4}} = -\frac{5}{16} \left(\frac{V_k'}{V_k} \right)^2 + \frac{1}{4} \frac{V_k''}{V_k} , \\ A_2 &= V_k + \frac{5}{16} \left(\frac{V_k'}{V_k} \right)^2 - \frac{1}{4} \frac{V_k''}{V_k} , \end{aligned} \tag{3.20b}$$

and let

$$f_2(r) = \frac{\phi_2^+(r)}{\phi_2^-(r)} = \mathrm{e}^{2 \int_{r_1}^{r} \sqrt{V_k} \, \mathrm{d}r'} . \tag{3.20c}$$

Re interval I_3. We define (in terms of the Airy function, cf. [9] and [10, Definition 7.6.8])

$$\phi_3^+(r) = \sqrt{\pi} \zeta^{-1} \mathrm{Ai}\big(-\zeta^2 (r - r_0) \big) ; \quad \zeta := |V_k'(r_0)|^{\frac{1}{6}} , \tag{3.21a}$$

$$\begin{aligned} \phi_3^-(r) = &\sqrt{\pi} \mathrm{e}^{\frac{\pi \mathrm{i}}{6}} \zeta^{-1} \mathrm{Ai}\big(-\zeta^2 \mathrm{e}^{\frac{2\pi \mathrm{i}}{3}} (r - r_0) \big) \\ &+ \sqrt{\pi} \mathrm{e}^{-\frac{\pi \mathrm{i}}{6}} \zeta^{-1} \mathrm{Ai}\big(-\zeta^2 \mathrm{e}^{-\frac{2\pi \mathrm{i}}{3}} (r - r_0) \big) , \end{aligned} \tag{3.21b}$$

compute

$$B_3(r) = V_k(r) - \big(V_k(r_0) + V_k'(r_0)(r - r_0)\big) = \int_{r_0}^{r} (r - \tilde r)V_k''(\tilde r)\,\mathrm{d}\tilde r\,, \tag{3.21c}$$

$$A_3(r) = V_k(r_0) + V_k'(r_0)(r - r_0)\,,$$

and let

$$f_3(r) = \begin{cases} \exp\big(-\frac{4}{3}\zeta^3(r_0 - r)^{\frac{3}{2}}\big)\,, & \text{if} \quad r < r_0\,; \\ 1\,, & \text{if} \quad r \geq r_0\,. \end{cases} \tag{3.21d}$$

Re interval I_4. We define

$$\phi_4^+(r) = (-V_k)^{-\frac{1}{4}} \sin\left(\int_{r_0}^{r} \sqrt{-V_k}\,\mathrm{d}r' + \frac{\pi}{4}\right), \tag{3.22a}$$

$$\phi_4^-(r) = (-V_k)^{-\frac{1}{4}} \cos\left(\int_{r_0}^{r} \sqrt{-V_k}\,\mathrm{d}r' + \frac{\pi}{4}\right)\,, \tag{3.22b}$$

compute

$$B_4 = -\big((-V_k)^{-\frac{1}{4}}\big)''(-V_k)^{\frac{1}{4}} = -\frac{5}{16}\left(\frac{V_k'}{V_k}\right)^2 + \frac{1}{4}\frac{V_k''}{V_k}\,, \tag{3.22c}$$

$$A_4 = V_k + \frac{5}{16}\left(\frac{V_k'}{V_k}\right)^2 - \frac{1}{4}\frac{V_k''}{V_k}\,,$$

and let

$$f_4 = 1\,. \tag{3.22d}$$

3.3.2. Details of proof of Proposition 3.3. We start implementing the scheme outlined in Subsubsection 3.3.1.

In the interval I_1 we shall use a standard comparison argument. With V_k replaced by $V = \frac{\tilde k(\tilde k+1)}{r^2}$, the regular solution is given by the expression $u = r^{\tilde k+1}$ and the corresponding Riccati equation

$$\psi' = V - \psi^2 \tag{3.23}$$

is solved by $\psi = \frac{\phi'}{\phi} = \frac{\tilde k+1}{r}$.

