Phase-Controlled Collapse and Revival of Entanglement of Two Interacting Qubits

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We demonstrate the strong dependence of the entanglement dynamics of two distinguishable qubits in a trap on the relative phase of the pulses used for excitation. We show that the population and entanglement exhibits collapses and full revivals when the initial distribution of phonons is a coherent state.

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One of the most promising systems to build a quantum computer is based on trapped ions [1,2]. Recently, several schemes of coherent manipulation of the quantum states of trapped ions have been developed [3–6]. Here we consider a scheme for creating and controlling entanglement of two qubits in a linear trap, or in other words, two two-level quantum systems coupled to a harmonic bath. In our setup, the two-qubit system must have independently addressable transitions. There are two different strategies to create the entanglement: by individually addressing each system or by means of simultaneous indistinguishable excitation. Both cases were found promising, since even hot ions were shown to be useful for quantum computation [7–9], owing to the independence of the effective coupling on the vibrational (phonon) quantum number.

The basic system underlying the two-qubit manipulation involves a four-level system in closed-loop configuration, shown in Fig. 1. When each coupling can be addressed independently, new forms of control are possible. Recently, we have shown that the relative phase between the pulses can be used to control population dynamics as well as to prepare entangled states [10], by virtue of quantum interference between two pathways connecting the initial and target states [11]. Obviously, population dynamics and entanglement depend on many parameters of the system. In most experimental setups (for instance, trapped ions) the system is addressed by means of fully overlapping cw fields, so that the Rabi flopping depends only on the pulse area. Then, it is still possible to gain a higher finesse in the manipulation of the quantum system by introducing an externally controllable relative phase. In this work we show that the relative phase between the pulses has far more important influence on the population dynamics and entangled state manipulation when the qubits are coupled to a harmonic trap. For properly chosen relative phases one can observe either Rabi oscillations according to the Mølmer-Sørensen scheme [7–9] (the relative phase is zero) or collapse and revival phenomena, as in the well-known Jaynes-Cummings model [12,13] (the phase is not equal to zero). Additionally, the phase could be used to control the time of operation of quantum gates.

Let us consider the dynamics of two distinguishable qubits in a one-dimensional harmonic trap. We assume that the two additional degrees of freedom are suppressed, and we neglect decoherence effects caused, for instance, by spontaneous decay. The collective motion of, e.g., atoms or ions will be defined by an effective harmonic trap potential, with the Hamiltonian

$$\hat{H}_0 = \nu\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right) + \sum_i \frac{E_2^{(i)}}{2}(I - \sigma_{zi}) + J\sigma_{z1}\sigma_{z2}, \quad (1)$$

where ν is the frequency of the vibrational motion, $E_2^{(i)}$ is the transition energy in the i qubit (for instance, the excited internal state of the ions or atoms), σ_{zi} are Pauli matrices, and $\hat{a}^\dagger$, $\hat{a}$ are the vibrational ladder operators (all parameters in atomic units, $\hbar = 1$). We allow here interaction between the qubits, which in a simple case can be treated as an effective spin-spin coupling Hamiltonian, where J is the coupling constant.

The interaction of the qubits with the external fields can be written in the following form: $V_i = -\sum_{j,i}\Omega_j(t) \times \cos[\omega_j t + \phi_j - \eta_j(\hat{a}^\dagger + \hat{a})]\sigma_{xi} + \text{H.c.}$, where ω_j, ϕ_j are the laser frequency and phase, $\Omega_j(t)$ is the Rabi frequency [14], $\eta_j = k_j/\sqrt{4m\omega_t}$ is the Lamb-Dicke parame-

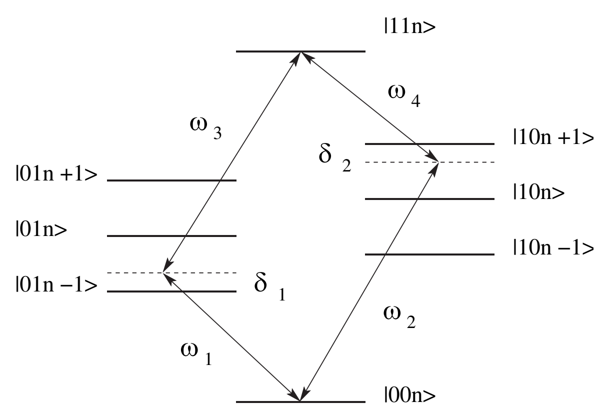

FIG. 1. Schematic of a four-level system for two qubits with distinguishable interactions in a linear trap.

ter, ω_t is the trap frequency, m is the ion mass, and k_j is the laser wave vector.

