Introduction to special functions

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1 Homographies

We set

$$\mathrm{Re}z := x = \tfrac{1}{2}(z+\overline{z}), \qquad \mathrm{Im}z := y = \tfrac{1}{2\mathrm{i}}(z-\overline{z}),$$

$$\mathbb{R}_+ := \{x \in \mathbb{R} \ : \ x > 0\}, \qquad \mathbb{C}_+ := \{z \in \mathbb{C} \ : \ \mathrm{Im}z > 0\},$$

$$\mathbb{R}^\times := \mathbb{R}\backslash\{0\}, \qquad \mathbb{C}^\times := \mathbb{C}\backslash\{0\},$$

$$K(z_0, r) := \{z \in \mathbb{C} \ : \ |z - z_0| < r\}, \quad z_0 \in \mathbb{C}, \ r > 0.$$

A transformation

$$\mathbb{C} \ni z \mapsto g(z) = az + b \in \mathbb{C} \tag{1.1}$$

where $a \neq 0$ is called affine.

Theorem 1.1 *Afine transformations are bijections of $\mathbb{C}$. If (z_1, z_2) are two distinct points in $\mathbb{C}$ and (w_1, w_2) are two distinct points in $\mathbb{C}$ then there exists a unique afine transformation g such that*

$$g(z_1) = w_1, \quad g(z_2) = w_2.$$

Afine transformations form a group, which we denote $\mathbb{C} \rtimes GL(1,\mathbb{C})$. Let

$$AGL(1,\mathbb{C}) := \left\{ \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} \ : \ a \in \mathbb{C}^\times, \ b \in \mathbb{C} \right\},$$

If

$$A = \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} \tag{1.2}$$

h_A denotes the transformation (1.1), then $AGL(1, \mathbb{C}) \ni A \mapsto h_A$ is an isomorphism of $AGL(1, \mathbb{C})$ onto the group of afine transformations.

For $z_1, z_2, z_3 \in \mathbb{C}$, $z_1 \neq z_2$, $z_2 \neq z_3$ define

$$(z_1, z_2; z_3) := \frac{z_1 - z_3}{z_2 - z_3}.$$

Then for any afine transformation g,

$$(g(z_1), g(z_2); g(z_3)) = (z_1, z_2; z_3).$$

Define

$$GL(n, \mathbb{C}) := \{A \in L(n, \mathbb{C}) \ : \ \det A \neq 0\}.$$

$$SL(n, \mathbb{C}) := \{A \in L(n, \mathbb{C}) \ : \ \det A = 1\}.$$

Analogously define $GL(n, \mathbb{R})$ and $SL(n, \mathbb{R})$. We have an isomorfizm

$$GL(n, \mathbb{C}) \ni A \mapsto (A^{-1})^{\mathrm{T}} \in GL(n, \mathbb{C}).$$

For

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \tag{1.3}$$

we have

$$(A^{-1})^{\mathrm{T}} = (ad - bc)^{-1} \begin{bmatrix} d & -c \\ -b & a \end{bmatrix}.$$

If $A \in GL(2, \mathbb{C})$ i $\lambda^2 = \det A$, then

$$\lambda^{-1} A \in SL(2, \mathbb{C}).$$

For $A \in SL(2, \mathbb{C})$

$$(A^{-1})^{\mathrm{T}} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} A \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

The center of $SL(2, \mathbb{C})$ is the two-element group $Z(SL(2, \mathbb{C})) = \{1, -1\}$. We set $PSL(2, \mathbb{C}) := SL(2, \mathbb{C})/\{1, -1\}$.

1.1 Riemann sphere

Definition 1.2 *The Riemann sphere is the set*

$$\tilde{\mathbb{C}} := \mathbb{C} \cup \{\infty\}.$$

We say that $\Omega \subset \tilde{\mathbb{C}}$ is open in $\tilde{\mathbb{C}}$, if $\Omega \cap \mathbb{C}$ is open in $\mathbb{C}$ and if $\infty \in \Omega$, then there exists $R > 0$ such that

$$\mathbb{C} \backslash K(0, R) \subset \Omega.$$

In $\mathbb{C}^2 \backslash \{0\}$ we introduce the relation

$$w \sim v \;\Leftrightarrow\; \exists_{\lambda \in \mathbb{C}^\times}\; \lambda w = v.$$

It is an equivalence relation. $(\mathbb{C}^2 \backslash \{0\}) / \sim$ is denoted $\mathbb{C}\mathrm{P}$ and can be identified with $\tilde{\mathbb{C}}$:

$$\mathbb{C}\mathrm{P} \ni \mathbb{C}^\times \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \mapsto \frac{w_1}{w_2} \in \tilde{\mathbb{C}}.$$

1.2 Homografies

Definition 1.3 *A transformation $\tilde{\mathbb{C}} \ni z \mapsto h(z) \in \tilde{\mathbb{C}}$ is called a homography if it has a form*

$$h(z) = \begin{cases} \frac{az+b}{cz+d} & z \neq -\frac{d}{c}, \infty, \\ \infty & z = -\frac{d}{c}, \\ \frac{a}{c} & z = \infty. \end{cases} \quad , \tag{1.4}$$

for some $a, b, c, d \in \mathbb{C}$ i $ad - bc \neq 0$. Taką homografię będziemy też oznaczali h_A, where A is (1.3)*. We denote the set of homographies by* Homog*.*

Note that for $ad - bc = 0$, the transformation (1.4) reduces to a constant.

Theorem 1.4 *Homographies are bijections of $\tilde{\mathbb{C}}$ into itself. They form a group. The map*

$$GL(2, \mathbb{C}) \ni A \mapsto h_A \in \mathrm{Homog}$$

is a surjective homomorphism of groups. In other words,

$$h_{A_1} h_{A_2} = h_{A_1 A_2}.$$

We have

$$h_{A_1} = h_{A_2}$$

iff there exists $\lambda \in \mathbb{C} \backslash \{0\}$ such that

$$A_1 = \lambda A_2$$

Proof. If $c = 0$, then h is an afine transformation, hence a bijection. Assume that $c \neq 0$. It can be decomposed as

$$h = g_2 k g_1 \tag{1.5}$$

where

$$\begin{aligned} g_1(z) &= cz + d, \\ k(z) &= z^{-1}, \\ g_2(z) &= -\tfrac{ad-bc}{c} z + \tfrac{a}{c}. \end{aligned}$$

All these transformations are bijections, hence so is h. □

If $\lambda^2 = ac - bd \neq 0$, then replacing $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with $\frac{1}{\lambda} \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ we do not change the homography and make sure that it is parametrized by an element of $SL(2, \mathbb{C})$. For all homographies there are exactly to matrices in $SL(2, \mathbb{C})$ associated with this homography: $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $\begin{bmatrix} -a & -b \\ -c & -d \end{bmatrix}$.

Hence instead of parametrizing homographies with elements of $GL(2, \mathbb{C})$, it is better to use $SL(2, \mathbb{C})$, or even $PSL(2, \mathbb{C})$ so that

$$PSL(2, \mathbb{C}) \ni \pm A \mapsto h_{\pm A} \in \text{Homog}$$

is a group isomorphism.

The representation of $GL(2, \mathbb{C})$ in $\mathbb{C}^2$ and in $\tilde{\mathbb{C}}$ are naturally releated:

Theorem 1.5 *Let*

$$\mathbb{C}^2 \backslash \{(0,0)\} \ni \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \mapsto \pi \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} := \frac{z_1}{z_2} \in \tilde{\mathbb{C}}$$

Then

$$\pi \circ A = h_A \circ \pi$$

Proof.

$$\begin{aligned} \pi \circ A \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} &= \pi \begin{bmatrix} az_1 + bz_2 \\ cz_1 + dz_2 \end{bmatrix} \\ &= \frac{az_1 + bz_2}{cz_1 + dz_2} = \frac{a\frac{z_1}{z_2} + b}{c\frac{z_1}{z_2} + d} \\ &= h_A \left(\frac{z_1}{z_2} \right) \; = \; h_A \circ \pi \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}. \end{aligned}$$

□

1.3 Properties of homographies

Lemma 1.6 *Homographies preserving ∞ coincide with afine transformations.*

Lemma 1.7 *The homography*

$$h_1(z) := \frac{z - z_1}{z - z_2}\frac{z_3 - z_2}{z_3 - z_1}$$

transforms (z_1, z_2, z_3) *on* $(0, \infty, 1)$.

Theorem 1.8 *If* (z_1, z_2, z_3) *are three distinct points in* $\tilde{\mathbb{C}}$ *and* (w_1, w_2, w_3) *are three distinct points in* $\tilde{\mathbb{C}}$*, then there exists a unique homography* h *such that*

$$h(z_1) = w_1, \quad h(z_2) = w_2, \quad h(z_3) = w_3. \tag{1.6}$$

Proof. Let h_1 transform (z_1, z_2, z_3) on $(0, \infty, 1)$ and h_2 transform (w_1, w_2, w_3) on $(0, \infty, 1)$. Then we set $h := h_2^{-1}h_1$.

Let us show the uniqueness. First note that If $z_3 = w_3 = \infty$, then the uniquenss follows from 1.1.

Let g, k be homographies such that

$$g(0) = z_1,\ g(1) = z_2,\ g(\infty) = z_3, \qquad k(w_1) = 0,\ k(w_2) = 1,\ k(w_3) = \infty.$$

Then

$$kh_ig(0) = 0,\ kh_ig(1) = 1,\ kh_ig(\infty) = \infty,\ i = 1, 2.$$

Hence

$$kh_1g = kh_2g. \tag{1.7}$$

Multiply (1.7) from the left by k^{-1} and from the right by g^{-1}. We obtain $h_1 = h_2$. □

If z_1, z_2, z_3, z_4 is a quadruple of pairwise distinct elements of $\tilde{\mathbb{C}}$, then the number

$$(z_1, z_2; z_3, z_4) := \frac{(z_1 - z_3)(z_2 - z_4)}{(z_1 - z_4)(z_2 - z_3)}$$

is called the cross-ratio of this quadruple.

Theorem 1.9 *If* h *is a homography, then*

$$(z_1, z_2; z_3, z_4) = (h(z_1), h(z_2); h(z_3), h(z_4)).$$

If z_1, z_2, z_3, z_4 *and* w_1, w_2, w_3, w_4 *are two quadruples, then there exists a homography transforming one on the other iff* $(z_1, z_2; z_3, z_4) = (w_1, w_2; w_3, w_4)$

1.4 Generalized circles

Definition 1.10 *A compactified line in* $\tilde{\mathbb{C}}$ *is the set of the form* $\overline{L} := L \cup \{\infty\} \subset \tilde{\mathbb{C}}$*, where* L *is a line in* $\mathbb{C}$*. A generalized circle is a subset of* $\tilde{\mathbb{C}}$*, which is either a circle or a compactified line. The complements of generalized circles have two connected components. They are called generalized discs.*

Theorem 1.11 *Every generalized circle is given by an equation of the form*

$$\alpha_{11}\overline{z}z + \alpha_{12}z + \alpha_{21}\overline{z} + \alpha_{22} = 0, \tag{1.8}$$

where

$$\alpha_{11}, \alpha_{22} \in \mathbb{R},\ \ \alpha_{12} = \overline{\alpha_{21}} \in \mathbb{C},\ \ \alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21} < 0.$$

Proof. The circle of center z_0 and radius $r \in \mathbb{R}_+$ has the equation

$$|z - z_0|^2 - r^2 = 0,$$

or

$$\overline{z}z - z\overline{z}_0 - \overline{z}z_0 + \overline{z}_0 z_0 - r^2 = 0. \tag{1.9}$$

Every line can be written as $z = re + ite$, where $|e| = 1$ and $r \geq 0$. It equation is then

$$z\overline{e} + \overline{z}e - 2r = 0. \tag{1.10}$$

Both (1.9) as well as (1.10) have the form (1.8). □

Proposition 1.12 *Homographies transform generalized circles on generalized circles and generalized discs on generalized discs. A quadruple* z_1, z_2, z_3, z_4 *is located on a generalized circl iff* $(z_1, z_2; z_3, z_4) \in \mathbb{R}$*.*

Proof. By (1.5), it is enough to check this for afine transformations, which is obvious and for inversions. If $w = \frac{1}{z}$, then in the variable w the equation (1.8) is

$$\alpha_{11} + \alpha_{12}\overline{w} + \alpha_{21}w + \alpha_{22}\overline{w}w = 0,$$

and also has the form (1.8). □

1.5 Group $U(2)$

In $\mathbb{C}^2$ we define the canonical Hermitian scaar product

$$(z|w) = \overline{z_1}w_1 + \overline{z_2}w_2.$$

The unitary group is defined as

$$U(2) := \{A \in M(2,\mathbb{C}) \ : \ (Az|Aw) = (z|w), \ z,w \in \mathbb{C}^2\}.$$

Equivalently,: $A^*A = 1$. $U(2)$ consists of matrices satisfying

$$\begin{cases} |a|^2 + |c|^2 = 1, \\ a\overline{b} + c\overline{d} = 0, \\ |b|^2 + |d|^2 = 1. \end{cases} \tag{1.11}$$

Theorem 1.13 *(1) Matrices of $U(2)$ have the form*

$$\begin{bmatrix} a & b \\ -\lambda\overline{b} & \lambda\overline{a} \end{bmatrix}, \tag{1.12}$$

gdzie

$$|a|^2 + |b|^2 = 1, \quad |\lambda| = 1.$$

Besides, λ is the determinant of (1.12).
(2) Matrices of $SU(2)$ have the form

$$\begin{bmatrix} a & b \\ -\overline{b} & \overline{a} \end{bmatrix},$$

where

$$|a|^2 + |b|^2 = 1.$$

Proof. Let

$$\frac{a}{\overline{d}} = -\frac{c}{\overline{b}} = \overline{\lambda}.$$

Then $\overline{b} = -\overline{\lambda}c$, $\overline{d} = \overline{\lambda}a$. We insert this to the third formula of (1.11) obtaining

$$|\lambda|^2(|c|^2 + |a|^2) = 1$$

Hence $|\lambda| = 1$. This implies $c = -\lambda\overline{b}$. Therefore, a matrix in $U(2)$ has the form (1.12), where $|\lambda| = 1$. The determinant of (1.12) equals $\lambda(|a|^2 + |b|^2) = \lambda$. Hence for $SU(2)$ we have $\lambda = 1$. □

1.6 Grops $U(1,1)$ and $SL(2,\mathbb{R})$

Let

$$I_{1,1} := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

In $\mathbb{C}^2$ we define a Hermitian pseudoscalar product

$$(z|I_{1,1}w) = \overline{z_1}w_1 - \overline{z_2}w_2.$$

The pseudounitary group of signature $1,1$ is defined as

$$U(1,1) := \{A \in M(2,\mathbb{C}) \ : \ (Az|I_{1,1}Aw) = (z|I_{1,1}w), \quad z,w \in \mathbb{C}^2\}.$$

Equivalently,

$$A^* I_{1,1} A = I_{1,1},$$

or

$$\begin{cases} |a|^2 - |c|^2 = 1, \\ a\overline{b} - c\overline{d} = 0, \\ |b|^2 - |d|^2 = -1. \end{cases} \tag{1.13}$$

Theorem 1.14 *(1) Matrices in $U(1,1)$ have the form*

$$\begin{bmatrix} a & b \\ \lambda\overline{b} & \lambda\overline{a} \end{bmatrix}, \tag{1.14}$$

where

$$|\lambda| = 1, \qquad |a|^2 - |b|^2 = 1.$$

Besides, λ is the determinant of (1.14).
(2) Matrices of $SU(1,1)$ have the form

$$\begin{bmatrix} a & b \\ \overline{b} & \overline{a} \end{bmatrix},$$

where

$$|a|^2 - |b|^2 = 1.$$

Proof. Let

$$\frac{a}{\overline{d}} = \frac{c}{\overline{b}} = \overline{\lambda}.$$

Then $\overline{b} = \overline{\lambda}c$, $\overline{d} = \overline{\lambda}a$. We insert this into the third formula of (1.13) obtaining

$$|\lambda|^2(|c|^2 - |a|^2) = -1.$$

Hence, $|\lambda| = 1$. This implies $c = \lambda\overline{b}$. Therefore, every matrix in $U(1,1)$ has the form (1.14). The determinant of (1.14) equals $\lambda(|a|^2 - |b|^2) = \lambda$. □

Theorem 1.15 *Let*

$$B = \frac{1}{2}\begin{bmatrix} 1 & -\mathrm{i} \\ -\mathrm{i} & 1 \end{bmatrix}, \quad B^{-1} = \begin{bmatrix} 1 & \mathrm{i} \\ \mathrm{i} & 1 \end{bmatrix}. \tag{1.15}$$

Then

$$SL(2, \mathbb{R}) \ni A \mapsto BAB^{-1} \in SU(1,1),$$

is an isomorphism.

Proof. We have

$$BAB^{-1} = \frac{1}{2}\begin{bmatrix} a + \mathrm{i}b - \mathrm{i}c + d & \mathrm{i}a + b + c - \mathrm{i}d \\ -\mathrm{i}a + b + c + \mathrm{i}d & a - \mathrm{i}b + \mathrm{i}c + d \end{bmatrix} = \begin{bmatrix} a_1 & b_1 \\ \overline{b_1} & \overline{a}_1 \end{bmatrix}$$

and

$$|a_1|^2 - |b_1|^2 = ad - bc = 1$$

□

1.7 Homographies mapping antipodal pairs on antipodal pairs

In $\tilde{\mathbb{C}}$ we define the "antipodal conjugation"

$$j(z) := -\frac{1}{\bar{z}}.$$

For $z = re^{\mathrm{i}\phi}$ we have $j(z) = r^{-1}e^{\mathrm{i}(\phi+\pi)}$. Clearly, $j^2(z) = z$. A pair of points $\{z_1, z_2\}$ such that $j(z_1) = z_2$ (and hence also $j(z_2) = z_1$) is called an antipodal pair.

Theorem 1.16 *Homographie transforming antipodal pairs onto antipodal pairs, or equivalently, satisfying*

$$hj = jh$$

have the formi h_A dwith $A \in SU(2)$.

Proof. Let $A \in GL(2, \mathbb{C})$ and $h = h_A$. Then

$$j(h(z)) = j\left(\frac{az + b}{cz + d}\right) = \frac{\bar{d}j(z) - \bar{c}}{-\bar{b}j(z) + \bar{a}} = h(j(z)) = \left(\frac{aj(z) + b}{cj(z)1 + d}\right).$$

Hence,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \lambda \begin{bmatrix} \bar{d} & -\bar{c} \\ -\bar{b} & \bar{a} \end{bmatrix}$$

Therefore, $|\lambda| = 1$ and $c = -\lambda\bar{b}$, $d = \lambda\bar{a}$. We obtain an element of $U(2)$. □

1.8 Homographies preserving a generalized disc

Theorem 1.17 *All homographies transforming $\mathbb{C}_+$ in itself have the form h_A for $A \in SL(2, \mathbb{R})$.*

Proof. First note that h and h^{-1} are continuous. Hence the image of the boundary is the boundary of the image. $\tilde{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$ is the boundary of $\mathbb{C}_+$. Hence, $h(\overline{\mathbb{R}}) = \overline{\mathbb{R}}$. By Lemma 1.7 and the uniqueness we see that $h = h_A$ for some $A \in GL(2, \mathbb{R})$. We have

$$\begin{aligned} h_A(z) &= \tfrac{az+b}{cz+d} \\ &= \tfrac{ac|z|^2+(ad+cb)\mathrm{Re}z+bd+i(ad-cb)\mathrm{Im}z}{|cz+d|^2} \end{aligned}$$

Hence $h_A(\mathbb{C}_+) \subset \mathbb{C}_+$ iff $\det A > 0$. Therefore, we can replace A with $(\det A)^{-\frac{1}{2}} A \in SL(2, \mathbb{R})$. □

Theorem 1.18 *All homographies transforming the unit disc $\{z \ : \ |z| < 1\}$ into itself have the form h_A for $A \in SU(1, 1)$.*

Proof. Recall that

$$SL(2, \mathbb{R}) \ni A \mapsto BAB^{-1} = \tilde{A} \in SU(1, 1),$$

is and isomorphism, where B was defined in (1.15). Hence

$$h_{\tilde{A}} = h_B h_A (h_B)^{-1}.$$

Thus, by Theorem 1.17, it is enough to show that

$$h_B(z) = \frac{z - \mathrm{i}}{-\mathrm{i}z + 1}$$

transforms $\mathbb{C}_+$ onto $\{z \ : \ |z| < 1\}$. Indeed,

$$\left| \frac{x + \mathrm{i}y - \mathrm{i}}{1 + y - \mathrm{i}x} \right|^2 = \frac{x^2 + (y - 1)^2}{x^2 + (y + 1)^2} < 1 \ \Leftrightarrow \ z \in \mathbb{C}_+.$$

□

2 Separation of variables in the Helmholtz equation in 2 dimensions

2.1 Holomorphic functions

Let us identify $\mathbb{R}^2$ with $\mathbb{C}$ by the transformation $\mathbb{R}^2 \ni (x,y) \mapsto z := x + \mathrm{i}y \in \mathbb{C}$. We introduce the following operators acting on functions on $\mathbb{C}$

$$\partial_z := \frac{1}{2}(\partial_x - i\partial_y), \quad \partial_{\overline{z}} := \frac{1}{2}(\partial_x + i\partial_y).$$

Note that

$$\begin{aligned} \partial_z z = 1, &\qquad \partial_z \overline{z} = 0, \\ \partial_{\overline{z}} z = 0, &\qquad \partial_{\overline{z}} \overline{z} = 1, \\ \partial_x = \partial_z + \partial_{\overline{z}}, &\quad \partial_y = \mathrm{i}\partial_z - \mathrm{i}\partial_{\overline{z}}. \end{aligned}$$

$$\mathrm{d}f = \partial_z f \mathrm{d}z + \partial_{\overline{z}} f \mathrm{d}\overline{z}.$$

Proposition 2.1 *Let $\Omega \subset \mathbb{C}$ be open and Let*

$$\Omega \ni z = x + \mathrm{i}y \mapsto u + \mathrm{i}v = f \in \mathbb{C} \tag{2.1}$$

be a smooth function. The following conditions are equivalent

1. *For any $z \in \Omega$ there exists the derivative in the complex sense*

$$\lim_{h \to 0} \frac{f(z+h) - f(z)}{h} =: f'(z). \tag{2.2}$$

2. *The Cauchy-Riemann conditions hold*

$$\partial_x u = \partial_y v, \qquad \partial_y u = -\partial_x v. \tag{2.3}$$

3. *$\partial_{\overline{z}} f = 0$. If this is the case, then $\partial_z f = f'$ in the sense of (2.2).*

4. *For any $z \in \Omega$ there exists $r > 0$ such that f is given by a convergent power series in $K(z,r)$.*

If the above conditions hold we say that f is holomorphic in Ω. The word analytic is used as a synonym of holomorphic.

