Homogeneous rank one perturbations

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Abstract

A holomorphic family of closed operators with a rank one perturbation given by the function $x^{\frac{m}{2}}$ is studied. The operators can be used in a toy model of renormalization group.

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1 Introduction

Rank one perturbations can be used to illustrate various interesting mathematical concepts. For instance, they can be singular: the perturbation is not an operator and an infinite renormalization may be needed. Rank one perturbations are often applied to model physical phenomena.

Our paper is devoted to a special class of exactly solvable rank one perturbations, which are both singular and physically relevant. We consider the Hilbert space $L^2[0,\infty[$. The starting point is the operator of multiplication by $x \in [0,\infty[$, denoted by X. We try to perturb it by a rank one operator involving the function $x^{\frac{m}{2}}$. Thus, we try to define an operator formally given by

$$X + \lambda |x^{\frac{m}{2}}\rangle\langle x^{\frac{m}{2}}|. \tag{1.1}$$

Note that we allow m and λ to be complex. In particular, (1.1) is usually non-Hermitian. The function $x^{\frac{m}{2}}$ is never square integrable, and therefore, the perturbation is always singular.

(1.1) is very special. Formally, X is homogeneous of degree 1 and its perturbation is homogeneous of degree m. We will see that in order to define a closed operator on $L^2[0,\infty[$ one needs to restrict m by the condition $-1 < \mathrm{Re} m < 1$. Besides, a special treatment is needed in the case $m = 0$. One obtains two holomorphic families of closed operators, $H_{m,\lambda}$ and H_0^ρ. λ and ρ can be interpreted as coupling constants, which in the case $0 \leq \mathrm{Re} m < 1$ need an infinite renormalization.

The families of the operators that we introduce are exactly solvable in a rather strong sense: one can compute their resolvents, spectral projections and Møller (wave) operators. One can also describe their spectra, which can be quite curious.

In our opinion, the families $H_{m,\lambda}$ and H_0^ρ that we constructed are quite instructive. One can argue that they provide excelent material for exercises in a semi-advanced course on operator theory, or even quantum physics. They illustrate various sophisticated concepts related to operators in Hilbert spaces (singular perturbations of various kinds, scattering theory). They can also be treated as toy models of some important ideas of theoretical physics such as renormalization group flows and breaking of scaling symmetry.

We believe, that in some form these operators show up in many contexts in mathematics and theoretical physics, especially when we deal with scaling symmetry. Below we briefly describe one situation where these operators are present.

As shown in [4], for $\mathrm{Re} m > -1$ one can define a holomorphic family of closed Schrödinger operators on $L^2[0,\infty[$ homogeneous of degree -2 given formally by

$$\tilde{H}_m = -\partial_x^2 + \Big(-\frac{1}{4} + m^2\Big)\frac{1}{x^2}. \tag{1.2}$$

(As compared with the notation of [4], we add a tilde to distinguish from the operators considered in this paper). These operators have contiuous spectrum in $[0,\infty[$ of multiplicity 1. They can be diagonalized with help of the so-called Hankel transformation $\mathcal{F}_m$, whose kernel has a simple expression in terms of the Bessel function J_m.

As shown in [7], for $-1 < \mathrm{Re} m < 1$ there exists a two-parameter holomorphic family of closed operators that can be associated with the differential expression on the right hand side of (1.2). They correspond to mixed boundary conditions at zero and are denoted $\tilde{H}_{m,\kappa}$. The case $m = 0$ needs special treatment, and one introduces a family of $\tilde{H}_0^{\nu}$. As we show in our paper, the operators $\tilde{H}_{m,\kappa}$, resp. $\tilde{H}_0^{\nu}$, are equivalent (similar) to the operators $H_{m,\lambda}$ and H_0^{ρ}, where κ and ν are linked by a simple relation with λ and ρ, see Thms 4.1 and 4.2.

The operators $\tilde{H}_{m,\kappa}$ and $\tilde{H}_0^{\nu}$ are very well motivated—they constitute natural classes of Schrödinger operators, which are relevant for many problems in mathematical physics. However, their theory looks complicated–it requires the knowledge of some special functions, more precisly, Bessel-type functions and the Gamma function. On the other hand, the theory of $H_{m,\lambda}$ and H_0^{ρ} does not involve special functions at all—it uses only trigonometric functions and the logarithm.

The paper is organized as follows. In Section 2 we recall the theory of sigular rank one perturbations. It is sometimes called the Aronszajn-Donoghue theory and goes back to [2, 5, 3]. It is described in particular in [1, 6, 12]. We discuss also the scattering theory in the context of rank one perturbations. Here the basic reference is [13].

Note, however, that we do not assume that the perturbation is self-adjoint, and most of the literature on this subject is restricted to the self-adjoint case. A notable exception are the articles [10, 11], where non-self-adjoint perturbations of self-adjoint operators are studied.

Section 3 is the main part of our paper. Here we construct and study the operators $H_{m,\lambda}$ and H_0^{ρ}.

In Section 4 we describe the relationship of the operators $H_{m,\lambda}$ and H_0^{ρ} with Schrödinger operators with inverse square potentials $\tilde{H}_{m,\kappa}$ and $\tilde{H}_0^{\nu}$. There exists large literature for such Schrödinger operators, see eg. [8], it is however usually restricted to the self-adjoint case. The general case is studied in [4] and especially [7].

In the appendix we collect some integrals that are used in our paper.

2 General theory of rank one perturbations

2.1 Preliminaries

We consider the Hilbert space $L^2[0,\infty[$ with the scalar product

$$(f|g) := \int_0^\infty \overline{f(x)} g(x) \mathrm{d}x. \tag{2.1}$$

In addition, it is also equipped with the bilinear form

$$\langle f|g\rangle := \int_0^\infty f(x)g(x)\mathrm{d}x, \tag{2.2}$$

Thus we use round brackets for the sesquilinear scalar product and angular brackets for the closely related bilinear form. Note that in some sense the latter plays a more important role in our paper (and in similar exactly solvable problems) than the former.

If B is an operator then B^* denotes the usual Hermitian adjoint of B, whereas $B^\#$ denotes the *transpose* of B, that is, its adjoint w.r.t. the (2.2). Clearly, if B is a bounded linear operator with

$$\big(Bf\big)(k) := \int_0^\infty B(k,x)f(x)\mathrm{d}x,$$

then

$$\big(B^*f\big)(x) = \int_0^\infty \overline{B(k,x)}f(k)\mathrm{d}k,$$

while

$$\big(B^\# f\big)(x) = \int_0^\infty B(k,x)f(k)\mathrm{d}k.$$

An operator B is self-adjoint if $B = B^*$. We will say that it is *self-transposed* if

$$B^\# = B. \tag{2.3}$$

It is useful to note that a holomorphic function of a self-transposed operator is self-transposed.

It is convenient to use sometimes Dirac's "bra-ket" notation. For a function $f \in L^2[0,\infty[$, we have the operator $|f\rangle : \mathbb{C} \to L^2[0,\infty[$ given by

$$\mathbb{C} \ni z \mapsto |f\rangle z := zf \in L^2[0,\infty[\tag{2.4}$$

and its transpose $\langle f| := |f\rangle^\# : L^2[0,\infty[\to \mathbb{C}$ given by

$$L^2[0,\infty[\ni v \mapsto \langle f|v = \int f(x)v(x)\mathrm{d}x. \tag{2.5}$$

We will also use the same notation in the case f is not square integrable—then $\langle f|$ is an unbounded operator and $|f\rangle$ is an unbounded form with appropriate domains.

