An introduction to rigorous formulations of quantum field theory

Daniel Ranard
Perimeter Institute for Theoretical Physics
Advisors: Ryszard Kostecki and Lucien Hardy

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Abstract

No robust mathematical formalism exists for nonperturbative quantum field theory. However, the attempt to rigorously formulate field theory may help one understand its structure. Multiple approaches to axiomatization are discussed, with an emphasis on the different conceptual pictures that inspire each approach.

Contents

1 Introduction

1.1 Personal perspective

What is quantum field theory? Rather than ask how nature truly acts, simply ask: what is this theory? For a moment, strip the physical theory of its interpretation. What remains is the abstract mathematical arena in which one performs calculations. The theory of general relativity becomes geometry on a Lorentzian manifold; quantum theory becomes the analysis of Hilbert spaces and self-adjoint operators. What, then is quantum field theory?

The appropriate arena for calculations remains unclear. Many physicists believe that quantum field theory is precisely a quantum theory. Indeed, we use the language of Hilbert spaces and operators to conduct calculations. But to a mathematician, there is rarely an actual Hilbert space or operator in sight. Path integrals, too, are understood by analogy rather than precise mathematics. Under the critical eye, quantum field theory amounts to a set of rules for manipulating formal integrals and operator symbols to obtain scattering amplitudes. Of course, the critic is wrong: physicists wield a variety of heuristic notions about the perceived mathematical structures beneath these rules.

Many physicists feel that these loose notions are sufficient. Maybe we understand our theory perfectly well, so that rigor is an unnecessary luxury. On the other hand, I take the view that mathematics is clarity; mathematics is knowledge of structure. If we seek strong mathematical foundations, it is not because we are over-anxious about ϵ's and δ's. When Born realized that Heisenberg's inscrutable calculations were best cast in the language of matrices, he found a more suitable arena for quantum mechanics. And when von Neumann identified Hilbert spaces as the appropriate setting for both matrix and wave mechanics, the new formalism not only clarified calculations but also fostered insight on the logic behind the theory.

Yet rigorous formalism often comes only after the discovery process, not during or before. Is quantum field theory unready for axiomatics? For instance, 't Hooft suggested that if Wightman's rigid axioms for field theory had gripped the physics community in the 1950s, physicists may have missed new ideas like Yang-Mills and string theory. Still, rigor need not stifle the imagination. The appropriate mathematical setting will even suggest new avenues for generalization.

Critics of rigorous approaches also ask, why try to formalize a theory which may be only effective? And while any physical theory is necessarily an approximation, the structure of the theory need not be "approximate." At the least, we should be able to formalize the perturbative renormalization techniques used everyday. In fact, there is already significant work in this direction [1]. Nonetheless, few believe quantum field theory has a mathematical structure revealed in its entirety by perturbative expansions. For example, the expected mass gap in 4D Yang-Mills theory may be "invisible to perturbation theory" [2].

Finally, rigor will also foster a fruitful relationship with mathematics. As Witten speculated, "...the relation between mathematics and physics will remain unsatisfactory unless the program of constructive field theory is resumed in some form" [2].

1.2 Goals and outline

In what follows, I survey multiple rigorous approaches to quantum field theory (QFT). The material requires modest familiarity with QFT as conventionally taught – the first few chapters of [3], for example. Modest

breadth of mathematical knowledge will also be helpful. Otherwise, this paper should serve as a gateway for the uninitiated.

I will begin by sketching various heuristic pictures of QFT. These pictures are alternately associated with the Schrödinger wavefunctional, the algebra of local observables, the path integral, and the Fock space. Each of these emphasizes a different way of thinking about field theory, and each indicates a different avenue for formalization. The reader already comfortable with the relationships between the wavefunctional, the path integral, and canonical quantization may skip this section. After these sketches, we identify the problems with our naive mathematical expressions that prevent us from easily formalizing the theory. Then I present different sets of axioms for QFT: those of Wightman (for the standard Hilbert space picture), Haag-Kastler (for the algebraic picture), and Osterwalder-Schrader (for the Euclidean path integral picture). Finally, I discuss difficulties and outlook.

I probably will not spoil the ending if I reveal now: the approaches surveyed ultimately fail to model the field theories we use. Each approach presents coherent axioms but few working examples. The axioms might begin,"The mathematical data of a quantum field theory consists of a Hilbert space $\mathcal{H}$, a self-adjoint Hamiltonian operator H acting on $\mathcal{H}$," and so on. Given a particular quantum field theory with some spacetime and Lagrangian, the challenge is then to construct this theory within the language of the axioms. To complicate matters, the rigorous theory may not have an object that explicitly looks like the familiar Lagrangian – we might never write down a ϕ^4. So how do we recognize the that a rigorous formulation describes what we call ϕ^4-theory? It seems all we may ask is that the scattering amplitudes in our reformulation agree with perturbative calculations. Ideally, the rigorous theory will also allow us to prove non-perturbative facts inaccessible to perturbative techniques.

To date, constructions exist for free theories (the scalar bosonic Klein-Gordon field, the spin-1/2 Dirac field) as well as some interacting theories in lower dimensions. What, then, do the axioms achieve? First, we can prove a few interesting results about any model that does satisfy the axioms. For instance, the spin-statistics and CPT theorems follow from both the Haag-Kastler and Wightman axioms. More importantly, each approach provides a new perspective on field theory. Yet still, there are few examples of well-defined interacting theories. Why have axiomatic constructions failed? Without specifying an answer, let the failure simply be a warning. The warning is: no one knows the exact way to think about quantum field theory. Some pictures are more naive or inconsistent, and some are more robust, but none is necessarily correct.

2 Sketches of quantum field theory

We begin with a quick tour of different pictures of quantum field theory, without attempting to make them rigorous. Most of the expressions we write down will be naive, formal expressions, indicated with the symbol $\approx$. The expressions will be misleading if taken literally. Usually, the technical difficulties will concern convergence or topology, but these issues may indicate profound problems.

2.1 Locality through tensor products

To arrive at our first picture, we ask: where is the notion of spatial locality in quantum mechanics? When we model a single particle with Hilbert space $L^2(\mathbb{R}^3)$, the $\mathbb{R}^3$ represents space, but it is hard to define a notion of locality in this setting. When we measure the position $\hat{\mathbf{x}}$ of the particle, this measurement appears to act

globally, probing the wavefunction over all of space before collapsing it to a point. And for a free particle, wavefunctions with compact support may appear localized, but they will time-evolve to have infinite support within any nonzero time. Note that our Hilbert space $L^2(R)^3$ naively looks like a direct sum over space,

$$L^2(R) \approx \text{span}(\{|\mathbf{x}\rangle, \mathbf{x} \in \mathbb{R}^3\}) \approx \bigoplus_{\mathbb{R}^3} \mathbb{C}. \tag{1}$$

Evidently, the direct sum is not the appropriate structure for locality. Meanwhile, in the general quantum theory of Hilbert spaces and observables, where one may model experiments without an explicit notion of space, there is still an abstract notion of locality. This notion is usually encoded by tensor products. A physical system with subsystems A and B has Hilbert space $\mathcal{H}_\mathcal{A} \otimes \mathcal{H}_\mathcal{B}$, where the tensor factors $\mathcal{H}_\mathcal{A}$ and $\mathcal{H}_\mathcal{B}$ are the Hilbert spaces associated with the individual subsystems. Observables of the form $O_A \otimes \mathbb{1}$ are dubbed local to A. In particular, locality means locality with respect to a subsystem. These subsystems may represent different spatial regions, or they may be more abstract. Under the standard interpretation, observers local to a subsystem will only have access to the corresponding local observables. One may also turn this around: the so-called locality of an observer is determined by which observables she can access.