2.2 Cauchy-Riemann conditions and conformal transformations

Let

$$\Omega \ni z = x + \mathrm{i}y \mapsto u + \mathrm{i}v = f \in \mathbb{C} \tag{2.4}$$

be a function. We say that it is a conformal map if the following holds. If $[0,1] \ni \tau \mapsto (x_i(\tau), y_i(\tau))), i = 1,2$, are two curves starting at $(x_1(0), y_1(0))) = (x_2(0), y_2(0)))$, and $[0,1] \ni \tau \mapsto (u_i(\tau), v_i(\tau)))$ are their images, then

$$\frac{(\partial_\tau x_1(0), \partial_\tau y_1(0)) \cdot (\partial_\tau x_2(0), \partial_\tau y_2(0))}{\| (\partial_\tau x_1(0), \partial_\tau y_1(0)) \| \| (\partial_\tau x_2(0), \partial_\tau y_2(0)) \|}$$
$$= \frac{(\partial_\tau u_1(0), \partial_\tau v_1(0)) \cdot (\partial_\tau u_2(0), \partial_\tau v_2(0))}{\| (\partial_\tau u_1(0), \partial_\tau v_1(0)) \| \| (\partial_\tau u_2(0), \partial_\tau v_2(0)) \|}$$

Let (2.4) be analytic. The analyticity is equivalent to the Cauchy-Riemann conditions, which imply

$$(\partial_x^2 + \partial_y^2)u = (\partial_x^2 + \partial_y^2)v = 0, \tag{2.5}$$
$$(\partial_x u)^2 + (\partial_y u)^2 = (\partial_x v)^2 + (\partial_y v)^2 = (\partial_x u)^2 + (\partial_x v)^2 = |\partial_z f|^2, \tag{2.6}$$
$$\partial_x u \partial_x v + \partial_y u \partial_y v = \partial_x u \partial_x v - \partial_x v \partial_x u = 0. \tag{2.7}$$

(2.5) mean that the functions u and v are harmonic. The conditions (2.6) an (2.7) mean that the vectors $(\partial_x u, \partial_y u)$ and $(\partial_x v, \partial_y v)$ are orthogonal and of the same length. The determinant of the matrix

$$\begin{bmatrix} \partial_x u & \partial_y u \\ \partial_x v & \partial_y v \end{bmatrix} \tag{2.8}$$

is $|\partial_z f|^2$. Hence this is a (proper) rotation matrix times $|\partial_z f|$. Therefore, if $|\partial_z f| \neq 0$, then the transformation (2.4) preserves angles—is conformal.

2.3 Antiholomorphic functions

We say that a function $z \mapsto f$ is antiholomorphic if $z \mapsto \overline{f(z)}$ is holomorphic. Clearly, it is equivalent to $\partial_z f = 0$. The Cauchy-Riemann conditions are replaced by

$$\partial_x u = -\partial_y v, \qquad \partial_y u = \partial_x v. \tag{2.9}$$

The matrix (2.8) has the determinant $-|\partial_z f|^2$. Hence it is proportional to and improper rotation.

If f is anti-holomorphic and $|\partial_{\bar{z}} f| \neq 0$, then f is conformal.

2.4 Helmholtz equation in Cartesian coordinates

The Helmholtz equation in 2 dimensions has the form

$$(\partial_x^2 + \partial_y^2 + E)g, \tag{2.10}$$

where E is a parameter. It has many solutions. To distinguish interesting solutions one has to add boundary conditions, e.g. the Dirichlet conditions on the boundary of a domain Ω.

In Cartesian coordinates it is convenient to solve the Helmholtz equation in a rectangle, e.g. $[0, A] \times [0, B]$. We then use the ansatz

$$g(x, y) = p(x)q(y),$$

and we obtain

$$\frac{1}{p(x)}\left(\partial_x^2 + E\right)p(x) = -\frac{1}{q(y)}\partial_y^2 q(y). \tag{2.11}$$

The lhs does not depend on y and the rhs does not depend on x. Hence (2.11) equals a constant C, which leads to

$$\begin{aligned}(\partial_x^2 + E - C)p(x) &= 0,\\ (\partial_y^2 + C)q(y) &= 0.\end{aligned}$$

The Dirichlet boundary conditions mean

$$p(0) = p(A) = 0, \quad q(0) = q(B) = 0,$$

which yields

$$p(x) = \sin n\pi\frac{x}{A}, \quad q(y) = \sin m\pi\frac{y}{B}, \quad E - C = \frac{n^2\pi^2}{A^2}, \quad C = \frac{m^2\pi^2}{B^2}.$$

In particular, $E = \pi^2(\frac{n^2}{A^2} + \frac{m^2}{B^2})$.

If Ω is a disc or an anulus, or their sector, it is more convenient to use polar coordinates. More generally, in the coordinates $u(x, y), v(x, y)$ it is convenient to solve differential equations in a domain $\Omega = \{(x, y) \ : \ u_0 < u(x, y) < u_1, \ v_0 < v(x, y) < v_1\}$.

2.5 Change of coordinates in the Laplacian

Let Ω be an open subset of $\mathbb{R}^2$ and

$$\Omega \ni (x, y) \mapsto (u, v) \in \mathbb{R}^2 \tag{2.12}$$

be a smooth transformation. We then have

$$\begin{aligned}(\partial_x^2+\partial_y^2)g &= \partial_x(\partial_x u\partial_u g+\partial_x v\partial_v g)+\partial_y(\partial_y u\partial_u g+\partial_y v\partial_v g)\\ &= (\partial_x^2+\partial_y^2)u\ \partial_u g+(\partial_x^2+\partial_y^2)v\ \partial_v g\\ &\quad +(\partial_x u\partial_x u+\partial_y u\partial_y u)\partial_u^2 g\\ &\quad +2(\partial_x u\partial_x v+\partial_y u\partial_y v)\partial_u\partial_v g,\\ &\quad +(\partial_x v\partial_x v+\partial_y v\partial_y v)\partial_v^2 g.\end{aligned} \tag{2.13}$$

Assume now that $z = x + \mathrm{i}y \mapsto u + \mathrm{i}v = f$ is an analytic function. The Cauchy-Riemann conditions imply (2.5), (2.6) and (2.7). Inserting this in (2.13) we obtain

$$(\partial_x^2+\partial_y^2)g = |\partial_z f|^2(\partial_u^2+\partial_v^2)g. \tag{2.14}$$

2.6 Helmholtz equation in polar coordinates

Let us analyze the change of coordinates (2.12) to the Helmoholtz equation. It will be more convenient to focus on the inverse transformation

$$(u, v) \mapsto (x, y) \tag{2.15}$$

and their interpretation in terms of the complex variable

$$f = u + \mathrm{i}v \mapsto x + \mathrm{i}y = z.$$

The equation (2.14) can then be rewritten as

$$|\partial_f z|^2(\partial_x^2+\partial_y^2) = \partial_u^2+\partial_v^2. \tag{2.16}$$

Consider the function $z = \mathrm{e}^f$, that means,

$$x = \mathrm{e}^u\cos v, \quad y = \mathrm{e}^u \sin v.$$

Then $\partial_f z = \mathrm{e}^f$, hence $|\partial_f z|^2 = \mathrm{e}^{2u}$. Therefore,

$$\partial_x^2+\partial_y^2 = \mathrm{e}^{-2u}(\partial_u^2+\partial_v^2). \tag{2.17}$$

To reduce it to a better known form, we substitute $r = \mathrm{e}^u$ and we rename v into ϕ. We then have $\partial_u = r\partial_r$ and (2.17) reduces to the well-known expression of the Laplacian in polar coordinates:

$$\partial_x^2+\partial_y^2 = \partial_r^2+\frac{1}{r}\partial_r+\frac{1}{r^2}\partial_\phi^2. \tag{2.18}$$

Let us now apply the following ansatz:

$$g(r,\phi) = p(r)q(r). \tag{2.19}$$

Then the Helmholtz equation

$$\left(r^2\partial_r^2 + r\partial_r + \partial_\phi^2 + r^2E\right) p(r)q(\phi)$$

after dividing by $p(r)q(\phi)$ can be rewritten as

$$\frac{1}{p(r)}\left(r^2\partial_r^2 + r\partial_r + r^2E\right) p(r) = -\frac{1}{q(\phi)}\partial_\phi^2 q(\phi) \tag{2.20}$$

The lhs of (2.20) does not depend on r and the rhs does not depend on ϕ. Therefore, (2.20) equals a constant, which we can call C. (2.20) separates into two equations

$$\left(r^2\partial_r^2 + r\partial_r - C + r^2E\right) p(r) \quad = \quad 0, \tag{2.21}$$

$$\left(\partial_\phi^2 + C\right) q(\phi) \tag{2.22}$$

(2.22) is solved by

$$q(\phi) = \mathrm{e}^{\mathrm{i}m\phi}, \quad m^2 = C.$$

Hence (2.21) can be rewritten as

$$\left(r^2\partial_r^2 + r\partial_r - m^2 + r^2E\right) p(r) = 0. \tag{2.23}$$

It can be reduced to the (standard or modified) Bessel equation.

2.7 Helmholtz equation in parabolic coordinates

Consider now the system of coordinates generated by the analytic transformation $z = \frac{f^2}{2}$. We have then

$$x = \frac{u^2 - v^2}{2}, \quad y = uv,$$

$\partial_f z = f$, $|\partial_f z|^2 = u^2 + v^2$. We can restrict the coordinates to $v > 0$. The Laplacian is transformed as follows:

$$\partial_x^2 + \partial_y^2 = (u^2 + v^2)^{-1}(\partial_u^2 + \partial_v^2).$$

Hence the Helmholtz equation in parabolic coordinates has the form

$$\left(\partial_u^2 + \partial_v^2 + E(u^2 + v^2)\right)g = 0. \tag{2.24}$$

The ansatz

$$g(u,v) = p(u)q(v). \tag{2.25}$$

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$$\frac{1}{p(u)}\left(\partial_u^2 + Eu^2\right)p(u) = -\frac{1}{q(v)}\left(\partial_v^2 + Ev^2\right)q(v) = C \tag{2.26}$$

which can be written as

$$\left(\partial_u^2 + Eu^2 - C\right)p(u) \;=\; 0, \tag{2.27}$$

$$\left(\partial_v^2 + Ev^2 + C\right)q(v) \;=\; 0, \tag{2.28}$$

which reduce to the harmonic oscillator eigenequation, also called the Weber equation.

2.8 Helmholtz equation in elliptic-hyperbolic coordinates

Consider now the coordinate system generated by the analytic function $z = \cosh f$. We then have

$$x = \cosh u \cos v, \quad y = \sinh u \sin v,$$

$\partial_f z = \sinh f = \sinh u \cos v + \mathrm{i} \cosh u \sin v$, $|\partial_f z|^2 = \sinh^2 u + \sin^2 v$. The Laplacian is transformed as follows:

$$\partial_x^2 + \partial_y^2 = (\sinh^2 u + \sin^2 v)^{-1}(\partial_u^2 + \partial_v^2).$$

The Helmholtz equation becomes

$$(\partial_u^2 + \partial_v^2 + E(\sinh^2 u + \sin^2 v))g. \tag{2.29}$$

The ansatz (2.25) yields

$$\frac{1}{p(u)}\left(\partial_u^2 + E\sinh^2 u\right)p(u) = -\frac{1}{q(v)}\left(\partial_v^2 + E\sin^2 v\right)q(v) = C \tag{2.30}$$

which can be transformed into

$$\left(\partial_u^2 + E\sinh^2 u - C\right)p(u) \;=\; 0, \tag{2.31}$$

$$\left(\partial_v^2 + E\sin^2 v + C\right)q(v) \;=\; 0, \tag{2.32}$$

and reduces to the (standard or modified) Mathieu equation.

2.9 Transformations in $\mathbb{R}^2$ separating the Helmholtz equation

Let

$$(x, y) \mapsto (u, v) \in \mathbb{R}^2$$

be a transformation such that $z = x + \mathrm{i}y \mapsto u + \mathrm{i}v = f$ is analytic. The Helmholtz equation

$$(\partial_x^2 + \partial_y^2)g = Eg,$$

in the new coordinates is

$$(\partial_u^2 + \partial_v^2)g = |\partial_f z|^2 Eg.$$

This equation separates if $|\partial_f z|^2$ has the form

$$|\partial_f z|^2 = a(u) + b(v).$$

This is equivalent to the condition

$$\partial_u \partial_v |\partial_f z|^2 = 0.$$

We are using now the variable $f = u + \mathrm{i}v$, so the analogs of the operators ∂_z i $\partial_{\overline{z}}$ are now

$$\partial_f := \frac{1}{2}(\partial_u - i\partial_v), \quad \partial_{\overline{f}} := \frac{1}{2}(\partial_u + i\partial_v).$$

Using $\overline{\partial_f z} = \partial_{\overline{f}} \overline{z}$ and

$$\partial_u = \partial_f + \partial_{\overline{f}}, \qquad \partial_v = \mathrm{i}\partial_f - \mathrm{i}\partial_{\overline{f}},$$

we obtain

$$\begin{aligned} 0 = \partial_u \partial_v |\partial_f z|^2 &= \mathrm{i}(\partial_f^2 - \partial_{\overline{f}}^2)\partial_f z \partial_{\overline{f}} \overline{z} \\ &= \mathrm{i}\left(\partial_f^3 z \partial_{\overline{f}} \overline{z} - \partial_f z \partial_{\overline{f}}^3 \overline{z}\right). \end{aligned}$$

Hence,

$$\frac{\partial_f^3 z}{\partial_f z} = \frac{\partial_{\overline{f}}^3 \overline{z}}{\partial_{\overline{f}} \overline{z}}. \tag{2.33}$$

The lhs of (2.33) is holomorphic and the rhs is antiholomorphic. Hence (2.33) equals a constant, which we call D and we obtain the equation

$$\partial_f^3 z = D \partial_f z. \tag{2.34}$$

Let us classify solutions of (2.34) up to translations $z \mapsto z + a$, rotations $z \mapsto \mathrm{e}^{\mathrm{i}\alpha} z$ and scaling $z \mapsto \lambda z$.

(1) $D = 0$. Then $z = Af^2 + Bf + C$. If $A = 0$, it is a trivial change of coordinates. If $A \neq 0$, this reduces to $z = f^2$.

(2) $D \neq 0$, $z = A\mathrm{e}^{\sqrt{D}f}$. The change of coordinates reduces to $z = \mathrm{e}^f$.

(3) $D \neq 0$, $z = A\mathrm{e}^{\sqrt{D}f} + B\mathrm{e}^{-\sqrt{D}f}$. The change of coordinates reduces to $z = \cosh f$.

3 Euler's Gamma function

3.1 The Gamma function as a generalization of the factorial and the II Euler's integral

We define the Gamma function by the II Euler's integral:

$$\Gamma(z) := \int_0^\infty \mathrm{e}^{-t} t^{z-1} \mathrm{d}t \tag{3.1}$$

$$= 2\int_0^\infty \mathrm{e}^{-\xi^2} \xi^{2z-1} \mathrm{d}\xi, \quad \mathrm{Re} z > 0, \tag{3.2}$$

t^{z-1} denotes the principal branch of the power function. We introduce also the Pochhammer symbol:

$$\begin{aligned} (a)_0 &:= 1, \\ (a)_n &:= a(a+1)\dots(a+n-1), && n = 0,1,2,\dots \\ (a)_n &:= \tfrac{1}{(a+n)\dots(a-1)}, && n = \dots,-2,-1. \end{aligned}$$

Obviously, $(1)_n = n!$.

Theorem 3.1 *The following identities hold:*

$$\Gamma(z+1) = z\Gamma(z), \tag{3.3}$$

$$\Gamma(n+1) = n!, \quad n = 0,1,2,\dots, \tag{3.4}$$

$$\Gamma(z+n) = (z)_n \Gamma(z), \quad n \in \mathbb{Z}. \tag{3.5}$$

Proof. (3.3) follows by integration by parts. (3.4) follows from (3.3) and $\Gamma(1) = 1$. □

Define the set

$$\Omega_n := \{z \ : \ \mathrm{Re} z > -n\} \backslash \{0, -1, \dots, -n+1\}$$

and the function

$$\Omega_n \ni z \mapsto \Gamma_n(z) := \frac{\Gamma(z+n)}{(z)_n}.$$

If $n > m$, then

$$\Gamma_n(z) = \Gamma_m(z), \;\; z \in \Omega_m,$$

which follows from the identity (3.74) On the set

$$\bigcup_{n=1}^{\infty} \Omega_n = \mathbb{C}\backslash\{0, -1, -2, \dots\}$$

we define

$$\Gamma(z) := \Gamma_n(z), \;\; z \in \Omega_n.$$

Thus defined function Γ is the maximal analytic extension of the function $\Gamma(z)$ defined with the integral (3.2).

There exist alternative ways of extending the Gamma function:

Theorem 3.2 (Prym decomposition)

$$\Gamma(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n+z)} + \int_1^{\infty} \mathrm{e}^{-t} t^{z-1} \mathrm{d}t, \qquad z \in \mathbb{C}\backslash\{0, -1, -2, \dots\}.$$

Theorem 3.3 (Cauchy-Saalschütz formula)

$$\Gamma(z) = \int_0^{\infty} t^{z-1} \Big(\mathrm{e}^{-t} - \sum_{k=0}^{n} \frac{(-t)^k}{k!} \Big) \mathrm{d}t, \qquad -1-n < \mathrm{Re} z. \tag{3.6}$$

Proof. Let $\Gamma_n(z)$ be the rhs of (3.6). First we check that it is an analytic function on $-1-n < \mathrm{Re}z$. Let us use the induction. Clearly,

$$\Gamma_{-1}(z) = \Gamma(z), \quad 0 < \mathrm{Re}z.$$

Integrating by parts we obtain

$$\Gamma_n(z) = \frac{t^z}{z} \Big(\mathrm{e}^{-t} - \sum_{k=0}^{n} \frac{(-t)^k}{k!} \Big) \Big|_0^{\infty} + \frac{1}{z} \int_0^{\infty} t^z \Big(\mathrm{e}^{-t} - \sum_{k=0}^{n-1} \frac{(-t)^k}{k!} \Big) \mathrm{d}t \tag{3.7}$$

$$= \frac{1}{z} \Gamma_{n-1}(z+1) = \frac{1}{z} \Gamma(z+1) = \Gamma(z), \tag{3.8}$$

where at the end we used the induction assumption, and then identity (3.13). □

3.2 I Euler's integral and the Beta function

Theorem 3.4 (1st Euler's integral)

$$\frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)} = \int_0^1 t^{u-1}(1-t)^{v-1}\mathrm{d}t \tag{3.9}$$

$$= 2\int_0^{\frac{\pi}{2}} \cos^{2u-1}\phi \sin^{2v-1}\phi \mathrm{d}\phi, \quad \mathrm{Re}u > 0,\ \mathrm{Re}v > 0.$$

$$\frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)} \frac{\sin\pi u}{\sin\pi(u+v)} = \frac{\Gamma(1-u-v)\Gamma(v)}{\Gamma(1-u)} = \int_0^\infty (t+1)^{u-1}t^{v-1}\mathrm{d}t \tag{3.10}$$

$$= 2\int_0^\infty \cosh^{2u-1}\theta \sinh^{2v-1}\theta, \quad \mathrm{Re}v > 0,\ \mathrm{Re}(1-u-v) > 0.$$

Proof. Substituting $t = \frac{1}{s} + 1$ into (3.9) and using (3.14) we obtain (3.10). Let us prove (3.9)

We have

$$\Gamma(u)\Gamma(v) = 4\int_0^\infty \int_0^\infty \mathrm{e}^{-\xi^2-\eta^2}\xi^{2u-1}\eta^{2v-1}\mathrm{d}\xi\mathrm{d}\eta. \tag{3.11}$$

We use polar coordinates

$$\xi = r\cos\phi,\ \eta = r\sin\phi.$$

Thus (3.11) equals

$$4\int_0^\infty \mathrm{e}^{-r^2} r^{2u+2v-1}\mathrm{d}r \int_0^{\pi/2} \cos^{2u-1}\phi \sin^{2v-1}\phi \mathrm{d}\phi \tag{3.12}$$

$$= \Gamma(u+v)\int_0^1 t^{u-1}(1-t)^{v-1}\mathrm{d}t. \tag{3.13}$$

(In the last step we substituted $t = \cos^2\phi$). □

Motivated by (3.9) one often introduces the so called Beta function:

$$B(u,v) := \frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)}.$$

Theorem 3.5 *The following identities hold:*

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin\pi z}, \tag{3.14}$$

$$\Gamma(1/2) = \sqrt{\pi} \tag{3.15}$$

Proof. Assume for the moment that $0 < \mathrm{Re} z < 1$. Consider the holomorphic function $\mathbb{C}\backslash[0,1] \ni t \mapsto f(t) = t^{z-1}(t-1)^{-z}$ (The functions t^{z-1} and $(t-1)^{-z}$ are understood in terms of their principal branches defined resp. on $\mathbb{C}\backslash[-\infty, 0[$ i $\mathbb{C}\backslash[-\infty, 1[$. Hence $f(t)$ is defined a priori on $\mathbb{C}\backslash[-\infty, 1[$, but it analytically extends to $\mathbb{C}\backslash[0,1]$).