We fix $\epsilon_1 \in]0,1]$ (actually $\epsilon_1 > 0$ can be chosen arbitrarily) and notice the following uniform bound in $r \in I_1$:

$$V_k(r) = \frac{k(k+1)}{r^2}\big(1 + O(k^{-\epsilon_1})\big)\,. \tag{3.24}$$

Using (3.24), we can find $C > 0$ such that with $k^\pm := k(1 \pm Ck^{-\epsilon_1})$ and $V_k^\pm(r) := \frac{k^\pm(k^\pm+1)}{r^2}$ there are estimates

$$V_k(r)\begin{cases} \leq V_k^+(r) \\ \geq V_k^-(r) \end{cases}, \quad r \in I_1\,.$$

Now, by using [2, Theorem 1.8] and the Riccati equation, it follows that the regular solution u of (3.15) is positive in I_1, and that $v := \frac{u'}{u}$ obeys the bounds

$$v(r)\begin{cases}\le \frac{k^+ +1}{r}\\ \ge \frac{k^- +1}{r}\end{cases},\quad r\in I_1\,.$$

We conclude the uniform bound

$$v(r)=\frac{k+1}{r}\big(1+O(k^{-\epsilon_1})\big)\,,\quad r\in I_1\,. \tag{3.25}$$

The connection formula at $r=r_1$ reads

$$c_j\begin{pmatrix}1\\ v\end{pmatrix}_{r=r_{j-1}}=\begin{pmatrix}a_j^+\phi_j^+ + a_j^-\phi_j^-\\ a_j^+(\phi_j^+)' + a_j^-(\phi_j^-)'\end{pmatrix}_{r=r_{j-1}}\,,\quad j=2\,. \tag{3.26}$$

Obviously, (3.26) is solved for the coefficients by

$$\begin{pmatrix}a_j^+\\ a_j^-\end{pmatrix}_{r=r_{j-1}}=\frac{c_j}{W(\phi_j^-,\phi_j^+)}\begin{pmatrix}(-\phi_j^-)'+\phi_j^- v\\ (\phi_j^+)'-\phi_j^+ v\end{pmatrix}_{r=r_{j-1}}\,,\quad j=2\,. \tag{3.27}$$

Next, from (3.20a) we compute

$$(\phi_2^\pm)'=\left(\pm\sqrt{V_k}-\frac{1}{4}\frac{V_k'}{V_k}\right)\phi_2^\pm\,. \tag{3.28}$$

We substitute these expressions and (3.25) in the right hand side of (3.27) and obtain

$$\begin{pmatrix}a_2^+(r_1)\\ a_2^-(r_1)\end{pmatrix}=c_2\frac{2k}{r_1}\begin{pmatrix}1+O\big(k^{-\epsilon_1}\big)\\ O\big(k^{-\epsilon_1}\big)\end{pmatrix}. \tag{3.29}$$

To apply (3.19), we notice that

$$T_2N_2T_2^{-1}=B_2\phi_2^-\phi_2^+\begin{pmatrix}1&1\\-1&-1\end{pmatrix}=B_2O\big(V_k^{-\frac12}\big)\,.$$

Whence (for the first inequality below we assume that the integral is bounded in k so that the inequality $\exp x-1\le Cx$ applies – this will be justified by (3.31)),

$$\begin{aligned}
&\left\|T_2(r_2)\left\{\begin{pmatrix}a_2^+\\ a_2^-\end{pmatrix}(r_2)-\begin{pmatrix}a_2^+\\ a_2^-\end{pmatrix}(r_1)\right\}\right\|\\
&\quad=\left\{\left(\exp\int_{r_1}^{r_2}\left|\left(-\frac{5}{16}\left(\frac{V_k'}{V_k}\right)^2+\frac14\frac{V_k''}{V_k}\right)O\big(V_k^{-\frac12}\big)\right|\mathrm{d}r'\right)-1\right\}O\left(\frac{k}{r_1}\right)\\
&\quad\le C_1\frac{k}{r_1}r_0\int_{r_1/r_0}^{r_2/r_0}\left(\frac{\big(V_k'\big)^2}{V_k^{\frac52}}+\frac{|V_k''|}{V_k^{\frac32}}\right)\mathrm{d}s\quad(\text{changing variables } r'=r_0s)\\
&\quad\le C_2r_1^{-1}\int_{r_1/r_0}^{r_2/r_0}\left(\frac{s^{-6}}{(s^{-2}-s^{-\mu})^{\frac52}}+\frac{s^{-4}}{(s^{-2}-s^{-\mu})^{\frac32}}\right)\mathrm{d}s\\
&\quad=C_2r_1^{-1}\left(\int_{1/2}^{r_2/r_0}\cdots\,\mathrm{d}s+\int_{r_1/r_0}^{1/2}\cdots\,\mathrm{d}s\right)
\end{aligned}$$

