Vector quantum fields

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Various approaches to quantization of vector fields, both massive (Proca) and massless (Maxwell), are described. The massless limit of the Proca fields is discussed.

1. Introduction

In these notes I would like to discuss quantization of free vector fields on a flat Minkowski space – both massless Maxwell fields and massive Proca fields. This subject, especially in the massless case, is covered in many standard textbooks.[2,8–11,13] Therefore the reader may wonder why I consider such a topic.

In my opinion, this subject has several interesting and confusing aspects, which are not sufficiently discussed in the literature. Therefore, I would like to discuss it in this short note, based on my more comprehensive manuscript.[4]

First, there is more than one way to formulate its classical theory. One option is to initially reduce the constraints and then formulate the theory. Or one can start with the unreduced theory and then reduce.

Second, there are several ways to formulate the quantized theory of vector fields. If one starts from the unreduced classical formalism, one can use either the formalism with a (positive definite) Hilbert space or with an indefinite scalar product. One could argue that the former formalism is more physical. Nowadays it is rarely presented in the literature, however it is implicit in older works such as,[9] and in a more mathematical form in.[12] The latter approach, known under the name of the Gupta-Bleuler formalism, dates back only to the early 50's.[6,7] It is nowadays much more popular in the literature. However, it is based on an obviously nonphysical trick. At the end the physical theory does not depend on the approach that we chose. However the route to the physical theory is quite different depending on the approach.

Third, the limit $m \searrow 0$ is for vector fields rather subtle. Indeed, for $m > 0$ the fields have 3 spin degrees of freedom, whereas for $m = 0$ only 2. In practice, physical quantities are continuous wrt m, including the continuity at zero even though the number of spin degree of freedom is discontinuous.

Note that both the massive and massless theories are Poincaré covariant. I will argue, however, that to describe the limit $m^2 \searrow 0$ one has to choose a frame of reference. Then the longitudinal spin decouples from the theory and the transversal degrees of freedom converge to the potential in the Coulomb gauge, which is not manifestly covariant. Thus the Poincaré covariance is broken twice, and the resulting massless theory is covariant!

My note will start with a discussion of scalar quantum fields. Of course, this subject is quite straightforward and covered in all textbooks on quantum field theory. I do this in order to fix the notation and describe the general philosophy of quantizing free fields.[1,5] According to this philosophy one starts with a classical Hamiltonian systems, and then describes its quantization. Classical Hamiltonians for free fields are always quadratic – therefore quantization is rather straightforward and the main problem is the infinite number of degrees of freedom.

One can sometimes hear the statement that the topic of free quantum fields is very simple, because it is just the 2nd quantization of a unitary representation of the Poincaré group – the spin 0 representation in the case of scalar fields. This view does not capture the most important feature of the fields – their locality. Actually, even the simple topic of scalar quantum fields can be formulated in a few equivalent but seemingly different ways. Some of them we describe, because we need them in the slightly more complicated case of vector fields.

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2. Scalar fields

2.1. *Notation*

We equip the Minkowski space $\mathbb{R}^{1,3}$ with the pseudo-Euclidean form of signature $(-+++)$

$$x_\mu x^\mu = g_{\mu\nu} x^\mu x^\nu = -(x^0)^2 + \sum_{i=1}^{3} (x^i)^2.$$

$\mathbb{R}^3$ we will typically denote the *spatial part* of the Minkowski space obtained by setting $x^0 = 0$. If $x \in \mathbb{R}^{1,3}$, then $\vec{x}$ will denote the projection of x onto its spatial $\mathbb{R}^3$.

A function on $\mathbb{R}^{1,3}$ is called *space-compact* iff there exists a compact $K \subset \mathbb{R}^{1,3}$ such that supp$f$ is contined in the causal shadow of K.

2.2. *Classical theory*

Let $\mathcal{Y}_{\mathrm{KG}}$ denote the *space of real smooth space-compact solutions of the Klein-Gordon equation*

$$(-\Box + m^2)\zeta = 0.$$

Let $\dot{\zeta}$ denote the time derivative of ζ.

The space $\mathcal{Y}_{\rm KG}$ has a natural *symplectic form*

$$\zeta_1\omega\zeta_2 = \int \Big(-\dot\zeta_1(t,\vec{x})\zeta_2(t,\vec{x}) + \zeta_1(t,\vec{x})\dot\zeta_2(t,\vec{x})\Big)\,{\rm d}\vec{x}.$$

The Poincaré group $\mathbb{R}^{1,3}\rtimes O(1,3)$ acts on $\mathcal{Y}_{\rm KG}$ and $\mathbb{C}\mathcal{Y}_{\rm KG}$ by

$$r_{(a,\Lambda)}\zeta(x) := \zeta\left((a,\Lambda)^{-1}x\right).$$

For $x\in\mathbb{R}^{1,3}$, the *classical fields* $\phi(x)$, $\pi(x)$ are defined as the functionals on $\mathcal{Y}_{\rm KG}$ given by

$$\langle\phi(x)|\zeta\rangle := \zeta(x), \qquad \langle\pi(x)|\zeta\rangle := \dot\zeta(x).$$

Clearly,

$$(-\Box + m^2)\phi(x) = 0, \quad \dot\phi(x) = \pi(x).$$

The symplectic structure on the space $\mathcal{Y}_{\rm KG}$ leads to a *Poisson bracket* on functions on $\mathcal{Y}_{\rm KG}$:

$$\begin{aligned}\{\phi(t,\vec{x}),\phi(t,\vec{y})\} = \{\pi(t,\vec{x}),\pi(t,\vec{y})\} &= 0,\\ \{\phi(t,\vec{x}),\pi(t,\vec{y})\} &= \delta(\vec{x}-\vec{y}).\end{aligned}$$

We can smear the fields with help of the *pairing given by the symplectic form.*

$$\phi(\!(\zeta)\!) = \int \Big(-\overline{\dot\zeta(t,\vec{x})}\phi(t,\vec{x}) + \overline{\zeta(t,\vec{x})}\pi(t,\vec{x})\Big)\,{\rm d}\vec{x},$$

where $\zeta\in\mathbb{C}\mathcal{Y}_{\rm KG}$. Note that

$$\{\phi(\!(\zeta_1)\!),\phi(\!(\zeta_2)\!)\} = \overline{\zeta}_1\omega\overline{\zeta}_2.$$

We introduce the *(total) Hamiltonian*, which is a quadratic function on $\mathcal{Y}_{\rm KG}$:

$$H := \int \frac{1}{2}\Big(\pi(t,\vec{x})^2 + \big(\vec\partial\phi(t,\vec{x})\big)^2 + m^2\phi(t,\vec{x})^2\Big)\,{\rm d}\vec{x}.$$

H generates the dynamics:

$$\dot\phi(x) = \{H,\phi(x)\}, \quad \dot\pi(x) = \{H,\pi(x)\}.$$

Set $\varepsilon(\vec{k}) := \sqrt{\vec{k}^2+m^2}$. For $\big(\pm\varepsilon(\vec{k}),\vec{k}\big)$, the *plane wave* $|k)$ is defined as

$$(x|k) = \frac{1}{(2\pi)^{3/2}\sqrt{2\varepsilon(\vec{k})}}{\rm e}^{{\rm i}kx}. \tag{1}$$

The *plane wave functionals* act on $\mathcal{Y}_{\rm KG}$ and are defined by

$$\begin{aligned}a(k) &= {\rm i}\phi(\!(|k)\!)),\\ a^*(k) &= -{\rm i}\phi(\!(|-k)\!)).\end{aligned}$$

The fields can be written in terms of $a^*(k)$, $a(k)$:

$$\phi(x) = (2\pi)^{-\frac{3}{2}}\int\frac{{\rm d}\vec{k}}{\sqrt{2\varepsilon(\vec{k})}}\left({\rm e}^{{\rm i}kx}a(k) + {\rm e}^{-{\rm i}kx}a^*(k)\right).$$

$a(k)$, $a^*(k)$ diagonalize simultaneously the Hamiltonian and symplectic form:

$$H = \int \mathrm{d}\vec{k}\varepsilon(\vec{k})a^*(k)a(k),$$
$$\mathrm{i}\omega = \int \mathrm{d}\vec{k}a^*(k) \wedge a(k).$$

$\mathcal{W}_{\mathrm{KG}}^{(+)}$ will denote the subspace of $\mathbb{C}\mathcal{Y}_{\mathrm{KG}}$ consisting of *positive frequency solutions*, that is,

$$\mathcal{W}_{\mathrm{KG}}^{(+)} := \{g \in \mathbb{C}\mathcal{Y}_{\mathrm{KG}} \ : \ \langle a^*(k)|g\rangle = 0\}.$$

For $g_1, g_2 \in \mathcal{W}_{\mathrm{KG}}^{(+)}$ we define the scalar product

$$(g_1|g_2) := \mathrm{i}\overline{g}_1\omega g_2 = \int \overline{\langle a(k)|g_1\rangle}\langle a(k)|g_2\rangle \mathrm{d}\vec{k}$$

The *Hilbert space of positive energy solutions* is denoted $\mathcal{Z}_{\mathrm{KG}}$, and is the completion of $\mathcal{W}_{\mathrm{KG}}^{(+)}$ in this scalar product.