Owing to the interaction between the qubits, the transition frequencies of the four-level system are different in general, depending on the specific system realization. As particular examples, the interaction between qubits gives rise to blockade effects known as dipole blockade in atomic systems [15] and it also takes place in semiconductor quantum dots [16]. In a very general approach, we consider here excitation of the trapped two-qubit system by four off-resonant fields driving the following transitions: $|00n\rangle \overset{\Omega_1(t)}{\longleftrightarrow} |01n-1\rangle \overset{\Omega_3(t)}{\longleftrightarrow} |11n\rangle$ and $|00n\rangle \overset{\Omega_2(t)}{\longleftrightarrow} |01n+1\rangle \overset{\Omega_4(t)}{\longleftrightarrow} |11n\rangle$ (Fig. 1), where "0" or "1" denotes the qubit state and n is the vibrational quantum number. The Hamiltonian for the total wave function $|\psi\rangle = a_1|00n\rangle + a_2|11n\rangle + b_1|01n-1\rangle + b_2|10n+1\rangle$ in the rotating wave approximation has the form

$$H = -\frac{1}{2}\begin{pmatrix} 0 & \Omega_{1,n} & \Omega_{2,n+1} & 0 \\ \Omega_{1,n}^* & -2\delta_1 & 0 & \Omega_{3,n} \\ \Omega_{2,n+1}^* & 0 & -2\delta_2 & \Omega_{4,n+1} \\ 0 & \Omega_{3,n}^* & \Omega_{4,n+1}^* & 0 \end{pmatrix}, \qquad (2)$$

where $\Omega_{i,n} = \eta_i \Omega_{i,0}(t) e^{i\phi_i}\sqrt{n}$, and $\delta_{1,2}$ are the detunings including energy level shifts due to spin-spin interaction.

By choosing the phases $b_1 \to b_1 e^{-i\phi_1}$, $b_2 \to b_2 e^{-i\phi_2}$, and $a_2 \to a_2 e^{-i(\phi_1+\phi_3)}$, after adiabatic elimination (off-resonant excitation) of the b_i amplitudes, we obtain the following equation in the case of completely overlapped pulses, $\Omega_{i,0}(t) = \Omega_0(t)$:

$$H = -\frac{\eta^2\Omega_0^2(t)}{4}\begin{pmatrix} \frac{n}{\delta_1} + \frac{n+1}{\delta_2} & \frac{n}{\delta_1} + \frac{n+1}{\delta_2}e^{i\phi} \\ \frac{n}{\delta_1} + \frac{n+1}{\delta_2}e^{-i\phi} & \frac{n}{\delta_1} + \frac{n+1}{\delta_2} \end{pmatrix}, \qquad (3)$$

where $\phi = \phi_4 + \phi_2 - \phi_1 - \phi_3$ is the effective phase difference between the two distinct two-photon couplings, and we assume $\eta_1 \approx \eta_2 = \eta$.

Finally, choosing detunings as $\delta_2 = -\delta_1 = \delta_0$ we obtain

$$H = -\frac{\eta^2\Omega_0^2(t)}{4\delta_0}\begin{pmatrix} 1 & i\alpha^* \\ -i\alpha & 1 \end{pmatrix}, \qquad (4)$$

where $\alpha = [(2n+1)\sin(\phi/2) + i\cos(\phi/2)]$.

In Eq. (4) the ac Stark shifts do not depend on the vibrational quantum number, but the effective Rabi frequency is still a function of n. At $\phi = 0$, Eq. (4) reproduces the well-known Mølmer-Sørensen Hamiltonian of trapped ions [7–9], while at $\phi = \pi$ the effective Rabi frequency is linearly proportional to $2n+1$. The coupling between states $|00n\rangle$ and $|11n\rangle$ depends on the relative phase, ϕ. Only at $\phi = 0$ the coupling between the internal states does not depend on the motional states. As a result, one observes Rabi oscillations between the ground and the excited electronic states even if the motional state is not a single Fock state.