$$\leq C_3 r_1^{-1} \max\left(\int_{1/2}^{r_2/r_0} (1-s^{2-\mu})^{-\frac{5}{2}}\,\mathrm{d}s, \int_{r_1/r_0}^{1/2} s^{-1}\,\mathrm{d}s\right)$$

$$\leq C_4 k^{\frac{3}{2}\epsilon_2-1}\,\frac{k}{r_1}\,; \tag{3.30}$$

we need here

$$\frac{3}{2}\epsilon_2 - 1 < 0\,. \tag{3.31}$$

We conclude by combining (3.29) and (3.30):

$$\begin{pmatrix} a_2^+(r_2) \\ a_2^-(r_2) \end{pmatrix} = c_2 \frac{2k}{r_1} \begin{pmatrix} 1 + O(k^{-\epsilon_1}) + O(k^{\frac{3}{2}\epsilon_2-1}) \\ O(k^{-\epsilon_1}) + \mathrm{e}^{2\int_{r_1}^{r_2}\sqrt{V_k}\,\mathrm{d}r'} O(k^{\frac{3}{2}\epsilon_2-1}) \end{pmatrix}. \tag{3.32}$$

Next we repeat the above procedure passing from the interval I_2 to I_3.

The first issue is the connection formula (3.26) with $j=2$ replaced by $j=3$. The left hand side can be estimated using (3.28), (3.32) and the following estimates (where (3.31) is used):

$$\sqrt{V_k(r_2)} = \frac{\sqrt{k(k+1)}}{r_2}\big(1-(1-k^{-\epsilon_2})^{2-\mu}\big)^{\frac{1}{2}}$$

$$= \frac{k}{r_0}(2-\mu)^{\frac{1}{2}} k^{-\frac{\epsilon_2}{2}}\big(1+O(k^{-\epsilon_2})\big)\,, \tag{3.33}$$

$$\frac{V_k'(r_2)}{V_k(r_2)} = \frac{O\Big(\frac{k^2}{r_2^3}\Big)}{V_k(r_2)} = r_2^{-1} O\big(k^{\epsilon_2}\big)\,. \tag{3.34}$$

Notice that (3.33) dominates (3.34) (by (3.31) again), so that

$$\left(\sqrt{V_k} - \frac{1}{4}\frac{V_k'}{V_k}\right)(r_2) = (2-\mu)^{\frac{1}{2}}\frac{k}{r_0} k^{-\frac{\epsilon_2}{2}}\big(1+O(k^{-\epsilon_2})\big)\,.$$

We conclude that

$$v(r_2) = \frac{(\phi_2^+)'(r_2)}{\phi_2^+(r_2)}\big(1+O(k^{\frac{3}{2}\epsilon_2-1})\big)$$

$$= (2-\mu)^{\frac{1}{2}}\frac{k}{r_0}k^{-\frac{\epsilon_2}{2}}\big(1+O(k^{-\epsilon_2})+O(k^{\frac{3}{2}\epsilon_2-1})\big)\,. \tag{3.35}$$

By (3.26) and (3.27) with $j=2$ replaced by $j=3$, up to multiplication by a positive constant,

$$\begin{pmatrix} a_3^+ \\ a_3^- \end{pmatrix}_{r=r_2} = \begin{pmatrix} (-\phi_3^-)' + \phi_3^- v \\ (\phi_3^+)' - \phi_3^+ v \end{pmatrix}_{r=r_2}\,. \tag{3.36}$$