We have a natural identification $\mathcal{Y}_{\mathrm{KG}} \ni \zeta \mapsto \zeta^{(+)} \in \mathcal{W}_{\mathrm{KG}}^{(+)}$ given by

$$\zeta = \zeta^{(+)} + \overline{\zeta^{(+)}}.$$

This identification allows us to define a real scalar product on $\mathcal{Y}_{\mathrm{KG}}$:

$$\begin{aligned}\langle\zeta_1|\zeta_2\rangle_{\mathcal{Y}} &:= \mathrm{Re}(\zeta_1^{(+)}|\zeta_2^{(+)}) \\ &= \int\int \dot{\zeta}_1(0,\vec{x})(-\mathrm{i})D^{(+)}(0,\vec{x}-\vec{y})\dot{\zeta}_2(0,\vec{y})\mathrm{d}\vec{x}\mathrm{d}\vec{y} \\ &\quad + \int\int \zeta_1(0,\vec{x})(-\Delta_{\vec{x}}+m^2)(-\mathrm{i})D^{(+)}(0,\vec{x}-\vec{y})\zeta_2(0,\vec{y})\mathrm{d}\vec{x}\mathrm{d}\vec{y},\end{aligned}$$

where $D^{(+)}$ is the standard positive frequency solution of the Klein-Gordon equation, sometimes called the *Wightman function*.

2.3. *Quantization*

Let us describe the quantization of the Klein-Gordon equation. We will use the "hat" to denote the quantized objects. It can be formulated in an axiomatized way. There are several equivalent formulations – we will present three of them.

We start with an axiomatization which is the closest to the presentation given in most standard textbooks on QFT. We want to construct a Hilbert space, a self-adjoint operator and normalized vector $(\mathcal{H}, \hat{H}, \Omega)$ such that Ω is the unique ground state of $\hat{H}$. We will assume that $H\Omega = 0$. We also want to have a self-adjoint operator valued distribution

$$\mathbb{R}^{1,3} \ni x \mapsto \hat{\phi}(x), \tag{2}$$

such that, with $\hat{\pi}(x) := \dot{\hat{\phi}}(x)$,

(1) $(-\Box + m^2)\hat{\phi}(x) = 0$,

(2) $[\hat{\phi}(0,\vec{x}),\hat{\phi}(0,\vec{y})] = [\hat{\pi}(0,\vec{x}),\hat{\pi}(0,\vec{y})] = 0,$
$[\hat{\phi}(0,\vec{x}),\hat{\pi}(0,\vec{y})] = \mathrm{i}\delta(\vec{x}-\vec{y}).$
(3) $\mathrm{e}^{\mathrm{i}t\hat{H}}\hat{\phi}(x^0,\vec{x})\mathrm{e}^{-\mathrm{i}t\hat{H}} = \hat{\phi}(x^0+t,\vec{x}).$
(4) Ω is cyclic for $\hat{\phi}(x)$.

The above problem has a solution, which is unique up to a unitary equivalence. Let us describe this solution.

For the Hilbert space we choose the bosonic Fock space $\mathcal{H} = \Gamma_{\mathrm{s}}(\mathcal{Z}_{\mathrm{KG}})$ and for Ω the Fock vacuum. Suppose the annihilation operator is denoted by $\hat{a}(k)$. Then

$$\hat{\phi}(x) := (2\pi)^{-\frac{3}{2}} \int \frac{\mathrm{d}\vec{k}}{\sqrt{2\varepsilon(\vec{k})}} \left(\mathrm{e}^{\mathrm{i}kx}\hat{a}(k) + \mathrm{e}^{-\mathrm{i}kx}\hat{a}^*(k)\right)$$

and

$$\hat{H} := \int \hat{a}^*(k)\hat{a}(k)\varepsilon(\vec{k})\mathrm{d}\vec{k}$$

satisfy our requirements.

The orthochronous Poincaré group $\mathbb{R}^{1,3} \rtimes O^{\uparrow}(1,3)$ is automatically unitarily implemented on $\mathcal{H}$, even though we did not demand it beforehand.

There exists an alternative equivalent formulation of the quantization program, which uses *smeared fields* instead of point fields. It avoids the use of distributions on the Minkowski space and uses more directly the natural degrees of freedom of the fields.

Again, we want to construct $(\mathcal{H},\hat{H},\Omega)$ as above and a linear function

$$\mathcal{Y}_{\mathrm{KG}} \ni \zeta \mapsto \hat{\phi}(\!(\zeta)\!)$$

with values in self-adjoint operators such that

(1) $[\hat{\phi}(\!(\zeta_1)\!),\hat{\phi}(\!(\zeta_2)\!)] = \mathrm{i}\zeta_1\omega\zeta_2.$
(2) $\hat{\phi}(\!(r_{(t,\vec{0})}\zeta)\!) = \mathrm{e}^{\mathrm{i}t\hat{H}}\hat{\phi}(\!(\zeta)\!)\mathrm{e}^{-\mathrm{i}t\hat{H}}.$
(3) Ω is cyclic for the algebra generated by $\hat{\phi}(\!(\zeta)\!)$.

One can pass between these two versions of the quantization by

$$\hat{\phi}(\!(\zeta)\!) = \int \Big(-\dot{\zeta}(t,\vec{x})\hat{\phi}(t,\vec{x}) + \zeta(t,\vec{x})\hat{\pi}(t,\vec{x})\Big)\,\mathrm{d}\vec{x}. \tag{3}$$

Let us mention yet another equivalent approach to quantization, using the language of C^**-algebras*. This approach has its own conceptual advantages and is popular in a part of the mathematical physics community.

Let $\mathrm{CCR}(\mathcal{Y}_{\mathrm{KG}})$ denote the *(Weyl)* C^**-algebra of the CCR over* $\mathcal{Y}_{\mathrm{KG}}$. By definition, it is generated by $W(\zeta)$, $\zeta \in \mathcal{Y}_{\mathrm{KG}}$, such that

$$W(\zeta_1)W(\zeta_2) = \mathrm{e}^{-\mathrm{i}\frac{\zeta_1\omega\zeta_2}{2}}W(\zeta_1+\zeta_2), \quad W(\zeta)^* = W(-\zeta).$$

$\mathbb{R}^{1,3} \rtimes O^{\uparrow}(1,3)$ acts on $\mathrm{CCR}(\mathcal{Y}_{\mathrm{KG}})$ by $*$-automorphisms defined by

$$\hat{r}_{(a,\Lambda)}\left(W(\zeta)\right) := W\left(r_{(a,\Lambda)}(\zeta)\right).$$

We are looking for a cyclic representation of this algebra with the time evolution generated by a positive Hamiltonian.

The solution is provided by the state on on $\mathrm{CCR}(\mathcal{Y}_{\mathrm{KG}})$ defined by

$$\psi\big(W(\zeta)\big) = \exp\Big(-\frac{1}{2}\langle y|y\rangle_{\mathcal{Y}}\Big).$$

Let $(\mathcal{H}_\psi, \pi_\psi, \Omega_\psi)$ be the GNS representation generated by the state ψ. Then this representation has the required properties. $\mathcal{H}_\psi$ can be identified with $\Gamma_{\mathrm{s}}(\mathcal{Z}_{\mathrm{KG}})$ and the fields are related to the Weyl operators by

$$\pi_\psi(W(\zeta)) = \mathrm{e}^{\mathrm{i}\hat{\phi}(\!(\zeta)\!)}.$$

Note that there are other formulations of quantization of free fields. In particular, instead of starting with the Hamiltonian formalism with fields satisfying the equations of motion (the *on shell approach*) one can start with the Lagrangian formalism where at the beginning fields are not subject to the equations of motion (the *off-shell* approach) see eg.[3] In our notes, however, we restrict ourselves to the on-shell approach.

3. Massive photons

3.1. *Classical theory based on the Proca equation*

Let $\mathcal{Y}_{\mathrm{Pr}}$ denote the set of real smooth space-compact solutions of the Proca equation

$$-\partial^\mu(\partial_\mu\zeta_\nu - \partial_\nu\zeta_\mu) + m^2\zeta_\nu(x) = 0. \tag{1}$$

Note that (1) is equivalent to the Klein-Gordon equation together with the Lorentz condition

$$\begin{aligned}(-\Box + m^2)\zeta_\nu &= 0,\\ \partial^\nu\zeta_\nu &= 0.\end{aligned}$$

$\mathcal{Y}_{\mathrm{Pr}}$ is a symplectic space with the *symplectic form*

$$\begin{aligned}&\zeta_1\omega_{\mathrm{Pr}}\zeta_2\\ &= \int\Big(-\Big(\dot{\vec{\zeta}}_1(t,\vec{x}) - \vec{\partial}\zeta_{10}(t,\vec{x})\Big)\vec{\zeta}_2(t,\vec{x}) + \vec{\zeta}_1(t,\vec{x})\Big(\dot{\vec{\zeta}}_2(t,\vec{x}) - \vec{\partial}\zeta_{10}(t,\vec{x})\Big)\Big)\,\mathrm{d}\vec{x}.\end{aligned}$$

The Poincaré group $\mathbb{R}^{1,3} \rtimes O(1,3)$ acts on $\mathcal{Y}_{\mathrm{Pr}}$ by

$$r_{(a,\Lambda)}\zeta_\mu(x) := \Lambda_\mu^\nu\zeta_\nu\left((a,\Lambda)^{-1}x\right).$$