In QFT, we seek a quantum theory that encodes spatial locality, a prominent aspect of our experience. Our first picture of QFT is then a Hilbert space with a subsystem assigned to every point in space. With this structure, we may consider observers local to different regions in space. Which Hilbert space should we use to represent the individual subsystems? Most generally, we might choose the generic countably-infinite-dimensional Hilbert space. (There is only one, up to isomorphism; see Section 3.1.) Equivalently, we accord a continuous degree of freedom to each point in space: at every point $\mathbf{x} \in \mathbb{R}^3$ we have a Hilbert space $\mathcal{H}_\mathbf{x}$ with a continuum of naive basis elements $\{|z\rangle_\mathbf{x}, z \in \mathbb{R}\}$. Ignoring the difficulties of an uncountably-indexed tensor product, we arrive at the Hilbert space

$$\begin{aligned} \mathcal{H} &\approx \bigotimes_{\mathbf{x}\in\mathbb{R}^3} \mathcal{H}_\mathbf{x} \\ \mathcal{H}_\mathbf{x} &\approx \text{span}(\{|z\rangle_\mathbf{x}, z \in \mathbb{R}\} \approx L^2(\mathbb{R}). \end{aligned} \tag{2}$$

Though I presented loose physical motivation for this picture, I will not claim the reasoning was sound. Some authors present conventional QFT as the inevitable marriage of special relativity and quantum theory [4, 5], often starting from a particle-based perspective and arriving at fields after developing some particle-based notion of locality. But it is difficult to make strong claims at the interface of different physical theories unless both are placed within a common axiomatic setting. Here, we are more interested in the structure of the theory, regardless of precise motivations.

So far, our notion of locality is spatial, not spatio-temporal. We imagine working in the Schrödinger picture, where we have time-independent observables. Without ambiguity, an operator $O_\mathbf{x} : \mathcal{H}_\mathbf{x} \to \mathcal{H}_\mathbf{x}$ is understood also as a local operator $O_\mathbf{x} : \mathcal{H} \to \mathcal{H}$. If an observer is at location $\mathbf{x}$, she may make measurements with observables $O_\mathbf{x}$. Meanwhile, states evolve in time and may be entangled. When we later venture from the Schrödinger picture, terminology will become more complicated.

2.2 Schrödinger (wavefunctional) representation

We have our Hilbert space; where are the fields? First we find a basis for $\mathcal{H}$. Consider a finite tensor product $\bigotimes_{i\in S} \mathcal{A}_i$ of finite-dimensional Hilbert space $\mathcal{A}_i = \text{span}(\{|j\rangle_i, j \in T\}$. (Thus index sets S and T are finite.) Then we may formally identify the basis of the composite space as the set of maps $\{\phi : S \to T\}$,

$$\bigotimes_{i\in S} \text{span}(\{|j\rangle_i, j \in T\} = \text{span}(\{\phi : S \to T)\}. \tag{3}$$

In other words, the basis for a tensor product may be given as the Cartesian product of the bases of the tensor factors. For the Hilbert space $\mathcal{H}$ of Equation (2), S and T correspond to $\mathbb{R}^3$ and $\mathbb{R}$. By analogy, we might say that maps $\{\phi : \mathbb{R}^3 \to \mathbb{R}\}$ form a basis for $\mathcal{H}$. If we do not want our states to vary wildly between nearby tensor factors, we might restrict our basis to $C(\mathbb{R}^3)$, the space of continuous maps $\mathbb{R}^3 \to \mathbb{R}$. Then

$$\mathcal{H} \approx \bigotimes_{\mathbf{x}\in\mathbb{R}^3} \mathcal{H}_{\mathbf{x}} \approx \text{span}(C(\mathbb{R}^n)). \tag{4}$$

We might call $\phi : \mathbb{R}^3 \to \mathbb{R}$ a field configuration, although the nomenclature is nonstandard. Field configurations should not be confused with quantum field operators. We might also call $C(\mathbb{R}^3)$ the configuration space, in analogy to ordinary quantum mechanics. (There, we use the Hilbert space $L^2(\mathbb{R}^3)$ to model a particle with classical configuration space $\mathbb{R}^3$.)

If we had a measure μ on the configuration space, then we could define $\mathcal{H}$ as the space of square-integrable complex-valued functions of the field,

$$\mathcal{H} \approx L^2(C(\mathbb{R}^3), \mu). \tag{5}$$

In fact, if we had a true measure, the above expression would be well-defined. (See Section 3.5 for an introduction to this topic.) You might call an element of this space a wavefunctional, in analogy with the wavefunction of non-relativistic quantum mechanics. In ordinary quantum mechanics, an element $\psi \in L^2(\mathbb{R}^3)$ may be specified by a wavefunction $\psi : \mathbb{R}^3 \to \mathbb{C}$, which gives a complex number for each point $\mathbf{x}$. Meanwhile, an element $\Psi \in L^2(C(\mathbb{R}^n))$ may be specified by a wavefunctional $\Psi[\phi]$, which gives a complex number for each field configuration $\phi : \mathbb{R}^3 \to \mathbb{R}$. The naive basis elements $|\mathbf{x}\rangle$ of $L^2(\mathbb{R})$ are analogous to the elements $|\phi\rangle$ of $\mathcal{H}$.

We have now met the so-called Schrödinger representation or wavefunctional picture of QFT. Though less common than other formalisms, the Schrödinger representation is sometimes used for field theory on curved spacetime. For a non-rigorous but comprehensive exposition of the formalism including QED, a good reference is Hatfield [6]. One may be tempted to think of the wavefunctional $\Psi[\phi]$ as a probability amplitude for some classical field ϕ to be found in various configurations, directly analogous to the position wavefunction of non-relativistic quantum mechanics. However, this interpretation is not promising. Would this global wavefunctional collapse when measured by any observer? Does all interference then disappear? Instead, we will focus on the local observables $O_{\mathbf{x}}$.

Let us explore the tensor product structure (TPS) given by Equation (2) from the wavefunctional perspective. (We will call this the spatial TPS, in contrast to other tensor product structures considered later.) Consider a partition of space into two disjoint regions, $\mathbb{R}^3 = A \bigsqcup B$. Then the configuration space splits into a direct sum of vector spaces, $C(\mathbb{R}^3) = C(A) \oplus C(B)$, which is a Cartesian product of underlying sets.