Note that (2.4) and (2.5) are consistent with the notation for (2.2)—both use angular brackets. In principle, we could also use the Dirac's bras and kets suggested by the scalar product (2.1), involving round brackets,

$$|f) := |f\rangle, \quad (f| := \langle \overline{f}|, \tag{2.6}$$

but we prefer to use the notation associated with (2.2).

2.2 Construction

Let X denote the (unbounded) operator on $L^2[0,\infty[$ given by

$$Xv(x) := xv(x), \tag{2.7}$$

$$v \in \mathrm{Dom}(X) = \Big\{ v \in L^2[0,\infty[\quad | \quad \int |v(x)|^2 x^2 \mathrm{d}x < \infty \Big\}. \tag{2.8}$$

Let h_2, h_1 be measurable functions on $[0,\infty[$. Consistently with the notation introduced in (2.4) and (2.5), we will write $|h_2\rangle\langle h_1|$ for the (possibly unbounded) quadratic form given by

$$\big(w|h_2\rangle\langle h_1|v\big) := \int \overline{w(x)} h_2(x)\mathrm{d}x \int h_1(y)v(y)\mathrm{d}y, \tag{2.9}$$

for w, v in the (obvious) domain of $|h_2\rangle\langle h_1|$. (Note the absence of the complex conjugation on h_1).

It is well known that in some situations

$$H_\lambda := X + \lambda |h_2\rangle\langle h_1| \tag{2.10}$$

can be interpreted as an operator, possibly after an appropriate renormalization of the coupling constant λ. This is sometimes called the Aronszajn-Donoghue theory, and is described e.g. in [1, 6, 12]. We will need a somewhat non-standard version of this theory, because our rank one perturbation does not have to be Hermitian. Therefore, we describe it in some detail.

One can consider three cases of the Aronszajn-Donoghue theory, with an increasing level of difficulty. The first case is elementary:

Assumption I. $h_1, h_2 \in L^2$.

Then $|h_2\rangle\langle h_1|$ is a bounded operator. Therefore, H_λ is well defined on $\mathrm{Dom}(X)$, and we can easily compute the resolvent of H_λ. In fact, define

$$g_\infty(z) \quad := \quad \langle h_1|(z-X)^{-1}|h_2\rangle = \int_0^\infty h_1(x)(z-x)^{-1}h_2(x)\mathrm{d}x, \tag{2.11}$$

$$g_\lambda(z) \quad = \quad -\lambda^{-1} + g_\infty(z). \tag{2.12}$$

Then

$$\mathrm{sp} H_\lambda \subset \{z \ : \ g_\lambda(z) = 0\} \cup [0,\infty[, \tag{2.13}$$

and for such z such that $g_\lambda(z) \neq 0$, $z \notin [0,\infty[$,

$$(z - H_\lambda)^{-1} = R_\lambda(z), \tag{2.14}$$

where

$$R_\lambda(z) = (z-X)^{-1} \tag{2.15}$$

$$\qquad - g_\lambda(z)^{-1}(z-X)^{-1}|h_2\rangle\langle h_1|(z-X)^{-1}, \quad \lambda \neq 0; \tag{2.16}$$

$$R_0(z) := (z-X)^{-1}. \tag{2.17}$$

Consider now

Assumption II. $\frac{h_1}{1+X}, \frac{h_2}{1+X} \in L^2, \quad \frac{h_1 h_2}{1+X} \in L^1.$

Then it is easy to check that g_λ and $R_\lambda(z)$ are still well defined. Besides, $R_\lambda(z)$ satisfies the resolvent equation, has zero kernel and dense range. Hence, by the theory of pseudoresolvents [9], there exists a closed operator H_λ such that (2.14) is true. Note that $H_0 = X$ and often we can include $\lambda = \infty$.

Finally, consider

Assumption III. $\frac{h_1}{1+X}, \frac{h_2}{1+X} \in L^2.$

Then g_λ is in general ill defined. Instead, we consider the equation

$$\partial_z g(z) = -\langle h_1|(z-X)^{-2}|h_2\rangle. \tag{2.18}$$

If one of solutions of (2.18) is called g^0, then all other are given by

$$g^\rho(z) := \rho + g^0(z), \tag{2.19}$$

for some $\rho \in \mathbb{C}$. We set

$$R^\rho(z) := (z-X)^{-1} \tag{2.20}$$

$$- g^\rho(z)^{-1}(z-X)^{-1}|h_2\rangle\langle h_1|(z-X)^{-1}, \tag{2.21}$$

$$R^\infty(z) := (z-X)^{-1}. \tag{2.22}$$

Again, $R^\rho(z)$ is a pseudoresolvent and by [9] there exists a unique family of operators H^ρ such that

$$(z-H^\rho)^{-1} = R^\rho(z). \tag{2.23}$$

We have thus constructed a family of operators. Under Assumption I or II, it can be written as H_λ, where $\lambda \in \mathbb{C} \cup \{\infty\}$ has the meaning of a coupling constant. Under Assumption III, in general, the coupling constant may lose its meaning, and we may be forced to use the parametrization H^ρ, where again $\rho \in \mathbb{C} \cup \{\infty\}$. (In practice, however, as we will see, the notation H_λ could be natural even if Assumption II does not hold).

2.3 Point spectrum

Untill the end of this section we suppose that Assumption III is satisfied. We consider an operator H of the form H_λ or H^ρ, as described above. Thus,

$$(z-H)^{-1} = (z-X)^{-1} \tag{2.24}$$

$$- g(z)^{-1}(z-X)^{-1}|h_2\rangle\langle h_1|(z-X)^{-1}, \tag{2.25}$$

where $g = g_\lambda$ or $g = g^\rho$.

It is easy to see that the spectrum of H consists of $[0, \infty[$ and eigenvalues at

$$\{w \in \mathbb{C}\backslash[0, \infty[: \ 0 = g(w)\}. \tag{2.26}$$

If $w \in \mathbb{C}\backslash[0, \infty[$ is an eigenvalue with $\langle h_1|(w - X)^{-2}|h_2\rangle \neq 0$, then it is simple and the corresponding eigenprojection is given by the formula

$$\mathbb{1}_{\{w\}}(H) = \frac{(w - X)^{-1}|h_2\rangle\langle h_1|(w - X)^{-1}}{\langle h_1|(w - X)^{-2}|h_2\rangle}. \tag{2.27}$$

2.4 Dilations

Before we continue, let us say a few words about the group of dilations

$$U_\tau f(x); = \mathrm{e}^{\frac{\tau}{2}} f(\mathrm{e}^\tau x), \quad \tau \in \mathbb{R}. \tag{2.28}$$

It can be written as $U_\tau = \mathrm{e}^{\mathrm{i}\tau A}$, where the generator of dilations is

$$A := \frac{1}{2\mathrm{i}}(x\partial_x + \partial_x x). \tag{2.29}$$

We say that H is homogeneous of degree p if $U_\tau H U_\tau^* = \mathrm{e}^{pt} H$. For instance, X is homogeneous of degree 1.

It is easy to see that an operator B on $L^2(\mathbb{R}_+)$ has the integral kernel

$$B(x, y) = \frac{1}{\sqrt{xy}} \phi\Big(\ln \frac{x}{y}\Big), \tag{2.30}$$

iff $B = \hat{\phi}(A)$, where

$$\hat{\phi}(\xi) = \int \phi(t) \mathrm{e}^{-\mathrm{i}t\xi} \mathrm{d}t, \tag{2.31}$$

see e.g. [4]. For example, by (A.3),

$$\frac{\pm 2\pi \mathrm{i}}{(\mathrm{e}^{\pm 2\pi A} + \mathbb{1})} \quad \text{has the kernel} \quad \frac{1}{(x - y \mp \mathrm{i}0)}. \tag{2.32}$$

2.5 Essential spectrum

It follows from the Weyl Theorem that the essential spectrum of H is $[0, \infty[$. Detailed study of the essential spectrum requires technical assumptions on the perturbation [13]. In this section, we will limit ourselves to a heuristic theory, without specifying precise assumptions. It will be possible to justify these formulas in the concrete situation considered in our paper.