Introduce the functionals $A_\mu(x)$, called *potentials*, acting on $\zeta \in \mathcal{Y}_{\mathrm{Pr}}$ as

$$\langle A_\mu(x)|\zeta\rangle := \zeta_\mu(x).$$

We also introduce the *field tensor* and the *electric field vector*:

$$\begin{aligned}F_{\mu\nu}(x) &:= \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x),\\ E_i(x) &:= F_{0i}(x) \;=\; \dot{A}_i - \partial_i A_0.\end{aligned}$$

The symplectic form leads to a *Poisson bracket* on functions on $\mathcal{Y}_{\rm Pr}$:

$$\{A_i(t,\vec{x}),A_j(t,\vec{y})\}=\{E_i(t,\vec{x}),E_j(t,\vec{y})\}=0,$$
$$\{A_i(t,\vec{x}),E_j(t,\vec{y})\}=\delta_{ij}\delta(\vec{x}-\vec{y}). \tag{2}$$

For $\zeta\in\mathcal{Y}_{\rm Pr}$, the corresonding *spatially smeared potential* is the functional on $\mathcal{Y}_{\rm Pr}$ given by

$$A(\!(\zeta)\!)=\int\Big(-\overline{(\dot{\vec{\zeta}}(t,\vec{x})-\partial\zeta^0(t,\vec{x}))}\vec{A}(t,\vec{x})+\overline{\vec{\zeta}(t,\vec{x})}\vec{E}(t,\vec{x})\Big)\,{\rm d}\vec{x}. \tag{3}$$

Note that

$$\{A(\!(\zeta_1)\!),A(\!(\zeta_2)\!)\}=\overline{\zeta}_1\omega_{\rm Pr}\overline{\zeta}_2.$$

The Hamiltonian generates the equations of motion:

$$H:=\int\Big(\frac{1}{2}\vec{E}^2(t,\vec{x})+\frac{1}{2m^2}({\rm div}\vec{E})^2(t,\vec{x})+({\rm r\vec{o}t}\vec{A})^2(t,\vec{x})$$
$$+\frac{m^2}{2}\vec{A}^2(t,\vec{x})\Big){\rm d}\vec{x}. \tag{4}$$

For $\vec{k}\in\mathbb{R}^3$, $\vec{k}\neq\vec{0}$ fix two spatial vectors $\vec{e}_1(\vec{k}),\vec{e}_2(\vec{k})$ that form an oriented orthonormal basis of the plane orthogonal to $\vec{k}$. Define

$$\vec{e}(\vec{k},\pm1):=\frac{1}{\sqrt{2}}\left(\vec{e}_1(\vec{k})\pm{\rm i}\vec{e}_2(\vec{k})\right).$$

Introduce

$$u(k,0):=\Big(\frac{|\vec{k}|}{m},\frac{\varepsilon(\vec{k})\vec{k}}{m|\vec{k}|}\Big), \tag{5}$$
$$u(k,\pm1):=\left(0,\vec{e}(\vec{k},\pm1)\right). \tag{6}$$

A *plane wave* $|k,\sigma)$ is defined as

$$(x|k,\sigma)=\frac{1}{(2\pi)^{3/2}\sqrt{2\varepsilon(\vec{k})}}u_\mu(k,\sigma){\rm e}^{{\rm i}kx}, \tag{7}$$

We also introduce the plane wave functionals

$$a(k,\sigma)=-{\rm i}A(\!(|k,\sigma))\!)$$
$$a^*(k,\sigma)={\rm i}A(\!(|-k,\sigma))\!).$$

The potentials can be written in terms of $a^*(k,\sigma)$, $a(k,\sigma)$ as

$$A_\mu(x)=(2\pi)^{-\frac{3}{2}}\sum_{\sigma=0,\pm1}\int\frac{{\rm d}\vec{k}}{\sqrt{2\varepsilon(\vec{k})}}\left(u_\mu(k,\sigma){\rm e}^{{\rm i}kx}a(k,\sigma)+\overline{u_\mu(k,\sigma)}{\rm e}^{-{\rm i}kx}a^*(k,\sigma)\right).$$

We have accomplished the diagonalization of the Hamiltonian:

$$H = \sum_{\sigma=0,\pm1} \int \mathrm{d}\vec{k}\varepsilon(\vec{k}) a^*(k,\sigma) a(k,\sigma),$$
$$\mathrm{i}\omega_{\mathrm{Pr}} = \sum_{\sigma=0,\pm1} \int a^*(k,\sigma) \wedge a(k,\sigma) \mathrm{d}\vec{k}.$$

$\mathcal{W}_{\mathrm{Pr}}^{(+)}$ will denote the subspace of $\mathbb{C}\mathcal{Y}_{\mathrm{Pr}}$ consisting of *positive frequency solutions*:

$$\mathcal{W}_{\mathrm{Pr}}^{(+)} := \{g \in \mathbb{C}\mathcal{Y}_{\mathrm{Pr}} \ : \ \langle a^*(k,\sigma)|g\rangle = 0, \quad \sigma = \pm, 0\}.$$

For $g_1, g_2 \in \mathcal{W}_{\mathrm{Pr}}^{(+)}$ we define the scalar product

$$(g_1|g_2) := -\mathrm{i}\overline{g}_1 \omega_{\mathrm{Pr}} g_2 = \sum_{\sigma=0,\pm1} \int \overline{\langle a(k,\sigma)|g_1\rangle} \langle a(k,\sigma)|g_2\rangle \mathrm{d}\vec{k}$$

We set $\mathcal{Z}_{\mathrm{Pr}}$ to be the completion of $\mathcal{W}_{\mathrm{Pr}}^{(+)}$ in this scalar product.

3.2. *Quantization of the Proca equation*

We want to construct a Hilbert space $\mathcal{H}$, a self-adjoint operator $\hat{H}$ having a ground state Ω with $H\Omega = 0$ and a self-adjoint operator-valued distribution $\mathbb{R}^{1,3} \ni x \mapsto \hat{A}_\mu(x)$ such that, setting $\vec{\hat{E}} = \dot{\vec{\hat{A}}} - \vec{\partial}\hat{A}_0$, we have

(1) $-\partial^\mu(\partial_\mu \hat{A}_\nu - \partial_\nu \hat{A}_\mu) + m^2 \hat{A}_\nu(x) = 0,$
(2) $[\hat{A}_i(0,\vec{x}), \hat{A}_j(0,\vec{y})] = [\hat{E}_i(0,\vec{x}), \hat{E}_j(0,\vec{y})] = 0,$
$[\hat{A}_i(0,\vec{x}), \hat{E}_j(0,\vec{y})] = \mathrm{i}\delta_{ij}\delta(\vec{x}-\vec{y}),$
(3) $\mathrm{e}^{\mathrm{i}t\hat{H}} \hat{A}_\mu(x^0,\vec{x}) \mathrm{e}^{-\mathrm{i}t\hat{H}} = \hat{A}_\mu(x^0+t,\vec{x}),$
(4) Ω is cyclic for $\hat{A}_\mu(x)$.

The above problem has a solution, which is unique up to unitary equivalence, which we describe below.

For the Hilbert space we should take the bosonic Fock space $\mathcal{H} = \Gamma_{\mathrm{s}}(\mathcal{Z}_{\mathrm{Pr}})$ and for Ω the Fock vacuum. The annihilation operator is denoted by $\hat{a}(k,\sigma)$. The *quantized potentials* are

$$\hat{A}_\mu(x) = (2\pi)^{-\frac{3}{2}} \int \frac{\mathrm{d}\vec{k}}{\sqrt{2\varepsilon(\vec{k})}} \sum_{\sigma=0,\pm1} \Big(u_\mu(k,\sigma)\mathrm{e}^{\mathrm{i}kx}\hat{a}(k,\sigma) + \overline{u_\mu(k,\sigma)}\mathrm{e}^{-\mathrm{i}kx}\hat{a}^*(k,\sigma) \Big)$$

The *quantum Hamiltonian* is

$$\hat{H} = \sum_{\sigma=0,\pm1} \int \varepsilon(\vec{k}) \hat{a}^*(k,\sigma)\hat{a}(k,\sigma) \mathrm{d}\vec{k}.$$

3.3. *Classical theory based on the Klein-Gordon equation*

In an alternative approach to vector fields one considers first the Klein-Gordon equation on functions with values in $\mathbb{R}^{1,3}$:

$$(-\Box + m^2)\zeta_\mu(x) = 0 \tag{8}$$

(without the Lorentz condition). The space of smooth real space-compact solutions of (8) will be denoted by $\mathcal{Y}_{\rm vec}$. We have a symplectic form on $\mathcal{Y}_{\rm vec}$

$$\zeta_1\omega_{\rm vec}\zeta_2 = \int \Big(-\dot{\zeta}_{1\nu}(t,\vec{x})\zeta_2^\nu(t,\vec{x}) + \zeta_{1\nu}(t,\vec{x})\dot{\zeta}_2^\nu(t,\vec{x})\Big)\,{\rm d}\vec{x}.$$

$A^\mu(x)$ is the linear functional on $\mathcal{Y}_{\rm vec}$ given by

$$\langle A^\mu(x)|\zeta\rangle := \zeta^\mu(x).$$

We clearly have

$$(-\Box + m^2)A_\mu(x) = 0. \tag{9}$$

The conjugate variable is $\Pi_\mu(x) := \dot{A}_\mu(x)$. The Poisson structure is given by the equal time brackets

$$\begin{aligned}\{A_\mu(t,\vec{x}), A_\nu(t,\vec{y})\} \;&=\; \{\Pi_\mu(t,\vec{x}), \Pi_\nu(t,\vec{y})\} = 0,\\ &\;\; \{A_\mu(t,\vec{x}), \Pi_\nu(t,\vec{y})\} = g_{\mu\nu}\delta(\vec{x}-\vec{y}).\end{aligned}$$

The Hamiltonian is

$$H = \int \Big(\frac{1}{2}\Pi_\mu(t,\vec{x})\Pi^\mu(t,\vec{x}) + \frac{1}{2}A_{\mu,i}(t,\vec{x})A^{\mu,i}(t,\vec{x}) + \frac{m^2}{2}A_\mu(t,\vec{x})A^\mu(t,\vec{x})\Big){\rm d}\vec{x}.$$

The Hamiltonian is unbounded from below.