The solution of the Schrödinger equation with the Hamiltonian of Eq. (4) for arbitrary phase is $a_1 = \cos[\varepsilon_n S(t)]$ and $a_2 = \alpha\sin[\varepsilon_n S(t)]/\varepsilon_n$, where $S(t) = \eta^2\int_0^t dt'\Omega_0^2(t')/4\delta_0$ and $\varepsilon_n = [1 + 4n(n+1)\sin^2(\phi/2)]^{1/2}$.

To demonstrate the effect of the relative phase we now consider the cw regime, that is, when $\Omega_0(t) = \Omega_0$ is time independent. In general, when the initial state of the phonons is not a single Fock state, one has to average the results over the corresponding state distribution. Here we consider two particular situations: coherent and thermal state of the phonons. Averaging using the coherent state distribution, $P_c(n) = e^{-\bar{n}}\bar{n}^n/n!$, where $\bar{n}$ is the average number of phonons, we obtain for the population inversion

$$W(t) = \sum_{n=0}^{\infty} P_c(n)(|a_1|^2 - |a_2|^2) = \sum_{n=0}^{\infty} P_c(n)\cos[2gt\varepsilon_n], \qquad (5)$$

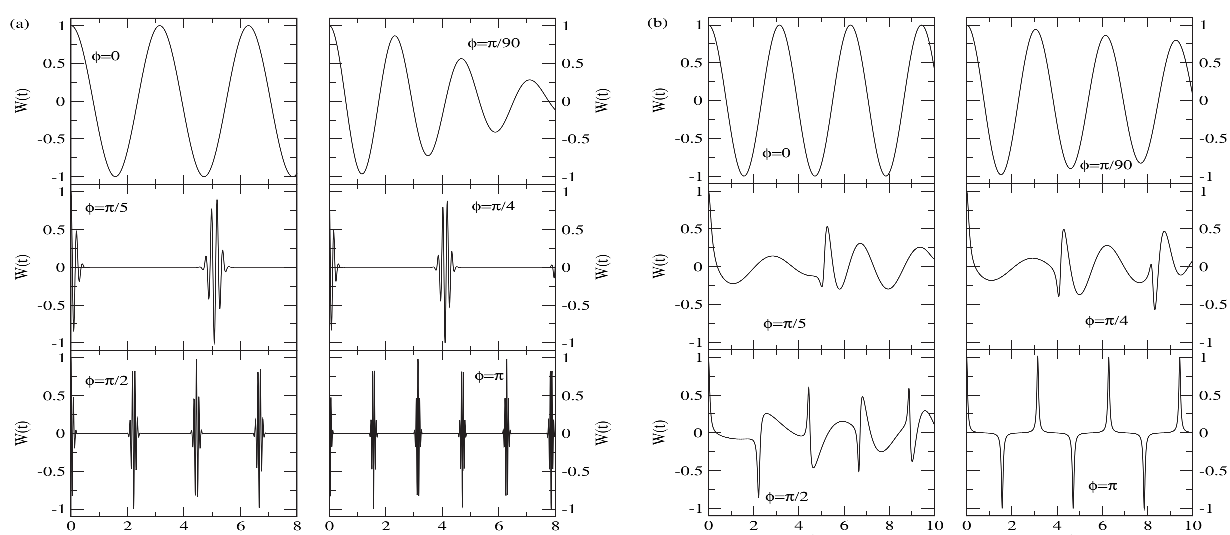

FIG. 2. Population inversion dynamics, $W(t)$, of the qubits at various relative phases, ϕ: (a) for the initial coherent state of phonons with average number $\bar{n} = 25$; (b) for the initial thermal distribution of phonons with average number $\bar{n} = 5$.

where $g = \eta^2\Omega_0^2/4\delta_0$. It can be seen that at $\phi = 0$ the dynamics of the system does not depend on the vibrational quantum number, as it was shown in Refs. [7–9], and we observe simple Rabi oscillations with frequency defined by $2g$. However, in general the Rabi frequencies depend on the vibrational quantum number n, $G_n = 2g\varepsilon_n$.

To our knowledge, there is no general analytic solution for the summations in Eq. (5). However, in the limit of $\bar{n} \gg 1$ the summation can be done exactly, and we obtain the analytic form for population inversion $W(t) = e^{-2\bar{n}\sin^2(\tau/2)}\cos(\bar{n}\sin\tau)$, where $\tau = 4gt\sin(\phi/2)$. The envelope function, $e^{-2\bar{n}\sin^2(\tau/2)}$, shows that all revivals in this model imply the exact regeneration of the initial value, that is, full revivals.