First, we will check that the kernel of $\mathbb{1}_{[0,\infty[}(H)$, that is, of the spectral projection of H onto $[0, \infty[$ is given by the following formula:

$$\begin{aligned} \mathbb{1}_{[0,\infty[}(H)(x, y) \quad &= \quad \delta(x - y) && (2.33) \\ &+ \frac{h_2(x)h_1(y)}{(x - y - \mathrm{i}0)} \Big(\frac{1}{g(y + \mathrm{i}0)} - \frac{1}{g(x + \mathrm{i}0)}\Big) && (2.34) \\ + \frac{1}{2\pi\mathrm{i}} \frac{h_2(x)h_1(y)}{(x - y - \mathrm{i}0)} &\int \mathrm{d}s \frac{h_1(s)h_2(s)}{g(s + \mathrm{i}0)g(s - \mathrm{i}0)} \Big(\frac{1}{(x - s - \mathrm{i}0)} + \frac{1}{(s - y - \mathrm{i}0)}\Big). \end{aligned}$$

To see this we use the Stone formula

$$\mathbb{1}_{[0,\infty[}(H) = \mathrm{w-}\lim_{\epsilon\searrow 0} \frac{1}{2\pi\mathrm{i}} \int_0^\infty \mathrm{d}s\Big((s-\mathrm{i}\epsilon-H)^{-1}-(s+\mathrm{i}\epsilon-H)^{-1}\Big). \tag{2.35}$$

Thus

$$\begin{aligned}
& \mathbb{1}_{[0,\infty[}(H)(x,y) \\
= & \frac{1}{2\pi\mathrm{i}}\int_0^\infty \mathrm{d}s\Big(\frac{1}{(s-\mathrm{i}0-x)}\delta(x-y) - \frac{1}{(s+\mathrm{i}0-x)}\delta(x-y)\Big) \\
+ & \frac{1}{2\pi\mathrm{i}}\int_0^\infty \mathrm{d}s\Big(\frac{h_2(x)h_1(y)}{g(s-\mathrm{i}0)(s-\mathrm{i}0-x)(s-\mathrm{i}0-y)} - \frac{h_2(x)h_1(y)}{g(s+\mathrm{i}0)(s+\mathrm{i}0-x)(s+\mathrm{i}0-y)}\Big) \\
= & \int_0^\infty \mathrm{d}s\delta(s-x)\delta(x-y) \\
& +\frac{1}{2\pi\mathrm{i}}\int_0^\infty \mathrm{d}s\frac{h_2(x)h_1(y)}{g(s-\mathrm{i}0)(s-\mathrm{i}0-x)}\Big(\frac{1}{(s-\mathrm{i}0-y)} - \frac{1}{(s+\mathrm{i}0-y)}\Big) \\
& +\frac{1}{2\pi\mathrm{i}}\int_0^\infty \mathrm{d}s\Big(\frac{1}{(s-\mathrm{i}0-x)} - \frac{1}{(s+\mathrm{i}0-x)}\Big)\frac{h_2(x)h_1(y)}{(s+\mathrm{i}0-y)g(s+\mathrm{i}0)} \\
& +\frac{1}{2\pi\mathrm{i}}\int_0^\infty \mathrm{d}s\Big(\frac{1}{g(s-\mathrm{i}0)} - \frac{1}{g(s+\mathrm{i}0)}\Big)\frac{h_2(x)h_1(y)}{(s-\mathrm{i}0-x)(s+\mathrm{i}0-y)} \\
= & \delta(x-y) \\
& -\int_0^\infty \mathrm{d}s\Big(\frac{h_2(x)h_1(y)}{g(s-\mathrm{i}0)(s-\mathrm{i}0-x)}\delta(s-y) + \delta(s-x)\frac{h_2(x)h_1(y)}{(s+\mathrm{i}0-y)g(s+\mathrm{i}0)}\Big) \\
& +\frac{1}{2\pi\mathrm{i}}\int_0^\infty \mathrm{d}s\frac{h_1(s)h_2(s)}{g(s-\mathrm{i}0)g(s+\mathrm{i}0)}\frac{h_2(x)h_1(y)}{(x-y+\mathrm{i}0)}\Big(\frac{1}{(s-\mathrm{i}0-x)} - \frac{1}{(s+\mathrm{i}0-y)}\Big).
\end{aligned}$$

2.6 Møller operators I

For the purpose of our paper, we define the Møller operators between X and H, denoted $W^\pm(H,X)$ and $W^\pm(X,H)$, by describing their kernels:

$$W^\pm(H,X;x,y) = \delta(x-y) + \frac{h_2(x)h_1(y)}{(x-y\pm\mathrm{i}0)g(y\mp\mathrm{i}0)}, \tag{2.36}$$

$$W^\pm(X,H;x,y) = \delta(x-y) + \frac{h_2(x)h_1(y)}{g(x\pm\mathrm{i}0)(y-x\mp\mathrm{i}0)}. \tag{2.37}$$

To motivate the definitions (2.36) and (2.37) recall that in the literature on stationary scattering theory, e.g. [13], the Møller operators are often introduced as follows:

$$W^\pm(H,X) := \mathrm{w-}\lim_{\epsilon\searrow 0}\frac{\epsilon}{\pi}\int \mathrm{d}s(s\mp\mathrm{i}\epsilon-H)^{-1}(s\pm\mathrm{i}\epsilon-X)^{-1}, \tag{2.38}$$

$$W^\pm(X,H) := \mathrm{w-}\lim_{\epsilon\searrow 0}\frac{\epsilon}{\pi}\int \mathrm{d}s(s\mp\mathrm{i}\epsilon-X)^{-1}(s\pm\mathrm{i}\epsilon-H)^{-1}. \tag{2.39}$$

If (2.38) and (2.39) exist, then a formal computation shows that their kernels are given by (2.36) and (2.37).

In general, there is no guarantee that $W^{\pm}(H,X)$ and $W^{\pm}(X,H)$ exist as bounded operators. If this is the case, we expect the following properties:

$$W^{\pm}(X,H)W^{\pm}(H,X) = \mathbb{1}, \tag{2.40}$$
$$W^{\pm}(H,X)W^{\pm}(X,H) = \mathbb{1}_{[0,\infty[}(H), \tag{2.41}$$
$$W^{\pm}(H,X)X = HW^{\pm}(H,X). \tag{2.42}$$

A rigorous derivation of (2.40), (2.41) and (2.42) for some classes of perturbations can be found in [13]. It is not very difficult to derive these identities on a formal level.