Introduce two subspaces of $\mathcal{Y}_{\rm vec}$

$$\begin{aligned}\mathcal{Y}_{\rm Lor} &:= \{\zeta \in \mathcal{Y}_{\rm vec} \;:\; \partial_\mu\zeta^\mu = 0\},\\ \mathcal{Y}_{\rm sc} &:= \{\zeta \in \mathcal{Y}_{\rm vec} \;:\; \zeta^\mu = \partial^\mu\chi, \quad \chi \in \mathcal{Y}_{\rm KG}\}.\end{aligned}$$

Note that $\mathcal{Y}_{\rm vec} = \mathcal{Y}_{\rm Lor} \oplus \mathcal{Y}_{\rm sc}$ is a decomposition into symplectically orthogonal subspaces each preserved by the Poincaré group. If $\zeta \in \mathcal{Y}_{\rm vec}$, then its projection onto $\mathcal{Y}_{\rm sc}$ is

$$\zeta_{\rm sc}^\mu := \frac{1}{m^2}\partial^\mu\partial_\nu\zeta^\nu.$$

Elements of $\mathcal{Y}_{\rm Lor}$ satisfy the Proca equation, so that we can make the identification

$$\mathcal{Y}_{\rm Lor} = \mathcal{Y}_{\rm Pr}.$$

On $\mathcal{Y}_{\rm Lor}$ the forms $\omega_{\rm vec}$ and $\omega_{\rm Pr}$ coincide.

Clearly, we are back with the theory introduced at the beginning. In particular, the Hamiltonian restricted to $\mathcal{Y}_{\rm Lor}$ is now positive.

In order to diagonalize the Hamiltonian, besides the polarization vectors $u(k,\sigma)$ with $\sigma = 0, \pm 1$, we will need the scalar polarization vectors:

$$u(k, \mathrm{sc}) := \frac{1}{m}(\varepsilon(\vec{k}), \vec{k}).$$

The potentials can be decomposed as

$$A_\mu(x) = (2\pi)^{-\frac{3}{2}} \sum_\sigma \int \frac{\mathrm{d}\vec{k}}{\sqrt{2\varepsilon(\vec{k})}} \left(u_\mu(k,\sigma)\mathrm{e}^{\mathrm{i}kx} a(k,\sigma) + \overline{u_\mu(k,\sigma)}\mathrm{e}^{-\mathrm{i}kx} a^*(k,\sigma) \right).$$

Clearly, the restriction to $\mathcal{Y}_{\mathrm{Lor}}$ amounts to dropping all scalar components.

We diagonalize the Hamiltonian

$$H = \sum_{\sigma=0,\pm1} \int \mathrm{d}\vec{k}\varepsilon(\vec{k}) a^*(k,\sigma) a(k,\sigma) - \int \mathrm{d}\vec{k}\varepsilon(\vec{k}) a^*(k,\mathrm{sc}) a(k,\mathrm{sc}).$$

$\mathcal{W}_{\mathrm{vec}}^{(+)}$ will denote the subspace of $\mathbb{C}\mathcal{Y}_{\mathrm{vec}}$ consisting of *positive frequency solutions*:

$$\mathcal{W}_{\mathrm{vec}}^{(+)} := \{ g \in \mathbb{C}\mathcal{Y}_{\mathrm{Pr}} \ : \ a^*(k,\sigma) g = 0, \ \sigma = \pm, 0, \mathrm{sc} \}.$$

For $g_1, g_2 \in \mathcal{W}_{\mathrm{vec}}^{(+)}$ we have a natural scalar product

$$\begin{aligned} (g_1|g_2) &:= \mathrm{i}\overline{g}_1 \omega_{\mathrm{vec}} g_2 \\ &= \sum_{\sigma=0,\pm1} \int \overline{\langle a(k,\sigma)|g_1\rangle} \langle a(k,\sigma)|g_2\rangle \mathrm{d}\vec{k} - \int \overline{\langle a(k,\mathrm{sc})|g_1\rangle} \langle a(k,\mathrm{sc})|g_2\rangle \mathrm{d}\vec{k} \\ &= \int g^{\mu\nu} \overline{\langle a_\mu(k)|g_1\rangle} \langle a_\nu(k)|g_2\rangle \mathrm{d}\vec{k}. \end{aligned}$$

Unfortunately, the above definition gives an indefinite scalar product. We can also introduce a positive definite scalar product, which unfortunately is not covariant:

$$(g_1|g_2)_+ := \sum_\mu \int \overline{\langle a_\mu(k)|g_1\rangle} \langle a_\mu(k)|g_2\rangle \mathrm{d}\vec{k}.$$

The positive frequency space $\mathcal{W}_{\mathrm{vec}}^{(+)}$ completed in the norm given by $(\cdot|\cdot)_+$ will be called $\mathcal{Z}_{\mathrm{vec}}$.

$\mathcal{W}_{\mathrm{vec}}^{(+)}$ can be in the obvious way decomposed into the direct sum of orthogonal subspaces $\mathcal{W}_{\mathrm{Lor}}^{(+)}$ and $\mathcal{W}_{\mathrm{sc}}^{(+)}$. On $\mathcal{W}_{\mathrm{Lor}}^{(+)}$ the scalar product (10) is positive definite, on $\mathcal{W}_{\mathrm{sc}}^{(+)}$ it is negative definite. Their completions will be denoted $\mathcal{Z}_{\mathrm{Lor}}$ and $\mathcal{Z}_{\mathrm{sc}}$.

As usual, any $\zeta \in \mathcal{Y}_{\mathrm{vec}}$ can be projected on $\mathcal{W}_{\mathrm{vec}}^{(+)}$ which allows us to define a real scalar product on $\mathcal{Y}_{\mathrm{vec}}$:

$$\begin{aligned} \langle \zeta_1|\zeta_2\rangle_{\mathcal{Y}} &:= \mathrm{Re}(\zeta_1^{(+)}|\zeta_2^{(+)}) \\ &= \int\int \dot{\zeta}_{1\mu}(0,\vec{x})(-\mathrm{i})D^{(+)}(0,\vec{x}-\vec{y})\dot{\zeta}_2^\mu(0,\vec{y})\mathrm{d}\vec{x}\mathrm{d}\vec{y} \\ &\quad + \int\int \zeta_{1\mu}(0,\vec{x})(-\Delta_{\vec{x}}+m^2)(-\mathrm{i})D^{(+)}(0,\vec{x}-\vec{y})\zeta_2^\mu(0,\vec{y})\mathrm{d}\vec{x}\mathrm{d}\vec{y}. \end{aligned}$$

Again, this scalar product is positive definite on $\mathcal{Y}_{\mathrm{Lor}}$ and negative definite on $\mathcal{Y}_{\mathrm{sc}}$.

3.4. *Quantizations based on the Klein-Gordon equation*

There exist at least two methods of quantization, which use the symplectic space $\mathcal{Y}_{\mathrm{vec}}$ introduced in (8) as the starting point.

(1) The first insists on using only *positive definite Hilbert spaces*. Unfortunately, the Hamiltonian turns out to be unbounded from below.
(2) In the *Gupta-Bleuler approach* the potentials $\hat{A}^\mu(x)$ evolve with positive frequencies. Unfortunately, it uses an indefinite scalar product.

In the quantization on a positive definite Hilbert space we use

$$\Gamma_{\mathrm{s}}(\mathcal{Z}_{\mathrm{Lor}} \oplus \overline{\mathcal{Z}}_{\mathrm{sc}}) \tag{10}$$

equipped with a positive definite scalar product. More explicitly, we replace $a(k,\sigma)$ with $\hat{a}(k,\sigma)$ for $\sigma = 0, \pm 1$. We replace $a(k, \mathrm{sc})$ with $\hat{b}^*(k, \mathrm{sc})$. They satisfy the standard commutation relations

$$[\hat{a}(k,\sigma), \hat{a}^*(k',\sigma')] = \delta_{\sigma,\sigma'}\delta(\vec{k}-\vec{k}'),$$
$$[\hat{b}(k,\mathrm{sc}), \hat{b}^*(k',\mathrm{sc})] = \delta(\vec{k}-\vec{k}').$$