It remains to examine the asymptotics of $\phi_3^{\pm}$ and their derivatives at r_2. For that we notice the asymptotics as $r - r_0 \to -\infty$, cf. [9, Appendix B] and [10, (7.6.20)],

$$\phi_3^+ = \frac{\exp\left(-\frac{2}{3}\zeta^3(r_0-r)^{\frac{3}{2}}\right)}{2\zeta^{\frac{3}{2}}(r_0-r)^{\frac{1}{4}}}\Big(1+O\big(\zeta^{-3}(r_0-r)^{-\frac{3}{2}}\big)\Big)\,, \tag{3.37a}$$

$$(\phi_3^+)' = \zeta^3(r_0-r)^{\frac{1}{2}}\frac{\exp\left(-\frac{2}{3}\zeta^3(r_0-r)^{\frac{3}{2}}\right)}{2\zeta^{\frac{3}{2}}(r_0-r)^{\frac{1}{4}}}\Big(1+O\big(\zeta^{-3}(r_0-r)^{-\frac{3}{2}}\big)\Big)\,, \tag{3.37b}$$

$$\phi_3^- = \frac{\exp\left(\frac{2}{3}\zeta^3(r_0-r)^{\frac{3}{2}}\right)}{\zeta^{\frac{3}{2}}(r_0-r)^{\frac{1}{4}}}\Big(1+O\big(\zeta^{-3}(r_0-r)^{-\frac{3}{2}}\big)\Big)\,, \tag{3.37c}$$

$$(\phi_3^-)' = -\zeta^3(r_0-r)^{\frac{1}{2}}\frac{\exp\left(\frac{2}{3}\zeta^3(r_0-r)^{\frac{3}{2}}\right)}{\zeta^{\frac{3}{2}}(r_0-r)^{\frac{1}{4}}}\Big(1+O\big(\zeta^{-3}(r_0-r)^{-\frac{3}{2}}\big)\Big)\,. \tag{3.37d}$$

Since $\zeta^3(r_0-r_2)^{\frac{3}{2}} \asymp \sqrt{2-\mu}\,k^{1-\frac{3}{2}\frac{\epsilon_2}{2-\mu}}$, cf. (3.13), these asymptotics are applicable. By the same computation, (3.35) can be rewritten as

$$v(r_2) = \zeta^3(r_0-r_2)^{\frac{1}{2}}\big(1+O(k^{-\epsilon_2})+O(k^{\frac{3}{2}\epsilon_2-1})\big)\,. \tag{3.38}$$

Whence, in conjunction (3.36), we obtain (up to multiplication by a positive constant)

$$\begin{pmatrix} a_3^+(r_2) \\ a_3^-(r_2)\end{pmatrix} = \begin{pmatrix} \exp\left(\frac{2}{3}\zeta^3(r_0-r_2)^{\frac{3}{2}}\right)\big(1+O(k^{-\epsilon_2})+O(k^{\frac{3}{2}\epsilon_2-1})\big) \\ \exp\left(-\frac{2}{3}\zeta^3(r_0-r_2)^{\frac{3}{2}}\right)\big(O(k^{-\epsilon_2})+O(k^{\frac{3}{2}\epsilon_2-1})\big)\end{pmatrix}. \tag{3.39}$$

Next, to apply (3.19) with $j = 3$ we need the following asymptotics of $\phi_3^{\pm}$ and their derivatives as $r - r_0 \to +\infty$, cf. [9, Appendix B] and [10, (7.6.20) and (7.6.21)]:

$$\phi_3^+ = \zeta^{-\frac{3}{2}}(r-r_0)^{-\frac{1}{4}}\left(\sin\left(\frac{2}{3}\zeta^3(r-r_0)^{\frac{3}{2}}+\frac{\pi}{4}\right)+O\big(\zeta^{-3}(r-r_0)^{-\frac{3}{2}}\big)\right), \tag{3.40a}$$

$$(\phi_3^+)' = \zeta^{\frac{3}{2}}(r-r_0)^{\frac{1}{4}}\left(\cos\left(\frac{2}{3}\zeta^3(r-r_0)^{\frac{3}{2}}+\frac{\pi}{4}\right)+O\big(\zeta^{-3}(r-r_0)^{-\frac{3}{2}}\big)\right), \tag{3.40b}$$

$$\phi_3^- = \zeta^{-\frac{3}{2}}(r-r_0)^{-\frac{1}{4}}\left(\cos\left(\frac{2}{3}\zeta^3(r-r_0)^{\frac{3}{2}}+\frac{\pi}{4}\right)+O\big(\zeta^{-3}(r-r_0)^{-\frac{3}{2}}\big)\right), \tag{3.40c}$$

$$(\phi_3^-)' = -\zeta^{\frac{3}{2}}(r-r_0)^{\frac{1}{4}}\left(\sin\left(\frac{2}{3}\zeta^3(r-r_0)^{\frac{3}{2}}+\frac{\pi}{4}\right)+O\big(\zeta^{-3}(r-r_0)^{-\frac{3}{2}}\big)\right). \tag{3.40d}$$