In turn, $L^2(\cdot)$ of a Cartesian product of sets (in particular, a direct sum of vector spaces) yields a tensor product – recall, for instance, that $L^2(\mathbb{R}^2) = L^2(\mathbb{R} \oplus \mathbb{R}) = L^2(\mathbb{R}) \otimes L^2(\mathbb{R})$. So we have

$$\mathcal{H} \approx L^2(C(\mathbb{R})) \approx L^2(C(A)) \otimes L^2(C(B)). \tag{6}$$

In the notation of Equation (2), the space $L^2(C(A))$ corresponds to $\mathcal{H}_A \approx \bigotimes_{\mathbf{x} \in A} \mathcal{H}_{\mathbf{x}}$, and likewise for $L^2(C(B))$. More generally, to any region A we assign a Hilbert space $\mathcal{H}_A$, and for any two disjoint regions A and B we have

$$\mathcal{H}_{A \sqcup B} = \mathcal{H}_A \otimes \mathcal{H}_B \tag{7}$$

This identity should also be evident directly from Equation (2).

To further understand the TPS of the Schrödinger representation, we can ask what it means for the wavefunctional to be in an unentangled product state. A product state is a simple tensor, a state of the form $a \otimes b \otimes c \ldots$. From Equation (2), a product state $\Psi \in \mathcal{H}$ should be of the form

$$\Psi \approx \bigotimes_{\mathbf{x} \in \mathbb{R}^3} f_{\mathbf{x}}, \tag{8}$$

where $f_{\mathbf{x}} : \mathbb{R} \to \mathbb{C}, z_{\mathbf{x}} \mapsto f_{\mathbf{x}}(z_{\mathbf{x}})$ is understood to be an element of $H_{\mathbf{x}}$, in the sense of a wavefunction. Translating to the Schrödinger representation, an unentangled state is of the form

$$\Psi[\phi] \approx \prod_{\mathbf{x} \in \mathbb{R}^3} f_{\mathbf{x}}(\phi(\mathbf{x})). \tag{9}$$

For instance, if $\Psi[\phi]$ is the "δ-function" wavefunctional concentrated at the field configuration $\phi = \phi_0$, we have $f_{\mathbf{x}} \approx \delta(\phi(\mathbf{x}) - \phi_0(\mathbf{x}))$. Hence the corresponding state $|\phi_0\rangle$ is a product state.

We would like to introduce local operators, local with respect to a point in space. Generically, an operator local to $\mathbf{x} = \mathbf{x}_0$ will be of the form $\bigotimes_{\mathbf{x} \in \mathbb{R}^3 \setminus \{\mathbf{x}_0\}} \bigotimes O_{\mathbf{x}_0}$, for some operator $O_{\mathbf{x}_0} : H_{\mathbf{x}_0} \to H_{\mathbf{x}_0}$. Again, we may also refer to $O_{\mathbf{x}_0}$ as an operator on $\mathcal{H}$. Recalling $\mathcal{H}_{\mathbf{x}} \approx \mathrm{span}(\{|z\rangle_x, z \in \mathbb{R}\}$, introduce the multiplication and differentiation operators

$$\begin{aligned} \hat{z}_{\mathbf{x}} &: \mathcal{H}_{\mathbf{x}} \to \mathcal{H}_{\mathbf{x}},\ f_{\mathbf{x}}(z_{\mathbf{x}}) \mapsto z_{\mathbf{x}} f_{\mathbf{x}}(z_{\mathbf{x}}) \\ -i\frac{\hat{\partial}}{\partial z_{\mathbf{x}}} &: \mathcal{H}_{\mathbf{x}} \to \mathcal{H}_{\mathbf{x}},\ f_{\mathbf{x}}(z_{\mathbf{x}}) \mapsto -i\frac{\partial}{\partial z_{\mathbf{x}}} f_{\mathbf{x}}(z_{\mathbf{x}}). \end{aligned} \tag{10}$$

We may consider these as local operators on $\mathcal{H}$, where they act on wavefunctionals like

$$\begin{aligned} \left(\hat{z}_{\mathbf{x}} \Psi\right)[\phi] &= \phi(\mathbf{x}) \Psi[\phi] \\ \left(-i\frac{\hat{\partial}}{\partial z_{\mathbf{x}}} \Psi\right)[\phi] &= -i\frac{\delta}{\delta \phi(\mathbf{x})} \Psi[\phi] \end{aligned} \tag{11}$$

and where the last expression denotes a functional derivative.

Inspired by the above, define

$$\begin{aligned}\hat{\phi}(\mathbf{x}) &\equiv \hat{z}_{\mathbf{x}} \\ \hat{\pi}(\mathbf{x}) &\equiv -i\frac{\hat{\partial}}{\partial z_{\mathbf{x}}} \equiv -i\frac{\hat{\delta}}{\delta\phi(\mathbf{x})}.\end{aligned} \tag{12}$$

Call $\phi(\mathbf{x})$ a field operator and $\pi(\mathbf{x})$ a conjugate momentum field operator. (I now take the liberty of dropping the hat notation for operators.) We might alternatively call them operator-valued fields, because we have an operator for each point $\mathbf{x} \in \mathbb{R}^3$. The field operators act on $\mathcal{H}_{\mathbf{x}}$ analogously to the traditional position and momentum operators of quantum mechanics, and they satisfy the canonical commutation relations (CCR):

$$\begin{aligned}[\phi(\mathbf{x}), \phi(\mathbf{y})] &= 0 \\ [\pi(\mathbf{x}), \pi(\mathbf{y})] &= 0 \\ [\phi(\mathbf{x}), \pi(\mathbf{y})] &\approx i\delta^3(\mathbf{x}-\mathbf{y}).\end{aligned} \tag{13}$$

For fixed $\mathbf{x} \in \mathbb{R}^3$, algebraic combinations of $\pi(\mathbf{x})$ and $\phi(\mathbf{x})$ should generate the algebra of all operators local to $\mathcal{H}_{\mathbf{x}}$. For one reason why, consider the operators

$$\begin{aligned}\alpha_{\mathbf{x}} &= \tfrac{1}{\sqrt{2}}(\phi(\mathbf{x}) + i\pi(\mathbf{x})) \\ \alpha^+_{\mathbf{x}} &= \tfrac{1}{\sqrt{2}}(\phi(\mathbf{x}) - i\pi(\mathbf{x})).\end{aligned} \tag{14}$$

These act like ladder operators on $\mathcal{H}_{\mathbf{x}}$, raising and lowering the energy eigenstates of the simple harmonic oscillator Hamiltonian $H_{\mathbf{x}} = \alpha^+_{\mathbf{x}}\alpha_{\mathbf{x}} + \frac{1}{2} = \frac{1}{2}(\phi(\mathbf{x})^2 + \pi(\mathbf{x})^2)$. Any operator may then be constructed by algebraic combinations of the raising and lowering operators. (This fact may still require some thought.)

When we arrive at the more conventional picture of QFT in Section 2.6, we will find that the $\phi(\mathbf{x})$ operators are precisely the conventional field operators. Meanwhile, if we began with the conventional picture of QFT using field operators $\phi(\mathbf{x})$, we could then reconstruct the states $|\phi_0\rangle \in L^2(C(\mathbb{R}^3))$ by asking for the simultaneous eigenstates of operators $\phi(\mathbf{x})$ with corresponding eigenvalues $\phi_0(x)$. (The field operators $\phi(\mathbf{x})$ form a complete set of commuting observables.)