Let $\gamma = [0, 1^+, 0^+]$ be the contour called the *bone*. The residuum at infinity of f is -1, hence

$$2\mathrm{i}\pi = -\, 2\pi\mathrm{i}\mathrm{Res} f(\infty) = \int_\gamma f(t)\mathrm{d}t \tag{3.16}$$

$$= (\mathrm{e}^{\mathrm{i}\pi z} - \mathrm{e}^{-\mathrm{i}\pi z}) \int_0^1 t^{z-1}(1-t)^{-z}\mathrm{d}t \tag{3.17}$$

$$= (2\mathrm{i}\sin\pi z)\mathrm{B}(z, 1-z) = (2\mathrm{i}\sin\pi z)\Gamma(z)\Gamma(1-z). \tag{3.18}$$

This implies (3.14) for $0 < \mathrm{Re} z < 1$. We extend it to all $z \in \mathbb{C}$ by analyticity.

Substituting in (3.14) $z = 1/2$ we obtain

$$\Gamma^2(1/2) = \pi.$$

We know that

$$\Gamma(z) > 0, \ z > 0.$$

Ths implies (3.15). □

Corollary 3.6 (Gauss integral) *If* $\mathrm{Re} a > 0$*, then*

$$\int_{-\infty}^{\infty} \mathrm{e}^{-at^2}\mathrm{d}t = \sqrt{\frac{\pi}{a}}. \tag{3.19}$$

Proof. Changing the variables in (3.15) we obtain

$$\sqrt{\pi} = \Gamma(1/2) = \int_{]-\infty,\infty[} \mathrm{e}^{-t^2}\mathrm{d}t \tag{3.20}$$

$$= \int_{]-\sqrt{a}\infty,\sqrt{a}\infty[} \mathrm{e}^{-t^2}\mathrm{d}t \tag{3.21}$$

$$= \sqrt{a}\int_{]-\infty,\infty[} \mathrm{e}^{-as^2}\mathrm{d}s. \tag{3.22}$$

□

Corollary 3.7 (Fresnel integral) *We have*

$$\lim_{R\to\infty}\int_{-R}^{R} \mathrm{e}^{\pm \mathrm{i}x^2}\mathrm{d}x = \mathrm{e}^{\pm\mathrm{i}\frac{\pi}{4}}\sqrt{\pi}.$$

Proof. We integrate on the sides of the triangle $0, R, R+\mathrm{i}R$. On the vertical side we have

$$\int_0^R \mathrm{e}^{-R^2+y^2}\mathrm{d}y = \int_0^1 \mathrm{e}^{-R^2(1-t^2)}R\mathrm{d}t \to 0,$$

using the Lebesgue Theorem. □

Note that the function $\Gamma(z)$ has 1st order poles at $z = 0, -1, \ldots$ with the residues

$$\mathrm{Res}\Gamma(-n) = \lim_{z\to -n}\Gamma(z)(z+n) \tag{3.23}$$

$$= \frac{(z+n)\pi}{\Gamma(1-z)\sin\pi z} = \frac{(-1)^n}{n!}. \tag{3.24}$$

Theorem 3.8 (The Legendre duplication formula)

$$2^{2z-1}\Gamma(z)\Gamma\left(z+1/2\right) = \sqrt{\pi}\Gamma(2z),$$

Proof.

$$\frac{\Gamma(z)^2}{\Gamma(2z)} = \int_0^1 t^{z-1}(1-t)^{z-1}\mathrm{d}t = 2\int_0^{1/2} t^{z-1}(1-t)^{z-1}\mathrm{d}t.$$

Substituting $s = 4t(1-t)$ we obtain

$$2^{1-2z}\int_0^1 s^{z-1}(1-s)^{-1/2}\mathrm{d}s = 2^{1-2z}\frac{\Gamma(z)\Gamma(\frac{1}{2})}{\Gamma(z+\frac{1}{2})}.$$

□

There exists a generalization of the above identity called the Gauss multiplication formula, which we will prove later:

$$\Gamma(nz) = (2\pi)^{\frac{1-n}{2}} n^{nz-\frac{1}{2}}\prod_{k=0}^{n-1}\Gamma\Big(z+\frac{k}{n}\Big). \tag{3.25}$$

3.3 The Gamma function and integrals in the complex domain

Theorem 3.9 (The Hankel Formula)

$$\frac{1}{\Gamma(z+1)} = \frac{1}{2\pi \mathrm{i}} \int\limits_{[-\infty,0^+,-\infty[} \mathrm{e}^s s^{-z-1} \mathrm{d}s. \tag{3.26}$$

Proof. Assume temporarily that $\mathrm{Re} z < 0$.

$$\int\limits_{[-\infty,0^+,-\infty[} \mathrm{e}^s s^{-z-1} \mathrm{d}s = \mathrm{e}^{-\mathrm{i}\pi(-z-1)} \int_{]-\infty,0]} \mathrm{e}^s (-s)^{-z-1} \mathrm{d}s \tag{3.27}$$

$$+ \mathrm{e}^{\mathrm{i}\pi(-z-1)} \int_{[0,-\infty[} \mathrm{e}^s (-s)^{-z-1} \mathrm{d}s \tag{3.28}$$

$$= \left(\mathrm{e}^{-\mathrm{i}\pi(-z-1)} - \mathrm{e}^{\mathrm{i}\pi(-z-1)}\right) \int_0^\infty \mathrm{e}^{-t} t^{-z-1} \mathrm{d}t \tag{3.29}$$

$$= \mathrm{i}2\sin(-\pi z)\Gamma(-z) = \frac{2\pi \mathrm{i}}{\Gamma(z+1)} \tag{3.30}$$

Then we extend the identity to all z by analytic continuation. □

Theorem 3.10

$$\frac{\Gamma(u+v+1)}{\Gamma(u+1)\Gamma(v+1)} = \frac{1}{2\pi \mathrm{i}} \int_{]-\infty,0^+,-\infty[} t^{-u-1}(1-t)^{-v-1} \mathrm{d}t \tag{3.31}$$

$$= \frac{1}{2\pi \mathrm{i}} \int_{]\infty,1^-,\infty[} t^{-u-1}(1-t)^{-v-1} \mathrm{d}t, \quad u+v+1 > 0.$$

Proof. Note that $]-\infty, 0^+, -\infty[$ and $]\infty, 1^-, \infty[$ yield the same integral:

$$\int\limits_{]\infty,0^-,\infty[} t^{-u-1}(1-t)^{-v-1} \mathrm{d}t \tag{3.32}$$

$$= (-\mathrm{e}^{-\mathrm{i}\pi(v+1)} + \mathrm{e}^{\mathrm{i}\pi(v+1)}) \int_1^\infty t^{-u-1}(t-1)^{-v-1} \mathrm{d}t \tag{3.33}$$

$$= -2\mathrm{i}\sin\pi v \frac{\Gamma(-v)\Gamma(1+u+v)}{\Gamma(1+u)} = 2\mathrm{i}\pi \frac{\Gamma(1+u+v)}{\Gamma(1+u)\Gamma(1+v)}. \tag{3.34}$$

□

If $u+v =\in \mathbb{Z}$, then a loop encircling 1 and 0 counterclockwise is located on the Riemann surface of the function $t^{u-1}(t-1)^{v-1}$. We obtain the identity

Theorem 3.11 *For $n \in \mathbb{Z}$ we have*

$$\frac{\Gamma(u)}{\Gamma(n+1)\Gamma(u-n)} = \frac{(u-1)\dots(u-n)}{n!} \tag{3.35}$$

$$= \frac{1}{2\pi \mathrm{i}} \int_{[0,1^+,0^+]} t^{u-1}(t-1)^{n-u}\mathrm{d}t, \tag{3.36}$$

Proof. Gdy zastosujemy homografię $t = -s^{-1}$ to dostaniemy

$$\frac{1}{2\pi\mathrm{i}} \int_{[0,1^+,0^+]} t^{u-1}(t-1)^{n-u}\mathrm{d}t = \frac{1}{2\pi\mathrm{i}} \int_{[0^+]} s^{n+1}(1-s)^{n-u}\mathrm{d}s \tag{3.37}$$

$$= \frac{1}{n!}\left(\frac{\mathrm{d}}{\mathrm{d}s}\right)^n (1-s)^{n-u}\big|_{s=0} = \frac{(u-1)\dots(u-n)}{n!}. \tag{3.38}$$

□

Consider now the function $(-t)^{u-1}(t-1)^{v-1}$, where both powers are understood in the sense of the prinbcipal branches. Its domain is $\mathbb{C}\backslash\mathbb{R}$. It is a domain consisting of two connected componennts on which

$$(-t)^{u-1}(t-1)^{v-1} = \begin{cases} t^{u-1}(1-t)^{v-1}\mathrm{e}^{-\mathrm{i}\pi(u-1)}\mathrm{e}^{\mathrm{i}\pi(v-1)}, & \mathrm{Im}t > 0; \\ t^{u-1}(1-t)^{v-1}\mathrm{e}^{\mathrm{i}\pi(u-1)}\mathrm{e}^{-\mathrm{i}\pi(v-1)}, & \mathrm{Im}t < 0. \end{cases} \tag{3.39}$$

Consider the contour „double eight" $[0,1^+,0^-,1^-,0^+]$. It starts on the lower sheet of the Riemann surface of the function $(-t)^{u-1}(t-1)^{v-1}$, on which the first interval $[0,1]$ is situated. It is easy to see that $[0,1^+,0^-,1^-,0^+]$ is a closed curve on this Riemann surface. Riemanna. Note in passing that the third interval on this contour is on the upper sheet mentioned in (3.39). The integral of $(-t)^{u-1}(t-1)^{v-1}$ on the double eight can be expressed by the Gamma function:

Theorem 3.12

$$\frac{1}{\Gamma(u+v)\Gamma(1-u)\Gamma(1-v)} \tag{3.40}$$

$$= \frac{1}{(2\pi)^2} \int_{[0,1^+,0^-,1^-,0^+]} (-t)^{u-1}(t-1)^{v-1}\mathrm{d}t, \tag{3.41}$$

Proof. Assume Reu, Re$v > 0$.

$$\int\limits_{[0,1^+,0^-,1^-,0^+]} (-t)^{u-1}(t-1)^{v-1}\mathrm{d}t \tag{3.42}$$

$$= \left(\mathrm{e}^{\mathrm{i}\pi(u-1)}\mathrm{e}^{-\mathrm{i}\pi(v-1)} - \mathrm{e}^{\mathrm{i}\pi(u-1)}\mathrm{e}^{\mathrm{i}\pi(v-1)} + \mathrm{e}^{-\mathrm{i}\pi(u-1)}\mathrm{e}^{\mathrm{i}\pi(v-1)} - \mathrm{e}^{-\mathrm{i}\pi(u-1)}\mathrm{e}^{-\mathrm{i}\pi(v-1)}\right)$$

$$\times \int_0^1 t^{u-1}(1-t)^{v-1}\mathrm{d}t \tag{3.43}$$

$$= -\,(\mathrm{e}^{\mathrm{i}\pi u} - \mathrm{e}^{-\mathrm{i}\pi u})(\mathrm{e}^{\mathrm{i}\pi v} - \mathrm{e}^{-\mathrm{i}\pi v})B(u,v) \tag{3.44}$$

$$= -\,(2\mathrm{i}\sin\pi v)(2i\sin\pi u)\frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)} \tag{3.45}$$

$$=(2\pi)^2\frac{1}{\Gamma(u+v)\Gamma(1-u)\Gamma(1-v)} \tag{3.46}$$

By analytic continuation the identity is extended to all u, v. □

3.4 Infinite products

Let us first recall the basic facts about series.

If there exists a finite limit $I := \lim\limits_{n\to\infty}\sum_{j=1}^n b_j$, we say that the series $\sum\limits_{j=1}^{\infty} b_j$ is *convergent conditionally* and we write $I = \sum\limits_{j=1}^{\infty} b_j$.

If $\sum\limits_{j=1}^{\infty} |b_j| < \infty$, we say that the series $\sum\limits_{j=1}^{\infty} b_j$ is convergent absolutely. One can show that absolute convergence implies conditional convergence, also after a change of the order of terms in the series, and its value does not deepend on the order.

Lemma 3.13

$$\sum_{n=1}^{\infty}|a_n| < \infty. \tag{3.47}$$

iff only a finite number of terms a_j equals -1 and

$$\sum_{n=1}^{\infty}|\log(1+a_n)| < \infty, \tag{3.48}$$

where in the series (3.48) we removed all j with $a_j = -1$. (In the above formula by $|\log(1+a)|$ we understand the principal branch of the logarithm extended by continuity to a function on na $\mathbb{C}\backslash\{-1\}$, which is unambiguous).

Proof. Suppose (3.47) is true. Then $\lim_{j\to\infty} a_j = 0$ and therefore a finite number of terms a_j equals -1. Besides, outside a finite number of indices,

$$|a_j| \leq \frac{1}{2}. \tag{3.49}$$

$\mathbb{C}\backslash\{-1\} \ni t \mapsto \left|\frac{\log(1+t)}{t}\right|$ is a positive continuous function. Hence for $|t| \leq \frac{1}{2}$ there exist $0 < C_1 \leq C_2$, such that

$$C_1 \leq \left|\frac{\log(1+t)}{t}\right| \leq C_2.$$

Hence

$$|\log(1+a_n)| \leq C_2|a_n|.$$

To prove the converse implication, it suffices to assume that all a_j differ from -1. (3.48) implies $\lim_{n\to\infty}\log(a_n+1) = 0$, and hence (3.49) holds outside of a finite number of indices, and then

$$|a_n| \leq C_1^{-1}|\log(1+a_n)|.$$

□

Suppose now that $(a_j)_{j=0}^{\infty}$ is a sequence of complex numbers such that none of a_j equals -1. If there exists a finite limit $I := \lim\limits_{n\to\infty}\prod\limits_{j=1}^{n}(1+a_j)$, then we say that the infinite product $\prod\limits_{j=1}^{\infty}(1+a_j)$ is conditionally convergent and we write $I = \prod\limits_{j=1}^{\infty}(1+a_j)$.

We say that the infinite product $\prod\limits_{j=1}^{\infty}(1+a_j)$ is absolutely convergent iff (3.47) is true. Thus taking the logarithm of an absolutely convergent infinite product we obtain an absolutely convergent series. Hence the value of an absolutely convergent infinite product does not depend on the order of terms.

3.5 Trigonometric functions as infinite products

Theorem 3.14 *We have*

$$\sum_{j=-\infty}^{\infty} \frac{1}{(z-j)^2} = \frac{\pi^2}{\sin^2 \pi z}, \tag{3.50}$$

$$\frac{1}{z} + 2\sum_{j=1}^{\infty} \frac{z}{z^2-j^2} = \lim_{n\to\infty} \sum_{j=-n}^{n} \frac{1}{z+j} = \frac{\pi \cos \pi z}{\sin \pi z}, \tag{3.51}$$

$$z\prod_{j=1}^{\infty}\Big(1-\frac{z^2}{j^2}\Big) = \frac{\sin \pi z}{\pi}. \tag{3.52}$$

The infinite product in (3.52) *is absolutely convergent.*

Proof.

$$\sum_{j=-\infty}^{\infty} \frac{1}{(z-j)^2} - \frac{\pi^2}{\sin^2 \pi z}$$

is an entire function. It is periodic with period 1 and converges to zero for $|\mathrm{Im} z| \to \infty$. Hence it is bounded. Therefore, by the Liouville Theorem it is zero. This proves (3.50).

By (3.50), the derivative of

$$\frac{1}{z} + 2\sum_{j=1}^{\infty} \frac{z}{z^2-j^2} - \frac{\pi \cos \pi z}{\sin \pi z}, \tag{3.53}$$

is zero. (3.53) is an entire odd constant function, hence it is zero. This proves (3.51).

By (3.51) we have

$$\frac{\mathrm{d}}{\mathrm{d}z}\log\Big(z\prod_{j=1}^{\infty}(1-\frac{z^2}{j^2})\Big) = \frac{\mathrm{d}}{\mathrm{d}z}\log\Big(\frac{\sin \pi z}{\pi}\Big). \tag{3.54}$$

Therefore,

$$z\prod_{j=1}^{\infty}(1-\frac{z^2}{j^2}) = C\frac{\sin \pi z}{\pi}. \tag{3.55}$$

Comparing the drivatives of both sides in (3.55) at zero we obtain $C = 1$. This proves (3.52). $\square$

3.6 The Gamma function and infinite products

We define the Euler-Mascheroni constant

$$\gamma = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k} - \log n\right) = 1 + \sum_{j=2}^{\infty}\Big(\tfrac{1}{j} + \log\big(1 - \tfrac{1}{j}\big)\Big) \sim 0,577\ldots.$$

Theorem 3.15 (The Gauss formula)

$$\Gamma(z) = \lim_{n\to\infty}\frac{n!n^z}{z(z+1)\cdots(z+n)}$$

(The Weierstrass formula)

$$\frac{1}{\Gamma(z)} = z\mathrm{e}^{\gamma z}\prod_{n=1}^{\infty}\Big(1+\frac{z}{n}\Big)\exp\Big(-\frac{z}{n}\Big).$$

The above product is absolutely convergent.

Lemma 3.16 *For* $0 \le t \le n$ *we have*

$$0 \le (1-\tfrac{t}{n})^n \le \mathrm{e}^{-t}, \qquad \lim_{n\to\infty}(1-\tfrac{t}{n})^n = \mathrm{e}^{-t}.$$

Proof. The function $f_n(t) := \mathrm{e}^t(1-\frac{t}{n})^n$ satisfies $f_n(n) = 0$, $f_n(0) = 1$

$$f_n'(t) = -\mathrm{e}^t(1-\tfrac{t}{n})^{n-1}\tfrac{t}{n} \le 0.$$

Hence, $0 \le f_n(t) \le 1$. □

Proof of Thm 3.15. We have

$$\int_0^1 (1-\beta)^n\beta^{z-1}\mathrm{d}\beta = \frac{\Gamma(n+1)\Gamma(z)}{\Gamma(z+n+1)} = \frac{n!}{z(z+1)\cdots(z+n)}.$$

Therefore,

$$\int_0^n\Big(1-\frac{t}{n}\Big)^n t^{z-1}\mathrm{d}t = \frac{n!n^z}{z(z+1)\cdots(z+n)}. \tag{3.56}$$

But for $0 \le t \le \infty$

$$\lim_{n\to\infty}\theta(n-t)\Big(1-\frac{t}{n}\Big)^n t^{z-1} = \mathrm{e}^{-t}t^{z-1}. \tag{3.57}$$

By Lemma 3.16, we can apply the Lebesgue Dominated Convergence Theorem with the dominating function $\mathrm{e}^{-t}t^{\mathrm{Re}z-1}$. Hence,

$$\lim_{n\to\infty}\int_0^n \Big(1-\frac{t}{n}\Big)^n t^{z-1}\mathrm{d}t = \int_0^\infty \mathrm{e}^{-t}t^{z-1}\mathrm{d}t. \tag{3.58}$$

This proves the Gauss formula for $\mathrm{Re}z > 0$.

To show the Weierstrass formula we note

$$z\mathrm{e}^{\gamma z}\prod_{n=1}^{\infty}\Big(1+\frac{z}{n}\Big)\exp(-\frac{z}{n}). \tag{3.59}$$

$$=z\lim_{n\to\infty}\exp z\Big(\sum_{k=1}^{n}\frac{1}{k}-\log n\Big)\prod_{k=1}^{n}\Big(1+\frac{z}{k}\Big)\exp(-\frac{z}{k}) \tag{3.60}$$

$$=\lim_{n\to\infty}n^{-z}z\prod_{k=1}^{n}(1+\frac{z}{k})=\lim_{n\to\infty}\frac{n^{-z}z(z+1)\cdots(z+n)}{n!}. \tag{3.61}$$

□

The Gauss or Weierstrass formulas can be used to prove the results from previous sections. For instance, using the Gauss formula we obtain

$$\begin{aligned}\Gamma(z+1) &= \lim_{n\to\infty}\frac{n!n^{z+1}}{(z+1)\cdots(z+n+1)}\\ &=z\lim_{n\to\infty}\Big(\frac{n}{n+1}\Big)^{z+1}\lim_{n\to\infty}\frac{(n+1)!(n+1)^z}{z(z+1)\cdots(z+n+1)}=z\Gamma(z).\end{aligned}$$

Using the Weierstrass formula we get

$$\begin{aligned}\frac{1}{\Gamma(z)\Gamma(1-z)} &= \frac{-1}{z\Gamma(z)\Gamma(-z)}\\ &= z\prod_{n=1}^{\infty}\Big(1-\frac{z^2}{n^2}\Big)=\frac{\sin\pi z}{\pi}.\end{aligned}$$

Proof of the Gauss multiplication formula (3.25). Let

$$G(z)=\prod_{k=0}^{m-1}\Gamma\Big(z+\frac{k}{m}\Big).$$

Using the Gauss formula we obtain

$$\frac{1}{G(z)} = \lim_{n\to\infty} \frac{\prod\limits_{n=0}^{nm+m-1}(mz+k)}{(n!)^m n^{mz+\frac{1}{2}(m-1)} m^{m(n+1)}},$$

$$\Gamma(mz) = \lim_{n\to\infty} \frac{(mn)!(mn)^{mz}}{\prod\limits_{k=0}^{mn}(mz+k)}.$$

Hence,

$$\frac{\Gamma(mz)}{G(z)} = \lim_{n\to\infty} \frac{(mn)!m^{mz-m(n+1)}n^{-\frac{1}{2}(m-1)}}{(n!)^m} \prod_{k=mn+1}^{mn+m-1}(mz+k)$$

$$= \lim_{n\to\infty} \frac{(mn)!m^{mz-mn-1)}n^{\frac{1}{2}(m-1)}}{(n!)^m} = (2\pi)^{\frac{1}{2}(m-1)} m^{mz-\frac{1}{2}},$$

where we first used

$$\lim_{n\to\infty} \prod_{k=nm+1}^{nm+m-1} \frac{mz+k}{n} = m^{m-1},$$

and then we applied the Stirling formula, see Corollary 3.28. □

3.7 A few integrals with a parameter

Proposition 3.17 *Let $t \mapsto f(t)$ be holomorphic for $|\arg t| < \alpha$. Suppose that $\epsilon > 0$ and*

$$|f(z)| \le C|z|^{-\epsilon}, \qquad |f(z) - f(0)| \le C|z|^{\epsilon}.$$

Then for $|\arg z| < \alpha$

$$\int_0^\infty (f(t) - f(zt))\frac{\mathrm{d}t}{t} = f(0)\log z.$$

Proof.