Let us give a formal derivation of (2.40):

$$W^{\pm}(X,H)W^{\pm}(H,X)(x,y) \tag{2.43}$$
$$=\delta(x-y) + \frac{h_2(x)h_1(y)}{g_\lambda(x\pm \mathrm{i}0)(y-x\mp \mathrm{i}0)} + \frac{h_2(x)h_1(y)}{(x-y\pm \mathrm{i}0)g_\lambda(y\mp \mathrm{i}0)} \tag{2.44}$$
$$+\int \mathrm{d}t \frac{h_2(x)h_1(t)h_2(t)h_1(y)}{g_\lambda(x\pm \mathrm{i}0)(t-x\mp \mathrm{i}0)(t-y\pm \mathrm{i}0)g_\lambda(y\mp \mathrm{i}0)}. \tag{2.45}$$

Now

$$\int \mathrm{d}t \frac{h_1(t)h_2(t)}{(t-x\mp \mathrm{i}0)(t-y\pm \mathrm{i}0)} \tag{2.46}$$
$$=\int \mathrm{d}t \frac{h_1(t)h_2(t)}{(y-x\mp \mathrm{i}0)}\Big(-\frac{1}{(t-x\mp \mathrm{i}0)}+\frac{1}{(t-y\pm \mathrm{i}0)}\Big) \tag{2.47}$$
$$=\frac{1}{(y-x\mp \mathrm{i}0)}\big(-g_\lambda(x\pm \mathrm{i}0)+g_\lambda(y\mp \mathrm{i}0)\big). \tag{2.48}$$

Therefore, (2.43) is $\delta(x-y)$.

We omit the derivation of (2.41), which is similar to that of (2.40), although somewhat more difficult, since we need to use (2.33).

To obtain (2.42), we will compute that

$$W^{\pm}(H,X)(z-X)^{-1} = (z-H)^{-1}W^{\pm}(H,X). \tag{2.49}$$

Indeed,

$$
\begin{aligned}
&(z-H)^{-1}W^{\pm}(H,X)(x,y) && (2.50)\\
=&\frac{\delta(x-y)}{(z-x)}-\frac{h_2(x)h_1(y)}{g_\lambda(z)(z-x)(z-y)} && (2.51)\\
&+\frac{h_2(x)h_1(y)}{(z-x)(x-y\pm \mathrm{i}0)g_\lambda(y\mp \mathrm{i}0)} && (2.52)\\
&-\int\frac{h_2(x)h_1(t)h_2(t)h_1(y)}{g_\lambda(z)(z-x)(z-t)(t-y\pm \mathrm{i}0)g_\lambda(y\mp \mathrm{i}0)}\mathrm{d}t && (2.53)\\
=&\frac{\delta(x-y)}{(z-y)}-\frac{h_2(x)h_1(y)}{g_\lambda(z)(z-x)(z-y)} && (2.54)\\
&+\frac{h_2(x)h_1(y)}{(z-x)(z-y)g_\lambda(y\pm \mathrm{i}0)} && (2.55)\\
&+\frac{h_2(x)h_1(y)}{(x-y\pm \mathrm{i}0)(z-y)g_\lambda(y\mp \mathrm{i}0)} && (2.56)\\
&-\int\frac{h_2(x)h_1(t)h_2(t)h_1(y)}{g_\lambda(z)(z-x)(z-t)(z-y)g_\lambda(y\mp \mathrm{i}0)}\mathrm{d}t && (2.57)\\
&-\int\frac{h_2(x)h_1(t)h_2(t)h_1(y)}{g_\lambda(z)(z-x)(t-y\pm \mathrm{i}0)(z-y)g_\lambda(y\mp \mathrm{i}0)}\mathrm{d}t. && (2.58)
\end{aligned}
$$

To handle (2.57)+(2.58) we note that

$$\int\frac{h_1(t)h_2(t)}{(z-t)}\mathrm{d}t-\int\frac{h_1(t)h_2(t)}{(y-t\mp \mathrm{i}0)}\mathrm{d}t=g_\lambda(z)-g(y\mp \mathrm{i}0). \qquad (2.59)$$

Therefore, (2.57)+(2.58) cancels the second term of (2.54) and (2.55). Thus, (2.50) equals

$$
\begin{aligned}
&\frac{\delta(x-y)}{(z-y)}+\frac{h_2(x)h_1(y)}{(x-y\pm \mathrm{i}0)(z-y)g_\lambda(y\mp \mathrm{i}0)} && (2.60)\\
=&W^{\pm}(H,X)(z-X)^{-1}(x,y), && (2.61)
\end{aligned}
$$

which proves (2.49).

2.7 Møller operators II

Let us now consider $H=H_\lambda$.

We can rewrite (2.36) in terms of $W^{\pm}(H_0,X)=1\!\!1$ and $W^{\pm}(H_\infty,X)$:

$$
\begin{aligned}
W^{\pm}(H_\lambda,X) \;=\; &\frac{1}{\big(1-\lambda g_\infty(X\mp \mathrm{i}0)\big)}\\
&-W^{\pm}(H_\infty,X)\frac{\lambda g_\infty(X\mp \mathrm{i}0)}{\big(1-\lambda g_\infty(X\mp \mathrm{i}0)\big)} \qquad (2.62)
\end{aligned}
$$

Then we consider H^ρ. In general, there is no analog of (2.62). Instead, using (2.32) one obtains the following compact formula for $W^\pm(H^\rho, X)$:

$$W^\pm(H^\rho, X) = 1\!\!1 \mp h_2(X) \frac{2\pi\mathrm{i}}{(\mathrm{e}^{\pm 2\pi A} + 1\!\!1)} \frac{h_1(X)}{\big(\rho + g^0(X \mp \mathrm{i}0)\big)}. \qquad (2.63)$$

2.8 Self-transposed and self-adjoint cases

In applications, it often happens that one of the following two conditions holds:

$$h_1 = h_2 =: h, \quad \text{resp.} \quad \overline{h}_1 = h_2 =: h, \qquad (2.64)$$

(The former is the case of our paper; the latter is in most of the literature). Then Assumptions I, II, III slightly simplify and can be rewritten in terms of the scale of Hilbert spaces associated with the positive operator $1 + X$:
Assumption I. $h \in L^2[0, \infty[$.
Assumption II. $h \in (1 + X)^{\frac{1}{2}} L^2[0, \infty[$.
Assumption III. $h \in (1 + X) L^2[0, \infty[$.
Moreover, the family H_λ is self-transposed, resp. self-adjoint, that is

$$H_\lambda^\# = H_\lambda, \quad \text{resp.} \quad H_\lambda^* = H_\lambda,$$

and the Møller operators satisfy

$$\big(W^\pm(H, X)\big)^\# = W^\mp(X, H), \quad \text{resp.} \ \big(W^\pm(H, X)\big)^* = W^\pm(X, H). \qquad (2.65)$$

To see (2.65) it is enough to look at the kernels (2.36) and (2.37).

3 Family of rank one perturbations

3.1 Construction

We still consider $L^2[0, \infty[$ with the operator X defined as in (2.8). Let $m \in \mathbb{C}$ and

$$h_m(x) := x^{\frac{m}{2}}.$$

We would like to define an operator formally given by

$$H_{m,\lambda} := X + \lambda |h_m\rangle\langle h_m|. \qquad (3.1)$$

We check that for $-1 < \mathrm{Re} m < 0$ Assumption II is satisfied. Therefore the construction described in Section 2 allows us to define a closed operator $H_{m,\lambda}$ for m in this range and $\lambda \in \mathbb{C} \cup \{\infty\}$. Using (A.4), we compute for $z \in \mathbb{C} \backslash [0, \infty[$:

$$\langle h_m | (z - X)^{-1} | h_m \rangle^{-1} \qquad (3.2)$$

$$= \int_0^\infty x^m (z - x)^{-1} \mathrm{d}x \qquad (3.3)$$

$$= -(-z)^m \int_0^\infty \Big(\frac{x}{-z}\Big)^m \Big(1 + \frac{x}{(-z)}\Big)^{-1} \frac{\mathrm{d}x}{(-z)} \qquad (3.4)$$

$$= (-z)^m \frac{\pi}{\sin \pi m}. \qquad (3.5)$$

Next note that Assumption III is satisfied for $-1 < \mathrm{Re}m < 1$. Moreover, equation

$$\partial_z g(z) = -\langle h_m|(z-X)^{-2}|h_m\rangle^{-1} = -(-z)^{m-1}\frac{\pi m}{\sin \pi m}$$

can be solved, obtaining the following solutions

$$g_\lambda(z) \ := \ -\lambda^{-1} + (-z)^m \frac{\pi}{\sin \pi m}, \quad m \neq 0, \tag{3.6}$$
$$g^\rho(z) \ = \ \rho - \ln(-z), \qquad m = 0. \tag{3.7}$$

Note that only for $m = 0$ we use the "superindex notation" and we do not use the "coupling constant" λ. For $m \neq 0$ we keep the "coupling constant" λ, even though for $\mathrm{Re}m \geq 0$ it has lost its meaning described by (3.1).