In particular,

$$T_3N_3T_3^{-1} = B_3\zeta^{-2}O(k^0) \quad \text{uniformly in} \quad r \in I_3\,.$$

In conjunction with (3.19), (3.13) and the fact that

$$V_k''(r) = O\big(k^{-\frac{4+2\mu}{2-\mu}}\big) \quad \text{uniformly in} \quad r \in I_3\,, \tag{3.41}$$

we obtain

$$\begin{aligned}
&\left\| T_3(r_3)\left\{\begin{pmatrix} a_3^+ \\ a_3^- \end{pmatrix}(r_3) - \begin{pmatrix} a_3^+ \\ a_3^- \end{pmatrix}(r_2)\right\}\right\| \\
&\le C_1\big((r_3-r_0)^3+(r_0-r_2)^3\big)k^{-\frac{4+2\mu}{2-\mu}}\,k^{\frac{2}{3}\frac{1+\mu}{2-\mu}}a_3^+(r_2) \\
&\le C_2 k^{\frac{4}{3}-3\min(\epsilon_2,\epsilon_3)}a_3^+(r_2)\,; \qquad (3.42)
\end{aligned}$$

here we need

$$\frac{4}{3}-3\min(\epsilon_2,\epsilon_3)<0\,, \qquad (3.43)$$

cf. (3.31). At this point let us for convenience take $\epsilon_3=\epsilon_2$, so that (3.43) simplifies and in conjunction with (3.31) leads to the single requirement

$$\frac{2}{3}>\epsilon_2=\epsilon_3>\frac{4}{9}\,. \qquad (3.44)$$

We conclude that (up to multiplication by the positive constant $a_3^+(r_2)$)

$$\begin{pmatrix} a_3^+(r_3) \\ a_3^-(r_3) \end{pmatrix} = \begin{pmatrix} 1+O(k^{\frac{4}{3}-3\epsilon_2}) \\ O(k^{\frac{4}{3}-3\epsilon_2}) \end{pmatrix}. \qquad (3.45)$$

Next we need to study the connection formula passing from I_3 to I_4; a little linear algebra takes it to the form

$$\begin{pmatrix} a_4^+ \\ a_4^- \end{pmatrix} = \begin{pmatrix} W(\phi_4^-,\phi_3^+) & W(\phi_4^-,\phi_3^-) \\ W(\phi_3^+,\phi_4^+) & W(\phi_3^-,\phi_4^+) \end{pmatrix}\begin{pmatrix} a_3^+ \\ a_3^- \end{pmatrix}, \quad r=r_3\,.$$

So we need to compute the appearing Wronskians. To this end we note the following uniform asymptotics for $r\in[r_0,r_3]$, which are readily obtained from (3.13) and (3.41) (recall that by now $\epsilon_3=\epsilon_2$):

$$V_k(r)=V_k'(r_0)(r-r_0)\big(1+O(k^{-\epsilon_2})\big)\,, \qquad (3.46a)$$

$$V_k'(r)=V_k'(r_0)\big(1+O(k^{-\epsilon_2})\big)\,, \qquad (3.46b)$$

$$\sqrt{-V_k(r)}=\zeta^3(r-r_0)^{\frac{1}{2}}\big(1+O(k^{-\epsilon_2})\big)\,, \qquad (3.46c)$$

$$\begin{aligned}
\int_{r_0}^{r}\sqrt{-V_k(r')}\,\mathrm{d}r' &= \frac{2}{3}\zeta^3(r-r_0)^{\frac{3}{2}}\big(1+O(k^{-\epsilon_2})\big) \\
&= \frac{2}{3}\zeta^3(r-r_0)^{\frac{3}{2}}+O(k^{1-\frac{5}{2}\epsilon_2})\,. \qquad (3.46d)
\end{aligned}$$