$$\int_r^R (f(t) - f(zt))\frac{\mathrm{d}t}{t} = \Big(\int_{[r,R]} + \int_{[zR,zr]}\Big) f(t)\frac{\mathrm{d}t}{t}$$

$$= \Big(\int_{[r,zr]} + \int_{[zR,R]}\Big) f(t)\frac{\mathrm{d}t}{t}$$

$$\to \int_{[r,zr]} f(0)\frac{\mathrm{d}t}{t} = f(0)\log(z),$$

where in the final step $r \to 0$, $R \to \infty$. □

Here is a real version of the above proposition:

Proposition 3.18 *Let $f(t)$ be measurable on $[0, \infty[$ such that*

$$\int_1^\infty |f(t)| \frac{\mathrm{d}t}{t} < \infty, \qquad \int_0^1 |f(t) - f(0)| \frac{\mathrm{d}t}{t} < \infty.$$

Then for $z \in [0, \infty[$

$$\int_0^\infty (f(t) - f(zt)) \frac{\mathrm{d}t}{t} = f(0) \log z.$$

Corollary 3.19 *We have*

$$\log z = \int_0^\infty (\mathrm{e}^{-t} - \mathrm{e}^{-zt}) \frac{\mathrm{d}t}{t}. \tag{3.62}$$

Proposition 3.20 *We have the following integral representation of the Euler-Mascheroni constant:*

$$\gamma = \int_0^\infty \Big(\frac{1}{\mathrm{e}^t - 1} - \frac{1}{t\mathrm{e}^t} \Big) \mathrm{d}t. \tag{3.63}$$

Proof. We have

$$\int_0^\infty \frac{(1 - \mathrm{e}^{-nt})}{(\mathrm{e}^t - 1)} \mathrm{d}t = \sum_{j=1}^n \int_0^\infty \mathrm{e}^{-jt} \mathrm{d}t = \sum_{j=1}^n \frac{1}{j},$$
$$\int_0^\infty (1 - \mathrm{e}^{-nt}) \frac{\mathrm{d}t}{\mathrm{e}^t t} = \log(n+1).$$

Hence,

$$\gamma = \lim_{n\to\infty} \Big(1 + \frac{1}{2} + \cdots + \frac{1}{n} - \log(n+1) \Big) \tag{3.64}$$

$$= \lim_{n\to\infty} \int_0^\infty \Big(\frac{(1 - \mathrm{e}^{-nt})}{(\mathrm{e}^t - 1)} - \frac{(1 - \mathrm{e}^{-nt})}{t\mathrm{e}^t} \Big) \mathrm{d}t \tag{3.65}$$

$$= \int_0^\infty \Big(\frac{1}{\mathrm{e}^t - 1} - \frac{1}{t\mathrm{e}^t} \Big) \mathrm{d}t + \lim_{n\to\infty} \int_0^\infty \Big(\frac{1}{\mathrm{e}^t - 1} - \frac{1}{\mathrm{e}^t t} \Big) \mathrm{e}^{-tn} \mathrm{d}t. \tag{3.66}$$

Now the second term on (3.66) is zero. □

Proposition 3.21 (The Pringsheim formula)

$$\frac{1}{2}+\frac{1}{2}\log\frac{1}{2}=\int_0^\infty\Big(\frac{1}{1-\mathrm{e}^{-t}}-\frac{1}{t}-\frac{1}{2}\Big)\frac{\mathrm{e}^{-\frac{1}{2}t}}{t}\mathrm{d}t \tag{3.67}$$

$$=\int_0^\infty\Big(\frac{1}{\mathrm{e}^{t}-1}-\frac{1}{t}+\frac{1}{2}\Big)\frac{\mathrm{e}^{-\frac{1}{2}t}}{t}\mathrm{d}t. \tag{3.68}$$

Proof. Set

$$I:=\int_0^\infty\Big(\frac{1}{\mathrm{e}^{t}-1}-\frac{1}{t}+\frac{1}{2}\Big)\frac{\mathrm{e}^{-t}}{t}\mathrm{d}t \tag{3.69}$$

$$J:=\int_0^\infty\Big(\frac{1}{\mathrm{e}^{t}-1}-\frac{1}{t}+\frac{1}{2}\Big)\frac{\mathrm{e}^{-\frac{1}{2}t}}{t}\mathrm{d}t. \tag{3.70}$$

Note that

$$\frac{1}{\mathrm{e}^{t}-1}-\frac{1}{t}+\frac{1}{2}\sim\frac{1}{12}t,$$

hence both integrands are continuous at 0 and therefore both I and J are well defined. Change of coordinates yields an alternative expression for I:

$$I:=\int_0^\infty\Big(\frac{1}{\mathrm{e}^{\frac{t}{2}}-1}-\frac{2}{t}+\frac{1}{2}\Big)\frac{\mathrm{e}^{-\frac{t}{2}}}{t}\mathrm{d}t. \tag{3.71}$$

Now,

$$\begin{aligned}
J&=I+(J-I)\\
&=\int_0^\infty\Big(\frac{1}{\mathrm{e}^{t}-1}-\frac{1}{t}+\frac{1}{2}\Big)\frac{\mathrm{e}^{-t}}{t}\mathrm{d}t+\int_0^\infty\Big(\frac{1}{\mathrm{e}^{t}-1}-\frac{1}{\mathrm{e}^{\frac{t}{2}}-1}+\frac{1}{t}\Big)\frac{\mathrm{e}^{-\frac{t}{2}}}{t}\mathrm{d}t\\
&=\int_0^\infty\Big(\frac{\mathrm{e}^{-\frac{t}{2}}-\mathrm{e}^{-t}}{t}-\frac{1}{2}\mathrm{e}^{-t}\Big)\frac{\mathrm{d}t}{t}\\
&=-\int_0^\infty\frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{\mathrm{e}^{-\frac{t}{2}}-\mathrm{e}^{-t}}{t}\Big)\mathrm{d}t+\int_0^\infty\frac{(\mathrm{e}^{-t}-\mathrm{e}^{-\frac{t}{2}})}{2t}\mathrm{d}t\\
&=\frac{1}{2}+\frac{1}{2}\log\frac{1}{2}.
\end{aligned}$$

□

3.8 The logaritmic derivative of the Gamma function

The Weierstrass formula implies immediately that

$$\log\Gamma(z) = -\gamma z - \log z + \sum_{j=1}^{\infty}\Big(\frac{z}{j} - \log\Big(1+\frac{z}{j}\Big)\Big), \tag{3.72}$$

$$\partial_z \log\Gamma(z) = -\gamma + \sum_{j=0}^{\infty}\Big(\frac{1}{j+1} - \frac{1}{j+z}\Big), \tag{3.73}$$

$$\partial_z^2 \log\Gamma(z) = \sum_{j=0}^{\infty}\frac{1}{(j+z)^2}. \tag{3.74}$$

We also have

$$\log\Gamma(1) = 0, \quad \log\Gamma\Big(\frac{1}{2}\Big) = \frac{1}{2}\log\pi, \quad \partial_z\log\Gamma(1) = -\gamma. \tag{3.75}$$

Proposition 3.22

$$\Gamma(n+1+\epsilon) = n!\bigg(1+\epsilon\Big(-\gamma+\sum_{j=1}^{n}\frac{1}{j}\Big)\bigg) + O(\epsilon^2), \quad n = 1,2,\dots, \tag{3.76}$$

$$\Gamma(-n+\epsilon) = \frac{(-1)^n}{n!}\Big(\epsilon^{-1} - \gamma + \sum_{j=1}^{n}\frac{1}{j}\Big) + O(\epsilon), \quad n = 1,2,\dots. \tag{3.77}$$

Proof. First we note

$$\partial_z\log\Gamma(n+1) = -\gamma + \sum_{j=1}^{n}\frac{1}{j}, \quad n = 1,2,\dots$$

But $\Gamma'(z) = \Gamma(z)\partial_z\log\Gamma(z)$ and $\Gamma(n+1) = n!$. This shows (3.76).

Next,

$$\partial_z\Big(\log\Gamma(z) + \frac{1}{z+n}\Big)\Big|_{z=-n} = -\gamma + \sum_{j=1}^{n}\frac{1}{j}.$$

But

$$\partial_z\Big(\Gamma(z)(z+n)\Big) = (z+n)\Gamma'(z)+\Gamma(z) = (z+n)\Gamma(z)\partial_z\Big(\log\Gamma(z)+(z+n)^{-1}\Big).$$

□

3.9 Asymptotic series

Suppose that a function f is defined on a set $K(z_0, r) \cap \{\alpha_1 < \arg(z - z_0) < \alpha_2\}$. We write

$$f(z) \sim \sum_{j=0}^{\infty} a_j (z - z_0)^j,$$

if for any n there exists C_n such that

$$\Big| f(z) - \sum_{j=0}^{n} a_j (z - z_0)^j \Big| \leq C_n |z - z_0|^{n+1}.$$

Obviously, if a function is given by a convergent series, then it is asymptotic to this series: if $f(z) = \sum_{j=0}^{\infty} a_j (z - z_0)^j$ for $z \in K(z_0, r)$, then $f(z) \sim \sum_{j=0}^{\infty} a_j (z - z_0)^j$.

Example 3.23 *For* $-\frac{\pi}{2} + \epsilon < \arg z < \frac{\pi}{2} - \epsilon$

$$\mathrm{e}^{-\frac{1}{z}} \sim \sum_{j=0}^{\infty} 0 z^j.$$

Example 3.24 *For* $-\frac{\pi}{4} + \epsilon < \arg z < \frac{\pi}{4} - \epsilon$ *and* $-\frac{\pi}{4} + \epsilon < \arg -z < \frac{\pi}{4} - \epsilon$

$$\mathrm{e}^{-\frac{1}{z^2}} \sim \sum_{j=0}^{\infty} 0 z^j.$$

In particular, all the derivatives of $\mathbb{R} \ni x \to \mathrm{e}^{-\frac{1}{x^2}}$ *at zero vanish.*

Example 3.25 (The error function)

$$\mathrm{Erf}(z) := \int_0^z \mathrm{e}^{-t^2} \mathrm{d}t.$$

Clearly, $\lim_{z \to \infty} \mathrm{Erf}(z) = \frac{1}{2}\sqrt{\pi}$.

Proposition 3.26 *For* $-\frac{\pi}{2} + \epsilon < \arg z < \frac{\pi}{2} - \epsilon$

$$\begin{aligned} \frac{1}{2}\sqrt{\pi} - \mathrm{Erf}(z) &= \int_z^{\infty} \mathrm{e}^{-t^2} \mathrm{d}t \\ &\sim \frac{\mathrm{e}^{-z^2}}{2z} \Big(1 + \sum_{k=1}^{\infty} (-1)^k \frac{1 \cdot 3 \cdots (2k-1)}{(2z^2)^k} \Big). \end{aligned} \tag{3.78}$$

Proof. For simplicity, let us restrict ourselves to $z > 0$. We integrate by parts:

$$\begin{aligned}\int_z^\infty \mathrm{e}^{-t^2}\mathrm{d}t &= -\frac{1}{2}\int_z^\infty \left(\partial_t \mathrm{e}^{-t^2}\right)t^{-1}\mathrm{d}t\\ &= \frac{1}{2z}\mathrm{e}^{-z^2} - \frac{1}{2}\int_z^\infty \mathrm{e}^{-t^2}t^{-2}\mathrm{d}t\\ &= \frac{1}{2z}\mathrm{e}^{-z^2} + \frac{1}{2^2}\int_z^\infty \left(\partial_t \mathrm{e}^{-t^2}\right)t^{-3}\mathrm{d}t\\ &= \frac{1}{2z}\mathrm{e}^{-z^2} - \frac{1}{2^2z^3}\mathrm{e}^{-z^2} + \frac{3}{2^2}\int_z^\infty \mathrm{e}^{-t^2}t^{-4}\mathrm{d}t.\end{aligned}$$

Then we estimate:

$$\Big|\int_z^\infty \mathrm{e}^{-t^2}t^{-4}\mathrm{d}t\Big| \leq \mathrm{e}^{-z^2}\int_z^\infty t^{-4}\mathrm{d}t = \mathrm{e}^{-z^2}\frac{1}{4}z^{-3}.$$

Therefore,

$$\frac{\sqrt{\pi}}{2} - \mathrm{Erf}(z) = \mathrm{e}^{-z^2}\Big(\frac{1}{2z} + O\Big(\frac{1}{z^3}\Big)\Big).$$

Continuing like this, we obtain the expansion (3.78). □

3.10 Binet's identities

Theorem 3.27 (1st Binet's identity)

$$\begin{aligned}\log\Gamma(z) =& \Big(z - \frac{1}{2}\Big)\log z - z + \frac{1}{2}\log 2\pi\\ &+ \int_0^\infty \Big(\frac{1}{2} + \frac{1}{\mathrm{e}^t - 1} - \frac{1}{t}\Big)\mathrm{e}^{-zt}\frac{\mathrm{d}t}{t};\end{aligned} \tag{3.79}$$

$$\partial_z \log\Gamma(z) = \log z + \int_0^\infty \Big(\frac{1}{1-\mathrm{e}^{-t}} - \frac{1}{t}\Big)\mathrm{e}^{-zt}\mathrm{d}t; \tag{3.80}$$

$$\partial_z^2 \log\Gamma(z) = \int_0^\infty \frac{t\mathrm{e}^{-tz}}{1-\mathrm{e}^{-t}}\mathrm{d}t. \tag{3.81}$$

Note that the above integrals are convergent. In particular, the integrands are continuous at 0:

$$\lim_{t\to 0}\Big(\frac{1}{2} + \frac{1}{\mathrm{e}^t - 1} - \frac{1}{t}\Big)\frac{1}{t} = \frac{1}{12}, \tag{3.82}$$

$$\lim_{t\to 0}\Big(\frac{1}{1-\mathrm{e}^{-t}} - \frac{1}{t}\Big) = \frac{1}{2}, \tag{3.83}$$

$$\lim_{t\to 0}\frac{t}{1-\mathrm{e}^{-t}} = 1. \tag{3.84}$$

Proof. First we prove (3.81). Using (3.74) we obtain

$$\begin{aligned}\partial_z^2 \log\Gamma(z) &= \sum_{n=0}^{\infty} \frac{1}{(n+z)^2} \\ &= \sum_{n=0}^{\infty} \int_0^{\infty} \mathrm{e}^{-t(z+n)} t\mathrm{d}t \\ &= \int_0^{\infty} \frac{t\mathrm{e}^{-tz}}{1-\mathrm{e}^{-t}} \mathrm{d}t. \qquad (3.85)\end{aligned}$$

This proves (3.81). Hence,

$$\begin{aligned}\partial_z^2 \log\Gamma(z) &= \int_0^{\infty} \mathrm{e}^{-tz}\mathrm{d}t + \int_0^{\infty} \Big(\frac{1}{1-\mathrm{e}^{-t}} - \frac{1}{t}\Big) t\mathrm{e}^{-tz}\mathrm{d}t \\ &= \frac{1}{z} + \int_0^{\infty} \Big(\frac{1}{1-\mathrm{e}^{-t}} - \frac{1}{t}\Big) t\mathrm{e}^{-tz}\mathrm{d}t.\end{aligned}$$

We integrate this, using $\partial_z \log\Gamma(1) = -\gamma$:

$$\partial_z \log\Gamma(z) = \partial_z \log\Gamma(1) + \int_1^z \partial_y^2 \log\Gamma(y)\mathrm{d}y \qquad (3.86)$$

$$\begin{aligned}&= -\gamma + \int_1^z \frac{1}{y}\mathrm{d}y + \int_1^z \int_0^{\infty} \Big(\frac{1}{1-\mathrm{e}^{-t}} - \frac{1}{t}\Big) t\mathrm{e}^{-ty}\mathrm{d}t\mathrm{d}y \qquad (3.87) \\ &= -\gamma + \log z - \int_0^{\infty} \Big(\frac{1}{1-\mathrm{e}^{-t}} - \frac{1}{t}\Big) \mathrm{e}^{-ty}\mathrm{d}t\Big|_{y=1}^{y=z} \qquad (3.88) \\ &= \log z - \int_0^{\infty} \Big(\frac{1}{1-\mathrm{e}^{-t}} - \frac{1}{t}\Big) \mathrm{e}^{-tz}\mathrm{d}t, \qquad (3.89)\end{aligned}$$

where in the last step we used the integral representation of γ. This proves (3.80). Therefore,

$$\partial_z \log\Gamma(z) = \log z - \frac{1}{2z} - \int_0^{\infty} \Big(\frac{1}{1-\mathrm{e}^{-t}} - \frac{1}{t} - \frac{1}{2}\Big) \mathrm{e}^{-tz}\mathrm{d}t.$$

Hence,

$$\log\Gamma(z) = \log\Gamma\Big(\frac{1}{2}\Big) + \int_{\frac{1}{2}}^{z} \partial_y \log\Gamma(y)\mathrm{d}y$$
$$= \frac{1}{2}\log\pi + \int_{\frac{1}{2}}^{z} \log y\mathrm{d}y - \frac{1}{2}\int_{\frac{1}{2}}^{z} \frac{1}{y}\mathrm{d}y - \int_{\frac{1}{2}}^{z}\int_0^\infty \Big(\frac{1}{1-\mathrm{e}^{-t}} - \frac{1}{t} - \frac{1}{2}\Big)\mathrm{e}^{-ty}\mathrm{d}t\mathrm{d}y$$
$$= \frac{1}{2}\log\pi + z - \log z - \frac{1}{2}\log\frac{1}{2} + \frac{1}{2} - \frac{1}{2}\log z + \frac{1}{2}\log\frac{1}{2}$$
$$+ \int_0^\infty \Big(\frac{1}{1-\mathrm{e}^{-t}} - \frac{1}{t} - \frac{1}{2}\Big)\mathrm{e}^{-ty}\frac{\mathrm{d}t}{t}\Big|_{y=\frac{1}{2}}^{y=z}$$
$$= \Big(z - \frac{1}{2}\Big)\log z - z + \frac{1}{2}\log 2\pi + \int_0^\infty \Big(\frac{1}{1-\mathrm{e}^{-t}} - \frac{1}{t} - \frac{1}{2}\Big)\mathrm{e}^{-tz}\frac{\mathrm{d}t}{t},$$

where in the last step we used the Pringsheim formula. This proves (3.79).
□

Corollary 3.28 *Let $\epsilon > 0$ and $|\arg z| < \pi - \epsilon$.*
(1) The **Stirling identity**

$$\lim_{z\to\infty}\Big(\log\Gamma(z) - \big((z - \frac{1}{2})\log z - z + \frac{1}{2}\log 2\pi\big)\Big) = 0,$$
$$\lim_{z\to\infty}\frac{\Gamma(z)}{z^{z-\frac{1}{2}}\mathrm{e}^{-z}\sqrt{2\pi}} = 1.$$

(2) Let

$$f(t) = \frac{1}{1-\mathrm{e}^{-t}} - \frac{1}{t} - \frac{1}{2}.$$

Then f is bounded together with all its derivatives for $t \in [0,\infty[$ and

$$f(t) = \sum_{n=1}^\infty f_n t^n, \quad |t| < 2\pi.$$

Moreover, the following asymptotic expansion of the Gamma function holds:

$$\left|\log\Gamma(z) - \left(\Big(z - \frac{1}{2}\Big)\log z - z + \frac{1}{2}\log 2\pi - \sum_{j=1}^n \frac{(j-1)!f_j}{z^{-j}}\right)\right| \le C|z|^{-n-1}.$$

Theorem 3.29 (Plana identity) *Let $m \le n$ be integers, $\phi(z)$ an analytic function, $|\phi(z)| \le \mathrm{e}^{(1-\epsilon)|\mathrm{Im}z|}$ for $\epsilon > 0$ and $m \le \mathrm{Re}z \le n$. Then*

$$\frac{1}{2}\phi(m) + \phi(m+1) + \cdots + \phi(n-1) + \frac{1}{2}\phi(n) \tag{3.90}$$
$$= \int_m^n \phi(z)\mathrm{d}z - \mathrm{i}\int_0^\infty \frac{\phi(n+\mathrm{i}y) - \phi(n-\mathrm{i}y)}{\mathrm{e}^{2\pi y} - 1}\mathrm{d}y + \mathrm{i}\int_0^\infty \frac{\phi(m+\mathrm{i}y) - \phi(m-\mathrm{i}y)}{\mathrm{e}^{2\pi y} - 1}\mathrm{d}y.$$

Proof. Introduce the contours

$$\gamma_+ = [m^-, (m+1)^-, \dots, (n-1)^-, n^-, n+\mathrm{i}R, m+\mathrm{i}R, m],$$
$$\gamma_- = [m^+, (m+1)^+, \dots, (n-1)^+, n^+, n-\mathrm{i}R, m-\mathrm{i}R, m].$$

Using

$$\mathrm{Res}\frac{\phi(z)}{\mathrm{e}^{\pm 2\pi \mathrm{i}z}-1}\Big|_{z=k} = \mp\frac{1}{2\pi\mathrm{i}}\phi(k), \quad k\in\mathbb{Z},$$

we obtain

$$\begin{aligned} 0 \quad &= \quad \int_{\gamma_+}\frac{\phi(z)}{\mathrm{e}^{-2\pi\mathrm{i}z}-1}\mathrm{d}z \qquad (3.91)\\ \underset{R\to\infty}{\to} \quad & -\mathrm{i}\int_0^\infty \frac{\phi(m+\mathrm{i}y)}{\mathrm{e}^{2\pi y}-1}\mathrm{d}y + \mathrm{i}\int_0^\infty\frac{\phi(n+\mathrm{i}y)}{\mathrm{e}^{2\pi y}-1}\mathrm{d}y\\ & +\mathcal{P}\int_m^n\frac{\phi(x)}{\mathrm{e}^{-2\pi\mathrm{i}x}-1}\mathrm{d}x + \frac14\phi(m) + \frac12\sum_{j=m+1}^{n-1}\phi(j) + \frac14\phi(n);\\ 0 \quad &= \quad \int_{\gamma_-}\frac{\phi(z)}{\mathrm{e}^{2\pi\mathrm{i}z}-1}\mathrm{d}z \qquad (3.92)\\ \underset{R\to\infty}{\to} \quad & \mathrm{i}\int_0^\infty \frac{\phi(m-\mathrm{i}y)}{\mathrm{e}^{2\pi y}-1}\mathrm{d}y - \mathrm{i}\int_0^\infty\frac{\phi(n-\mathrm{i}y)}{\mathrm{e}^{2\pi y}-1}\mathrm{d}y\\ & +\mathcal{P}\int_m^n\frac{\phi(x)}{\mathrm{e}^{2\pi\mathrm{i}x}-1}\mathrm{d}x + \frac14\phi(m) + \frac12\sum_{j=m+1}^{n-1}\phi(j) + \frac14\phi(n). \end{aligned}$$