The following theorem summarizes our construction:

Theorem 3.1 (1) *For any $-1 < \mathrm{Re}m < 1$, $m \neq 0$, $\lambda \in \mathbb{C} \cup \{\infty\}$, there exists a unique closed operator $H_{m,\lambda}$ such that*

$$\begin{aligned}(z - H_{m,\lambda})^{-1} \ &= \ (z-X)^{-1} \\ &+\Big(\lambda^{-1} - (-z)^m \frac{\pi}{\sin \pi m}\Big)^{-1} (z-X)^{-1}|h_m\rangle\langle h_m|(z-X)^{-1}.\end{aligned} \tag{3.8}$$

In particular, $H_{m,0} = X$.

(2) *For any $\rho \in \mathbb{C} \cup \{\infty\}$, there exists a unique closed operator H_0^ρ such that*

$$\begin{aligned}(z - H_0^\rho)^{-1} \ &= \ (z-X)^{-1} \\ &-\big(\rho + \ln(-z)\big)^{-1}(z-X)^{-1}|h_0\rangle\langle h_0|(z-X)^{-1}.\end{aligned} \tag{3.9}$$

In particular, $H_0^\infty = X$.

Note that the operators $H_{m,\lambda}$ and H_0^ρ are self-transposed.

It will be convenient to introduce the shorthand

$$\varsigma(m,\lambda) = \varsigma := \lambda \frac{\pi}{\sin \pi m}. \tag{3.10}$$

Then we can rewrite (3.8) as

$$\begin{aligned}(z - H_{m,\lambda})^{-1} \ &= \ \frac{1}{1-\varsigma(-z)^m}(z-X)^{-1} \\ &-\frac{\varsigma(-z)^m}{1-\varsigma(-z)^m}(z - H_{m,\infty})^{-1}.\end{aligned} \tag{3.11}$$

It is possible to include both $H_{m,\lambda}$ and H_0^ρ in a single analytic family of closed operators (see [9]).

Theorem 3.2 *For $m \neq 0$, set $\lambda(m,\rho) := \frac{m}{1-m\rho}$. Then*

$$(\rho, m) \mapsto \begin{cases} H_{m,\lambda(m,\rho)}, & m \neq 0; \\ H_0^\rho, & m = 0; \end{cases} \tag{3.12}$$

is an analytic family (their resolvents depend analytically on (ρ, m)).

Proof. We have

$$\rho(m, \lambda) := m^{-1} - \lambda^{-1}. \tag{3.13}$$

Hence, for small m

$$\begin{aligned} g_\lambda(z) &= -\lambda^{-1} + \mathrm{e}^{m\ln(-z)} \frac{\pi}{\sin \pi m} && (3.14) \\ &= -\lambda^{-1} + \big(1 + m\ln(-z)\big)m^{-1} + O(m) && (3.15) \\ &= \rho(m, \lambda) + \ln(-z) + O(m). && (3.16) \end{aligned}$$

□

3.2 Toy model of the renormalization group

The group of dilations ("the renormalization group") acts on our operators in a simple way:

$$\begin{aligned} U_\tau H_{m,\lambda} U_\tau^{-1} &= \mathrm{e}^\tau H_{m,\mathrm{e}^{\tau m}\lambda}, && (3.17) \\ U_\tau H_0^\rho U_\tau^{-1} &= \mathrm{e}^\tau H_0^{\rho+\tau}. && (3.18) \end{aligned}$$

We will show that an appropriately renormalized operator of the form X plus a rather arbitrary rank one perturbation is driven by the scaling to one of the operators that we consider in our paper.

Theorem 3.3 *Suppose that $h \in L^2[0, \infty[$ has a compact support and $h = x^{\frac{m}{2}}$ close to $x = 0$. Set*

$$H(\lambda) := X + \lambda |h\rangle\langle h|. \tag{3.19}$$

We then have the following statements, (where lim *denotes the norm resolvent limit):*

(1) *For $-1 < \mathrm{Re} m < 0$,*

$$\lim_{\tau \to -\infty} \mathrm{e}^{-\tau} U_\tau H(\lambda \mathrm{e}^{-m\tau}) U_\tau^{-1} = H_{m,\lambda}. \tag{3.20}$$

(2) *For $m = 0$,*

$$\lim_{\tau \to -\infty} \mathrm{e}^{-\tau} U_\tau H\Big(\frac{1}{\tau}\Big) U_\tau^{-1} = H_0^\nu, \tag{3.21}$$

where

$$\nu : \;= \int_0^\infty \ln(y)(h^2)'(y)\mathrm{d}y. \tag{3.22}$$

(3) *For* $0 \leq \mathrm{Re} m < 1$, $m \neq 0$,

$$\lim_{\tau \to -\infty} \mathrm{e}^{-\tau} U_\tau H\Big(\frac{\lambda}{\mathrm{e}^{m\tau} - \alpha\lambda}\Big) U_\tau^{-1} = H_{m,\lambda}, \tag{3.23}$$

where

$$\alpha := \int_0^\infty \frac{h^2(x)}{x} \mathrm{d}x. \tag{3.24}$$

Proof. We will prove only the case $0 \leq \mathrm{Re} m < 1$, $m \neq 0$. The other cases are easier. Set $\lambda_\tau := \frac{\lambda}{\mathrm{e}^{m\tau} - \alpha\lambda}$.

$$\begin{aligned}
&\big(z - \mathrm{e}^{-\tau} U_\tau H(\lambda_\tau) U_{-\tau}\big)^{-1} \\
= \ & \mathrm{e}^{\tau} U_\tau \big(z\mathrm{e}^{\tau} - H(\lambda_\tau)\big)^{-1} U_{-\tau} \\
= \ & \mathrm{e}^{\tau} U_\tau \big(z\mathrm{e}^{\tau} - X\big)^{-1} U_{-\tau} \\
& - \Big(- \lambda_\tau^{-1} + \int_0^\infty h(x)^2 (z\mathrm{e}^{\tau} - x)^{-1} \mathrm{d}x \Big)^{-1} \\
& \times \mathrm{e}^{\tau} U_\tau (z\mathrm{e}^{\tau} - X\big)^{-1} |h\rangle\langle h| (z\mathrm{e}^{\tau} - X\big)^{-1} U_{-\tau} \\
= \ & \mathrm{I} - \mathrm{II}^{-1} \times \mathrm{III}.
\end{aligned}$$