$\hat{a}(k,\sigma)$, $\hat{b}(k,\sigma)$ kill the vacuum:

$$\hat{a}(k,\sigma)\Omega = \hat{b}(k,\mathrm{sc})\Omega = 0.$$

The quantized potentials and Hamiltonian become

$$\hat{A}_\mu(x) = (2\pi)^{-\frac{3}{2}} \sum_{\sigma=0,\pm1} \int \frac{\mathrm{d}\vec{k}}{\sqrt{2\varepsilon(\vec{k})}} \Big(u_\mu(k,\sigma)\mathrm{e}^{\mathrm{i}kx}\hat{a}(k,\sigma) + \overline{u_\mu(k,\sigma)}\mathrm{e}^{-\mathrm{i}kx}\hat{a}^*(k,\sigma)\Big)$$
$$+(2\pi)^{-\frac{3}{2}} \int \frac{\mathrm{d}\vec{k}}{\sqrt{2\varepsilon(\vec{k})}} \Big(u_\mu(k,\mathrm{sc})\mathrm{e}^{\mathrm{i}kx}\hat{b}^*(k,\mathrm{sc}) + \overline{u_\mu(k,\mathrm{sc})}\mathrm{e}^{-\mathrm{i}kx}\hat{b}(k,\mathrm{sc})\Big),$$

$$\hat{H} = \sum_{\sigma=0,\pm1} \int \mathrm{d}\vec{k}\varepsilon(\vec{k})\hat{a}^*(k,\sigma)\hat{a}(k,\sigma) - \int \mathrm{d}\vec{k}\varepsilon(\vec{k})\hat{b}^*(k,\mathrm{sc})\hat{b}(k,\mathrm{sc}).$$

Vectors built by applying fields satisfying the Lorentz condition to the vacuum will be called *physical.* Equivalently, physical vectors are elements of the Fock space built on $\mathcal{W}_{\mathrm{Lor}}^{(+)}$. After the completion the physical space coincides with $\Gamma_{\mathrm{s}}(\mathcal{Z}_{\mathrm{Lor}})$. Thus we obtain the same space as in the method "first reduce, then quantize".

It will be convenient to reformulate this method in the C^*-algebraic language. Let $\mathrm{CCR}(\mathcal{Y}_{\mathrm{vec}})$ denote the *(Weyl) C^*-algebra of the CCR over* $\mathcal{Y}_{\mathrm{vec}}$, that is, the C^*-algebra generated by $W(\zeta)$, $\zeta \in \mathcal{Y}_{\mathrm{vec}}$, such that

$$W(\zeta_1)W(\zeta_2) = \mathrm{e}^{-\mathrm{i}\frac{\zeta_1\omega_{\mathrm{vec}}\zeta_2}{2}}W(\zeta_1+\zeta_2), \quad W(\zeta)^* = W(-\zeta).$$

Choose the state on $\mathrm{CCR}(\mathcal{Y}_{\mathrm{vec}})$ defined by

$$\psi\big(W(\zeta)\big) = \exp\Big(-\frac{1}{2}\langle\zeta_1|\zeta_2\rangle_{\mathcal{Y}} - \frac{1}{m^2}\langle\partial_\mu\zeta^\mu|\partial_\nu\zeta^\nu\rangle_{\mathcal{Y}}\Big) \tag{11}$$

Let $(\mathcal{H}_\psi, \pi_\psi, \Omega_\psi)$ be the GNS representation generated by the state ψ. $\mathcal{H}_\psi$ can be identified with $\Gamma_{\rm s}(\mathcal{Z}_{\rm Lor} \oplus \overline{\mathcal{Z}}_{\rm sc})$ and the fields are related to the Weyl operators by

$$\pi_\psi(W(\zeta)) = \mathrm{e}^{\mathrm{i}\hat{A}(\!(\zeta)\!)}.$$

The Gupta-Bleuler approach also uses the symplectic space $\mathcal{Y}_{\rm vec}$ as the basic input. It follows almost verbatim the usual steps of quantization of the Klein-Gordon equation. We introduce the bosonic Fock space $\Gamma_{\rm s}(\mathcal{Z}_{\rm vec})$, which has an indefinite scalar product and can be viewed as a Krein space.

We replace $a(k,\sigma)$ by $\hat{a}(k,\sigma)$. The commutation relations have a wrong sign for the scalar component:

$$[\hat{a}(k,\sigma), \hat{a}^*(k',\sigma')] = \kappa_{\sigma,\sigma'}\delta(\vec{k}-\vec{k}').$$

The annihillation operators kill the vacuum:

$$\hat{a}(k,\sigma)\Omega = 0.$$

The expressions for the Hamiltonian and potentials are the same as in the classical case:

$$\hat{H} = \sum_{\sigma=0,\pm1} \int \mathrm{d}\vec{k}\varepsilon(\vec{k})\hat{a}^*(k,\sigma)\hat{a}(k,\sigma) - \int \mathrm{d}\vec{k}\varepsilon(\vec{k})\hat{a}^*(k,\mathrm{sc})\hat{a}(k,\mathrm{sc}).$$

$$\hat{A}_\mu(x) = (2\pi)^{-\frac{3}{2}} \sum_\sigma \int \frac{\mathrm{d}\vec{k}}{\sqrt{2\varepsilon(\vec{k})}} \Big(u_\mu(k,\sigma)\mathrm{e}^{\mathrm{i}kx}\hat{a}(k,\sigma) + \overline{u_\mu(k,\sigma)}\mathrm{e}^{-\mathrm{i}kx}\hat{a}^*(k,\sigma)\Big).$$

Similarly as in the previous method, vectors created by applying fields satisfying the Lorentz condition to the vacuum will be called *physical*. On the space of physical vectors the scalar product is positive definite and after the completion coincides with $\Gamma_{\rm s}(\mathcal{Z}_{\rm Lor})$.

4. Massless photons

4.1. *Classical theory based on the Maxwell equation*

If we set $m = 0$ in the Proca equation we obtain

$$-\partial_\mu \left(\partial^\mu \zeta^\nu(x) - \partial^\nu \zeta^\mu(x)\right) = 0, \tag{1}$$

which we will call the *Maxwell equation*. It has to be treated separately from the Proca equation, because of the *gauge invariance*. It is invariant w.r.t. the replacement of ζ_μ with $\zeta_\mu + \partial_\mu \chi$, where χ is an arbitrary smooth function on the space-time. This property poses problems both for the classical and quantum theory.

The space of smooth real space compact solutions of the Maxwell equation is denoted $\mathcal{Y}_{\widetilde{\rm Max}}$. It is equipped with a gauge-invariant presymplectic form:

$$\begin{aligned}&\zeta_1 \omega_{\widetilde{\rm Max}} \zeta_2 \\ &= \int \Big(-\Big(\dot{\vec{\zeta}}_1(t,\vec{x}) - \vec{\partial}\zeta_{10}(t,\vec{x})\Big)\vec{\zeta}_2(t,\vec{x}) + \vec{\zeta}_1(t,\vec{x})\Big(\dot{\vec{\zeta}}_2(t,\vec{x}) - \vec{\partial}\zeta_{10}(t,\vec{x})\Big)\Big)\mathrm{d}\vec{x}.\end{aligned}$$

We say that a solution ζ of the Maxwell equation is in the *Coulomb gauge* if

$$\zeta_0 = 0, \quad \mathrm{div}\vec{\zeta} = 0.$$

Note that every $\zeta \in \mathcal{Y}_{\widetilde{\mathrm{Max}}}$ is gauge-equivalent to a unique solution of the Maxwell equation in the Coulomb gauge, denoted by ζ^{Coul}.

We have three equivalent characterizations of solutions that physically describe the zero configuration, as described in the following easy proposition:

Proposition 4.1. *Let* $\zeta \in \mathcal{Y}_{\widetilde{\mathrm{Max}}}$. *We have the following equivalence:* $\zeta \in \mathrm{Ker}\omega_{\widetilde{\mathrm{Max}}} \;\Leftrightarrow\; \zeta^{\mathrm{Coul}} = 0 \;\Leftrightarrow\; \zeta = \partial\chi$.

Define $\mathcal{Y}_{\mathrm{Max}}$ to be $\mathcal{Y}_{\widetilde{\mathrm{Max}}}$ divided by the gauge equivalence. By Proposition 4.1 $\mathcal{Y}_{\mathrm{Max}}$ coincides with the *symplectic reduction* of $\mathcal{Y}_{\widetilde{\mathrm{Max}}}$:

$$\mathcal{Y}_{\mathrm{Max}} := \mathcal{Y}_{\widetilde{\mathrm{Max}}}/\mathrm{Ker}\omega_{\widetilde{\mathrm{Max}}}.$$

Obviously, it is equipped with a natural *symplectic form* ω_{Max}.

Let $A^\mu(x)$ denote the functional on $\mathcal{Y}_{\widetilde{\mathrm{Max}}}$ given by

$$\langle A^\mu(x)|\zeta\rangle := \zeta^\mu(x). \tag{2}$$

Obviously, $A^\mu(x)$ is not defined on $\mathcal{Y}_{\mathrm{Max}}$.

We introduce also the functional $A^{\mathrm{Coul}}_\mu(x)$ on $\mathcal{Y}_{\widetilde{\mathrm{Max}}}$, called the *classical potential in the Coulomb gauge*,

$$A^{\mathrm{Coul}}_0(x) := 0, \quad \vec{A}^{\mathrm{Coul}}(x) := \vec{A}(x) - \vec{\partial}\Delta^{-1}\mathrm{div}\vec{A}(x).$$

Note that

$$\langle A^{\mathrm{Coul}}_\mu(x)|\zeta\rangle = \langle A_\mu(x)|\zeta^{\mathrm{Coul}}\rangle = \zeta^{\mathrm{Coul}}_\mu(x).$$

$A^{\mathrm{Coul}}(x)$ does not depend on the gauge, hence can be interpreted as a functional on $\mathcal{Y}_{\mathrm{Max}}$. It is not, however, Lorentz covariant.

