The Categorical Foundations for General Relativity*

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Abstract

This work is an introduction to categorical foundations of General Relativity (GR). The Category Theory, born in the 1950s, is an abstract mathematical theory with constantly increasing area of influence and applications (to mention only algebraic topology, algebraic geometry and logic). Using this theory one can formulate differential geometry which does not use the notion of convergence, but—rehabilitating Leibniz's intuitions—is based on the infinitesimals, i.e. the infinitely small, nonzero nilpotent objects. Such formulation enables one to define all notions of differential geometry without the use of local coordinates and guarantees smoothness of all functions defined on a manifold. A characteristic feature of the theory of Synthetic Differential Geometry (SDG) is the necessity of performing all the reasonings and proofs in a constructive (intuitionistic) manner, that is, without the use of the rule of excluded middle. This agrees with the approach preferred in modern physics, since it does not assume the preexistence of any observable, including even geometry. In this work we consider the part of mathematical formalism of SDG which enables one to write down the Einstein equations. Some objects and notions (such as the Levi–Civita connection) had to be introduced or clarified. The original problem that we have dealt with is the physical meaning of the equations of GR expressed in this mathematical universe of discourse. It turns out that the categorical GR modelled in toposes (which are some categories allowing for interpretation of the language and intuitionistic logic) is not only some mathematical generalisation of the classical theory but also offers a new concept of the notion of *an event*. The topos-theoretic formulation of General Relativity introduced by Guts and Zvyagintsev [Guts and Zvyagintsev 1999] is analysed in details and a modified formulation is proposed. Moreover, it has been shown that demanding that the equations of GR hold on other stages of description of world then the classical set-theoretic space-time build with events concerned as points can be reduced to demanding that these equations have to be expressed solely in the terms of pure geometric objects. We have also noted the possibility of expressing the classical string theory as a certain subtheory of description of the geometry which appears in a natural way in the topos relativity. Since one of the important goals of this work was to show the broader context and motivation for theoretical physics to adapt the categorical and toposophical way of thinking, we begin and conclude it widely discussing the possible role of toposes as an alternative and more suitable universe of discourse and its relation to the commonly accepted set-theoretical paradigm.

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