Clearly,

$$\begin{aligned}
\mathrm{I} &= (z - X)^{-1}, \\
\mathrm{e}^{-\tau m} \mathrm{III} &= \mathrm{e}^{-\tau(m+1)} (z - X)^{-1} |U_\tau h\rangle\langle U_\tau h| (z - X)^{-1} \\
&\to (z - X)^{-1} |h_m\rangle\langle h_m| (z - X)^{-1}, \\
\mathrm{e}^{-m\tau} \mathrm{II} + \lambda^{-1} &= \mathrm{e}^{-m\tau} \int_0^\infty h(x)^2 \Big((\mathrm{e}^{\tau} z - x)^{-1} + x^{-1} \Big) \mathrm{d}x \\
&= \mathrm{e}^{-m\tau} (-z) \mathrm{e}^{\tau} \int_0^\infty \frac{h(x)^2 \mathrm{d}x}{x(x - z\mathrm{e}^{\tau})} \\
&= \mathrm{e}^{-m\tau} \int_0^\infty \frac{h(\mathrm{e}^{\tau}(-z)y)^2 \mathrm{d}y}{y(y+1)} \\
&\to (-z)^m \int_0^\infty \frac{y^{m-1} \mathrm{d}y}{(y+1)} \\
&= -(-z)^m \frac{\pi}{\sin \pi(m-1)} \\
&= (-z)^m \frac{\pi}{\sin \pi m}.
\end{aligned}$$

□

3.3 Point spectrum

The following theorem is analogous to the characterization of the point spectrum of the Bessel operator described in Theorem 5.2 of [7], and is the consequence of the same computation.

Theorem 3.4 (1) *$w \in \mathbb{C}\backslash[0,\infty[$ belongs to the point spectrum of $H_{m,\lambda}$ iff it satisfies the equation*

$$(-w)^{-m} = \varsigma. \tag{3.25}$$

The corresponding eigenprojection has the kernel

$$\mathbb{1}_{\{w\}}(H_{m,\lambda})(x,y) = \frac{\sin \pi m}{\pi m}(-w)^{-m+1}(w-x)^{-1}x^{\frac{m}{2}}y^{\frac{m}{2}}(w-y)^{-1}. \tag{3.26}$$

(2) *H_0^ρ possesses an eigenvalue iff $-\pi < \mathrm{Im}\rho < \pi$, and then it is $w = -\mathrm{e}^\rho$. The corresponding eigenprojection has the kernel*

$$\mathbb{1}_{\{w\}}(H_0^\rho)(x,y) = -w(w-x)^{-1}(w-y)^{-1}. \tag{3.27}$$

Let us stress that $\sigma_{\mathrm{p}}(H_{m,\lambda})$ depends in a complicated way on the parameters m and λ. There exists a complicated pattern of *phase transitions*, when some eigenvalues "disappear". This happens if

$$\pi \in \mathrm{Re}\frac{1}{m}\mathrm{Ln}(\varsigma), \quad \text{or} \quad -\pi \in \mathrm{Re}\frac{1}{m}\mathrm{Ln}(\varsigma), \tag{3.28}$$

where Ln denotes the multivalued logarithm function. A pair (m,λ) satisfying (3.28) will be called *exceptional*. For $m = 0$, we need a different condition. We say that $(0,\rho)$ is exceptional if

$$\mathrm{Im}\rho = -\pi \qquad \text{or} \qquad \mathrm{Im}\rho = \pi. \tag{3.29}$$

For a given m,λ all eigenvalues form a geometric sequence that lie on a logarithmic spiral. This spiral should be viewed as a curve on the Riemann surface of the logarithm, and only its "physical sheet" gives rise to eigenvalues. For m which are not purely imaginary, only a finite piece of the spiral is on the "physical sheet" and therefore the number of eigenvalues is finite.

If m is purely imaginary, this spiral degenerates to a half-line starting at the origin. Either the whole half-line is on the "physical sheet", and then the number of eigenvalues is infinite, or the half-line is "hidden on the non-physical sheet of the complex plane", and then there are no eigenvalues.

If m is real, the spiral degenerates to a circle. But then the operator has at most one eigenvalue.

Below we provide a characterization of $\#\sigma_{\mathrm{p}}(H_{m,\lambda})$, *i.e.* of the number of eigenvalues of $H_{m,\lambda}$. It is proven in [7], Proposition 5.3.

Proposition 3.5 *Let $m = m_{\mathrm{r}} + \mathrm{i}m_{\mathrm{i}} \in \mathbb{C}^\times$ with $|m_{\mathrm{r}}| < 1$.*

(i) Let $m_{\mathrm{r}} = 0$.

- *(a) If $\frac{\ln(|\varsigma|)}{m_{\mathrm{i}}} \in]-\pi,\pi[$, then $\#\sigma_{\mathrm{p}}(H_{m,\lambda}) = \infty$,*
- *(a) if $\frac{\ln(|\varsigma|)}{m_{\mathrm{i}}} \notin]-\pi,\pi[$ then $\#\sigma_{\mathrm{p}}(H_{m,\lambda}) = 0$.*

(ii) If $m_{\mathrm{r}} \neq 0$ and if $N \in \mathbb{N}$ satisfies $N < \frac{m_{\mathrm{r}}^2 + m_{\mathrm{i}}^2}{|m_{\mathrm{r}}|} \leq N+1$, then

$$\#\sigma_{\mathrm{p}}(H_{m,\lambda}) \in \{N, N+1\}.$$

3.4 Møller operators

First consider $m \neq 0$. We define the Møller operator

$$W^{\pm}_{m,\lambda} := W^{\pm}(H_{m,\lambda}, X), \tag{3.30}$$

as the operator with the kernel (2.36). Note that

$$W^{\mp\#}_{m,\lambda} = W^{\pm}(X, H_{m,\lambda}). \tag{3.31}$$

For $m \neq 0$ we have two distinct λ with $H_{m,\lambda}$ homogeneous of degree one. One of them is obviously $H_{m,0} = X$. The other is $H_{m,\infty}$. Therefore, the Møller operators $W^{\pm}_{m,\infty}$ are functions of A. The Møller operators $W_{m,\lambda}$ for all λ can be expressed in terms of $W^{\pm}_{m,\infty}$ and X. All this is described in the following theorem:

Theorem 3.6 *$W^{\pm}_{m,\infty}$ exist as bounded operators and*

$$W^{\pm}_{m,\infty} = \frac{\mathrm{e}^{\mp 2\pi A} + \mathrm{e}^{\mp \mathrm{i}m\pi}\mathbb{1}}{\mathrm{e}^{\mp 2\pi A} + \mathrm{e}^{\pm \mathrm{i}m\pi}\mathbb{1}}. \tag{3.32}$$

Besides, if (m, λ) is not exceptional, then $W^{\pm}_{m,\lambda}$ exist as bounded operators and are given by

$$W^{\pm}_{m,\lambda} = \frac{1}{\left(\mathbb{1} - \varsigma \mathrm{e}^{\pm \mathrm{i}\pi m} X^m\right)} \tag{3.33}$$

$$-W^{\pm}_{m,\infty} \frac{\varsigma \mathrm{e}^{\pm \mathrm{i}\pi m} X^m}{\left(\mathbb{1} - \varsigma \mathrm{e}^{\pm \mathrm{i}\pi m} X^m\right)}. \tag{3.34}$$