Armed with operators local to spatial points, we can write a local Hamiltonian. Just as in condensed matter formalism, we call a Hamiltonian local when it is the sum of operators local to subsystems. The time-evolution operator is then the tensor product of the time-evolution operators on each subsystem. Such Hamiltonians are sometimes called ultra-local, and they are usually uninteresting: nothing moves. More precisely, if observables local to different regions are uncorrelated, they will remain uncorrelated for all time. More general local Hamiltonians will also include interaction terms that couple neighboring subsystems; here, that means interaction terms acting on $\mathcal{H}_{\mathbf{x}} \otimes \mathcal{H}_{\mathbf{x}+\Delta\mathbf{x}}$. When we assign subsystems to spatial points, the notion of neighboring subsystems is vague, but we may imagine that operators like $\partial_x\phi(\mathbf{x})|_{\mathbf{x}=\mathbf{x}_0} = \lim_{\Delta\mathbf{x}\to 0}\frac{1}{\Delta\mathbf{x}}(\phi(\mathbf{x}_0+\Delta\mathbf{x}) - \phi(\mathbf{x}_0))$ couple neighboring points. The generic local Hamiltonian then looks like

$$H = \int_{\mathbf{x}\in\mathbb{R}^3} \{\text{combinations of } \phi(\mathbf{x}),\ \pi(\mathbf{x}),\ \text{and their spatial derivatives}\}\ \mathrm{d}^3\mathbf{x}. \tag{15}$$

For instance, we will soon discuss the "free" Hamiltonian of Equation (24). The term "free" is best

understood from the particle perspective of Section 2.6, where the free Hamiltonian is understood to describe free (non-interacting) particles moving through space. The term also indicates that the Hamiltonian is only quadratic in the field operators, whereas higher powers are considered interaction terms. One may formally calculate the vacuum wavefunctional for the free Hamiltonian, obtaining [6]

$$\Psi_0[\phi] \propto e^{-G[\phi]} \tag{16}$$

where

$$\begin{aligned} G[\phi] &= \textstyle\int_{\mathbb{R}^3\times\mathbb{R}^3} \phi(\mathbf{x}) g(\mathbf{x},\mathbf{y}) \phi(\mathbf{y}) \, \mathrm{d}^3\mathbf{x} \, \mathrm{d}^3\mathbf{y}, \\ g(x,y) &= \textstyle\int_{\mathbb{R}^3} \frac{1}{(2\pi)^3} \omega_{\mathbf{k}} e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})} \, \mathrm{d}^3\mathbf{k}, \\ \omega_{\mathbf{k}} &= \sqrt{\mathbf{k}^2 + m^2}. \end{aligned} \tag{17}$$

Recalling Equation (9), we see that the ground state would be a product state if $g(x,y) \propto \delta(x-y)$, but in fact the ground state here is highly entangled. This should be no surprise, given that the Hamiltonian couples neighboring subsystems. When physicists speak of vacuum entanglement in field theory, it is often in this sense, with respect to the spatial TPS.

2.3 Poincaré invariance and microcausality

In addition to being local, our Hamiltonian should satisfy what is called microcausality, and predictions should furthermore be Poincaré-invariant. These are conditions on the time-evolution specified by the Hamiltonian. To define microcausality, consider operators $O_{\mathbf{x}}$ and $O_{\mathbf{x}'}$ local to $\mathcal{H}_{\mathbf{x}}$ and $\mathcal{H}_{\mathbf{x}'}$ respectively. Until now, the development has been imagined in the Schrödinger picture; $O_{\mathbf{x}}$ and $O_{\mathbf{x}'}$ are static operators used to represent local operations defined for all time. Now let us work in the Heisenberg picture, where formally the states and operators of both pictures will coincide at $t = 0$. That is, $\phi(\mathbf{x}, t) \equiv U(t)^{-1}\phi(\mathbf{x})U(t)$ and $\pi(\mathbf{x}, t) \equiv U(t)^{-1}\pi(\mathbf{x})U(t)$ for time-evolution operator $U(t)$.

We must be careful when thinking about the locality of operators in Schrödinger and Heisenberg pictures. If the Schrödinger-picture operator $O_{\mathbf{x}}$ is local to $\mathbf{x}$ in the sense of the spatial TPS of Equation (2), then the Heisenberg-picture operator $O_{\mathbf{x}}(t)$ is also local to $\mathbf{x}$ when $t = 0$, precisely because $O_{\mathbf{x}}(t = 0) = O_{\mathbf{x}}$. But the operator $U^{-1}(t)O_{\mathbf{x}}U(t)$ is generically *not* local to $\mathbf{x}$ for some value of $t \neq 0$.

On the other hand, if we were thinking in the Heisenberg picture, we would call $O_{\mathbf{x}}(t)$ a local observable. That is, it models the Schrödinger-picture observable $O_{\mathbf{x}}$ used at time t by an observer local to $\mathbf{x}$. For a Heisenberg-picture state $|\psi\rangle$, or equivalently a Schrödinger-picture state $|\psi(t = 0)\rangle$, $\langle\psi| O_{\mathbf{x}} |\psi\rangle$ is the expectation value for an observer locally measuring $O_{\mathbf{x}}$ at time t. So when we call some operator local or non-local, we must have in mind which picture we are using. I will denote a Heisenberg-picture operator $O_{\mathbf{x}}(t) \equiv O_x$ for $x = (\mathbf{x}, t) \in \mathbb{R}^4$ and call this operator local to x. (Note the distinction between the symbol $x \in \mathbb{R}^4$ and the bold-faced $\mathbf{x} \in \mathbb{R}^3$, which will be used to indicate operators in different pictures.) We might simultaneously say that O_x is non-local to $\mathbf{x} \in \mathbb{R}^3$, where we temporarily consider $O_x = U(t)^{-1}O_{\mathbf{x}}U(t)$ as a Schrödinger-picture operator.

Although we will not pursue it, another way to deal with these different notions would be to define a TPS that evolves in time, so that the locality of an operator is only specified with respect to a certain time.

Some authors (see [7]) adopt the perspective that tensor product structures on Hilbert spaces are naturally induced by specifying commuting sub-algebras of accessible observables. Indeed, this perspective may make the most sense from an operational perspective, where we often take observables describing operational setups as primitive, rather than positing a pre-existing TPS on a Hilbert space. (In field theory, this perspective has the additional advantage that it gives us a new way to think about space: space "emerges" when we abstractly specify our algebra of observables along with which observables are accessible and compatible.) If the accessible observables are then evolving in time with the Heisenberg picture, the relevant TPS should as well. This time-dependent Heisenberg picture of locality would also prompt us to say that O_x is local to x for $x \in \mathbb{R}^4$, just as in the above paragraph.

With this notation, microcausality is the statement that

$$[O_x, O_{x'}] = 0 \text{ for } (x - x') \text{ spacelike.} \tag{18}$$

If an operator is local to $\mathbf{x}$, it will not stay local, but microcausality implies that the support of the operator grows no faster than the speed of light. Here, the support of an operator is the set of tensor factors on which the operator acts non-trivially, and we then associate these tensor factors with a region of space. In other words, the support of an operator is the smallest spatial region with respect to which the operator is local. Under the conventional interpretation adopted here, microcausality means that operations (unitary transformations and measurements) performed by local observers cannot affect measurement outcomes in a spacelike-separated region. The fact that local Hamiltonians most easily satisfy microcausality is the probably best motivation for local Hamiltonians.