We also introduce Lorentz covariant functionals $F_{\mu\nu}(x)$ on $\mathcal{Y}_{\widetilde{\mathrm{Max}}}$, called the *fields*:

$$\langle F_{\mu\nu}(x)|\zeta\rangle := \partial_\mu\zeta_\nu(x) - \partial_\nu\zeta_\mu(x).$$

They also do not depend on the gauge, hence can be interpreted as functionals on $\mathcal{Y}_{\mathrm{Max}}$.

We will write $E_i(x) = F_{0i}(x)$. Clearly, $\vec{E} = \partial_t\vec{A}^{\mathrm{Coul}}$ and

$$\mathrm{div}\vec{A}^{\mathrm{Coul}}(x) = 0, \quad \mathrm{div}\vec{E}(x) = 0. \tag{3}$$

In what follows we will usually drop the subscript Coul from $A^{\mathrm{Coul}}(x)$, even though this introduces a possible ambiguity with $A(x)$ acting on $\mathcal{Y}_{\widetilde{\mathrm{Max}}}$.

The symplectic structure on the space $\mathcal{Y}_{\mathrm{Max}}$ leads to a *Poisson bracket* on the level of functions on $\mathcal{Y}_{\mathrm{Max}}$:

$$\begin{aligned}\{A_i(t,\vec{x}), A_j(t,\vec{y})\} = \{E_i(t,\vec{x}), E_j(t,\vec{y})\} = 0,\\ \{A_i(t,\vec{x}), E_j(t,\vec{y})\} = \left(\delta_{ij} - \frac{\partial_i\partial_j}{\Delta}\right)\delta(\vec{x}-\vec{y}).\end{aligned}$$

For $\zeta \in \mathcal{Y}_{\rm Max}$ the corresponding *spatially smeared potential* is a functional on $\mathcal{Y}_{\rm Max}$ given by

$$A(\!(\zeta)\!) = \int \left(-\overline{\dot{\zeta}_\mu(t,\vec{x})}A^\mu(t,\vec{x}) + \overline{\zeta_\mu(t,\vec{x})}E^\mu(t,\vec{x})\right) \mathrm{d}\vec{x}.$$

Note that

$$\{A(\!(\zeta_1)\!), A(\!(\zeta_2)\!)\} = \overline{\zeta}_1\omega\overline{\zeta}_2.$$

$A(\!(\zeta)\!)$ depends on ζ only modulo gauge transformations and is Lorentz covariant. In the mass zero case the plane waves $|k,0)$ and the plane wave functionals $a^*(k,0)$ and $a(k,0)$ are ill defined. However the plane waves $k,\pm1)$ and the corresponding plane wave functionals $a^*(k,\pm1)$ and $a(k,\pm1)$ are well defined and can be used to express the potentials as

$$\begin{aligned} &A_\mu(x) \\ &= (2\pi)^{-\frac{3}{2}} \sum_{\sigma=\pm1} \int \frac{\mathrm{d}\vec{k}}{\sqrt{2\varepsilon(\vec{k})}} \left(u_\mu(x,\sigma)\mathrm{e}^{\mathrm{i}kx}a(k,\sigma) + \overline{u_\mu(x,\sigma)}\mathrm{e}^{-\mathrm{i}kx}a^*(k,\sigma)\right). \end{aligned}$$

Plane wave functionals diagonalize the Hamiltonian and the symplectic form:

$$\begin{aligned} H &= \sum_{\sigma=\pm1} \int \mathrm{d}\vec{k}\varepsilon(\vec{k})a^*(k,\sigma)a(k,\sigma), \\ \mathrm{i}\omega_{\rm Max} &= \sum_{\sigma=\pm1} \int a^*(k,\sigma) \wedge a(k,\sigma)\mathrm{d}\vec{k}. \end{aligned}$$

$\mathcal{W}_{\rm Max}^{(+)}$ will denote the subspace of $\mathbb{C}\mathcal{Y}_{\rm Max}$ consisting of classes of solutions that in the Coulomb gauge have positive frequencies.

For $g_1, g_2 \in \mathcal{W}_{\rm Max}^{(+)}$ we define the scalar product

$$(g_1|g_2) := \mathrm{i}\overline{g}_1\omega_{\rm Max}g_2 = \sum_{\sigma=\pm1} \int \overline{\langle a(k,\sigma)|g_1\rangle}\langle a(k,\sigma)|g_2\rangle \mathrm{d}\vec{k}.$$

The definition of $\mathcal{W}_{\rm Max}^{(+)}$ depends on the choice of coordinates. It is however easy to see that the space $\mathcal{W}_{\rm Max}^{(+)}$ is invariant w.r.t. $\mathbb{R}^{1,3} \rtimes O^\uparrow(1,3)$.

We set $\mathcal{Z}_{\rm Max}$ to be the completion of $\mathcal{W}_{\rm Max}^{(+)}$ in this scalar product.

We can identify $\mathcal{Y}_{\rm Max}$ with $\mathcal{W}_{\rm Max}^{(+)}$ and transport the scalar product onto $\mathcal{Y}_{\rm Max}$, which for ζ_1, ζ_2 is given by

$$\begin{aligned} \langle\zeta_1|\zeta_2\rangle_{\mathcal{Y}} &:= \mathrm{Re}(\zeta_1^{(+)}|\zeta_2^{(+)}) \\ &= \int\int (\dot{\zeta}_{1i}^{\rm Coul}(0,\vec{x})(-\mathrm{i})D^{(+)}(0,\vec{x}-\vec{y})\dot{\zeta}_{2i}^{\rm Coul}(0,\vec{y})\mathrm{d}\vec{x}\mathrm{d}\vec{y} \\ &+ \int\int \zeta_{1i}^{\rm Coul}(0,\vec{x})(-\Delta_{\vec{x}})(-\mathrm{i})D^{(+)}(0,\vec{x}-\vec{y})\zeta_{2i}^{\rm Coul}(0,\vec{y})\mathrm{d}\vec{x}\mathrm{d}\vec{y}. \end{aligned}$$

4.2. *Quantization based on the Maxwell equation*

The quantization of the Maxwell equation is similar to that of the Proca equation described in Subsect. 3.2. Condition (1) is replaced with

$$-\Box\hat{A}_i(x) = 0, \ \ \partial_i\hat{A}_i(x) = 0, \quad \hat{A}_0(x) = 0.$$

Condition (2) is replaced by

$$[\hat{A}_i(0,\vec{x}), A_j(0,\vec{y})] = [\hat{E}_i(0,\vec{x}), \hat{E}_j(0,\vec{y})] = 0,$$
$$[\hat{A}_i(0,\vec{x}), \hat{E}_j(0,\vec{y})] = \mathrm{i}\left(\delta_{ij} - \frac{\partial_i\partial_j}{\Delta}\right)\delta(\vec{x}-\vec{y}).$$

The above problem has a solution unique up to a unitary equivalence. We set $\mathcal{H} := \Gamma_{\mathrm{s}}(\mathcal{Z}_{\mathrm{Max}})$. The annihilation operators are denoted by $\hat{a}(k,\sigma)$. Ω will be the Fock vacuum. We set

$$\hat{A}_{,i}(x) := (2\pi)^{-\frac{3}{2}} \int \frac{\mathrm{d}\vec{k}}{\sqrt{2\varepsilon}} \sum_{\sigma=\pm 1} \Big(u_i(k,\sigma)\mathrm{e}^{\mathrm{i}kx}\hat{a}(k,\sigma) + \overline{u_i(k,\sigma)}\mathrm{e}^{-\mathrm{i}kx}\hat{a}^*(k,\sigma) \Big),$$

The quantum Hamiltonian is

$$\hat{H} := \sum_{\sigma=\pm 1} \int \hat{a}^*(k,\sigma)\hat{a}(k,\sigma)\varepsilon(\vec{k})\mathrm{d}\vec{k}.$$

The group $\mathbb{R}^{1,3} \rtimes O^{\uparrow}(1,3)$ acts on $\mathcal{H}$.

Here is the C^*-algebraic version of the above construction. Let $\mathrm{CCR}(\mathcal{Y}_{\widetilde{\mathrm{Max}}})$ denote the *(Weyl) C^*-algebra of canonical commutation relations* over $\mathcal{Y}_{\widetilde{\mathrm{Max}}}$. By definition, it is generated by $W(\zeta)$, $\zeta \in \mathcal{Y}_{\widetilde{\mathrm{Max}}}$, such that

$$W(\zeta_1)W(\zeta_2) = \mathrm{e}^{-\mathrm{i}\frac{\zeta_1\omega_{\widetilde{\mathrm{Max}}}\zeta_2}{2}} W(\zeta_1+\zeta_2), \quad W(\zeta)^* = W(-\zeta).$$

$\mathbb{R}^{1,3} \rtimes O^{\uparrow}(1,3)$ acts on $\mathrm{CCR}(\mathcal{Y}_{\widetilde{\mathrm{Max}}})$ by $*$-automorphisms defined by

$$\hat{r}_{(a,\Lambda)}\left(W(\zeta)\right) := W\left(r_{(a,\Lambda)}(\zeta)\right).$$

We are looking for a cyclic representation of this algebra with the time evolution generated by a positive Hamiltonian.