They satisfy

$$W^{\mp\#}_{m,\lambda} W^{\pm}_{m,\lambda} = \mathbb{1}, \tag{3.35}$$

$$W^{\pm}_{m,\lambda} W^{\mp\#}_{m,\lambda} = \mathbb{1}_{[0,\infty[}(H_{m,\lambda}), \tag{3.36}$$

$$W^{\pm}_{m,\lambda} X = H_{m,\lambda} W^{\pm}_{m,\lambda}. \tag{3.37}$$

Proof. By (2.36), the kernel of $W^{\pm}_{m,\infty}$ is given by

$$\begin{aligned} W^{\pm}_{m,\infty}(x,y) &= \delta(x-y) + \frac{\mathrm{e}^{\pm \mathrm{i}\pi m} \frac{\sin \pi m}{\pi} x^{-\frac{m}{2}} y^{\frac{m}{2}}}{x - y \mp \mathrm{i}0} \\ &= \delta(x-y) + \frac{\mathrm{e}^{\pm \mathrm{i}\pi m} \frac{\sin \pi m}{\pi}}{\sqrt{xy}} \left(\frac{y}{x}\right)^{\frac{m}{2}} \left(\left(\frac{x}{y}\right)^{\frac{1}{2}} - \left(\frac{y}{x}\right)^{\frac{1}{2}} \mp \mathrm{i}0\right)^{-1}. \end{aligned} \tag{3.38}$$

Now, by Subsection 2.4 and the formula (A.2), the operator with the kernel

$$\frac{1}{\sqrt{xy}} \left(\frac{y}{x}\right)^{\frac{m}{2}} \left(\left(\frac{x}{y}\right)^{\frac{1}{2}} - \left(\frac{y}{x}\right)^{\frac{1}{2}} \mp \mathrm{i}0\right)^{-1}$$

equals

$$\frac{\pm 2\pi \mathrm{i}}{\mathrm{e}^{\mp\pi \mathrm{i} m \pm 2\pi A} + \mathbb{1}}. \tag{3.39}$$

Therefore,

$$W^{\pm}_{m,\infty} = \mathbb{1} \pm \mathrm{e}^{\pm \mathrm{i}\pi m} \frac{\sin \pi m}{\pi} \frac{\pm 2\pi \mathrm{i}}{\mathrm{e}^{\mp\pi \mathrm{i} m \pm 2\pi A} + \mathbb{1}} \tag{3.40}$$

$$= \frac{\mathrm{e}^{\mp 2\pi A} + \mathrm{e}^{\mp \mathrm{i} m\pi}}{\mathrm{e}^{\mp 2\pi A} + \mathrm{e}^{\pm \mathrm{i} m\pi}}. \tag{3.41}$$

To see that (3.41) is bounded we use $-1 < \mathrm{Re} m < 1$.

If (m, λ) is not exceptional, then $\frac{1}{(\mathbb{1} - \varsigma \mathrm{e}^{\pm \mathrm{i}\pi m} X^m)}$ and $\frac{\varsigma \mathrm{e}^{\pm \mathrm{i}\pi m} X^m}{(\mathbb{1} - \varsigma \mathrm{e}^{\pm \mathrm{i}\pi m} X^m)}$ are bounded, and therefore the formula (2.62) defines $W^{\pm}_{m,\lambda}$ as a bounded operator.

(2.40), (2.41) and (2.42) rewritten using (3.30) and (3.31) yield (3.35), (3.36) and (3.37). □

Next, consider $m = 0$. We set

$$W^{\rho,\pm}_0 := W^{\pm}(H^{\rho}_0, X).$$

Note that

$$W^{\rho,\mp\#}_0 = W^{\pm}(X, H^{\rho}_0).$$

For $m = 0$, only $X = H^{\infty}_0$ is homogeneous of degree 1, therefore we do not have an analog of (3.34).

Theorem 3.7 *Suppose that ρ is not exceptional. Then the Møller operators $W^{\rho,\pm}_0$ exist as bounded operators and are given by*

$$W^{\rho,\pm}_0 = \mathbb{1} \mp \frac{2\pi \mathrm{i}}{(\mathrm{e}^{\pm 2\pi A} + \mathbb{1})} \frac{1}{(\ln X \mp \mathrm{i}\pi - \rho)}. \tag{3.42}$$

They satisfy

$$W^{\rho,\mp\#}_0 W^{\rho,\pm}_0 = \mathbb{1}, \tag{3.43}$$

$$W^{\rho,\pm}_0 W^{\rho,\mp\#}_0 = \mathbb{1}_{[0,\infty[}(H^{\rho}_0), \tag{3.44}$$

$$W^{\rho,\pm}_0 X = H^{\rho}_0 W^{\rho,\pm}_0. \tag{3.45}$$

Proof. Using the assumption that ρ is non-exceptional, we check that $\frac{1}{(\ln X \mp \mathrm{i}\pi - \rho)}$ is bounded. Now, the formula (2.63) proves (3.42) and the boundedness of $W^{\rho,\pm}_0$. □

4 Equivalence with Schrödinger operators with inverse square potentials

Recall that in [4] a holomorphic family of closed operators H_m was introduced. We change slightly the notation for these operators, and we will denote them by $\tilde{H}_m$ in this paper.

Thus, for Re$m > -1$, $\tilde{H}_m$ is the unique closed operators on $L^2[0,\infty[$ given on $C_c^\infty]0,\infty[$ by the differential expression

$$\tilde{H}_m = -\partial_x^2 + \Big(-\frac{1}{4} + m^2\Big)\frac{1}{x^2}, \tag{4.1}$$

such that functions in its domain behave as $x^{\frac{1}{2}+m}$ around zero. $\tilde{H}_m$ can be diagonalized with help of the so-called Hankel transformation $\mathcal{F}_m$, which is a bounded invertible involutive operator such that

$$\mathcal{F}_m \tilde{H}_m \mathcal{F}_m^{-1} = X^2, \tag{4.2}$$
$$\mathcal{F}_m A \mathcal{F}_m^{-1} = -A. \tag{4.3}$$

For $-1 < \mathrm{Re} m < 1$, two more general family of operators $H_{m,\kappa}$ and H_0^ν were constructed in [7]. In this paper they will be be denoted by $\tilde{H}_{m,\kappa}$ and $\tilde{H}_0^\nu$.

$\tilde{H}_{m,\kappa}$ is given by the differential expression on the right hand side of (4.1) with the boundary condition at zero $\kappa x^{\frac{1}{2}-m} + x^{\frac{1}{2}+m}$. $\tilde{H}_0^\nu$ is defined by (4.1) with $m = 0$ and the boundary conditions $x^{\frac{1}{2}}\log(x) + \nu x^{\frac{1}{2}}$. Note that

$$\tilde{H}_m = \tilde{H}_{m,0} = \tilde{H}_{-m,\infty}, \quad \tilde{H}_0 = \tilde{H}_0^\infty.$$

Define the unitary operator

$$(If)(x) := x^{-\frac{1}{4}} f(2\sqrt{x}). \tag{4.4}$$

Its inverse is

$$(I^{-1}f)(x) := \Big(\frac{y}{2}\Big)^{\frac{1}{2}} f\Big(\frac{y^2}{4}\Big). \tag{4.5}$$

Note that

$$I^{-1}XI = \frac{X^2}{4}, \tag{4.6}$$
$$I^{-1}AI = \frac{A}{2}. \tag{4.7}$$

Theorem 4.1 *We have*

$$\mathcal{F}_m^{-1} I^{-1} H_{m,\lambda} I \mathcal{F}_m = \frac{1}{4}\tilde{H}_{m,\kappa}, \tag{4.8}$$

where the pairs (m,λ) *and* (m,κ) *are linked by the relation*

$$\lambda\frac{\pi}{\sin(\pi m)} = \kappa\frac{\Gamma(m)}{\Gamma(-m)}, \tag{4.9}$$

(The relation (4.9) is equivalent to saying that the parameter ς *introduced in [7] (5.2) coincides with the parameter* ς *introduced in (3.10).)*