Meanwhile, defining Poincaré invariance of the theory is less straightforward. The difficulty is conceptual: in the framework described so far, there has been no indication of what a boosted observer will observe. We require additional theoretical or empirical input. Drawing on intuition from special relativity, we may suppose that if one observer performs an operation O_x local to $x \in \mathbb{R}^4$, another observer will consider it as some operation O'_{gx} local to gx, for the appropriate Poincaré transformation $g : \mathbb{R}^4 \to \mathbb{R}^4$. Here, g is an element of the Poincaré group $\mathcal{G} = \mathbb{R}^{1,3} \rtimes SO(1,3)$. Thus what one observer calls $\pi(x)$ or $\phi(x)$ another observer will identify as some combination of $\pi(gx)$ and $\phi(gx)$. More specifically, the change of reference frame will be encoded by an automorphism of the observables. These automorphisms may be given by a map

$$\alpha : G \to \mathrm{Aut}(\mathcal{A}) \tag{19}$$

$$g \mapsto \alpha_g \tag{20}$$

such that

$$\alpha_g(O_x) = O'_{gx} \tag{21}$$

This emphasis on the algebra of local observables and their transformations leads to the program of algebraic quantum field theory (AQFT). The traditional axioms for AQFT are the Haag-Kastler axioms, discussed in Section 4.2. Dually, we could consider specifying the Poincaré transformations of the Hilbert space. We might expect we can implement the automorphisms α_g as unitary elements U_g on the Hilbert space that act by conjugation on the observables. That is, we equip $\mathcal{H}$ with a unitary representation U of

the Poincaré group such that

$$\begin{aligned} U : G &\mapsto \{\text{Unitaries on } \mathcal{H}\} \\ g &\mapsto U(g) \\ U(g)^{-1} O_x U(g) &= \alpha_g(O_x) = O'_{gx}. \end{aligned} \tag{22}$$

(To allow for spin, we would need to consider projective representations of G, and we would also need to use a different Hilbert space $\mathcal{H}_x$. Here, we focus on scalar fields.)

We also need to specify an additional consistency requirement: time-translation via the representation of the Poincaré group should coincide with Hamiltonian time-evolution. That is, for time-translation $g_t \in G$, we require

$$U(g_t)^{-1} O_{\mathbf{x}_0,t_0} U(g_t) = e^{iHt} O_{\mathbf{x}_0,t_0} e^{-iHt} = O_{\mathbf{x},t_0+t}. \tag{23}$$

In other words, the notations $U(g_t)$ and $U(t) \equiv e^{-iHt}$ should coincide.

Meanwhile, in textbook development of field theory, we often begin with fields $\phi(\mathbf{x})$ and $\pi(\mathbf{x})$ that transform in a prescribed way under rotations and spatial translations. Then we write down some candidate Hamiltonian H in terms of our fields, possibly with interactions, and this H generates some time-evolution. Finally, H is called Poincaré-covariant if there is some representation of the Poincaré group for which H generates time-translations and for which the subgroup of rotations and spatial rotations acts as already prescribed. From this perspective, Poincaré covariance is a condition on H which may be checked for a given expression of H in terms of the field operators. Alternatively, one might begin with some representation of the Poincaré group, and then the Hamiltonian is defined as the generator of time-translation, eliminating the need for a further consistency check.

Note that a Lorentz-covariant Hamiltonian will *not* be Lorentz invariant. (Hence I say "Lorentz-covariant" Hamiltonian but "Lorentz-invariant" theory.) In other words, H will transform non-trivially under the Lorentz group. One way to see this is that H is the generator of time-translations, and time-translations and boosts do not commute in the Poincaré group. A Lorentz-covariant Hamiltonian will lead to a theory with Lorentz-invariant predictions, where here the meaning of Lorentz invariance is tautologically encoded by Equations (22) and (23).

The above discussion of Lorentz transformations may appear ad hoc. And perhaps it is, because special relativity has no obvious place in the framework of abstract quantum theory. The story of special relativity involves clocks, rulers, light, and bodies in motion. From this story, one derives the Lorentz transformations and how a moving body has different momenta measured by different observers. With this story in mind, perhaps Lorentz transformations in QFT are more satisfactorily approached from a particle picture. In that setting, we may define Lorentz transformations by how they act on a single-particle state $|\mathbf{k}\rangle$ of a given momentum $\mathbf{k} \in \mathbb{R}^3$. This definition will coincide with the one given above.

Regardless, we have our definition of Lorentz invariance, and it involves the interrelationship given by Equation (23) between the Hamiltonian and the representation of the Poincaré group. Now, we present an example of a Hamiltonian and Poincaré group representation that satisfy these requirements: the scalar field and the free Hamiltonian. By definition, the scalar field $\phi(x)$ transforms as $\phi(x) \mapsto \phi(\Lambda x)$ under Lorentz

transformations. The free Hamiltonian is defined by

$$H = \frac{1}{2}\int_{\mathbb{R}^3} \pi(\mathbf{x})^2 + m^2\phi(\mathbf{x})^2 + |\nabla\phi|^2 \ \mathrm{d}^3\mathbf{x}. \tag{24}$$

The $\pi(\mathbf{x})^2 + m^2\phi(\mathbf{x})^2$ term may be thought of as the simple harmonic oscillator Hamiltonian on subsystem $\mathcal{H}_{\mathbf{x}}$ with oscillating degree of freedom $|z\rangle_{\mathbf{x}} \in \mathcal{H}_{\mathbf{x}}$, while the $|\nabla\phi|^2$ term couples neighboring oscillators.

We also define $\pi(\mathbf{x}) \mapsto \pi(g\mathbf{x})$ for translations and rotations g. Then consistency with equations (22) and (23) already determines how the field $\pi(x)$ must transform under boosts. Specifically, one may calculate $\pi(x) = \partial_t\phi(x)$, so the Lorentz transformation properties of $\pi(x)$ are determined by those of ∂_t and $\phi(x)$. Finally, one may verify that the fields will satisfy the microcausality of Equation (18).

Using the same fields $\phi(\mathbf{x}), \pi(\mathbf{x})$ with the same transformation properties under translations and rotations, one may also define an interacting Hamiltonian such as

$$H = \frac{1}{2}\int_{\mathbb{R}^3} \pi(\mathbf{x})^2 + m^2\phi(\mathbf{x})^2 + \lambda\phi(\mathbf{x})^4 + |\nabla\phi|^2 \ \mathrm{d}^3\mathbf{x}. \tag{25}$$

Consistency with equations (22) and (23) will then determine how the fields transform under boosts, and the fields will satisfy microcausality.

We now pause to collect these notions into our first, loose definition for a scalar quantum field theory. The axioms should apply to free or interacting theories alike.

Informal axioms (Informal axioms for a real scalar QFT from a Hilbert space perspective)**.** *A quantum field theory consists of the following:*

1. *A Hilbert space $\mathcal{H}$,*
2. *self-adjoint field operators $\phi(\boldsymbol{x})$ and $\pi(\boldsymbol{x})$ for all $\boldsymbol{x} \in \mathbb{R}^3$, satisfying the canonical commutation relations of Equation (13),*
3. *unitary representation U of the Poinare group G, $U : g \mapsto U(g)$, and*
4. *Hamiltonian $\mathcal{H}$ that generates the time-translation of the Poincaré group, i.e. $U(g_t) = e^{-itH}$ for time-translation $g_t \in G$.*

The field operators and Poincaré group representation must further satisfy

$$U(g)^{-1}\phi(x)U(g) = \phi(gx) \tag{26}$$

for $x \in \mathbb{R}^4$ and Heisenberg-picture operators $\phi(x) = \phi(\boldsymbol{x}, t) = e^{itH}\phi(\boldsymbol{x})e^{-itH}$.