Consider the state on $\mathrm{CCR}(\mathcal{Y}_{\widetilde{\mathrm{Max}}})$ defined for $\zeta \in \mathcal{Y}_{\widetilde{\mathrm{Max}}}$ by

$$\psi\big(W(\zeta)\big) = \exp\big(-\frac{1}{2}\langle\zeta|\zeta\rangle_{\mathcal{Y}}\big).$$

Note that the state is gauge and Poincare invariant. Let $(\mathcal{H}_\psi, \pi_\psi, \Omega_\psi)$ be the GNS representation. $\mathcal{H}_\psi$ is naturally isomorphic to $\Gamma_{\mathrm{s}}(\mathcal{Z}_{\mathrm{Max}})$. Ω_ψ can be identified with the vector Ω. $\pi_\psi(W(\zeta))$ can be identified with $\mathrm{e}^{\mathrm{i}\hat{A}(\!(\zeta)\!)}$. In particular, if ζ_1 and ζ_2 are gauge equivalent, then $\hat{A}(\!(\zeta_1)\!) = \hat{A}(\!(\zeta_2)\!)$. However, $\hat{A}(x)$ (in the original sense, not in the Coulomb gauge) is not well defined.

4.3. *Classical theory based on the d'Alembert equation*

So far, our treatment of the massless vector fields was based on the Coulomb gauge, which depends on the choice of the temporal coordinate. One can ask whether massless vector fields can be studied in a manifestly covariant fashion.

Let Ξ be an arbitrary space-time function. The Maxwell equation allow us to impose a *generalized Lorentz condition*

$$\partial_\mu A^\mu = \Xi. \tag{4}$$

The Maxwell equation together with (4) imply

$$-\Box A^\mu = \partial^\mu \Xi. \tag{5}$$

The function Ξ has no physical meaning. Therefore it is natural to adopt the simplest choice $\Xi = 0$, that is the usual *Lorentz condition*, for which (5) reduces to the d'Alembert's equation for vector valued functions

$$-\Box A^\mu = 0.$$

We will discuss this approach in what follows.

Recall that the Proca equation is equivalent to the Klein-Gordon equation for vector fields together with the Lorentz condition. Therefore, one can first develop its theory on the symplectic space $\mathcal{Y}_{\rm vec}$, and then reduce it to the subpace $\mathcal{Y}_{\rm Lor}$, as described before.

One can follow a similar route for the Maxwell equation. However, there is a difference: the reduction by the Lorentz condition is insufficient, one has to make an additional reduction.

Anyway, let us start by introducing the space $\mathcal{Y}_{\rm vec}$, the form $\omega_{\rm vec}$, the subspace $\mathcal{Y}_{\rm Lor}$ and the potentials $A_\mu(x)$, as for the Proca equation except that now $m = 0$.

In the massive case $\mathcal{Y}_{\rm Lor}$ was symplectic (that means, the form $\omega_{\rm vec}$ restricted to $\mathcal{Y}_{\rm Lor}$ was nondegenerate). This is no longer true in the massless case. Instead, $\mathcal{Y}_{\rm Lor}$ is coisotropic. (That means, if ζ is symplectically orthogonal to $\mathcal{Y}_{\rm Lor}$, then $\zeta \in \mathcal{Y}_{\rm Lor}$).

$\mathcal{Y}_{\rm Lor}$ is a subspace of $\mathcal{Y}_{\widetilde{\rm Max}}$ and on $\mathcal{Y}_{\rm Lor}$ the forms $\omega_{\widetilde{\rm Max}}$ and $\omega_{\rm vec}$ coincide. The following fact is easy to show:

Proposition 4.2. *Any $\zeta \in \mathcal{Y}_{\widetilde{\rm Max}}$ is gauge equivalent to an element of $\mathcal{Y}_{\rm Lor}$.*

Therefore, the symplectically reduced $\mathcal{Y}_{\rm Lor}$ coincides with the symplectically reduced $\mathcal{Y}_{\widetilde{\rm Max}}$, that is, with $\mathcal{Y}_{\rm Max}$. This shows that both approaches to the Maxwell equation are equivalent.

$\mathcal{W}_{\rm Lor}^{(+)}$ will denote the subspace of $\mathbb{C}\mathcal{Y}_{\rm Lor}$ consisting of solutions that have positive frequencies.

For $g_1, g_2 \in \mathcal{W}_{\rm Lor}^{(+)}$ we define the scalar product

$$\begin{aligned}(g_1|g_2) &:= \mathrm{i}\overline{g}_1 \omega_{\rm vec} g_2 \\ &= \mathrm{i}\overline{g}_1^{\rm Coul} \omega_{\rm vec} g_2^{\rm Coul}.\end{aligned} \tag{6}$$

Note that the definition of (6) does not depend on the choice of coordinates and is invariant wrt. the group $\mathbb{R}^{1,3} \rtimes O^{\uparrow}(1,3)$.

The scalar product is positive semidefinite, but not strictly positive definite. Let $\mathcal{W}_{\mathrm{Lor},0}^{(+)}$ be the subspace of elements $\mathcal{W}_{\mathrm{Lor}}^{(+)}$ with a zero norm. $\mathcal{W}_{\mathrm{Lor},0}^{(+)}$ consists of pure gauges. The space $\mathcal{W}_{\mathrm{Lor}}^{(+)}/\mathcal{W}_{\mathrm{Lor},0}^{(+)}$ has a nondegenerate scalar product. Its completion is naturally isomorphic to the space $\mathcal{Z}_{\mathrm{Max}}$.

We have a natural identification of $\mathcal{Y}_{\mathrm{Lor}}$ with $\mathcal{W}_{\mathrm{Lor}}^{(+)}$ given by the obvious projection. For $\zeta \in \mathcal{Y}_{\mathrm{Lor}}$ we will denote by $\zeta^{(+)}$ the corresponding element of $\mathcal{W}_{\mathrm{Lor}}^{(+)}$. This identification allows us to define a positive semidefinite scalar product on $\mathcal{Y}_{\mathrm{Lor}}$:

$$\begin{aligned}\langle \zeta_1|\zeta_2\rangle_{\mathcal{Y}} &:= \mathrm{Re}(\zeta_1^{(+)}|\zeta_2^{(+)}) \\ &= \int\int \dot{\zeta}_{1i}^{\mathrm{Coul}}(0,\vec{x})(-\mathrm{i})D^{(+)}(0,\vec{x}-\vec{y})\dot{\zeta}_{2i}^{\mathrm{Coul}}(0,\vec{y})\mathrm{d}\vec{x}\mathrm{d}\vec{y} \\ &+ \int\int \zeta_{1i}^{\mathrm{Coul}}(0,\vec{x})(-\Delta_{\vec{x}})(-\mathrm{i})D^{(+)}(0,\vec{x}-\vec{y})\zeta_{2i}^{\mathrm{Coul}}(0,\vec{y})\mathrm{d}\vec{x}\mathrm{d}\vec{y}.\end{aligned}$$

4.4. *Quantizations based on the d'Alembert equation*

One can try to use the symplectic space $\mathcal{Y}_{\mathrm{vec}}$ of real vector valued solutions of the Klein-Gordon equation as the basis for quantization. In the literature, this starting point is employed by two approaches.

(1) The first, which we call the *approach with a subsidiary condition* has the advantage that it uses only positive definite Hilbert spaces. Unfortunately, in this approach there are problems with the potential $\hat{A}^{\mu}(x)$. Besides, the full Hilbert space turns out to be non-separable.
(2) In the *Gupta-Bleuler approach* the potentials $\hat{A}^{\mu}(x)$ are well defined and covariant. Unfortunately it uses indefinite scalar product spaces.

Let us start to describe the first approach. It is convenient to use the C^*-algebraic formulation described in Subsect. 3.4. In particular, $\mathrm{CCR}(\mathcal{Y}_{\mathrm{vec}})$, the *(Weyl) C^*-algebra of canonical commutation relations* over $\mathcal{Y}_{\mathrm{vec}}$, is well defined also for $m = 0$ and is invariant wrt the Poincaré group.

Strictly speaking, the spaces $\mathcal{Y}_{\mathrm{vec}}$ and hence the algebras $\mathrm{CCR}(\mathcal{Y}_{\mathrm{vec}})$ are different for various m. We can identify them by using the initial conditions at $t = 0$.