Proof. To avoid notational collision, we denote by $\tilde{W}^{\pm}_{m'm}$ the Møller operators, denoted by $W^{\pm}_{m,m'}$ in [7]. We quote some identities from [7] contained in Prop. 4.11, Prop. 4.9 and Equation (6.3):

$$\begin{aligned} \tilde{W}^{\pm}_{-m,m} &= \frac{\mathrm{e}^{\pm\pi A} + \mathrm{e}^{\mp\mathrm{i}\pi m}}{\mathrm{e}^{\pm\pi A} + \mathrm{e}^{\pm\mathrm{i}\pi m}}, &(4.10)\\ \big(k^2 + \tilde{H}_{-m}\big)^{-1} &= \tilde{W}^{\pm}_{-m,m}\big(k^2 + \tilde{H}_m\big)^{-1}\tilde{W}^{\pm -1}_{-m,m}, &(4.11)\\ \big(k^2 + \tilde{H}_{m,\kappa}\big)^{-1} &= \frac{1}{1-\varsigma(\frac{k}{2})^{2m}}\big(k^2 + \tilde{H}_m\big)^{-1} \\ &\quad -\frac{\varsigma\big(\frac{k}{2}\big)^{2m}}{1-\varsigma(\frac{k}{2})^{2m}}\big(k^2 + \tilde{H}_{-m}\big)^{-1}. &(4.12) \end{aligned}$$

On the other hand, by Theorem 3.6 we have

$$\begin{aligned} W^{\pm}_{m,\infty} &= \frac{\mathrm{e}^{\mp 2\pi A} + \mathrm{e}^{\mp\mathrm{i}m\pi}\mathbb{1}}{\mathrm{e}^{\mp 2\pi A} + \mathrm{e}^{\pm\mathrm{i}m\pi}\mathbb{1}}, &(4.13)\\ (z - H_{m,\infty})^{-1} &= W^{\pm}_{m,\infty}(z - X)^{-1}W^{\pm -1}_{m,\infty}, &(4.14)\\ (z - H_{m,\lambda})^{-1} &= \frac{1}{1-\varsigma(-z)^m}(z - X)^{-1} \\ &\quad -\frac{\varsigma(-z)^m}{1-\varsigma(-z)^m}(z - H_{m,\infty})^{-1}. &(4.15) \end{aligned}$$

Now (4.3), (4.7), (4.10) and (4.13) imply

$$\mathcal{F}_m^{-1} I^{-1} W^{\pm}_{m,\infty} I \mathcal{F}_m = \tilde{W}^{\pm}_{-m,m}. \tag{4.16}$$

Setting $-z = \frac{k^2}{4}$, using (4.14), (4.16), (4.2), (4.6) and (4.11) we check that

$$\mathcal{F}_m^{-1} I^{-1} (z - H_{m,\infty})^{-1} I \mathcal{F}_m = \Big(z - \frac{\tilde{H}_{-m}}{4}\Big)^{-1}. \tag{4.17}$$

Finally, (4.12) and (4.15) yield

$$\mathcal{F}_m^{-1} I^{-1} (z - H_{m,\lambda})^{-1} I \mathcal{F}_m = \Big(z - \frac{\tilde{H}_{m,\kappa}}{4}\Big)^{-1}. \tag{4.18}$$

□

Theorem 4.2 *We have*

$$\mathcal{F}_0^{-1} I^{-1} H_0^{\rho} I \mathcal{F}_0 = \frac{1}{4}\tilde{H}_0^{\nu}, \tag{4.19}$$

where $\rho = -2\nu$.

Proof. The $m = 0$ case will be reduced to the $m \neq 0$.

First note that the family

$$(m, \nu) \mapsto \begin{cases} \tilde{H}_{m, \frac{m\nu - 1}{m\nu + 1}}, & m \neq 0 \\ \tilde{H}_0^{\nu}, & m = 0; \end{cases} \tag{4.20}$$

is analytic (see [7], Remark 2.5). Hence, for $m \to 0$,

$$\tilde{H}_{m, -1+2m\nu} \to \tilde{H}_0^{\nu}. \tag{4.21}$$

Similarly, by Theorem 3.2, the family

$$(m, \rho) \mapsto \begin{cases} H_{m, \frac{m}{1-m\rho}}, & m \neq 0 \\ H_0^{\rho}, & m = 0 \end{cases} \tag{4.22}$$

is analytic. Hence, for $m \to 0$,

$$H_{m, m+m^2\rho} \to H_0^{\rho}. \tag{4.23}$$

Around $m = 0$, the condition (4.9) becomes

$$\frac{\lambda}{m} \simeq -\kappa\big(1 + O(m)\big). \tag{4.24}$$

Therefore, around $m = 0$, (4.8) implies

$$\mathcal{F}_m^{-1} I^{-1} H_{m, m+m^2\rho} I \mathcal{F}_m = \frac{1}{4} \tilde{H}_{m, -1+2m\nu+O(m^2)}, \tag{4.25}$$

where $\rho = -2\nu$. Passing to the limit $m \to 0$ in (4.25) we obtain (4.19). $\square$

A Some integrals

- For $-\frac{1}{2} < \mathrm{Re}a < -\frac{1}{2}$,

$$\int_{-\infty}^{\infty} \frac{\mathrm{e}^{at} \mathrm{d}t}{\mathrm{e}^{\frac{t}{2}} + \mathrm{e}^{-\frac{t}{2}}} = \frac{\pi}{\cos \pi a}. \tag{A.1}$$

  Indeed, we integrate the analytic function $f(t) := \frac{\mathrm{e}^{at}}{\mathrm{e}^{\frac{t}{2}} + \mathrm{e}^{-\frac{t}{2}}}$ over the rectangle $-R, R, R + 2\pi\mathrm{i}, -R + 2\pi\mathrm{i}$, use $f(t \pm 2\pi\mathrm{i}) = -\mathrm{e}^{2\pi\mathrm{i}a} f(t)$ and $\mathrm{Res} f(\pi\mathrm{i}) = \frac{1}{\mathrm{i}} \mathrm{e}^{\pi\mathrm{i}a}$.

- For the same a,

$$\int_{-\infty}^{\infty} \frac{\mathrm{e}^{at} \mathrm{d}t}{\mathrm{e}^{\frac{t}{2}} - \mathrm{e}^{-\frac{t}{2}} \mp \mathrm{i}0} = \frac{\pm 2\pi\mathrm{i}}{\mathrm{e}^{\pm 2\pi\mathrm{i}a} + 1}. \tag{A.2}$$

  To see this we shift the integration in (A.1) from $\mathbb{R}$ to $\mathbb{R} \pm \mathrm{i}\pi$ without crossing the singularity at $\pm\pi\mathrm{i}$.

- Setting $a = -\mathrm{i}\xi$ in (A.2) we obtain the following Fourier transform:

$$h(t) = \frac{1}{\left(\mathrm{e}^{\frac{t}{2}} - \mathrm{e}^{-\frac{t}{2}} \mp \mathrm{i}0\right)}, \quad \hat{h}(\xi) = \frac{\pm 2\pi\mathrm{i}}{(\mathrm{e}^{\pm 2\pi\xi} + 1)}. \tag{A.3}$$

- For $-1 < \mathrm{Re} m < 0$,

$$\int_0^\infty s^m (1+s)^{-1} \mathrm{d}s = -\frac{\pi}{\sin \pi m}. \tag{A.4}$$

Indeed, we set $m = a - \frac{1}{2}$ and $s = \mathrm{e}^t$ in (A.1).

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