Then we add (3.91) and (3.92), using the identity

$$(\mathrm{e}^{2\pi\mathrm{i}x}-1)^{-1} + (\mathrm{e}^{-2\pi\mathrm{i}x}-1)^{-1} = -1.$$

□

Theorem 3.30 (2nd Binet's identity)

$$\log\Gamma(z) = \Big(z-\frac12\Big)\log z - z + \frac12\log 2\pi, +2\int_0^\infty \frac{\arctan\frac{t}{z}}{\mathrm{e}^{2\pi t}-1}\mathrm{d}t, \qquad (3.93)$$
$$\partial_z\log\Gamma(z) = \log z - \frac{1}{2z} - 2\int_0^\infty\frac{t\mathrm{d}t}{(z^2+t^2)(\mathrm{e}^{2\pi t}-1)} \qquad (3.94)$$
$$\partial_z^2\log\Gamma(z) = \frac{1}{2z^2} + \frac1z + 4\int_0^\infty\frac{zt\mathrm{d}t}{(z^2+t^2)^2(\mathrm{e}^{2\pi t}-1)}. \qquad (3.95)$$

Proof. (3.95) follows from the Plana formula applied to $\phi(z) = (z + t)^{-2}$. Then we integrate it twice and we obtain

$$\log \Gamma(z) = A + Bz + (z - \frac{1}{2}) \log z + 2 \int_0^\infty \frac{\arctan \frac{t}{z}}{\mathrm{e}^{2\pi t} - 1} \mathrm{d}t.$$

Comparing with the 1st Binet's identity for $z \sim 0$ we get $A = \frac{1}{2} \log 2\pi$, $B = -1$. □

4 Homogeneous distributions in $d = 1$

4.1 Homogeneous distributions of order -1 and 0

The function $\frac{1}{x}$ is not in L^1_{loc}, therefore it does not define a regular distribution. However, it can be naturally interpreted as a distribution in several ways:

$$\mathcal{P} \int \frac{1}{x} \phi(x) \mathrm{d}x :== \lim_{\epsilon \searrow 0} \Big(\int_{-\infty}^{-\epsilon} + \int_{\epsilon}^{\infty} \Big) \frac{1}{x} \phi(x) \mathrm{d}x \tag{4.1}$$

$$= \Big(\int_{-\infty}^{a} + \int_{a}^{\infty} \Big) \frac{1}{x} \phi(x) \mathrm{d}x + \int_{-a}^{a} \frac{1}{x} \big(\phi(x) - \phi(0) \big) \mathrm{d}x, \tag{4.2}$$

$$\int \frac{\phi(x) \mathrm{d}x}{(x \pm \mathrm{i}0)} := \lim_{\epsilon \searrow 0} \int \frac{\phi(x) \mathrm{d}x}{(x \pm \mathrm{i}\epsilon)}. \tag{4.3}$$

The letter $\mathcal{P}$ stands for the *principal value* and indicates that it is not the usual integral. The Sochocki formula is relationship between three kinds of order -1 distributions:

$$\frac{1}{x \pm \mathrm{i}0} = \frac{1}{x} \mp \mathrm{i}\pi\delta(x).$$

Here is an equivalent definition of the principal value $\frac{1}{x}$: Equivalently,

$$\frac{1}{x} := \frac{1}{2} \Big(\frac{1}{(x + \mathrm{i}0)} + \frac{1}{(x - \mathrm{i}0)} \Big).$$

$$\begin{aligned}
\int \mathrm{e}^{-\mathrm{i}xk}\mathrm{d}x &= 2\pi\delta(k), && (4.4)\\
\int \theta(\pm x)\mathrm{e}^{-\mathrm{i}xk}\mathrm{d}x &= \frac{\mp \mathrm{i}}{k \mp \mathrm{i}0}, && (4.5)\\
\int \mathrm{sgn}(x)\mathrm{e}^{-\mathrm{i}xk}\mathrm{d}x &= -2\mathrm{i}\frac{1}{k}, && (4.6)\\
\int \delta(x)\mathrm{e}^{-\mathrm{i}xk}\mathrm{d}x &= 1, && (4.7)\\
\int \frac{\mathrm{e}^{-\mathrm{i}kx}}{x \pm \mathrm{i}0}\mathrm{d}x &= \mp 2\pi\mathrm{i}\theta(\pm k), && (4.8)\\
\int \frac{\mathrm{e}^{-\mathrm{i}kx}}{x}\mathrm{d}x &= -\pi\mathrm{i}\mathrm{sgn}(k), && (4.9)
\end{aligned}$$

$$\int \mathrm{e}^{-\mathrm{i}\xi s}(s-\lambda)^{-1}\mathrm{d}\ s \ = \ \begin{cases} -2\pi\mathrm{i}\theta(\xi)\mathrm{e}^{-\mathrm{i}\lambda\xi} & \mathrm{Im}\lambda < 0; \\ 2\pi\mathrm{i}\theta(-\xi)\mathrm{e}^{-\mathrm{i}\lambda\xi} & \mathrm{Im}\lambda > 0; \\ -\pi\mathrm{i}\mathrm{sgn}(\xi)\mathrm{e}^{-\mathrm{i}\lambda\xi}, & \mathrm{Im}\lambda = 0. \end{cases} \qquad (4.10)$$

4.2 Homogeneous distributions of integer order

Define for $n = 0, 1, 2, \ldots$

$$\frac{1}{x^{n+1}} := \frac{1}{2}\Big(\frac{1}{(x+\mathrm{i}0)^{n+1}} + \frac{1}{(x-\mathrm{i}0)^{n+1}}\Big).$$

Clearly

$$\partial_x^n \frac{1}{x} = (-1)^n n! \frac{1}{x^{n+1}}, \qquad x \cdot \frac{1}{x^{n+1}} = \frac{1}{x^n}. \qquad (4.11)$$

$$\int x^n e^{-ixk} dx = 2\pi i^n \delta^{(n)}(k), \tag{4.12}$$

$$\int x^n \theta(\pm x) e^{-ixk} dx = \pm \frac{(-i)^{n+1} n!}{(k \mp i0)^{n+1}}, \tag{4.13}$$

$$\int x^n \mathrm{sgn}(x) e^{-ixk} dx = 2(-i)^{n+1} n! \frac{1}{k^{n+1}}, \tag{4.14}$$

$$\int \delta^{(n)}(x) e^{-ixk} dx = i^n k^n, \tag{4.15}$$

$$\int \frac{e^{-ikx}}{(x \pm i0)^{n+1}} dx = \pm \frac{2\pi(-i)^{n+1}}{n!} k^n \theta(\pm k), \tag{4.16}$$

$$\int \frac{e^{-ikx}}{x^{n+1}} dx = \frac{\pi(-i)^{n+1}}{n!} k^n \mathrm{sgn}(k). \tag{4.17}$$

4.3 Homogeneous distributions of arbitrary order I

For any $\lambda \in \mathbb{C}$

$$(\pm ix + 0)^\lambda := \lim_{\epsilon \to 0} (\pm ix + \epsilon)^\lambda.$$

is a tempered distribution. If $\mathrm{Re}\lambda > -1$, then it is simply the distribution given by the locally integrable function

$$e^{\pm i \mathrm{sgn}(x) \frac{\pi}{2} \lambda} |x|^\lambda. \tag{4.18}$$

The functions

$$x_\pm^\lambda := (\pm x)^\lambda \theta(\pm x) \tag{4.19}$$

define distributions only for $\mathrm{Re}\lambda > -1$. We can extend them to $\lambda \in \mathbb{C}$ except for $\lambda = -1, -2, \dots$ by putting

$$x_\pm^\lambda := \frac{1}{2i \sin \pi\lambda} \Big(-e^{-i\frac{\pi}{2}\lambda} (\mp ix + 0)^\lambda + e^{i\frac{\pi}{2}\lambda} (\pm ix + 0)^\lambda \Big). \tag{4.20}$$

We will sometimes write $\frac{1}{x_\pm^{-\lambda}}$ instead of $x_\pm^\lambda$. We have

$$x_\pm^{\lambda+1} = x \cdot x_\pm^\lambda. \tag{4.21}$$

Instead of $x_\pm^\lambda$ it is often more convenient to consider

$$\rho_\pm^\lambda(x) := \frac{x_\pm^\lambda}{\Gamma(\lambda+1)} \tag{4.22}$$

$$= \frac{\Gamma(-\lambda)}{2\pi i} \Big(e^{-i\frac{\pi}{2}\lambda} (\mp ix + 0)^\lambda - e^{i\frac{\pi}{2}\lambda} (\pm ix + 0)^\lambda \Big). \tag{4.23}$$

Note that using (4.22) and (4.23) we have defined $\rho_\pm^\lambda$ for all $\lambda \in \mathbb{C}$.

Theorem 4.1 *The distributions* $\rho_\pm^\lambda$ *satisfy the recurrence relations*

$$\partial_x \rho_\pm^\lambda(x) = \pm\rho_\pm^{\lambda-1}(x).$$

At integers we have

$$\rho_\pm^n = \frac{x_\pm^n}{n!}, \quad n = 0, 1, \dots; \tag{4.24}$$
$$\rho_\pm^{-n-1} = (\pm 1)^n \delta^n(x), \quad n = 0, 1, \dots. \tag{4.25}$$

Their Fourier transforms are below:

$$\int \mathrm{e}^{-\mathrm{i}\xi x} \rho_\pm^\lambda(x)\mathrm{d}x = (\pm\mathrm{i}\xi + 0)^{-\lambda-1},$$

$$\int \mathrm{e}^{-\mathrm{i}\xi x} (\mp\mathrm{i}\xi + 0)^\lambda \mathrm{d}\xi = 2\pi\rho_\pm^{-\lambda-1}(x).$$

Proof. (4.25) follows from

$$\rho_\pm^{-n-1}(x) = \frac{(\mp 1)^{n+1} n!}{2\pi\mathrm{i}}\Big((x \pm \mathrm{i}0)^{-n-1} - (x \mp \mathrm{i}0)^{-n-1}\Big) \tag{4.26}$$
$$= -\frac{(\pm 1)^n}{2\pi\mathrm{i}} \partial_x^n \Big((x \pm \mathrm{i}0)^{-1} - (x \mp \mathrm{i}0)^{-1}\Big). \tag{4.27}$$

□

Theorem 4.2 *Let* $-n-1 < \mathrm{Re}\lambda$, $\lambda \notin \{\dots, -2, -1\}$. *Then for any* $a > 0$,

$$\int x_+^\lambda \phi(x)\mathrm{d}x = \int_a^\infty x^\lambda \phi(x)\mathrm{d}x + \int_0^a x^\lambda \Big(\phi(x) - \sum_{j=0}^{n-1} \frac{x^j}{j!}\phi^{(j)}(0)\Big)\mathrm{d}x + \sum_{j=0}^{n-1} a^{\lambda+j+1}\phi^{(j)}(0) \sum_{l=0}^{j} \frac{(-1)^l}{(j-l)!(\lambda+1)\cdots(\lambda+1+l)}. \tag{4.28}$$

If $-n-1 < \mathrm{Re}\lambda < -n$, *we can even go with* a *to infinity*

$$\int x_+^\lambda \phi(x)\mathrm{d}x = \int_0^\infty x^\lambda \Big(\phi(x) - \sum_{j=0}^{n-1} \frac{x^j}{j!}\phi^{(j)}(0)\Big)\mathrm{d}x. \tag{4.29}$$

Proof. We use induction. Suppose that the formula is true for λ

$$-\lambda \int x_+^{\lambda-1}\phi(x)\mathrm{d}x = \int x_+^{\lambda}\partial_x\phi(x)\mathrm{d}x \tag{4.30}$$

$$\begin{aligned} &= \int_a^\infty x^\lambda \partial_x \phi(x)\mathrm{d}x \qquad (4.31)\\ &+ \int_0^a x^\lambda \partial_x \Big(\phi(x) - \sum_{j=0}^{n} \frac{x^j}{j!}\phi^{(j)}(0)\Big)\mathrm{d}x \\ &+ \sum_{j=0}^{n-1} a^{\lambda+j+1}\phi^{(j+1)}(0) \sum_{l=0}^{j} \frac{(-1)^l}{(j-l)!(\lambda+1)\cdots(\lambda+1+l)}. \end{aligned}$$

Then we integrate by parts, obtaining the identity for $\lambda - 1$. □

4.4 Homogeneous distributions of arbitrary order II

We also can define even and odd homogeneous distributions:

$$|x|^\lambda = \frac{1}{2\cos(\frac{\pi}{2}\lambda)}\Big((-\mathrm{i}x+0)^\lambda + (\mathrm{i}x+0)^\lambda\Big), \tag{4.32}$$

$$|x|^\lambda \mathrm{sgn}(x) = \frac{1}{2\mathrm{i}\sin(\frac{\pi}{2}\lambda)}\Big(-(-\mathrm{i}x+0)^\lambda + (\mathrm{i}x+0)^\lambda\Big). \tag{4.33}$$

The Fourier transforms:

$$\int |k|^\lambda \mathrm{e}^{-\mathrm{i}xk}\mathrm{d}k = \pi^{\frac{1}{2}} \frac{\Gamma(\frac{\lambda+1}{2})}{\Gamma(-\frac{\lambda}{2})}\Big|\frac{x}{2}\Big|^{-\lambda-1}, \tag{4.34}$$

$$\int |k|^\lambda \mathrm{sgn}(k)\mathrm{e}^{-\mathrm{i}xk}\mathrm{d}k = -\mathrm{i}\pi^{\frac{1}{2}} \frac{\Gamma(\frac{\lambda+2}{2})}{\Gamma(\frac{1-\lambda}{2})}\Big|\frac{x}{2}\Big|^{-\lambda-1}\mathrm{sgn}(x), \tag{4.35}$$

Especially symmetric expressions for Fourier transforms are obtained if we introduce

$$\eta^{\lambda}_{\mathrm{ev}}(x) := \Gamma\Big(\frac{\lambda}{2}+\frac{1}{2}\Big)^{-1}\Big(\frac{x^2}{2}\Big)^{\frac{\lambda}{2}} \tag{4.36}$$

$$= (2\pi)^{-1}\Gamma\Big(-\frac{\lambda}{2}+\frac{1}{2}\Big)2^{-\frac{\lambda}{2}}\Big((\mathrm{i}x+0)^{\lambda}+(-\mathrm{i}x+0)^{\lambda}\Big) \tag{4.37}$$

$$= \frac{2^{\frac{\lambda}{2}}}{\sqrt{\pi}}\Gamma\Big(1+\frac{\lambda}{2}\Big)\Big(\rho^{\lambda}_{+}(x)+\rho^{\lambda}_{-}(x)\Big), \tag{4.38}$$

$$\eta^{\lambda}_{\mathrm{odd}}(x) := \Gamma\Big(\frac{\lambda}{2}+1\Big)^{-1}\Big(\frac{x^2}{2}\Big)^{\frac{\lambda+1}{2}}\frac{1}{x} \tag{4.39}$$

$$= \mathrm{i}(2\pi)^{-1}\Gamma\Big(-\frac{\lambda}{2}\Big)2^{-\frac{\lambda}{2}-\frac{1}{2}}\Big((\mathrm{i}x+0)^{\lambda}-(-\mathrm{i}x+0)^{\lambda}\Big) \tag{4.40}$$

$$= \frac{2^{\frac{\lambda}{2}-\frac{1}{2}}}{\sqrt{\pi}}\Gamma\Big(\frac{1}{2}+\frac{\lambda}{2}\Big)\Big(\rho^{\lambda}_{+}(x)-\rho^{\lambda}_{-}(x)\Big). \tag{4.41}$$

We then have the following relations:

$$\partial_x\eta^{\lambda}_{\mathrm{ev}} = \lambda\eta^{\lambda-1}_{\mathrm{odd}}, \qquad \partial_x\eta^{\lambda}_{\mathrm{odd}} = \eta^{\lambda-1}_{\mathrm{ev}}, \tag{4.42}$$

$$x\eta^{\lambda}_{\mathrm{ev}}(x) = (\lambda+1)\eta^{\lambda+1}_{\mathrm{odd}}(x), \qquad x\eta^{\lambda}_{\mathrm{odd}}(x) = \eta^{\lambda+1}_{\mathrm{ev}}(x); \tag{4.43}$$

$$\mathcal{F}\eta^{\lambda}_{\mathrm{ev}} = \eta^{-\lambda-1}_{\mathrm{ev}}, \qquad \mathcal{F}\eta^{\lambda}_{\mathrm{odd}} = -\mathrm{i}\eta^{-\lambda-1}_{\mathrm{odd}}; \tag{4.44}$$

$$\eta^{-1-2m}_{\mathrm{ev}}(x) = \frac{(-1)^m\sqrt{2}}{2^m\left(\frac{1}{2}\right)_m}\delta^{(2m)}(x), \quad m = 0,1,\dots; \tag{4.45}$$

$$\eta^{-2m}_{\mathrm{odd}}(x) = \frac{(-1)^m\sqrt{2}}{2^m\left(\frac{1}{2}\right)_m}\delta^{(2m-1)}(x), \quad m = 1,2,\dots. \tag{4.46}$$

4.5 Anomalous distributions of degree -1

We introduce distributions which extend $\frac{\theta(\pm k)}{|k|} = k_\pm^{-1}$ and $\frac{1}{|k|}$:

$$\int k_+^{-1}\phi(k)\mathrm{d}k := -\int_0^\infty \log(k)\phi^{(1)}(k)\mathrm{d}k \tag{4.47}$$

$$= \int_0^a \frac{\phi(k)-\phi(0)}{k}\mathrm{d}k + \log(a)\phi(0) + \int_a^\infty \frac{\phi(k)}{k}\mathrm{d}k \tag{4.48}$$

$$= \lim_{\epsilon\searrow 0}\Big(\int_\epsilon^\infty \frac{\phi(k)}{k} + \phi(0)\ln(\epsilon)\Big), \tag{4.49}$$

$$\int k_-^{-1}\phi(k)\mathrm{d}k := -\int_{-\infty}^0 \log(-k)\phi^{(1)}\mathrm{d}k \tag{4.50}$$

$$= -\int_{-a}^0 \frac{\phi(k)-\phi(0)}{k}\mathrm{d}k + \log(a)\phi(0) - \int_{-\infty}^{-a} \frac{\phi(k)}{k}\mathrm{d}k \tag{4.51}$$

$$= \lim_{\epsilon\searrow 0}\Big(-\int_{-\infty}^{-\epsilon} \frac{\phi(k)}{k} + \phi(0)\ln(\epsilon)\Big), \tag{4.52}$$

$$\frac{1}{|k|} = k_-^{-1} + k_+^{-1}. \tag{4.53}$$

We have

$$\frac{1}{k} = -k_-^{-1} + k_+^{-1},$$

For typographical reasons, sometimes we will write $\frac{1}{k_\pm}$ for $k_\pm^{-1}$

Proposition 4.3 *Here are the Fourier transform of various forms of* $\frac{1}{|k|}$ *and* *the logarthm:*

$$\int k_\pm^{-1}\mathrm{e}^{-\mathrm{i}xk}\mathrm{d}k = -\log(\pm\mathrm{i}x+0)-\gamma = -\log|x| \mp \frac{\mathrm{i}\pi}{2}\mathrm{sgn}(x)-\gamma, \tag{4.54}$$

$$\int \frac{1}{|k|}\mathrm{e}^{-\mathrm{i}xk}\mathrm{d}k = -2\log|x|-2\gamma, \tag{4.55}$$

$$\int \log|x|\mathrm{e}^{-\mathrm{i}xk}\mathrm{d}x = -\pi\frac{1}{|k|}-2\pi\gamma\delta(k), \tag{4.56}$$

$$\int \log(\pm\mathrm{i}x+0)\mathrm{e}^{-\mathrm{i}xk}\mathrm{d}x = -2\pi\frac{\theta(\mp k)}{|k|}-2\pi\gamma\delta(k), \tag{4.57}$$

$$\int \log(x\mp\mathrm{i}0)\mathrm{e}^{-\mathrm{i}xk}\mathrm{d}x = -2\pi k_\mp^{-1}+(-2\pi\gamma\mp\mathrm{i}\pi)\delta(k), \tag{4.58}$$

$$\int \log(x-\lambda)\mathrm{e}^{-\mathrm{i}xk}\mathrm{d}x = \mathrm{e}^{-\mathrm{i}\lambda k}\Big(-2\pi k_\mp^{-1}+(-2\pi\gamma\mp\mathrm{i}\pi)\delta(k)\Big), \quad \pm\mathrm{Im}\lambda>0. \tag{4.59}$$

Proof. We start from one of the formulas for the Euler constant. We change the variable from k to yk, with $y>0$:

$$\begin{aligned}
-\gamma &= \int_0^\infty \mathrm{e}^{-k}\log(k)\mathrm{d}k \\
&= \int_0^\infty \mathrm{e}^{-yk}\log(ky)\mathrm{d}(ky) \\
&= y\log(y)\int_0^\infty \mathrm{e}^{-ky}\mathrm{d}k + y\int_0^\infty \mathrm{e}^{-ky}\log(k)\mathrm{d}k \\
&= y\log(y)\frac{1}{y} - \int_0^1 \big(\partial_k(\mathrm{e}^{-ky}-1)\big)\log(k)\mathrm{d}k - \int_1^\infty \big(\partial_k\mathrm{e}^{-ky}\big)\log(k)\mathrm{d}k \\
&= \log(y)+\int_0^1\frac{\mathrm{e}^{-ky}-1}{k}\mathrm{d}k+\int_1^\infty\frac{\mathrm{e}^{-ky}}{k}\mathrm{d}k.
\end{aligned}$$

The rhs is analytic in y on the right halfplane. It is constant on the positive halfline. So it is constant on the whole halfplane. Therefore, we can replace y with $\mathrm{i}x$. This proves (4.54), which implies (**??**) and (4.55).