Recall that in the massive case

$$(\Omega|\hat{A}(\!(\zeta)\!)^2\Omega) = \langle\zeta|\zeta\rangle_{\mathcal{Y}} + \frac{2}{m^2}\langle\partial_\mu\zeta^\mu|\partial_\nu\zeta^\nu\rangle_{\mathcal{Y}}. \tag{7}$$

Clearly, if $m \searrow 0$, (7) is divergent to $+\infty$ for $\zeta \notin \mathcal{Y}_{\mathrm{Lor}}$ and equals $\langle\zeta|\zeta\rangle_{\mathcal{Y}}$ for $\zeta \in \mathcal{Y}_{\mathrm{Lor}}$. So, the following state on $\mathrm{CCR}(\mathcal{Y}_{\mathrm{vec}})$ is the limit of the state that was considered for the Proca equation:

$$\psi\big(W(\zeta)\big) = \begin{cases}\exp\Big(-\frac{1}{2}\langle\zeta|\zeta\rangle_{\mathcal{Y}}\Big), & \zeta \in \mathcal{Y}_{\mathrm{Lor}}, \\ 0, & \zeta \notin \mathcal{Y}_{\mathrm{Lor}}.\end{cases}$$

Let $(\mathcal{H}_\psi, \pi_\psi, \Omega_\psi)$ denote the *GNS representation* for this state. We have an identification

$$J : \mathcal{H}_\psi \to l^2\left(\mathcal{Y}_{\mathrm{vec}}/\mathcal{Y}_{\mathrm{Lor}}, \Gamma_{\mathrm{s}}(\mathcal{Z}_{\mathrm{Max}})\right). \tag{8}$$

To describe this identification, first note that $\mathcal{Y}_{\mathrm{vec}}/\mathcal{Y}_{\mathrm{Lor}}$ can be parametrized by smooth space-compact functions

$$\Xi = \partial_\mu \zeta^\mu,$$

which can be called the *values of the Lorentz condition.* For each Ξ choose $\zeta_\Xi \in \mathcal{Y}_{\mathrm{vec}}$ such that $\partial_\mu \zeta_\Xi^\mu = \Xi$. We demand that

$$\Big(J\pi_\psi\big(W(\zeta_\Xi)\big)\Omega_\psi\Big)(\Xi) = \begin{cases} \Omega, & \partial_\mu\zeta^\mu = \Xi, \\ 0, & \partial_\mu\zeta^\mu \neq \Xi. \end{cases}$$

Then J is given by

$$\left(J\pi_\psi\big(W(\zeta)\big)\Omega_\psi\right)(\Xi) = \begin{cases} \mathrm{e}^{\frac{\mathrm{i}}{2}\zeta\omega_{\mathrm{vec}}\zeta_\Xi}\mathrm{e}^{\mathrm{i}\hat{A}(\!(\zeta-\zeta_\Xi)\!)}\Omega, & \partial_\mu\zeta^\mu = \Xi, \\ 0, & \partial_\mu\zeta^\mu \neq \Xi. \end{cases}$$

Note that $\mathcal{H}_\psi$ is *non-separable* – it is a direct sum of *superselection sectors* corresponding to various values of the *Lorentz condition.*

Special role is played by the (separable) subspace corresponding to the Lorentz condition $\partial_\mu A^\mu(x) = 0$. On this subspace, the fields are equivalent to the usual quantization obtained by the method "first reduce, then quantize".

Note that $\pi_\psi(W(\zeta))$ maps between various sectors of (8) if $\partial_\mu\zeta^\mu \neq 0$. The unitary group $\mathbb{R} \ni t \mapsto \pi_\psi\left(W(t\zeta)\right)$ is strongly continuous if and only if $\partial_\mu\zeta^\mu = 0$. If this is the case, we can write $\pi_\psi(W(\zeta)) = \mathrm{e}^{\mathrm{i}\hat{A}(\!(\zeta)\!)}$. We have $\hat{A}(\!(\zeta_1)\!) = \hat{A}(\!(\zeta_2)\!)$ if in addition ζ_1 differs from ζ_2 by a pure gauge. $\hat{A}(\!(\zeta)\!)$ is ill defined if $\partial_\mu\zeta^\mu \neq 0$.

The approach that we described above, restricted to the 0th sector was typical for older presentations, eg. Jauch and Rohrlich.[9] However, without the language of C^*-algebras it is somewhat awkward to describe. One usually says that the Lorentz condition $\partial_\mu\hat{A}^\mu(x) = 0$ is enforced on the Hilbert space of states and constitutes a *subsidiary condition.*

The *Gupta-Bleuler* approach follows the same lines as in the massive case until we arrive at the algebraic Fock space built on $\mathcal{W}_{\mathrm{Lor}}^{(+)}$. As we know, the scalar product on $\mathcal{W}_{\mathrm{Lor}}^{(+)}$ is only semidefinite. We factor $\mathcal{W}_{\mathrm{Lor}}^{(+)}$ by the null space of its scalar product, obtaining $\mathcal{W}_{\mathrm{Max}}^{(+)}$. We complete it, obtaining $\mathcal{Z}_{\mathrm{Max}}$ and we take the corresponding Fock space $\Gamma_{\mathrm{s}}(\mathcal{Z}_{\mathrm{Max}})$ – thus the resulting physical theory is the same as with the other quantizations.

5. The $m \to 0$ limit

In order to understand the $m \to 0$ limit let us consider quantum vector field interacting with an external *conserved current*, that is a function $\mathbb{R}^{1,3} \ni x \mapsto J^\mu(x)$

satisfying

$$\partial_\mu J^\mu(x) = 0.$$

We assume that J^μ is a *Schwarz function on the space-time*. (Massive and massless) quantum fields interacing with J^μ satisfy the equation

$$-\partial^\mu\big(\partial_\mu \hat{A}_\nu(x) - \partial_\nu \hat{A}_\mu(x)\big) + m^2 \hat{A}_\nu(x) = J_\nu(x). \tag{9}$$

The corresponding *scattering operator* can be computed exactly. In particular, for a positive mass it is

$$\begin{aligned}
\hat{S} = \exp&\left(\frac{\mathrm{i}}{2}\int \mathrm{d}k \overline{J^i(k)} \frac{1}{m^2 + k^2 - \mathrm{i}0}\left(g_{ij} - \frac{k_i k_j}{m^2 + \vec{k}^2}\right) J^i(k)\right.\\
&\left.-\frac{\mathrm{i}}{2}\int \mathrm{d}k \frac{1}{\vec{k}^2 + m^2}|J^0(k)|^2\right)\\
&\times \exp\left(-\mathrm{i}\sum_{\sigma=0,\pm1}\int \mathrm{d}\vec{k} a^*(k,\sigma)\frac{\overline{u_\mu(k,\sigma)}}{\sqrt{2\varepsilon(\vec{k})}} J^\mu(k)\right)\\
&\times \exp\left(-\mathrm{i}\sum_{\sigma=0,\pm1}\int \mathrm{d}\vec{k} a(k,\sigma)\frac{u_\mu(k,\sigma)}{\sqrt{2\varepsilon(\vec{k})}} \overline{J^\mu(k)}\right)\\
= \hat{S}_{\mathrm{tr}} &\otimes \hat{S}_{\mathrm{lg}},
\end{aligned}$$

($J^\mu(k)$ denotes the Fourier transform of $J^\mu(x)$). Here the *transversal scattering operator* is

$$\begin{aligned}
\hat{S}_{\mathrm{tr}} = \exp&\left(\frac{\mathrm{i}}{2}\int \mathrm{d}k \overline{J^i(k)} \frac{1}{m^2 + k^2 - \mathrm{i}0}\left(g_{ij} - \frac{k_i k_j}{k^2}\right) J^j(k)\right.\\
&\left.-\frac{\mathrm{i}}{2}\int \mathrm{d}k \frac{1}{\vec{k}^2 + m^2}|J^0(k)|^2\right)\\
&\times \exp\left(-\mathrm{i}\sum_{\sigma=\pm1}\int \mathrm{d}\vec{k} a^*(k,\sigma)\frac{\overline{u_\mu(k,\sigma)}}{\sqrt{2\varepsilon(\vec{k})}} J^\mu(k)\right)\\
&\times \exp\left(-\mathrm{i}\sum_{\sigma=\pm1}\int \mathrm{d}\vec{k} a(k,\sigma)\frac{u_\mu(k,\sigma)}{\sqrt{2\varepsilon(\vec{k})}} \overline{J^\mu(k)}\right)
\end{aligned}$$

and converges as $m \searrow 0$ to the scattering operator for the massless theory in the Coulomb gauge. The *longitudinal scattering operator* is

$$\begin{aligned}\hat{S}_{\mathrm{lg}} = &\exp\left(\frac{\mathrm{i}}{2}m^2 \int \mathrm{d}k \overline{J^i(k)} \frac{1}{m^2 + k^2 - \mathrm{i}0} \frac{k_i k_j}{(m^2 + \vec{k}^2)\vec{k}^2} J^j(k)\right) \\ &\times \exp\left(-\mathrm{i} \int \mathrm{d}\vec{k} a^*(k,0) \frac{\overline{u_\mu(k,0)}}{\sqrt{2\varepsilon(\vec{k})}} J^\mu(k)\right) \\ &\times \exp\left(-\mathrm{i} \int \mathrm{d}\vec{k} a(k,0) \frac{u_\mu(k,0)}{\sqrt{2\varepsilon(\vec{k})}} \overline{J^\mu(k)}\right).\end{aligned}$$

This can be rewritten as

$$\begin{aligned}\hat{S}_{\mathrm{lg}} = &\exp\left(\frac{\mathrm{i}}{2}m^2 \int \mathrm{d}k \frac{|\vec{J} \cdot \vec{k}|^2}{(m^2 + k^2)(m^2 + \vec{k}^2)\vec{k}^2}\right) \\ &\times \exp\left(-\frac{1}{2} \int \mathrm{d}\vec{k} \frac{m^2 |J^0(k)|^2}{2\varepsilon(\vec{k})\vec{k}^2}\right) \\ &\times \exp\left(\mathrm{i} \int \mathrm{d}\vec{k} a^*(k,0) \frac{m J^0(k)}{|\vec{k}|\sqrt{2\varepsilon(\vec{k})}}\right) \\ &\times \exp\left(\mathrm{i} \int \mathrm{d}\vec{k} a(k,0) \frac{m \overline{J^0(k)}}{|\vec{k}|\sqrt{2\varepsilon(\vec{k})}}\right),\end{aligned}$$

where the integral on the first line should be understood as the principal value. Thus $\hat{S}_{\mathrm{lg}}$, under rather general circumstances, converges to $\mathbb{1}$ as $m \searrow 0$.

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