Using the analytic expressions for the probability amplitudes [Eq. (5)], we estimate the time period of the Rabi oscillations t_R, the collapse time t_c, and the interval between revivals t_r [17]. t_R is defined by the inverse of the Rabi frequency at $n = \bar{n}$, $G_{\bar{n}} = 2g\varepsilon_{\bar{n}}$. Therefore, we obtain $t_R \sim G_{\bar{n}}^{-1} = [2g\varepsilon_{\bar{n}}]^{-1}$. In the limit $\bar{n} \gg 1$, one obtains $t_R \sim 1/4g\bar{n}\sin(\phi/2)$.

The Rabi oscillations take place until a collapse time, when the oscillations related to different vibrational states become uncorrelated. Since the root-mean-square deviation for the coherent state $\langle\Delta n\rangle$ is equal to $\sqrt{\bar{n}}$, we estimate the collapse time using the condition $t_c(G_{\bar{n}+\sqrt{\bar{n}}} - G_{\bar{n}-\sqrt{\bar{n}}}) \sim 1$. Finally, for $\bar{n} \gg 1$, we obtain $t_c = [8g\sqrt{\bar{n}}\sin(\phi/2)]^{-1}$.

The revival of the oscillations takes place when the phases of the neighboring n's differ by 2π. Using $G_{\bar{n}}$ we find for the time interval between revivals $t_r = 2\pi m/[G_{\bar{n}} - G_{\bar{n}-1}] \sim \pi m/[2g\sin(\phi/2)]$, where $m = 1, 2, \ldots$. Figure 2(a) shows the population dynamics for different values of the relative phase after averaging over the coherent state distribution. The results are in perfect agreement with our estimates above.

In the case of a thermal distribution of the phonons, we have to perform the averaging over the one-mode Bose-Einstein distribution, $P_t(n) = \bar{n}^n/(1+\bar{n})^{n+1}$, where $\bar{n}$ is the average number of phonons. The analysis follows as previously, although the delocalization of the thermal distribution in terms of vibrational quantum number n (the main difference between coherent and thermal states) makes it impossible, in general, to apply any suitable approximation, even in the limit of $\bar{n} \gg 1$.

Figure 2(b) shows the population dynamics for different values of the relative phase after averaging over the thermal distribution. Here again, we observe Rabi oscillations with the frequency $2g$ at $\phi = 0$, but the population dynamics becomes absolutely uncorrelated at $\phi \neq 0$ on a longer time scale. It is interesting to see that as $\phi \to \pi$, the periodicity of the population dynamics (with frequency $2g$) is again restored. The time between peaks does not depend on the average number of phonons, and we observe perfect revival of the envelope function. However, the width of the peaks decreases when the average number of phonons increases.

To quantify the degree of entanglement, we construct the density matrix ρ and calculate the concurrence $C(t)$ [18–22]. According to the general expression for the case of two qubits A and B, $C(t) = \max(0, \sqrt{\lambda_1} - \sqrt{\lambda_2} - \sqrt{\lambda_3} - \sqrt{\lambda_4})$, where λ_i are the eigenvalues of the matrix $\varrho = \rho(\sigma_y^A \otimes \sigma_y^B)\rho^*(\sigma_y^A \otimes \sigma_y^B)$.

Figure 3 shows the dynamics of entanglement in the two-qubit system at various values of the relative phase. For coherent states the concurrence fully revives at any value of the relative phase [Fig. 3(a)]. The relative phase controls the revival time and the width of the reviving comb. The collapse of population inversion correlates