The above axioms are similar in spirit to the Wightman axioms of Section 4.1. Technicalities aside, the primary distinction of the Wightman axioms is the use of fields smeared over spacetime, as discussed in Section 3.3.

2.4 Experimental predictions

As we investigate and formalize quantum field theory, we are free to employ whichever mathematical structures we choose. But our search for the right structure is ill-posed unless we demand certain outputs from

our theory. These outputs are experimental predictions. For particle physicists using quantum field theory, the predictions are mostly about scattering. In the conventional Hilbert space framework, the data used to make these predictions is encoded in the (time-ordered) vacuum expectation values of field operators, $\langle\Omega|\,\mathcal{T}(\phi(x_n)...\phi(x_1)\,|\Omega\rangle$, where $\mathcal{T}(\cdot)$ denotes time ordering. This data be understood in terms of the map

$$(x_1, ..., x_n) \mapsto \langle\Omega|\,\mathcal{T}(\phi(x_n)...\phi(x_1)\,|\Omega\rangle\,, \tag{27}$$

also called an n-point correlation function. It is precisely these correlation functions that particle physicists use to make predictions about scattering experiments. To translate a correlation function into a scattering prediction, one uses the LSZ formula, which relies on the particle picture. Is this translation from theoretical formalism to experimental prediction consistent with the elements of interpretation already introduced? In particular, we have already discussed the interpretation of $\phi(x)$ and $\pi(x)$ as local operators available to an observer at position $\mathbf{x}$ and time t, and we have also specified how to model observers in different inertial frames. Are these interpretations consistent with the way in which the theory is actually used to produce scattering experiments? The answer is probably yes; for a related discussion, see Sections II.3 and VI in [13].

Meanwhile, which sort of data do we use to specify a particular quantum field theory? We typically specify a theory by some Hamiltonian and list of field or particle contents. For a real scalar field theory, this amounts to specifying the interaction term, such as ϕ^4, whereas in the Standard Model, we specify several fields and a complicated Lagrangian. Even if one develops a formalism without a Hilbert space or Hamiltonian, there will likely be some way of describing different models. Then our general framework for quantum field theory becomes a map

$$\text{QFT} : [\text{Hamiltonian or similar specification}] \;\mapsto\; \text{correlation functions}\,, \tag{28}$$

where the LHS is a specification such as ϕ^4 or the Standard Model Lagrangian. In each formulation of quantum field theory we must produce an object like the correlation functions. Of course, new theories of high-energy physics may characterize models and predictions in a completely different way, so that neither the LHS nor the RHS of the above retains the same character. But we might not call these theories QFT.

2.5 Path integral picture

From the picture of the wavefunctional and Hamiltonian, we may easily move to the path integral and Lagrangian formulation. The general procedure for writing a path integral in quantum theory proceeds as follows. For a Hilbert space $\mathcal{H}$ with Hamiltonian H and basis $\{|a_i\rangle\,, i \in S\}$ in the Schrödinger picture, we seek to find an expression for matrix elements $U(T)_{ij} = \langle a_i|\,U(T)\,|a_j\rangle$ of the time-evolution operator $U(T) = e^{-iHT}$. To do so, we write $U(T) \approx U(\epsilon)^n,\ \epsilon = \frac{T}{n}$. Then as $n \to \infty$,

$$U(T)_{ij} \approx (U(\epsilon)^n)_{ij} \approx \sum_{(k_1,...,k_n)\in S^n} U(\epsilon)_{ik_1}U(\epsilon)_{k_1k_2}...U(\epsilon)_{k_nj} \tag{29}$$

The element $(k_1, ..., k_n), \in S^n$ may be rewritten as a map $f : \{1, ..., n\} \to S$. For large n, we may reinterpret f as a path $f : [0, T] \to S$ where $k_m = f(\frac{m}{n}T)$. Hence we reinterpret the RHS above as a sum over paths. (The paths may be loosely interpreted as either trajectories in space or simply "paths" through the matrix

entries of $U(\epsilon)$ that constitute the product $U(\epsilon)_{k_1 k_2}...U(\epsilon)_{k_n j}$.) Meanwhile, the matrix element $U(\epsilon)_{k_m k_{m+1}}$ may be interpreted as a map $(k_m, k_{m+1}) \to \mathbb{C}$, or likewise a map $F : (f(t), f'(t)) \to \mathbb{C}$, recalling $f(\frac{m}{n}T) = k_m$. Then the time-evolution operator generically becomes the path integral

$$U(\epsilon)_{ij} \approx \int_{\substack{f \in C([0,T]) \\ f(0)=i, f(T)=j}} \prod_{t \in [0,T]} F[f(t), f'(t)] \mathcal{D}f. \tag{30}$$

also called a functional integral. The symbol $\mathcal{D}f$ indicates some (possibly non-existent) formalization of the limiting process used to produce this equation from the previous. Assuming F is nonzero, we may use $G = \log(F)$ to write

$$U(\epsilon)_{ij} \approx \int_{\substack{f \in C([0,T] \\ f(0)=i, f(T)=j}} \exp\left(\int_0^T G[f(t), f'(t)] \, \mathrm{d}t \right) \mathcal{D}f \tag{31}$$

Again, we have re-expressed the time-evolution operator as a sum over paths $f : [0, T] \mapsto S$, for Hilbert space H with a basis set indexed by S. The integrand is the exponential of a functional of f, and this functional is local in time.[1] If H is written in terms of operators q and p obeying the CCR, and if S labels eigenstates of q, one calculates that G is actually the Lagrangian $\mathcal{L}$ that corresponds classically to H via Legendre transformation.[2] We see this in the Feynman path integral for ordinary quantum mechanics.

The above may be extended to the case that $\mathcal{H}$ has a TPS given by $\mathcal{H} = \bigotimes_{x \in X} \mathcal{H}_{\mathbf{x}}$, where $\mathcal{H}_{\mathbf{x}}$ has basis set indexed by S. Then for H local with respect to the TPS, the time-evolution operator becomes an integral over "paths" $f : [0, T] \times X \mapsto S$, and the integrand is the exponential of a functional local in both time and X-space.