Due to (3.46c) and (3.46d), the asymptotics (3.40a)–(3.40d) at the point $r=r_3$ can be written in terms of

$$\theta := \int_{r_0}^{r_3}\sqrt{-V_k(r')}\,\mathrm{d}r'+\frac{\pi}{4}$$

as

$$\frac{\phi_3^+(r_3)}{(-V_k(r_3))^{-\frac{1}{4}}}=\sin\big(\theta+O(k^{1-\frac{5}{2}\epsilon_2})\big)+O(k^{-\epsilon_2})+O(k^{\frac{3}{2}\epsilon_2-1})\,, \qquad (3.47a)$$

$$\frac{(\phi_3^+)'(r_3)}{(-V_k(r_3))^{\frac{1}{4}}}=\cos\big(\theta+O(k^{1-\frac{5}{2}\epsilon_2})\big)+O(k^{-\epsilon_2})+O(k^{\frac{3}{2}\epsilon_2-1})\,, \qquad (3.47b)$$

$$\frac{\phi_3^-(r_3)}{(-V_k(r_3))^{-\frac{1}{4}}} = \cos\big(\theta + O(k^{1-\frac{5}{2}\epsilon_2})\big) + O(k^{-\epsilon_2}) + O(k^{\frac{3}{2}\epsilon_2-1})\,, \tag{3.47c}$$

$$\frac{(\phi_3^-)'(r_3)}{(-V_k(r_3))^{\frac{1}{4}}} = -\sin\big(\theta + O(k^{1-\frac{5}{2}\epsilon_2})\big) + O(k^{-\epsilon_2}) + O(k^{\frac{3}{2}\epsilon_2-1})\,. \tag{3.47d}$$

Next, using that

$$\frac{-V_k'}{(-V_k)^{\frac{3}{2}}}(r_3) = O(k^{\frac{3}{2}\epsilon_2-1})\,,$$

cf. (3.46a) and (3.46b), we obtain for the functions $\phi_4^\pm$

$$\phi_4^+(r_3) = \big(-V_k(r_3)\big)^{-\frac{1}{4}}\sin(\theta)\,, \tag{3.48a}$$

$$(\phi_4^+)'(r_3) = \big(-V_k(r_3)\big)^{\frac{1}{4}}\big(\cos(\theta) + O(k^{\frac{3}{2}\epsilon_2-1})\big)\,, \tag{3.48b}$$

$$\phi_4^-(r_3) = \big(-V_k(r_3)\big)^{-\frac{1}{4}}\cos(\theta)\,, \tag{3.48c}$$

$$(\phi_4^-)'(r_3) = -\big(-V_k(r_3)\big)^{\frac{1}{4}}\big(\sin(\theta) + O(k^{\frac{3}{2}\epsilon_2-1})\big)\,. \tag{3.48d}$$

The matrix of Wronskians is readily computed using (3.40a)–(3.40d) and (3.47a)–(3.47d). In combination with (3.45), we obtain (using in the second step (3.44))

$$\begin{aligned}\begin{pmatrix} a_4^+(r_3) - 1 \\ a_4^-(r_3)\end{pmatrix} &= O(k^{-\epsilon_2}) + O(k^{\frac{3}{2}\epsilon_2-1}) + O(k^{\frac{4}{3}-3\epsilon_2}) + O(k^{1-\frac{5}{2}\epsilon_2}) \\ &= O(k^{-\epsilon_2}) + O(k^{\frac{3}{2}\epsilon_2-1}) + O(k^{\frac{4}{3}-3\epsilon_2})\,. \end{aligned}\tag{3.49}$$

Now we estimate in I_4 using (3.49) (and mimicking partially (3.30))

$$\begin{aligned}
&\left\| \begin{pmatrix} a_4^+ \\ a_4^- \end{pmatrix}(r) - \begin{pmatrix} a_4^+ \\ a_4^- \end{pmatrix}(r_3) \right\| \\
&\le C_1 \left\{ \left(\exp \int_{r_3}^{r} \left| \left(-\frac{5}{16}\left(\frac{V_k'}{V_k}\right)^2 + \frac{1}{4}\frac{V_k''}{V_k}\right) O\big((-V_k)^{-\frac{1}{2}}\big) \right| \mathrm{d}r' \right) - 1 \right\} \\
&\le C_2 r_0 \int_{r_3/r_0}^{r/r_0} \left(\frac{\big(-V_k'\big)^2}{(-V_k)^{\frac{5}{2}}} + \frac{\big|-V_k''\big|}{(-V_k)^{\frac{3}{2}}} \right) \mathrm{d}s \quad \text{(changing variables } r' = r_0 s\text{)} \\
&\le C_3 r_0^{\mu/2-1} \int_{r_3/r_0}^{r/r_0} s^{\mu/2-2}\big((1-s^{\mu-2})^{-\frac{5}{2}} + (1-s^{\mu-2})^{-\frac{3}{2}}\big)\,\mathrm{d}s \\
&\le C_4 r_0^{\mu/2-1} \left(\int_2^\infty s^{\mu/2-2}\,\mathrm{d}s + \int_{r_3/r_0}^{2} (1-s^{\mu-2})^{-\frac{5}{2}}\,\mathrm{d}s \right) \\
&\le C_5 r_0^{\mu/2-1} k^{\frac{3}{2}\epsilon_2} \\
&= O(k^{\frac{3}{2}\epsilon_2-1})\,.
\end{aligned}\tag{3.50}$$