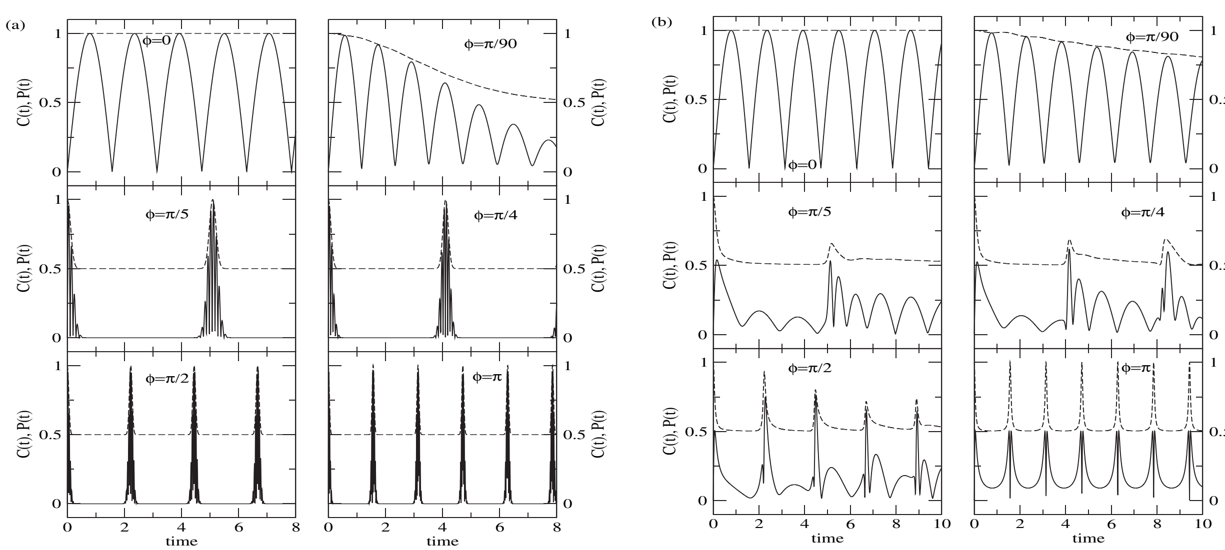

FIG. 3. Concurrence $C(t)$ (solid line) and Renyi entropy $P(t)$ (dashed line) in a system of two trapped qubits as a function of time at various relative phases, ϕ: (a) for the initial coherent state of phonons with average number $\bar{n} = 25$; (b) for the initial thermal distribution of phonons with average number $\bar{n} = 5$.

with the collapse of the concurrence, clearly revealing the decoherence of the system.

The degree of entanglement for the case of the thermal distribution behaves as in the coherent state at $\phi = 0$, or at very small values of the relative phase [compare Figs. 3(a) and 3(b)]. However, for larger ϕ the lack of correlation in the population dynamics is reflected in the concurrence. In this case, the correlation between the population inversion dynamics and the concurrence is less obvious than previously. In fact, the concurrence never reaches unity in contrast to the population inversion and the Renyi entropy, $P(t) = \mathrm{Tr}[\rho^2]$, which can be used as a measure of system purity. For the case of coherent states the Renyi entropy revives completely [see Fig. 3(a)], in agreement with our estimations. An interesting fact is that even for the thermal distribution at $\phi = \pi$ the entropy revives completely; that is, the system "becomes" pure [Fig. 3(b)].

In conclusion, we have demonstrated the fundamental role of the relative phase of the fields for creating entanglement in a two distinguishable qubit system of trapped atoms or ions. We have shown that, only in the case of zero relative phase, the dynamics of the system exhibits Rabi oscillations and does not depend on the motional states. In general, the dynamics is qualitatively different. There are collapses and revivals in the dynamics of the internal states of the qubits when the phonons of the trap are in a coherent state, and the dynamics is chaotic for a thermal distribution of the phonons, except when the relative phase is zero or π. Since coherent distributions experience full revivals with phase-controlled Rabi frequencies, we believe they could be used as two-qubit gates, with the additional advantage that the speed of the gate could be easily controlled.

The phase-induced collapse and revival of entanglement could be experimentally observed with current technology. The most difficult part in the setup is the ability to address independently the transitions in the qubit system. We believe that the first potential candidates are trapped ions in inhomogeneous magnetic fields [23,24] and Rydberg atoms in ponderomotive optical lattices [25]. The energy shifts due to the effective spin-spin coupling can vary from several Hz, as in the modified ion trap proposed by Mintert and Wunderlich [23], up to tens of MHz for Rydberg atoms [25]. By choosing magnetic sublevels of the ions (atoms) for the qubit, the effect of decoherence (spontaneous decay) can be neglected on the time scale of an order of few milliseconds or even seconds [7,8]. Using Rabi frequencies $\sim$100 kHz we predict up to six almost perfect revivals during 20–22 μs for a coherent state of phonons with average number $\bar{n} = 10$. Novel developments in quantum information processing based on quantum dots and superconducting qubits could also have suitable parameters to observe these effects [26]. The importance of the phase shows that additional care should be taken over the relative phase of the fields when the distinguishable qubits are not in a single Fock state. The phase should be locked to zero if one wants to apply a π-pulse technique for quantum logic operations, since even a small change in the phase introduces phonon-induced decoherence in the system dynamics.

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