Returning to our particular Hilbert space $\mathcal{H}$ with its spatial TPS, we apply the above statement with $S = \mathbb{R}$ and $X = \mathbb{R}^3$. The time-evolution operator becomes an integral over the space of fields configurations on a band of spacetime $[0, T] \times \mathbb{R}^3$, or the space $C([0, T] \times \mathbb{R}^3)$. Due to the canonical commutation relations of Equation (13), the local functional $C([0, T] \times \mathbb{R}^3) \to \mathbb{R}$ turns out to be the Lagrangian (density) $\mathcal{L}$ classically associated to the Hamiltonian $\mathcal{H}$. We write

$$\begin{aligned} \langle \phi_f | \, U(T) \, | \phi_i \rangle &\approx \int_{\substack{\varphi \in C([0,T] \times \mathbb{R}^3) \\ \text{s.t.} \varphi(0)=\phi_i, \varphi(T)=\phi_f}} e^{iS[\varphi]} \mathcal{D}\varphi \\ S[\phi] &\approx \int_{[0,T] \times \mathbb{R}^3} \mathcal{L}(\varphi(x), \nabla\varphi(x), \partial_t \varphi(x)) \, \mathrm{d}^4 x. \end{aligned} \tag{32}$$

This time-evolution operator will entangle the subsystems associated with different spatial regions, which may be traced to the fact that $\mathcal{H}$ is not ultralocal, due to the presence of the $|\nabla\phi(x)|^2$ term.

In the Schrödinger picture, we have the time-dependent wavefunctional

$$\Psi[\phi, t] = \int_{\phi_i \in C(\mathbb{R}^3)} \left(\int_{\substack{\varphi \in C([0,T] \times \mathbb{R}^3) \\ \text{s.t.} \varphi(0)=\phi_i, \varphi(T)=\phi}} e^{iS[\varphi]} \mathcal{D}\varphi \right) \Psi[\phi, 0] \mathcal{D}\phi_i \tag{33}$$

[1] A functional $A : C(\mathbb{R}^n, \mathbb{R}) \to \mathbb{R}$ is called local if it may be written as $A[f] = \int_{\mathbb{R}^n} a(f(x), f'(x), ..., f^{(n)}(x)) \, \mathrm{d}x$ for some function $a : \mathbb{R}^n \to \mathbb{R}$ with $n \in \mathbb{N}$.

[2] This may seem implausible: what does time-evolution by a quantum Hamiltonian have to do with classical mechanics or the Legendre transformation? But if $[x, p] = 1$ and $H = H(x, p)$, then we obtain Heisenberg-picture equations $\dot{x} = [x, H] = \frac{\partial H}{\partial p}$ and $\dot{p} = [p, H] = -\frac{\partial H}{\partial p}$, which exactly mirror Hamilton's equations. This fact is connected to the appearance of the classical Lagrangian in the path integral.

Of course, we have been careless with convergence, and the functional integration is not well-defined. However, imagine that we calculate the expression for $U(T)$ using a finite-volume spatial box B, so that the integral runs over field configurations on the finite spacetime region $B \times T$. Now discretize $B \times T$ into a finite lattice L with points $(x_i, t_j) \in L$. Then $U(T)$ of Equation (32) becomes an integral over a finite number of real variables $\phi_{x_i t_j}$ that represent field values at points in (x_i, t_j). Hence we have a finite-dimensional integral over $\mathbb{R}^{|L|}$. One might hope that this integral actually converges. Yet the integral contains factors almost identical to the following (where some irrelevant constants have been removed):

$$\int_{\phi_{x_i t_j} \in \mathbb{R}} e^{-iT\phi_{x_i t_j}^2} \, \mathrm{d}\phi_{x_i, t_j} \tag{34}$$

These integrals are ill-defined. On the other hand, if we imagine that $U(T)$ is analytic in T, then we may evaluate $U(T)$ for complex T. In the case that $\mathrm{Im}(T) < 0$, the above integral becomes well-defined; here, we happen to obtain $\sqrt{\pi}/\sqrt{iT}$ for function $\sqrt{(\cdot)}$ with branch cut along $\mathbb{R}^-$. Although the integral is not well-defined for $\mathrm{Im}(T) = 0$, we may formally assign it a value by analytic continuation from the expression with $\mathrm{Im}(T) < 0$. In the example, that would mean using the functional form $\sqrt{\pi}/\sqrt{iT}$ to assign a value to the integral for $T \in \mathbb{R}$. In this way, the discretized path integral becomes well-defined. In general, the formal continuum path integral will remain ill-defined. If we characterize the discretization by some cutoff parameter, the quantities defined by the discretized path integral will be well-defined but cutoff-dependent. Making sense of the cutoff-dependence is associated with Wilsonian picture of renormalization in perturbative quantum field theory.

For notational convenience, we define a parameter $\tau \equiv -iT$ and use T to denote the real, physical time. This procedure is called a Wick rotation, and τ is sometimes called imaginary time. The (discretized) integral expression for $U(\tau)$ is then well-defined. In this case, the continuum path integral becomes

$$\begin{aligned} \langle \phi_f | \, U(\tau) \, | \phi_i \rangle &\approx \langle \phi_f | \, U(-iT) \, | \phi_i \rangle \\ &\approx \int_{\substack{\varphi \in C([0,T] \times \mathbb{R}^3) \\ \text{s.t.} \varphi(0) = \phi_i, \varphi(T) = \phi_f}} e^{-S_E[\varphi]} \mathcal{D}\varphi \, , \\ S_E[\phi] &\approx \int_{[0,T] \times \mathbb{R}^3} \mathcal{L}(\varphi(x), \nabla\varphi(x), i\partial_t \varphi(x)) \, \mathrm{d}^4 x \end{aligned} \tag{35}$$

where $S_E[\phi]$ is called the Euclidean action. Note the changes in factors of i. Formally, the Euclidean action looks like the ordinary action but with Euclidean metric replacing Minkowski metric. An easy way to see how Equation (35) formally comes from Equation (32) is to work with the discretized versions of these expressions. Consider the discretized expression for Equation (32) – that is, the finite-dimensional integral described above. This expression will depend on T. If one replaces T with $\tau = -iT$, the resulting expression is then identical to what is found when discretizing Equation (35). As discussed, this discretized Euclidean path integral is a well-defined finite-dimensional integral for $\mathrm{Im}(\tau) < 0$. To evaluate the matrix elements of $U(T)$, we will need to use Equation (35) and analytically continue to $U(i\tau) = U(T)$.

We have defined what we mean by functional integral (at least in the discretized case) by using the Euclidean formulation. This definition will actually allow us to calculate the vacuum wavefunctional from the functional integral. To see how, note the following. In the Schrödinger picture, for a Hamiltonian H

with vacuum state $|\Omega\rangle$ and vacuum energy E_0, then for states $|\psi_i\rangle, |\psi_f\rangle$, the limit

$$\lim_{T\to\infty} \langle\psi_f| U(T) |\psi_i\rangle = \lim_{T\to\infty} \langle\psi_f| e^{-iHT} |\psi_i\rangle \tag{36}$$

is generically undefined. However, if we consider the analytically-continued time-evolution operator $U(T(1-i\epsilon))$ for $\epsilon > 0$, the limit

$$\begin{aligned}\lim_{T\to\infty} \langle\psi_f| U(T(1-i\epsilon)) |\psi_i\rangle &= \lim_{T\to\infty} \langle\psi_f| e^{-iHT(1-i\epsilon)} |\psi_i\rangle \\ &\approx \langle\psi_f|\Omega\rangle \langle\Omega|\psi_i\rangle e^{-iE_0T} e^{-\epsilon E_0 T}\end{aligned} \tag{37}$$