By inverting the Fourier transform we obtain (4.56) and (4.58). We can also get (4.58) from (4.56):

$$\int \log(x \mp \mathrm{i}0)\mathrm{e}^{-\mathrm{i}xk}\mathrm{d}x = \int \big(\log|x| \mp \mathrm{i}\pi\theta(-x)\big)\mathrm{e}^{-\mathrm{i}xk}\mathrm{d}x \tag{4.60}$$

$$= -\pi\frac{1}{|k|} - 2\pi\gamma\delta(k) \pm \pi\frac{1}{(k+\mathrm{i}0)} \tag{4.61}$$

$$= -2\pi\frac{1}{k_\mp} + (-2\pi\gamma \mp \mathrm{i}\pi)\delta(k). \tag{4.62}$$

□

Here is an alternative approach:

$$\frac{1}{k_\pm} = \lim_{\nu\searrow 0}\Big(\frac{1}{k_\pm^{1+\nu}} - \frac{1}{\nu}\delta(k)\Big), \tag{4.63}$$

$$\frac{1}{|k|} = \lim_{\nu\searrow 0}\Big(\frac{1}{|k|^{1+\nu}} - \frac{2}{\nu}\delta(k)\Big) \tag{4.64}$$

Here is the computation of the Fourier transform by this method:

$$\int \frac{\mathrm{e}^{-\mathrm{i}kx}}{k_\pm}\mathrm{d}k \approx \int \frac{\mathrm{e}^{-\mathrm{i}kx}}{k_\pm^{1+\nu}}\mathrm{d}k - \frac{1}{\nu} \tag{4.65}$$

$$= \Gamma(\nu)(\pm\mathrm{i}k+0)^{-\nu} - \frac{1}{\nu}, \tag{4.66}$$

$$\approx \Big(\frac{1}{\nu} - \gamma\Big)\big(1 - \nu\log(\pm\mathrm{i}k+0)\big) - \frac{1}{\nu}, \tag{4.67}$$

$$\approx -\log(\pm\mathrm{i}k+0) - \gamma. \tag{4.68}$$

4.6 Anomalous distributions of integral degree

Let $H_n := \frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{n}$ be the nth harmonic number. Define

$$k_\pm^{-n-1} := \frac{(\mp 1)^n}{n!}\partial_k^n k_\pm^{-1} + (\mp 1)^n H_n \frac{\delta^{(n)}(k)}{n!}, \tag{4.69}$$

$$\frac{\operatorname{sgn}(k)}{k^{n+1}} := k_+^{-n-1} + (-1)^n k_-^{-n-1}. \tag{4.70}$$

Theorem 4.4 *We have*

$$\frac{1}{x_\pm^{1+n}} = \lim_{\nu\to 0}\Big(\frac{1}{x_\pm^{1+n-\nu}} - \frac{(\mp 1)^n}{\nu}\frac{\delta^{(n)}(x)}{n!}\Big), \tag{4.71}$$

$$\frac{1}{|x|^{1+n}} = \lim_{\nu\to 0}\Big(\frac{1}{|x|^{1+n-\nu}} - \frac{\big((-1)^n+1\big)}{\nu}\frac{\delta^{(n)}(x)}{n!}\Big), \tag{4.72}$$

$$\frac{\operatorname{sgn}(x)}{|x|^{1+n}} = \lim_{\nu\to 0}\Big(\frac{\operatorname{sgn}(x)}{|x|^{1+n-\nu}} - \frac{\big((-1)^n-1\big)}{\nu}\frac{\delta^{(n)}(x)}{n!}\Big). \tag{4.73}$$

Proof. It is enough to consider only x_+^{-n-1}.

$$\int x_+^{-n-1+\nu}\phi(x)\mathrm{d}x = \int_0^\infty \frac{\left(\partial_x^{n+1}x^\nu\right)\phi(x)}{\nu(\nu-1)\cdots(\nu-n)}\mathrm{d}x \tag{4.74}$$

$$= \int_0^\infty \frac{x_+^\nu\phi^{(n+1)}(x)}{(-\nu)(1-\nu)\cdots(n-\nu)}\mathrm{d}x \tag{4.75}$$

$$= \int_0^\infty \frac{(x_+^\nu-1)\phi^{(n+1)}(x)}{(-\nu)(1-\nu)\cdots(n-\nu)}\mathrm{d}x \tag{4.76}$$

$$+ \int_0^\infty \frac{\phi^{(n+1)}(x)}{(-\nu)(1-\nu)\cdots(n-\nu)}\mathrm{d}x \tag{4.77}$$

$$= -\int_0^\infty \frac{\log(x)\phi^{(n+1)}(x)}{n!}\mathrm{d}x \tag{4.78}$$

$$+ \frac{1}{\nu}\frac{\phi^{(n)}(0)}{(1-\nu)\cdots(n-\nu)} \tag{4.79}$$

$$= \int x_+^{-n-1}\phi(x)\mathrm{d}x \tag{4.80}$$

$$+ \frac{1}{\nu}\frac{\phi^{(n)}(0)}{n!} + H_n\frac{\phi^{(n)}(0)}{n!} + O(\nu). \tag{4.81}$$

□

Theorem 4.5

$$\int x_+^{-n-1}\phi(x)\mathrm{d}x = \int_0^1 \frac{1}{x^{n+1}}\Big(\phi(x) - \sum_{j=0}^{n}\frac{x^j}{j!}\phi^{(j)}(0)\Big)\mathrm{d}x$$

$$+ \int_1^\infty \frac{1}{x^{n+1}}\Big(\phi(x) - \sum_{j=0}^{n-1}\frac{x^j}{j!}\phi^{(j)}(0)\Big)\mathrm{d}x. \tag{4.82}$$

Proof. Let $a > 0$. If we assume that $\mathrm{Re}\nu > -1$, then we can use (4.28) with n replaced with $n+1$:

$$\int \frac{1}{x_+^{n+1-\nu}}\phi(x)\mathrm{d}x \tag{4.83}$$

$$\begin{aligned}
=&\int_a^\infty \frac{1}{x^{n+1-\nu}}\phi(x)\mathrm{d}x \\
&+\int_0^a \frac{1}{x^{n+1-\nu}}\Big(\phi(x)-\sum_{j=0}^{n}\frac{x^j}{j!}\phi^{(j)}(0)\Big)\mathrm{d}x \\
&-\sum_{j=0}^{n} a^{-n+j+\nu}\phi^{(j)}(0)\sum_{l=0}^{j}\frac{1}{(j-l)!(n-l-\nu)\cdots(n-\nu)}.
\end{aligned} \tag{4.84}$$

The last term of the sum in (4.84) is

$$-a^{\nu}\phi^{(n)}(0)\sum_{l=0}^{n-1}\frac{1}{(n-l)!(n-l-\nu)\cdots(n-\nu)} \tag{4.85}$$

$$-a^{\nu}\phi^{(n)}(0)\frac{1}{(-\nu)\cdots(n-\nu)} \tag{4.86}$$

$$=-\phi^{(n)}(0)\frac{H_n}{n!} \tag{4.87}$$

$$+\frac{1}{\nu}\phi^{(n)}(0)\frac{1}{n!}+\log(a)\phi^{(n)}(0)\frac{1}{n!}+\phi^{(n)}(0)\frac{H_n}{n!}+O(\nu) \tag{4.88}$$

$$=\frac{1}{\nu}\phi^{(n)}(0)\frac{1}{n!}+\log(a)\phi^{(n)}(0)\frac{1}{n!}+O(\nu). \tag{4.89}$$

Thus we have proven that

$$\begin{aligned}
\int x_+^{-n-1}\phi(x)\mathrm{d}x =& \int_a^\infty \frac{1}{x^{n+1}}\phi(x)\mathrm{d}x \\
&+\int_0^a \frac{1}{x^{n+1}}\Big(\phi(x)-\sum_{j=0}^{n}\frac{x^j}{j!}\phi^{(j)}(0)\Big)\mathrm{d}x \\
&-\sum_{j=0}^{n-1} a^{-n+j}\phi^{(j)}(0)\sum_{l=0}^{j}\frac{1}{(j-l)!(n-l)\cdots n} \\
&+\log(a)\frac{\phi^{(n)}(0)}{n!}
\end{aligned} \tag{4.90}$$

Then we take $a\to\infty$, noting that

$$\int_1^a x^{-1}\mathrm{d}x=\log(a).$$

□

Proposition 4.6 *The Fourier transform:*

$$\int k_{\pm}^{-n-1}\mathrm{e}^{-\mathrm{i}xk}\mathrm{d}k = \frac{(\mp\mathrm{i}x)^n}{n!}\Big(-\log(\pm\mathrm{i}x+0)-\gamma+H_n\Big) \tag{4.91}$$

$$= \frac{(\mp\mathrm{i}x)^n}{n!}\Big(-\log|x|\mp\frac{\mathrm{i}\pi}{2}\mathrm{sgn}(x)-\gamma+H_n\Big) \tag{4.92}$$

Proof. We use (4.71):

$$\int_0^\infty \frac{\mathrm{e}^{-\mathrm{i}xk}}{k^{n+1}}\mathrm{d}k = \lim_{\nu\searrow 0}\Big(\int_0^\infty \frac{\mathrm{e}^{-\mathrm{i}xk}}{k^{1+n-\nu}}\mathrm{d}k - \frac{(\mp\mathrm{i}k)^n}{\nu n!}\Big) \tag{4.93}$$

$$= \lim_{\nu\searrow 0}\Big(\Gamma(-n+\nu)(\pm\mathrm{i}x+0)^{n-\nu} - \frac{(\mp\mathrm{i}k)^n}{\nu n!}\Big) \tag{4.94}$$

$$= \lim_{\nu\searrow 0}\left(\frac{(-1)^n}{n!}\Big(\frac{1}{\nu}-\gamma+H_n\Big)(\pm\mathrm{i}x)^n\big(1-\nu\log(\pm\mathrm{i}x+0)\big) - \frac{(\mp\mathrm{i}k)^n}{\nu n!}\right)$$

$$= \frac{(\mp\mathrm{i}x)^n}{n!}\big(-\gamma+H_n-\log(\pm\mathrm{i}x+0)\big). \tag{4.95}$$

□

4.7 Infrared regularized distributions

Theorem 4.7 *Let $n+1>2\alpha>n$. Then*

$$\frac{1}{k_+^{2\alpha}} = \lim_{m\to 0}\Big(\frac{\theta(k)}{(k^2+m^2)^\alpha} \tag{4.96}$$

$$-\sum_{j=0}^{n-1}\frac{\Gamma(\alpha-\frac{j}{2}-\frac{1}{2})\Gamma(\frac{j}{2}+\frac{1}{2})}{2m^{2\alpha-j-1}\Gamma(\alpha)j!}(-1)^j\delta^{(j)}(k)\Big). \tag{4.97}$$

Proof. Clearly,

$$\int\frac{\phi(k)}{k_+^{2\alpha}}\mathrm{d}k = \int_0^\infty\frac{1}{k^{2\alpha}}\Big(\phi(k)-\sum_{j=0}^{n-1}\frac{k^j}{j!}\phi^{(j)}(0)\Big)\mathrm{d}k \tag{4.98}$$

is the limit as $m\to 0$ of

$$\int_0^\infty\frac{1}{(k^2+m^2)^\alpha}\Big(\phi(k)-\sum_{j=0}^{n-1}\frac{k^j}{j!}\phi^{(j)}(0)\Big)\mathrm{d}k. \tag{4.99}$$

Now

$$\int_0^\infty \frac{k^j}{(k^2+m^2)^\alpha}\mathrm{d}k = \frac{\Gamma(\alpha-\frac{j}{2}-\frac{1}{2})\Gamma(\frac{j}{2}+\frac{1}{2})}{2m^{2\alpha-j-1}\Gamma(\alpha)} \tag{4.100}$$

□

Theorem 4.8 *Let $2p+1 > 2\alpha > 2p-1$, $p = 1, 2, \ldots$. Then*

$$\frac{1}{|k|^{2\alpha}} = \lim_{m\to 0}\Big(\frac{1}{(k^2+m^2)^\alpha} \tag{4.101}$$

$$-\sum_{l=0}^{p-1} \frac{\pi^{\frac{3}{2}} m^{-2\alpha+2l+1}(-1)^l}{\Gamma(\alpha)\sin\big(\pi(\alpha-\frac{1}{2})\big)2^{2l}l!\Gamma(\frac{3}{2}-\alpha+l)}\delta^{(2l)}(k)\Big). \tag{4.102}$$

Proof. Clearly,

$$\int \frac{\phi(k)}{k^{2\alpha}}\mathrm{d}k = \int \frac{1}{k^{2\alpha}}\Big(\phi(k) - \sum_{l=0}^{p-1}\frac{k^{2l}}{(2l)!}\phi^{(2l)}(0)\Big)\mathrm{d}k \tag{4.103}$$

is the limit as $m \to 0$ of

$$\int \frac{1}{(k^2+m^2)^\alpha}\Big(\phi(k) - \sum_{l=0}^{p-1}\frac{k^{2l}}{(2l)!}\phi^{(2l)}(0)\Big)\mathrm{d}k. \tag{4.104}$$

Now

$$\frac{1}{(2l)!}\int \frac{k^{2l}}{(k^2+m^2)^\alpha}\mathrm{d}k = \frac{m^{-2\alpha+2l+1}\Gamma(\alpha-l-\frac{1}{2})\Gamma(l+\frac{1}{2})}{\Gamma(\alpha)(2l)!} \tag{4.105}$$

$$= \frac{\pi^{\frac{3}{2}} m^{-2\alpha+2l+1}(-1)^l}{\Gamma(\alpha)\sin\big(\pi(\alpha-\frac{1}{2})\big)2^{2l}l!\Gamma(\frac{3}{2}-\alpha+l)}. \tag{4.106}$$

□

Theorem 4.9 *Let $n = 0, 1, \ldots$. Then*

$$\frac{1}{k_+^{n+1}} = \lim_{m\to 0}\Big(\frac{\theta(k)}{(k^2+m^2)^{\frac{n}{2}+\frac{1}{2}}} \tag{4.107}$$

$$-\sum_{j=0}^{n-1}\frac{m^{-n+j}\Gamma(\frac{n}{2}-\frac{j}{2})\Gamma(\frac{j}{2}+\frac{1}{2})}{2\Gamma(\alpha)j!}(-1)^j\delta^{(j)}(k)\Big) \tag{4.108}$$

$$-\begin{cases}\Big(\frac{1}{2}H_p + \log(m)\Big)\frac{1}{n!}(-1)^n\delta^n(k), & n = 2p+1;\\ \Big(\frac{1}{2}H_p(\frac{1}{2}) + \log(\frac{m}{2})\Big)\frac{1}{n!}(-1)^n\delta^{(n)}(k), & n = 2p.\end{cases} \tag{4.109}$$

Proof. Clearly,

$$\int \frac{\phi(k)}{k_+^{n+1}} \mathrm{d}k \tag{4.110}$$

is the limit as $m \to 0$ of

$$\int_0^\infty \frac{1}{(k^2+m^2)^{n+1}} \Big(\phi(k) - \sum_{j=0}^{n-1} \frac{k^j}{j!} \phi^{(j)}(0) \Big) \mathrm{d}k \tag{4.111}$$

$$- \int_0^1 \frac{k^n}{(k^2+m^2)^{\frac{n}{2}+\frac{1}{2}}} \frac{\phi^{(n)}(0)}{n!}. \tag{4.112}$$

Now for $n = 2p+1$ we have

$$\int_0^1 \frac{k^{2p+1}}{(k^2+m^2)^{p+1}} \mathrm{d}k \tag{4.113}$$

$$= - \sum_{j=1}^{p} \frac{k^{2j}}{2j(k^2+m^2)^j} \Big|_0^1 + \int_0^1 \frac{k}{k^2+m^2} \mathrm{d}k \tag{4.114}$$

$$= - \sum_{j=1}^{p} \frac{1}{2j(1+m^2)^j} + \frac{1}{2} \big(\log(1+m^2) - \log(m^2) \big) \tag{4.115}$$

$$= -\frac{1}{2} H_p - \log(m) + o(m^0). \tag{4.116}$$

For $n = 2p$ we compute

$$\int_0^1 \frac{k^{2p}}{(k^2+m^2)^{p+\frac{1}{2}}} \mathrm{d}k \tag{4.117}$$

$$= - \sum_{j=0}^{p-1} \frac{k^{2j+1}}{(2j+1)(k^2+m^2)^{j+\frac{1}{2}}} \Big|_0^1 + \int_0^1 \frac{1}{(k^2+m^2)^{\frac{1}{2}}} \mathrm{d}k \tag{4.118}$$

$$= - \sum_{j=1}^{p} \frac{1}{(2j+1)(1+m^2)^{j+\frac{1}{2}}} + \log \big(1 + \sqrt{1+m^2}\big) - \log(m) \tag{4.119}$$

$$= -\frac{1}{2} H_p\Big(\frac{1}{2}\Big) + \log(2) - \log(m) + o(m^0). \tag{4.120}$$

□

4.8 Distributions on halfline

We will denote by $C^\infty[0,\infty[$ smooth function having all right-sided derivatives at 0. We set

$$C_{\mathrm{N}}^\infty[0,\infty[:= \{\phi \in C^\infty[0,\infty[\ : \ \phi^{(2m+1)}(0) = 0, \ m = 0, 1, \dots\}, \qquad (4.121)$$

$$C_{\mathrm{D}}^\infty[0,\infty[:= \{\phi \in C^\infty[0,\infty[\ : \ \phi^{(2m)}(0) = 0, \ m = 0, 1, \dots\}. \qquad (4.122)$$

$\mathcal{S}_{\mathrm{N}}[0,\infty[$, $\mathcal{S}_{\mathrm{D}}[0,\infty[$ have obvious definitions. We set $\mathcal{S}_{\mathrm{N}}'[0,\infty[$, $\mathcal{S}_{\mathrm{D}}'[0,\infty[$ to be their duals.

Note that ∂_x and the multiplication by x map $\mathcal{S}_{\mathrm{N}}[0,\infty[$ into $\mathcal{S}_{\mathrm{D}}[0,\infty[$ and vice versa, as well as $\mathcal{S}_{\mathrm{N}}'[0,\infty[$ into $\mathcal{S}_{\mathrm{D}}'[0,\infty[$ and vice vera.

The cosine transformation with the kernel

$$\mathcal{F}_{\mathrm{N}}(x,k) := \sqrt{\frac{2}{\pi}} \cos(xk)$$

maps $\mathcal{S}_{\mathrm{N}}'[0,\infty[$ into into itself. We have

Likewise, the sine transformation with the kernel

$$\mathcal{F}_{\mathrm{N}}(x,k) := \sqrt{\frac{2}{\pi}} \sin(xk)$$

maps $\mathcal{S}_{\mathrm{D}}'[0,\infty[$ into into itself.

Let $I\phi(x) := \phi(-x)$. I maps $\mathcal{S}(\mathbb{R})$, as well as extends to a map of $\mathcal{S}'(\mathbb{R})$ into itself. We will write

$$\mathcal{S}_{\mathrm{ev}}(\mathbb{R}) := \{\phi \in \mathcal{S}(\mathbb{R}) \ : \ I\phi = \phi\}, \qquad (4.123)$$

$$\mathcal{S}_{\mathrm{ev}}'(\mathbb{R}) := \{\lambda \in \mathcal{S}'(\mathbb{R}) \ : \ I\lambda = \lambda\}, \qquad (4.124)$$

$$\mathcal{S}_{\mathrm{odd}}(\mathbb{R}) := \{\phi \in \mathcal{S}(\mathbb{R}) \ : \ I\phi = -\phi\}, \qquad (4.125)$$

$$\mathcal{S}_{\mathrm{odd}}'(\mathbb{R}) := \{\lambda \in \mathcal{S}'(\mathbb{R}) \ : \ I\lambda = -\lambda\}. \qquad (4.126)$$

If $\phi \in \mathcal{S}_{\mathrm{N}}[0,\infty[$, we set

$$\phi^{\mathrm{ev}}(x) := \begin{cases} \phi(x) & x \geq 0; \\ \phi(-x) & x \leq 0. \end{cases}$$

Note that $\phi^{\mathrm{ev}} \in \mathcal{S}_{\mathrm{ev}}(\mathbb{R})$.

If λ_{ev} is an even distribution in $\mathcal{S}'(\mathbb{R})$, then we can associate with it a distribution in $\mathcal{S}_{\mathrm{N}}'[0,\infty[$ by

$$\int_0^\infty \lambda_{\mathrm{N}}(x)\phi(x)\mathrm{d}x := \frac{1}{2}\int \lambda_{\mathrm{ev}}(x)\phi^{\mathrm{ev}}(x)\mathrm{d}x.$$

Similarly, if $\phi \in \mathcal{S}_{\rm D}[0,\infty[$, we set

$$\phi^{\rm odd}(x) := \begin{cases} \phi(x) & x \geq 0; \\ -\phi(-x) & x \leq 0. \end{cases}$$

Note that $\phi^{\rm odd} \in \mathcal{S}_{\rm odd}(\mathbb{R})$.

If $\lambda_{\rm odd}$ is an odd distribution in $\mathcal{S}'(\mathbb{R})$, then we can associate with it a distribution in $\mathcal{S}'_{\rm D}[0,\infty[$ by We set

$$\int_0^\infty \lambda_{\rm D}(x)\phi(x){\rm d}x := \frac{1}{2}\int \lambda_{\rm odd}(x)\phi^{\rm odd}(x){\rm d}x.$$

The usual Fourier transform $\mathcal{F}$ preserves $\mathcal{S}_{\rm ev}(\mathbb{R})$ and $\mathcal{S}_{\rm odd}(\mathbb{R})$. The Fourier transform on even distributions is closely related to the cosine transform and on odd distributions to the sine transform:

$$\mathcal{F}_{\rm N}\lambda_{\rm N} = (\mathcal{F}\lambda)_{\rm N}, \quad \lambda \in \mathcal{S}'_{\rm ev}(\mathbb{R}), \tag{4.127}$$

$$\mathcal{F}_{\rm D}\lambda_{\rm D} = {\rm i}(\mathcal{F}\lambda)_{\rm D}, \quad \lambda \in \mathcal{S}'_{\rm odd}(\mathbb{R}). \tag{4.128}$$

An example of an even distribution is $\eta_{\rm ev}$. Let $\eta_{\rm N}$ denote the corresponding distributtion in $\mathcal{S}'_{\rm N}[0,\infty[$.