By the same type of estimation we also deduce that for fixed k there exist $\epsilon_k > 0$ and $a_4^{\pm}(\infty) \in \mathbb{R}$ such that

$$a_4^{\pm}(r) = a_4^{\pm}(\infty) + O(r^{-\epsilon_k})\,.$$

By applying (3.50) with $r = \infty$ in combination with (3.49) (and using an elementary trigonometric formula), we conclude that (3.14) is true.

The general case. It remains to prove (3.14) under Conditions (1.3) and (1.4). All previous constructions and estimates carry over, so below we consider only some additional estimates that are needed. Denoting $U(r) = 2V(r) - 2\gamma r^{-\mu}$, the functions $\phi_j^{\pm}$ and f_j and the potentials A_j are exactly the same, while the potentials B_j are given as the old B_j plus U, $j = 2, 3, 4$.

Re interval I_1. We notice that (3.24) is valid (here with V_k defined upon replacing $2\gamma r^{-\mu} \to 2V$). Whence we can proceed exactly as before.

Re interval I_2. In addition to (3.30) we need the following estimation (assuming in the last step that $\frac{\mu}{2} + \epsilon < 1$):

$$\begin{aligned}
&\int_{r_1}^{r_2} |UO(V_k^{-\frac{1}{2}})|\,\mathrm{d}r' O\left(\frac{k}{r_1}\right)\\
&\le C_1 \frac{k}{r_1} r_0 \int_{r_1/r_0}^{r_2/r_0} \frac{r^{-1-\frac{\mu}{2}-\epsilon}}{V_k^{\frac{1}{2}}}\,\mathrm{d}s \quad (\text{changing variables } r = r_0 s)\\
&\le C_2 \frac{k}{r_1} \frac{r_0^{1-\frac{\mu}{2}-\epsilon}}{k} \int_{r_1/r_0}^{r_2/r_0} \frac{s^{-\frac{\mu}{2}-\epsilon}}{(1 - s^{2-\mu})^{\frac{1}{2}}}\,\mathrm{d}s\\
&\le C_3 k^{-\frac{2\epsilon}{2-\mu}} \frac{k}{r_1}\,.
\end{aligned} \tag{3.51}$$

Re interval I_3. In addition to (3.42) we need the following estimation

$$\begin{aligned}
&\int_{r_2}^{r_3} |U\zeta^{-2}|\,\mathrm{d}r'\\
&\le C_1 \int_{r_2}^{r_3} k^{\frac{2}{3}\frac{1+\mu}{2-\mu}} r'^{-1-\frac{\mu}{2}-\epsilon}\,\mathrm{d}r'\\
&\le C_2 k^{\frac{2}{3}\frac{1+\mu}{2-\mu}} r_0^{-\frac{\mu}{2}-\epsilon}(k^{-\epsilon_2} + k^{-\epsilon_3})\\
&\le C_3 k^{\frac{1}{3}-\epsilon_2-\frac{2\epsilon}{2-\mu}}\,.
\end{aligned} \tag{3.52}$$

Due to (3.44) the right hand side of (3.52) vanishes.