Likewise, an example of an odd distribution is $\eta_{\rm odd}$. Let $\eta_{\rm D}$ denote the corresponding distributtion in $\mathcal{S}'_{\rm D}[0,\infty[$.

We have

$$\mathcal{F}_{\rm N}\eta_{\rm N}^\lambda = \eta_{\rm N}^{-\lambda-1}, \qquad \mathcal{F}_{\rm D}\eta_{\rm D}^\lambda = \eta_{\rm D}^{-\lambda-1}; \tag{4.129}$$

5 Homogeneous distributions in arbitrary dimension

5.1 Sphere $\mathbb{S}^{d-1}$

Consider the Euclidean space $\mathbb{R}^d$. Introduce two varieties of spherical coordinates on a $d-1$-dimensional sphere

$$(\theta_{d-2},\dots,\theta_1,\phi) \in [0,\pi]\times\cdots\times[0,\pi]\times[0,2\pi[,$$

$$(w_{d-2},\dots,w_1,\phi) \in [0,\pi]\times\cdots\times[0,\pi]\times[0,2\pi[,$$

with $w_j = \cos\theta_j$, The spherical measure on $\mathbb{S}^{d-1}$ is

$$\begin{aligned} & \sin^{d-2}\theta_{d-2}{\rm d}\theta_{d-2}\cdots\sin\theta_1{\rm d}\theta_1{\rm d}\phi \\ = \; & (1-w_{d-2}^2)^{(d-3)/2}{\rm d}w_{d-2}\cdots{\rm d}w_1{\rm d}\phi. \end{aligned}$$

Theorem 5.1 *The area of the $d-1$-dimensional sphere is*

$$|\mathbb{S}_{d-1}| = \frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})},$$

or, in a more elementary form,

$$|\mathbb{S}_{2m+1}| = \frac{2\pi^{m+1}}{m!}, \quad m = 0, 1, \dots; \tag{5.1}$$

$$|\mathbb{S}_{2m}| = \frac{2\pi^{m}}{(\frac{1}{2})_m}, \quad m = 0, 1, \dots . \tag{5.2}$$

Proof. Method I. We compute in two ways the Gaussian integral: in the Cartesian coordinates

$$\int \mathrm{e}^{-x_1^2-\cdots-x_d^2}\mathrm{d}x_1\cdots\mathrm{d}x_d = \pi^{\frac{d}{2}},$$

and in spherical coordinates:

$$|\mathbb{S}_{d-1}|\int_0^\infty \mathrm{e}^{-r^2}r^{d-1}\mathrm{d}r = \frac{1}{2}\Gamma\Big(\frac{d}{2}\Big). \tag{5.3}$$

Method II. We compute the area of the sphere in the spherical coordinates:

$$|\mathbb{S}_{d-1}| = \int_0^\pi \sin^{d-2}\phi_{d-1}\mathrm{d}\phi_{d-1}\cdots\int_0^\pi \sin\phi_2\mathrm{d}\phi_2\int_0^{2\pi}\mathrm{d}\phi_1$$

Then we use

$$\int_0^\pi \sin^{k-1}\phi_k\mathrm{d}\phi_k = \frac{\sqrt{\pi}\Gamma(\frac{k-1}{2})}{\Gamma(\frac{k}{2})}, \quad k = 2, \dots, d-1; \quad \int_0^{2\pi}\mathrm{d}\phi_1 = 2\pi.$$

□

5.2 Homogeneous functions in arbitrary dimension

Theorem 5.2 *Let $-d < \lambda < 0$. Then on $\mathbb{R}^d$*

$$\int |x|^\lambda \mathrm{e}^{-\mathrm{i}x\xi}\mathrm{d}x = \pi^{\frac{d}{2}}\frac{\Gamma(\frac{\lambda+d}{2})}{\Gamma(-\frac{\lambda}{2})}\Big|\frac{\xi}{2}\Big|^{-\lambda-d}. \tag{5.4}$$

Proof. We use the spherical coordinates:

$$\begin{aligned}
&\int |x|^\lambda \mathrm{e}^{-\mathrm{i}x\xi}\mathrm{d}x && (5.5)\\
=\ &\int_0^\infty \mathrm{d}r \int_0^\pi \mathrm{d}\phi_{d-1} r^{\lambda+d-1}\mathrm{e}^{-\mathrm{i}r|\xi|\cos\phi_{d-1}} r^{\lambda+d-1}\sin^{d-2}\phi_{d-1}|\mathbb{S}_{d-2}| && (5.6)\\
=\ &\Gamma(\lambda+d)\int_0^{\frac{\pi}{2}} \Big((\mathrm{i}|\xi|\cos\phi_{d-1}+0)^{-\lambda-d} + (-\mathrm{i}|\xi|\cos\phi_{d-1}+0)^{-\lambda-d}\Big)\sin^{d-2}\phi_{d-1}\mathrm{d}\phi_{d-1}|\mathbb{S}_{d-2}|\\
=\ &\Gamma(\lambda+d)2\cos\Big(\frac{\lambda+d}{2}\pi\Big)|\xi|^{-\lambda-d}\int_0^{\frac{\pi}{2}}\cos^{-\lambda-d}\phi_{d-1}\sin^{d-2}\phi_{d-1}\mathrm{d}\phi_{d-1}|\mathbb{S}_{d-2}|.
\end{aligned}$$

Then we apply

$$\begin{aligned}
|\mathbb{S}_{d-2}| &= \frac{2\pi^{\frac{d-1}{2}}}{\Gamma(\frac{d-1}{2})},\\
\int_0^{\frac{\pi}{2}}\cos^{-\lambda-d}\phi_{d-1}\sin^{d-2}\phi_{d-1}\mathrm{d}\phi_{d-1} &= \frac{1}{2}\frac{\Gamma(\frac{-\lambda-d+1}{2})\Gamma(\frac{d-1}{2})}{\Gamma(-\frac{\lambda}{2})},\\
\Gamma(\lambda+d) &= \pi^{-\frac{1}{2}}2^{\lambda+d-1}\Gamma\Big(\frac{\lambda+d}{2}\Big)\Gamma\Big(\frac{\lambda+d+1}{2}\Big)\\
\cos\Big(\frac{\lambda+d}{2}\pi\Big) &= \frac{\pi}{\Gamma(\frac{\lambda+d+1}{2})\Gamma(\frac{-\lambda-d+1}{2})},
\end{aligned}$$

and we obtain (5.4) □

In order to express (5.4) in a more symmetric way, define

$$\eta^\lambda(x) := \frac{1}{\Gamma(\frac{\lambda+d}{2})}\Big(\frac{x^2}{2}\Big)^{\frac{\lambda}{2}}, \quad \lambda > -d.$$

We extend it to $\lambda \leq -d$ by setting

$$\eta^{\lambda-2m}(x) := \frac{(-2)^m}{\big(-\frac{\lambda}{2}\big)_m}\Delta^m\eta^\lambda(x). \tag{5.7}$$

Then

$$\mathcal{F}\eta^\lambda = \eta^{-\lambda-d}, \tag{5.8}$$
$$x^2\eta^\lambda = (\lambda+d)\eta^{\lambda+2}, \tag{5.9}$$
$$\Delta\eta^\lambda = \lambda\eta^{\lambda-2}. \tag{5.10}$$

5.3 Renormalizing the $|k|^{-d}$ function

Define the distribution $|k|^{-d}$ on $\mathbb{R}^d$:

$$\mathcal{P}\int |k|^{-d}\phi(k)\mathrm{d}k := \int_{|k|<1} |k|^{-d}\big(\phi(k)-\phi(0)\big)\mathrm{d}k + \int_{|k|>1} |k|^{-d}\phi(k)\mathrm{d}k..$$

Theorem 5.3 *We have an alternative definition of $|k|^{-d}$:*

$$|k|^{-d} = \lim_{\nu\searrow 0}\Big(|k|^{-d+\nu} - \frac{2\pi^{\frac{d}{2}}}{\nu\Gamma(\frac{d}{2})}\delta(k)\Big). \tag{5.11}$$

Here is its Fourier transform:

$$\begin{aligned}\frac{\Gamma(\frac{d}{2})}{2\pi^{\frac{d}{2}}}\int |k|^{-d}\mathrm{e}^{-\mathrm{i}kx}\mathrm{d}k &= -\log\Big(\frac{r}{2}\Big) + \frac{1}{2}\psi\Big(\frac{d}{2}\Big) - \frac{1}{2}\gamma\\ &= -\log r - \gamma, && d=1;\\ &= -\log\Big(\frac{r}{2}\Big) - \gamma, && d=2;\\ &= -\log r - \gamma + \frac{1}{2}H_m\Big(\frac{1}{2}\Big), && d=2m+1;\\ &= -\log\Big(\frac{r}{2}\Big) - \gamma + \frac{1}{2}H_m, && d=2(m+1).\end{aligned}$$

Proof.

$$\int_{|k|<1}|k|^{-d+\nu}\mathrm{d}k = \int_{|k|<1}|k|^{-1+\nu}\mathrm{d}|k||\mathbb{S}_{d-1}| = \frac{2\pi^{\frac{d}{2}}}{\nu\Gamma(\frac{d}{2})}. \tag{5.12}$$

This proves (5.11).

$$\int |k|^{-d+\nu}\mathrm{e}^{-\mathrm{i}kx}\mathrm{d}k \tag{5.13}$$

$$= \Big(\frac{r}{2}\Big)^{-\nu}\frac{\pi^{\frac{d}{2}}\Gamma(\frac{\nu}{2})}{\Gamma(\frac{d}{2}-\frac{\nu}{2})} \tag{5.14}$$

$$\approx \Big(1-\nu\log\big(\frac{r}{2}\big)\Big)\frac{\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})}\Big(1+\frac{\nu}{2}\psi\big(\frac{d}{2}\big)\Big)\Big(\frac{2}{\nu}-\gamma\Big)$$

$$\approx \frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})}\Big(\frac{1}{\nu} - \log\Big(\frac{r}{2}\Big) + \frac{1}{2}\psi\Big(\frac{d}{2}\Big) - \frac{1}{2}\gamma\Big). \tag{5.15}$$

□

6 The Laplace method

6.1 The basic Laplace method

We consider an integral

$$I(\lambda) = \int_a^b f(x)\mathrm{e}^{\lambda\phi(x)}\mathrm{d}x.$$

for large λ. We assume that f is once differentiable, ϕ is 3 times differentiable. We also assume that ϕ has a global maximum in $x_0 \in]a, b[$. Clearly, $\phi'(x_0) = 0$. We assume that $\phi''(x_0) \neq 0$. Obviously, $\phi''(x_0) < 0$.

In a neighborhood of x_0 we have

$$\phi(x) \approx \phi(x_0) + \frac{1}{2}\phi''(z_0)(x - x_0)^2. \tag{6.1}$$

The biggest contribution to the integral on the curve γ comes from a neighborhood of x_0. We obtain

$$\begin{aligned} I(\lambda) &= \int_{[a,b]} f(x)\mathrm{e}^{\lambda\phi(x)}\mathrm{d}x \\ &\approx \int_{-\infty}^{\infty} f(x_0)\mathrm{e}^{\lambda\phi(x_0)+\frac{\lambda}{2}\phi''(x_0)(x-x_0)^2}\mathrm{d}x \\ &= f(x_0)\mathrm{e}^{\lambda\phi(x_0)}\sqrt{\frac{2\pi}{-\lambda\phi''(x_0)}}. \end{aligned}$$

This can be formulated more rigorously:

Theorem 6.1 *Under the assumptions described above,*

$$\lim_{\lambda\to\infty} \frac{I(\lambda)}{f(x_0)\mathrm{e}^{\lambda\phi(x_0)}\sqrt{\frac{2\pi}{-\lambda\phi''(x_0)}}} = 1. \tag{6.2}$$

Proof. Without limiting the generality, we can assume that $\phi''(x_0) = -1$, $\phi(x_0) = 0$ and $x_0 = 0$. We will find $\epsilon > 0$ such that for $|x| < \epsilon$

$$|f(x) - f(x_0)| \leq c|x|, \tag{6.3}$$

$$\left|\phi(x) + \frac{x^2}{2}\right| \leq c|x|^3. \tag{6.4}$$

Now

$$I(\lambda) - \int_{-\infty}^{\infty} e^{-\lambda\frac{x^2}{2}} dx = I + II + III + IV, \tag{6.5}$$

$$I = \int_{]a,b[\setminus[-\epsilon,\epsilon[} f(x) e^{\lambda\phi(x)} dx, \tag{6.6}$$

$$II = \int_{-\epsilon}^{\epsilon} \big(f(x) - f(x_0)\big) e^{\lambda\phi(x)} dx, \tag{6.7}$$

$$III = \int_{-\epsilon}^{\epsilon} f(x_0)\Big(e^{\lambda\phi(x)} - e^{-\lambda\frac{x^2}{2}}\Big) dx, \tag{6.8}$$

$$IV = \int_{\mathbb{R}\setminus[-\epsilon,\epsilon]} e^{-\lambda\frac{x^2}{2}} dx. \tag{6.9}$$

Now, with various constants $c_i > 0$,

$$|I| \leq |b-a| e^{-\lambda c_1}, \tag{6.10}$$

$$|II| \leq c_2 \int |x| e^{\lambda\phi(x)} dx \leq c_2 \int |x| e^{-\lambda c_3 x^2} dx \leq \frac{c_4}{\lambda}, \tag{6.11}$$

$$|III| \leq c_5 \int_{-\epsilon}^{\epsilon} \lambda |x^3| e^{-\lambda c_6 x^2} dx = \frac{c_7}{\lambda}, \tag{6.12}$$

$$|IV| \leq e^{-\lambda c_8}. \tag{6.13}$$

To obtain (6.12) we used $|e^x - e^y| < |x-y| e^{\max(x,y)}$. □

6.2 The oscillatory Laplace method (the stationary phase method)

We consider now the integral

$$\int f(x) e^{i\lambda\psi(x)} dx.$$

for large λ. We assume that f and ψ are smooth and f is compactly supported. We also assume that ψ has a unique critical point $x_0 \in \mathrm{supp} f$, that is, $\psi'(x_0) = 0$. We assume that the critical point is non-degenerate, that is, $\psi''(x_0) \neq 0$.

$$\begin{aligned}
\int_{[a,b]} f(x) e^{i\lambda\psi(z)} dz &\approx f(x_0) e^{i\lambda\psi(x_0)} \sqrt{\frac{2\pi}{-i\lambda\psi''(x_0)}} \\
&= f(x_0) e^{i\lambda\psi(x_0)} e^{\frac{i\pi}{4}\mathrm{sgn}\psi''(x_0)} \sqrt{\frac{2\pi}{\lambda|\psi''(x_0)|}}.
\end{aligned}$$

The proof of the previous subsection is no longer valid. First, we have a problem with estimates far from the critical point. We can no longer use the exponential decay. Instead, we can use rapid oscillations, which lead to small errors by repeated integrations by parts. More precisely, suppose that χ is a cutoff function equal to 1 in $\left[x_0 - \frac{\epsilon}{2}, x_0 + \frac{\epsilon}{2}\right]$ and 0 outside of $[x_0 - \epsilon, x_0 + \epsilon]$. Set

$$f_1(x) := f(x)\chi(x), \qquad f_2(x) := f(x)\big(1 - \chi(x)\big).$$

Note that $|\psi'(x)| > c_0$ on $\mathrm{supp} f_2$. Now

$$\int_a^b f(x)\mathrm{e}^{\mathrm{i}\lambda\psi(x)}\mathrm{d}x \tag{6.14}$$

$$= \int_{x_0-\epsilon}^{x_0+\epsilon} f_1(x)\mathrm{e}^{\mathrm{i}\lambda\psi(x)}\mathrm{d}x + \int_a^b f_2(x)\mathrm{e}^{\mathrm{i}\lambda\psi(x)}\mathrm{d}x. \tag{6.15}$$

For the second term of (6.15) we use the identity

$$-\frac{\mathrm{i}}{\lambda\phi'(x)}\partial_x \mathrm{e}^{\mathrm{i}\lambda\phi(x)} = \mathrm{e}^{\mathrm{i}\lambda\phi(x)}, \tag{6.16}$$

$$\text{hence} \qquad \int_a^b f_2(x)\mathrm{e}^{\mathrm{i}\lambda\psi(x)}\mathrm{d}x = -\int_a^b f_2(x)\frac{\mathrm{i}}{\lambda\phi'(x)}\partial_x \mathrm{e}^{\mathrm{i}\lambda\psi(x)}\mathrm{d}x \tag{6.17}$$

$$= \int_a^b \mathrm{e}^{\mathrm{i}\lambda\psi(x)}\partial_x\Big(f_2(x)\frac{\mathrm{i}}{\lambda\phi'(x)}\Big)\mathrm{d}x \ = \ O(\lambda^{-1}). \tag{6.18}$$

Repeating integration by parts, we can show that this term is $O(\lambda^{-N})$ for any N. This trick is sometmes called the *non-stationary phase method.*

For the first term of (6.15), in order to improve error estimates it is useful to change the variable, so that for $|x - x_0| < \epsilon$

$$\psi(x) = \psi(x_0) - y^2. \tag{6.19}$$

Then we have

$$\int_{x_0-\epsilon}^{x_0+\epsilon} f_1(x)\mathrm{e}^{\mathrm{i}\lambda\psi(x)}\mathrm{d}x \tag{6.20}$$

$$= \int \tilde{f}_1(y)y\mathrm{e}^{\mathrm{i}\lambda\psi(x_0)}\mathrm{e}^{\mathrm{i}\lambda y^2}\mathrm{d}y + \int f(x_0)\mathrm{e}^{\mathrm{i}\lambda\psi(x_0)}\mathrm{e}^{\mathrm{i}\lambda y^2}\mathrm{d}y, \tag{6.21}$$

$$\tilde{f}_1(y) = \frac{1}{y}\Big(f_1\big(x(y)\big)\frac{\mathrm{d}x(y)}{\mathrm{d}y} - f(x_0)\Big). \tag{6.22}$$

Changing the variable $t = y^2$ and Integrating by parts we show that the first term of (6.21) is $O(\lambda^{-1})$.

6.3 The analytic Laplace method (the saddlepoint method)

Consider now the integral

$$I(\lambda) = \int_a^b f(x)\mathrm{e}^{\lambda\phi(x)}\mathrm{d}x$$

for large λ under analyticity conditions. We assume that f, ϕ extend to analytic functions on Ω, which is a neighborhood of $[a,b]$ in $\mathbb{C}$, and that we will find a path $\gamma \subset \Omega$ that connects a and b passing through a point z_0 where $\phi'(z_0) = 0$. We assume that at z_0 the function $\mathrm{Re}\phi$ restricted to γ has a maximum and $\phi''(z_0) \neq 0$. In a neighborhood of z_0 we have

$$\phi(z) \approx \phi(z_0) + \frac{1}{2}\phi''(z_0)(z-z_0)^2. \tag{6.23}$$

Let $|\psi| \leq \frac{\pi}{2}$ i $\phi''(z_0) = -|\phi''(z_0)|\mathrm{e}^{-\mathrm{i}2\psi}$. Let us introduce the coordinates

$$\mathbb{R}^2 \ni (t,s) \mapsto z = z_0 + (t+\mathrm{i}s)\mathrm{e}^{\mathrm{i}\psi}.$$

(6.23) can be rewritten as

$$\phi(z) \approx \phi(z_0) - \frac{1}{2}|\phi''(z_0)|\left(t^2 - s^2 + 2\mathrm{i}ts\right).$$

Hence level sets of $\mathrm{Re}\phi$ around z_0 presemble level sets on a saddlepoint (a mountain pass).