Re interval I_4**.** In addition to (3.50) we need the following estimation:

$$\begin{aligned}
&\int_{r_3}^{r} \left|UO\big((-V_k)^{-\frac{1}{2}}\big)\right| \mathrm{d}r' \\
&\le C_1 r_0^{-\epsilon} \int_{r_3/r_0}^{r/r_0} \frac{s^{-1-\epsilon}}{(1-s^{\mu-2})^{\frac{1}{2}}}\,\mathrm{d}s \quad \text{(changing variables } r' = r_0 s) \\
&\le C_2 k^{-\frac{2\epsilon}{2-\mu}}\,. \qquad (3.53)
\end{aligned}$$

This ends the proof of (3.14). □

3.4. End of proof of Proposition 3.2

We need the following elementary identity:

Lemma 3.4. *Let* $\mu < 2$*. Then*

$$\int_1^\infty \left(\sqrt{r^{-\mu} - r^{-2}} - \sqrt{r^{-\mu}}\right) \mathrm{d}r = \frac{2-\pi}{2-\mu}\,. \qquad (3.54)$$

Proof. We first substitute $r = s^{\frac{1}{\mu-2}}$ and then $s = \sin^2\phi$. Thus the left hand side of (3.54) equals

$$\begin{aligned}
&\frac{1}{2-\mu}\int_0^1 s^{-\frac{3}{2}}\left(\sqrt{1-s}-1\right)\mathrm{d}s = \frac{2}{2-\mu}\int_0^{\frac{\pi}{2}}\left(\frac{1-\cos\phi}{\sin^2\phi}-1\right)\mathrm{d}\phi \\
&= \frac{2}{2-\mu}\left(\frac{1-\cos\phi}{\sin\phi}-\phi\right)\Big|_0^{\pi/2} = \frac{2-\pi}{2-\mu}\,.
\end{aligned}$$

□

Proof of Proposition 3.2. Using Proposition 3.3 we calculate from (3.10)

$$\begin{aligned}
\sigma_k(0) &= \lim_{r\to\infty}\left(\int_{r_0}^{r}\sqrt{-V_k(\tilde r)}\mathrm{d}\tilde r + \frac{\pi}{4} - \int_{R_0}^{r}\sqrt{-2V(\tilde r)}\mathrm{d}\tilde r + \frac{k\pi}{2}\right) + o(k^0) \\
&= \int_{r_0}^{\infty}\left(\sqrt{-V_k(r)} - \sqrt{-V_0(r)}\right)\mathrm{d}r \\
&\quad + \int_{R_0}^{\infty}\left(\sqrt{-V_0(r)} - \sqrt{-2V(r)}\right)\mathrm{d}r \\
&\quad - \int_{R_0}^{r_0}\sqrt{-V_0(r)}\mathrm{d}r + \frac{(k+\frac{1}{2})\pi}{2} + o(k^0)\,.
\end{aligned}$$

Now (using Lemma 3.4)

$$\begin{aligned}
\int_{r_0}^{\infty}\left(\sqrt{-V_k(r)} - \sqrt{-V_0(r)}\right)\mathrm{d}r &= \sqrt{k(k+1)}\int_1^\infty\left(\sqrt{r^{-\mu}-r^{-2}} - \sqrt{r^{-\mu}}\right)\mathrm{d}r \\
&= \sqrt{k(k+1)}\,\frac{2-\pi}{2-\mu}\,; \\
\int_{r_0}^{R_0}\sqrt{-V_0(r)}\mathrm{d}r &= -\frac{2}{2-\mu}\sqrt{k(k+1)} + \frac{2\sqrt{2\gamma}}{2-\mu}R_0^{1-\frac{\mu}{2}}\,.
\end{aligned}$$

Thus,

$$\begin{aligned}\sigma_k(0) - \int_{R_0}^{\infty} \Big(\sqrt{-V_0(r)} - \sqrt{-2V(r)}\Big)\mathrm{d}r \\ &= -\sqrt{k(k+1)}\frac{\pi}{2-\mu} + \frac{(k+\frac{1}{2})\pi}{2} + \frac{2\sqrt{2\gamma}}{2-\mu}R_0^{1-\frac{\mu}{2}} + o(k^0) \\ &= -\frac{(k+\frac{1}{2})\pi\mu}{2(2-\mu)} + \frac{2\sqrt{2\gamma}}{2-\mu}R_0^{1-\frac{\mu}{2}} + o(k^0)\,. \qquad \square\end{aligned}$$

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Jan Dereziński
Deptartment of Math. Methods in Physics
Warsaw University
Hoża 74
PL-00-682, Warszawa
Poland
e-mail: `Jan.Derezinski@fuw.edu.pl`

Erik Skibsted
Institut for Matematiske Fag
Aarhus Universitet
Ny Munkegade
DK-8000 Aarhus C
Denmark
e-mail: `skibsted@imf.au.dk`

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