The biggest contribution to the integral on the curve γ comes from a neighborhood of z_0. Besides γ can be replaced with the line $\mathbb{R} \ni t \mapsto z_0 + \mathrm{e}^{\mathrm{i}\psi}t$. We obtain

$$\begin{aligned} I(\lambda) &= \int_\gamma f(z)\mathrm{e}^{\lambda\phi(z)}\mathrm{d}z \\ &\approx \int_{-\infty}^{\infty} f(z_0)\mathrm{e}^{\lambda\phi(z_0) - \frac{\lambda}{2}|\phi''(z_0)|t^2}\mathrm{e}^{\mathrm{i}\psi}\mathrm{d}t \\ &= f(z_0)\mathrm{e}^{\lambda\phi(z_0)}\sqrt{\frac{2\pi}{\lambda|\phi''(z_0)|}}\mathrm{e}^{\mathrm{i}\psi} \\ &= f(z_0)\mathrm{e}^{\lambda\phi(z_0)}\sqrt{\frac{2\pi}{-\lambda\phi''(z_0)}}. \end{aligned}$$

Arguing as in the previous subsections we can show

Theorem 6.2 *Ander the assumptions described above,*

$$\lim_{\lambda\to\infty} \frac{I(\lambda)}{f(z_0)\mathrm{e}^{\lambda\phi(z_0)}\sqrt{\frac{2\pi}{-\lambda\phi''(z_0)}}} = 1. \tag{6.24}$$

6.4 Asymptotics of the Gamma function at infinity by the saddlepoint method

Theorem 6.3 *Let $\epsilon > 0$. For $|\arg z| < \frac{\pi}{2} - \epsilon$ we have*

$$\lim_{z \to \infty} \frac{\Gamma(z+1)}{\sqrt{2\pi}\frac{z^{z+1/2}}{\mathrm{e}^z}} = 1. \tag{6.25}$$

Proof. We have

$$\Gamma(z+1) = \int_0^\infty \mathrm{e}^{-t} t^z \mathrm{d}t = z^{z+1} \int_0^\infty \mathrm{e}^{z\phi(u)} \mathrm{d}u,$$

where

$$\phi(u) = -u + \log u.$$

We compute:

$$\partial_u \phi(u) = -1 + \frac{1}{u}, \qquad \partial_u^2 \phi(u) = -\frac{1}{u^2}.$$

Hence $\phi(t)$ has a unique stationary point: for $u_0 = 1$. We have

$$\phi(u_0) = -1, \quad \partial_u^2 \phi(u_0) = -1.$$

Therefore, the saddlepoint method yields

$$\Gamma(z+1) \simeq z^{z+1} \mathrm{e}^{z\phi(u_0)} \int_{-\infty}^\infty \mathrm{e}^{\frac{z}{2}\partial_u^2\phi(u_0)(u-u_0)^2} \mathrm{d}u = \frac{z^z}{\mathrm{e}^z}\sqrt{2\pi z}.$$

□

6.5 Asymptotics of the Beta function at infinity by the saddlepoint method

The asymptotics of $\mathrm{B}(u,v)$ can be obtained from that of $\Gamma(z)$. It can be also computed directly by the saddlepoint method;

Theorem 6.4 *Let $\epsilon > 0$. For $|\arg u| < \frac{\pi}{2} - \epsilon$, $|\arg v| < \frac{\pi}{2} - \epsilon$, we have*

$$\lim_{u,v \to \infty} \frac{\mathrm{B}(u+1, v+1)}{\sqrt{2\pi}\frac{u^{u+1/2}v^{v+1/2}}{(u+v)^{u+v+3/2}}} = 1. \tag{6.26}$$

Proof. Clearly,

$$\mathrm{B}(u+1,v+1)=\int_0^1 \mathrm{e}^{\psi(t)}\mathrm{d}t,$$

where

$$\psi(t):=u\log t+v\log(1-t).$$

We compute:

$$\partial_t\psi(t)=\frac{u}{t}-\frac{v}{1-t},\qquad \partial_t^2\psi(t)=-\frac{u}{t^2}-\frac{v}{(1-t)^2}.$$

Hence $\psi(t)$ has a unique stationary pint: for $t_0=\frac{u}{u+v}$, and

$$\psi(t_0)=u\log\left(\frac{u}{u+v}\right)+v\log\left(\frac{v}{u+v}\right),\quad \partial_t^2\psi(t_0)=-\frac{(u+v)^3}{uv}.$$

If $\mathrm{Re}u>0$ and $\mathrm{Re}v>0$, then $\mathrm{Re}\psi(t)\to-\infty$ where t approaches 0 or 1. Thus it is easy to see that deforming the contour $[0,1]$ we can obtain a curve γ starting at 0, ending at 1 and passing through t_0 so that $\mathrm{Re}\psi(t)$ attains along this curve its maximum at t_0. Hence

$$\begin{aligned}\mathrm{B}(u+1,v+1)&=\int_\gamma \mathrm{e}^{\psi(t)}\mathrm{d}t\\ &\approx\int_\gamma \mathrm{e}^{\psi(t_0)+\frac{1}{2}\psi''(t_0)(t-t_0)^2}\mathrm{d}t\\ &\approx\left(\frac{u}{u+v}\right)^u\left(\frac{v}{u+v}\right)^v\left(\frac{2\pi uv}{(u+v)^3}\right)^{1/2}=\sqrt{2\pi}\frac{u^{u+1/2}v^{v+1/2}}{(u+v)^{u+v+3/2}}.\end{aligned}$$

□

6.6 Matrices

Let $c=[c_{ij}]$ be a $d\times d$ matrix. It defines a quadratic form defined for $x=[x_i]\in\mathbb{R}^d$ as

$$xcx=\sum_{i,j=1}^d x_ic_{ij}x_j.$$

Every matrix by a change of coordinates $y_j=\sum_{i=1}^d a_{ji}x_i$ can be reduced to a diagonal form:

$$xcx=\sum_{i=1}^d\lambda_i(y_i)^2.$$

The number of positive and negative terms on the diagonal does not depend on coordinates and defines the signatures of a matrix (d_+, d_-). Obviously, $d \geq d_+ + d_-$. The index of c is defined as $\mathrm{ind} c := d_+ - d_-$.

We say that c is non-degenerate if for any $x \in \mathbb{R}^d$, $x \neq 0$, there exists $y \in \mathbb{R}^d$ such that

$$ycx = \sum_{i,j=1}^{d} y_i c_{ij} x_j \neq 0.$$

Equivalently, $d_+ + d_- = d$.

Assume that $\mathbb{R}^d$ is equipped with the canonical scalar product $x \cdot y := \sum_{j=1}^{d} x_i y_i$. The matrix can be reduced to a diagonal form by an orthogonal transformation. The sequence $\lambda_1, \dots, \lambda_d$ up to a permutation does not depend on the choice of such an orthogonal transformation. The determinant does not change under such a transformation. Hence

$$\det[c_{ij}] = \prod_{i=1}^{d} \lambda_i.$$

We say that a matrix c is positive definite if for all $x \in \mathbb{R}^d$, $x \neq 0$,

$$xcx > 0.$$

Equivalently, $d_+ = d$.

6.7 Multidimensional Gauss and Fresnel integrals

Suppose that the matrix c is positive definite. Then

$$\int \mathrm{d}x \exp(-xcx) = \pi^{\frac{d}{2}} (\det c)^{-\frac{1}{2}}. \tag{6.27}$$

In fact, we make an orthogonal transformation diagonalizing c. We then have $\mathrm{d}x = \mathrm{d}x_1 \cdots \mathrm{d}x_d = \mathrm{d}y_1 \cdots \mathrm{d}y_d = \mathrm{d}y$ so that (6.27) is

$$\int \mathrm{d}y \exp\left(-\sum_i \lambda_i (y_i)^2\right) \ = \ \prod_{i=1}^{d} \int \mathrm{e}^{-\lambda_i (y_i)^2} \mathrm{d}y_i = \prod_{i=1}^{d} \sqrt{\frac{\pi}{\lambda_i}}.$$

If c is a nondegenerate matrix, then

$$\lim_{R\to\infty} \int_{|x|<R} \mathrm{d}x_1 \cdots \mathrm{d}x_d \exp(\mathrm{i}xcx) = \pi^{d/2} \mathrm{e}^{\mathrm{i}\frac{\pi}{4}\mathrm{ind} c} |\det c|^{-\frac{1}{2}}. \tag{6.28}$$

6.8 Multidimensional Laplace method

Consider the integral

$$I(\lambda) = \int_{\Theta} f(x) e^{\lambda \phi(x)} dx,$$

for large λ, where Θ is a subset of $\mathbb{R}^d$. We assume that ϕ possesses a global maximum inside Θ at the point $\tilde{x}$ belonging to the interior of Θ and is twice differentiable at $\tilde{x}$. We assume that the Hessian (the second derivative) of ϕ at $\tilde{x}$, denoted $\nabla^2\phi(\tilde{x})$, is negative definite. Then $\nabla\phi(\tilde{x}) = 0$ and

$$\begin{aligned} I(\lambda) &\approx \int_{\mathbb{R}^d} f(\tilde{x}) \exp\left(\lambda\phi(\tilde{x}) + \frac{\lambda}{2}\sum_{i,j=1}^{d} \nabla_i\nabla_j\phi(\tilde{x})(x_i - \tilde{x}_i)(x_j - \tilde{x}_j)\right) dx \\ &= f(\tilde{x}) e^{\lambda\phi(\tilde{x})} \left(\frac{2\pi}{\lambda}\right)^{\frac{d}{2}} \left(\det(-\nabla^2\phi(\tilde{x}))\right)^{-\frac{1}{2}}. \end{aligned}$$

Note that we did not require nor used analyticity of the integrand. Of course, the saddlepoint method has also a multidimensional analytic version, in the spirit of Theorem 6.2, which we will not describe here.

6.9 Multidimensional stationary phase method

We assume that f and ϕ are sufficiently smooth. For large λ we consider an integral of the form

$$I(\lambda) = \int_{\Theta} f(x) e^{i\lambda \phi(x)} dx,$$

where Θ is a subset of $\mathbb{R}^d$. We assume that ϕ possesses a unique critical point in Θ at point $\tilde{x}$ belonging to the interior of Θ. This means $\nabla\phi(\tilde{x}) = 0$. We assume that the Hessian of ϕ at $\tilde{x}$, denoted by $\nabla^2\phi(\tilde{x})$, is nondegenerate. Then

$$\begin{aligned} I(\lambda) &\approx \int_{\mathbb{R}^d} f(\tilde{x}) \exp\left(i\lambda\phi(\tilde{x}) + \frac{i\lambda}{2}\sum_{i,j=1}^{d} \nabla_i\nabla_j\phi(\tilde{x})(x_i - \tilde{x}_i)(x_j - \tilde{x}_j)\right) dx \\ &= f(\tilde{x}) e^{i\frac{\pi}{4}\operatorname{ind}\nabla^2\phi(\tilde{x})} e^{i\lambda\phi(\tilde{x})} \left(\frac{2\pi}{\lambda}\right)^{\frac{d}{2}} \left|\det\nabla^2\phi(\tilde{x}))\right|^{-\frac{1}{2}}. \end{aligned}$$

6.10 The diffusion equation and the Schrödinger equation

The following equations differ only by the presence of the imaginary unit: the free Schrödinger equation

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\Psi_t(x) = -\frac{1}{2m}\Delta\Psi_t(x).$$

and the diffusion equation (also called the heat eqution)

$$\frac{\mathrm{d}}{\mathrm{d}t}f_t(x) = \kappa\Delta f_t(x).$$

Introduce the momentum operator $p_i = -\mathrm{i}\nabla_{x_i}$.. Then $-\Delta = p^2$. The Schrödinger equation can be generalized to a dispersive Schrödinger equation, where ω is an arbitrary smooth function:

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\Psi_t(x) = \omega(p)\Psi_t(x).$$

It is solved by

$$\Psi_t = \mathrm{e}^{\mathrm{i}t\omega(p)}\Psi_0.$$

Let us introduce the Fourier transformation in the unitary convention:

$$\hat{\Psi}(\xi) = (2\pi)^{-\frac{d}{2}}\int \Psi(x)\mathrm{e}^{-\mathrm{i}x\xi}\mathrm{d}x,$$

$$\Psi(x) = (2\pi)^{-\frac{d}{2}}\int \hat{\Psi}(\xi)\mathrm{e}^{\mathrm{i}x\xi}\mathrm{d}\xi.$$

It diagonalizes the momentum:

$$\widehat{p\Psi}(\xi) = \xi\hat{\Psi}(\xi).$$

More generally,

$$\widehat{\omega(p)\Psi}(\xi) = \omega(\xi)\hat{\Psi}(\xi).$$

Therefore,

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\hat{\Psi}_t(\xi) = \omega(\xi)\hat{\Psi}_t(\xi),$$

$$\hat{\Psi}_t(\xi) = \mathrm{e}^{-\mathrm{i}t\omega(\xi)}\hat{\Psi}_0(\xi).$$

Mamy $\int |\Psi_t|^2(x)\mathrm{d}x = \int |\Psi_0|^2(x)\mathrm{d}x$.

In the position representation

$$\Psi_t(x) = \int U_t(x-y)\Psi_0(y)\mathrm{d}y, \tag{6.29}$$

$$U_t(x) = (2\pi)^{-d}\int \mathrm{e}^{-\mathrm{i}t\omega(\xi)+\mathrm{i}x\xi}\mathrm{d}\xi. \tag{6.30}$$

In particular, for the free Schrödinger equation with $m = 1$

$$\Psi_t(x) = \int (2\pi t\mathrm{i})^{-\frac{d}{2}}\mathrm{e}^{\frac{\mathrm{i}(x-y)^2}{2t}}\Psi_0(y)\mathrm{d}y.$$

For the difusion we obtain

$$f_t(x) = \int (4\pi\kappa t)^{-\frac{d}{2}}\mathrm{e}^{-\frac{(x-y)^2}{4\kappa t}}f_0(y)\mathrm{d}y.$$

Note that

(1) $\int f_t(x)\mathrm{d}x = \int f_0(x)\mathrm{d}x$;

(2) $f_0 \geq 0$ implies $f_t \geq 0$;

(3) $\int |f_t|^2(x)\mathrm{d}x = \int |f_0|^2(x)\mathrm{d}x$.

6.11 Legendre transformation

Let Ω be an open convex subset of $\mathbb{R}^d$ and

$$\Omega \ni \xi \mapsto \omega(\xi) \in \mathbb{R} \tag{6.31}$$

a convex C^2 function. More precisely, we assume that for distinct $\xi_1, \xi_2 \in \Omega$, $\xi_1 \neq \xi_2$, $0 < \tau < 1$,

$$\tau\omega(\xi_1) + (1-\tau)\omega(\xi_2) > \omega\left(\tau\xi_1 + (1-\tau)\xi_2\right). \tag{6.32}$$

Then

$$\Omega \ni \xi \mapsto v(\xi) := \nabla\omega(\xi) \in \mathbb{R}^d \tag{6.33}$$

is an injective function. Let $\tilde{\Omega}$ be the image of (6.33). We can define the function

$$\tilde{\Omega} \ni v \mapsto \xi(v) \in \Omega$$

inverse to (6.31). The Legendre transform of ω is defined as

$$\tilde{\omega}(v) := v\xi(v) - \omega(\xi(v)).$$

Theorem 6.5 (1) $\nabla\tilde\omega(v) = \xi(v)$.

(2) $\nabla^2\tilde\omega(v) = \nabla_v\xi(v) = \left(\nabla_\xi^2\omega(\xi(v))\right)^{-1}$. *Hence $\tilde\omega$ is convex.*

(3) $\tilde{\tilde\omega}(\xi) = \omega(\xi)$.

Proof. (1)

$$\nabla_v\tilde\omega(v) = \xi(v) + v\nabla_v\xi(v) - \nabla_\xi\omega(\xi(v))\nabla_v\xi(v) = \xi(v).$$

(2)

$$\nabla_v^2\tilde\omega(v) = \nabla_v\xi(v) = \nabla_\xi v(\xi(v))^{-1} = \left(\nabla_\xi^2\omega(\xi(v))\right)^{-1}.$$

(3)

$$\tilde{\tilde\omega}(\xi) = \xi v(\xi) - v(\xi)\xi(v(\xi)) + \omega(\xi(v(\xi))) = \omega(v).$$

□

Example 6.6 (1) $\Omega = \mathbb{R}^d$, $\omega(\xi) = \frac{1}{2}(\xi - a)m^{-1}(\xi - a) + v$, $\quad \tilde\Omega = \mathbb{R}^d$, $\tilde\omega(v) = \frac{1}{2}\xi m\xi + a\xi - v$.

(2) $\Omega = \mathbb{R}^d$, $\omega(\xi) = \sqrt{\xi^2 + m^2}$, $\qquad \tilde\Omega = \{v \in \mathbb{R}^d \;:\; |v| < 1\}$, $\tilde\omega(\xi) = -m\sqrt{1 - v^2}$.

(3) $\Omega = \mathbb{R}$, $\omega(\xi) = \mathrm{e}^\xi$, $\quad \tilde\Omega =]0, \infty[$, $\tilde\omega(v) = v\log v - v$.

6.12 Dispersive semiclassical Schrödinger equation

Introduce a small parameter $\hbar$. Let us change slightly the definition of the momentum and energy:

$$p_i = -\mathrm{i}\hbar\nabla_{x_i}, \quad E = \mathrm{i}\hbar\partial_t.$$

The dispersive Schrödinger equation in the semiclassical setting is

$$\mathrm{i}\hbar\frac{\mathrm{d}}{\mathrm{d}t}\Psi_t(x) = \omega(p)\Psi_t(x). \tag{6.34}$$

It is solved by

$$\Psi_t = \mathrm{e}^{\frac{\mathrm{i}t\omega(p)}{\hbar}}\Psi_0.$$

It is also conveneient to introduce the semiclassical Fourier transformation:

$$\begin{aligned}\hat\Psi(\xi) &= (2\pi\hbar)^{-\frac{d}{2}}\int \Psi(x)\mathrm{e}^{-\frac{\mathrm{i}x\xi}{\hbar}}\mathrm{d}x,\\ \Psi(x) &= (2\pi\hbar)^{-\frac{d}{2}}\int \hat\Psi(\xi)\mathrm{e}^{\frac{\mathrm{i}x\xi}{\hbar}}\mathrm{d}\xi.\end{aligned}$$

Here are its properties:

$$\begin{aligned}\int |\Psi(x)|^2 \mathrm{d}x &= \int |\Psi(\xi)|^2 \mathrm{d}\xi,\\ \widehat{\omega(p)\Psi}(\xi) &= \omega(\xi)\hat{\Psi}(\xi).\end{aligned}$$

We have

$$\Psi_t(x) = \int U_t(x-y)\Psi_0(y)\mathrm{d}y, \tag{6.35}$$

$$U_t(x) = (2\pi\hbar)^{-d}\int \mathrm{e}^{\frac{-\mathrm{i}t\omega(\xi)+\mathrm{i}(x-y)\xi}{\hbar}}\mathrm{d}\xi. \tag{6.36}$$

6.13 Semiclassical limit of dispersive evolution

Assume that $\Psi_t(x)$ evolves according to (6.34). We would like to find the propagation for small $\hbar$ in function of $\hat{\Psi}_0$

Let $v(\xi)$ and $\tilde{\omega}(x)$ be defined as in Subsect. 6.11. Write $\Psi_t(x)$ in the form

$$\Psi_t(x) = (2\pi\hbar)^{-\frac{d}{2}}\int \exp\left(\frac{\mathrm{i}\psi_t(x,\xi)}{\hbar}\right)\hat{\Psi}_0(\xi)\mathrm{d}\xi,$$

where

$$\psi_t(x,\xi) = -t\omega(\xi) + x\xi.$$

We use the stationary phase method:

$$\nabla_\xi\psi(x,\xi) = -t\nabla_\xi\omega(\xi) + x.$$

Hence the critical point is at

$$\tfrac{x}{t} = \nabla_\xi\omega(\xi) =: v(\xi), \tag{6.37}$$

$$\psi\left(x,\xi(\tfrac{x}{t})\right) = x\xi(\tfrac{x}{t}) - t\omega\left(\xi(\tfrac{x}{t})\right) =: t\tilde{\omega}(\tfrac{x}{t}). \tag{6.38}$$

where we introduced the *group velocity* $v(\xi)$ and the *Legendre transform* of $\omega(\xi)$, that is, $\tilde{\omega}(v)$. Besides,

$$\nabla_\xi^2\psi(x,\xi) = -t\nabla^2\omega(\xi) = -t\nabla_\xi v(\xi).$$

The stationary phase method yields

$$\begin{aligned}\Psi_t(x) \approx\ & (2\pi\hbar)^{-\frac{d}{2}}(2\pi\hbar)^{\frac{d}{2}}\exp\left(\mathrm{i}\frac{\pi}{4}\mathrm{ind}\nabla_\xi v(\xi(\tfrac{x}{t})\right) \qquad (6.39)\\ & \times t^{-\frac{d}{2}}\left|\det\nabla_v\xi(\tfrac{x}{t})\right|^{\frac{1}{2}}\exp\left(\frac{\mathrm{i}t}{\hbar}\tilde{\omega}(\tfrac{x}{t})\right)\hat{\Psi}_0\left(\xi(\tfrac{x}{t})\right).\end{aligned}$$

Thus the wave packet with momentum ξ travels with group velocity $v(\xi)$ and its phase is given by the Legendre transform of ω. Note that the L^2 norm of the rhs of (6.39) does not depend on time.

6.14 Wave and Klein-Gordon equation

The following equation for $m \neq 0$ is called the Klein-Gordon equation and for $m = 0$ the d'Alembert or wave equation:

$$\partial_t^2 \Psi(t,x) = (\Delta - m^2)\Psi(t,x). \tag{6.40}$$

Theorem 6.7 *$\Psi(t,\cdot)$ and $\partial_t\Psi(t,\cdot)$ for $t=0$ determine uniquely a solution of (6.40) in all times by the formula*

$$\Psi(t) = -\partial_t G(t)\Psi(0) + G(t)\partial_t\Psi(0),$$

where the propagator $G(t)$ is given by

$$G(t) \quad = \quad (-\Delta + m^2)^{-\frac{1}{2}} \sin\left(t\sqrt{-\Delta + m^2}\right).$$

Proof. (6.40) can be rewritten in the form

$$\left(\mathrm{i}\partial_t - \sqrt{-\Delta + m^2}\right)\left(\mathrm{i}\partial_t + \sqrt{-\Delta + m^2}\right)\Psi = 0.$$

If Ψ is its solution, we introduce

$$\Psi_\pm = \frac{1}{2}\left(1 \mp \mathrm{i}(-\Delta + m^2)^{-\frac{1}{2}}\partial_t\right)\Psi.$$

Then $\Psi = \Psi_+ + \Psi_-$ and

$$\left(\mathrm{i}\partial_t \pm \sqrt{-\Delta + m^2}\right)\Psi_\pm = 0.$$

Hence

$$\Psi_\pm(t) = \mathrm{e}^{\pm\mathrm{i}\sqrt{-\Delta+m^2}}\Psi_\pm(0).$$

Therefore,

$$\begin{aligned}
\Psi(t) \quad &= \quad \mathrm{e}^{\mathrm{i}\sqrt{-\Delta+m^2}}\Psi_+(0) + \mathrm{e}^{-\mathrm{i}\sqrt{-\Delta+m^2}}\Psi_-(0) \\
&= \quad \frac{1}{2}(\mathrm{e}^{\mathrm{i}t\sqrt{-\Delta+m^2}} + \mathrm{e}^{-\mathrm{i}t\sqrt{-\Delta+m^2}})\Psi(0) \\
&\quad\; +\frac{\mathrm{i}}{2}\left(-(\Delta + m^2)^{-\frac{1}{2}}\mathrm{e}^{\mathrm{i}t\sqrt{-\Delta+m^2}} + (-\Delta + m^2)^{-\frac{1}{2}}\mathrm{e}^{-\mathrm{i}t\sqrt{-\Delta+m^2}}\right)\partial_t\Psi(0).
\end{aligned}$$

□