appears well-defined. (It may be computed by expanding $|\psi_i\rangle$ in terms of energy eigenstates.) Now, in our case, the path integral expression for $U(T)$ is actually *defined* for $\mathrm{Im}(T) < 0$ and then analytically continued to $T \in \mathbb{R}$. So we expect that the path integral expression for $\langle\psi_f| U(T) |\psi_i\rangle$ to act as in Equation (37) when $\epsilon \to 0$, yielding $\langle\psi_f|\Omega\rangle \langle\Omega|\psi_i\rangle e^{-iE_0T}$. In other words, the path integral expression for $U(T)$ should be proportional to $|\Omega\rangle \langle\Omega|$ as $T \to \infty$. Then we may write

$$\Psi_0[\phi] \propto \int_{\substack{\varphi\in C([-\infty,0]\times\mathbb{R}^3) \\ \text{s.t.}\varphi(0)=\phi}} e^{iS[\varphi]} \mathcal{D}\varphi \tag{38}$$

where we integrate over the half-spacetime $C([-\infty, 0] \times \mathbb{R}^3)$. In this form especially, one may see the vacuum is invariant up to phase under time-evolution by Equation (32): time-evolution merely extends the domain of integration in Equation (38) to $C([-\infty, t] \times \mathbb{R}^3)$, but this yields the same wavefunctional. If we add a constant to the Hamiltonian so that the vacuum energy is zero, the phase factor disappears.

Repeating the above reasoning for $U(-iT)$, one finds a similar Euclidean expression for the vacuum wavefunctional,

$$\Psi_0[\phi] \propto \int_{\substack{\varphi\in C([-\infty,0]\times\mathbb{R}^3) \\ \text{s.t.}\varphi(0)=\phi}} e^{-S_E[\varphi]} \mathcal{D}\varphi \tag{39}$$

We may use our path integral expression for the vacuum to calculate correlation functions. Consider the vacuum expectation of field operators in the Heisenberg picture, $\langle\Omega| \phi(\mathbf{x}_n, t_n)...\phi(\mathbf{x}_1, t_1) |\Omega\rangle$, where we assume $t_n > ... > t_1$. Then

$$\langle\Omega| \phi(\mathbf{x}_n, t_n)...\phi(\mathbf{x}_1, t_1) |\Omega\rangle = \langle\Omega| \phi(\mathbf{x}_n) U(t_n - t_{n-1})...U(t_2 - t_1)\phi(\mathbf{x}_1) |\Omega\rangle \tag{40}$$

where we move to Schrödinger-picture operators on the RHS. Inserting partitions of unity and using Equation (32) to replace the time-evolution operators, one eventually obtains

$$\begin{aligned}\langle\Omega| \mathcal{T}(\phi(\mathbf{x}_n t_n)...\phi(\mathbf{x}_1, t_1)) |\Omega\rangle &= ... \\ &= \frac{\int_{\varphi\in C(\mathbb{R}^4)} e^{iS[\varphi]} \varphi(\mathbf{x}_n, t_n)...\varphi(\mathbf{x}_1, t_1)\mathcal{D}\varphi}{\int_{\varphi\in C(\mathbb{R}^4)} e^{iS[\varphi]} \mathcal{D}\varphi}.\end{aligned} \tag{41}$$

The denominator comes from the modulus of the RHS of Equation (38) and provides the right normalization.

Alternatively, we may use the Euclidean formulation. One finds

$$\langle\Omega|\,\mathcal{T}(\phi(\mathbf{x}_n,-it_n)...\phi(\mathbf{x}_1,-it_1))\,|\Omega\rangle = ... \\ = \frac{\int_{\varphi\in C(\mathbb{R}^4)} e^{-S_E[\varphi]}\varphi(\mathbf{x}_n,t_n)...\varphi(\mathbf{x}_1,t_1)\mathcal{D}\varphi}{\int_{\varphi\in C(\mathbb{R}^4)} e^{-S_E[\varphi]}\mathcal{D}\varphi}. \tag{42}$$

Again, to interpret this, we first assume the correlation function on the LHS of Equation (42) is some analytic function with domain $(\mathbf{x}_1,t_1,...,\mathbf{x}_n,t_n)$. Then we analytically continue the domain to $(\mathbf{x}_1,-it_1,...,\mathbf{x}_n,-it_n)$. Meanwhile, the discretized and finite-volume version of the RHS is well-defined, and we write the formal continuum limit. Nothing particularly strange has happened. In general, when one cannot calculate values of some analytic function $f:\mathbb{C}\to\mathbb{C}$ on the subset $\mathbb{R}\subset\mathbb{C}$ of the domain, it may be easier to find an expression for f restricted to another part of the domain. Afterward, one may analytically continue the function back to the subset $\mathbb{R}\subset\mathbb{C}$. This is a well-posed calculational technique for analytic functions.

The path integral picture is sometimes referred to as the Lagrangian picture, emphasizing the presence of the Lagrangian in the path integral. Poincaré-invariance is more manifest in this picture. Recall from Section 2.3 that when given fields with specified Poincaré transformation properties, it is nontrivial to check whether a particular Hamiltonian expressed in terms of the fields is appropriately covariant, but this verification is needed for a Lorentz-invariant theory. On the other hand, consider a Lagrangian that is Poincaré-invariant, in the sense that the expression for the action will remain unchanged under a coordinate transformation $x\mapsto\Lambda x$. We may use the Lagrangian to define correlation functions (without every worrying about a Hilbert space, for instance), and then the Poincaré-invariance of the Lagrangian will directly imply the Poincaré-invariance of the correlation functions. Thus the experimental predictions of a theory defined by a Poincaré-invariant Lagrangian are automatically invariant. Note that similarly, invariance of the Euclidean action under Euclidean transformations on spacetime will directly imply the Poincaré-invariance of the correlation functions.

2.6 Particles and the Fock space

The vacuum may also be understood by adopting a new basis for the configuration space of fields. We use basis $\tilde{\phi}(\mathbf{k})\equiv\int_{\mathbb{R}^3}\frac{1}{(2\pi)^3}\phi(\mathbf{x})e^{-i\mathbf{k}\cdot\mathbf{x}}\,\mathrm{d}^3\mathbf{x}$. This new basis for the configuration space also defines a new TPS for $\mathcal{H}$, here called the $\mathbf{k}$-space TPS, different from the spatial TPS considered before. (By analogy, it may help to consider how changing basis on $\mathbb{R}^2$ changes the TPS associated with $L^2(\mathbb{R}^2)=L^2(\mathbb{R})\otimes L^2(\mathbb{R})$.) In the new $\mathbf{k}$-space TPS, there is a tensor factor associated with every point in $\mathbf{k}$-space, with degree of freedom $\tilde{\phi}(k)$. Because $\tilde{\phi}(k)$ is complex, there are actually two real degrees of freedom, $\mathrm{Re}(\tilde{\phi})$ and $\mathrm{Im}(\tilde{\phi})$, associated with each point $\mathbf{k}$. We could write

$$\mathcal{H}\approx\bigotimes_{\mathbf{k}\in\mathbb{R}^3}\mathcal{H}_{\mathbf{k}} \tag{43}$$

where $\mathcal{H}_{\mathbf{k}}=\mathbb{C}^2$. There are associated operators $\tilde{\phi}(\mathbf{k})\equiv\int_{\mathbb{R}^3}\frac{1}{(2\pi)^3}\phi(\mathbf{x})e^{-i\mathbf{k}\cdot\mathbf{x}}\,\mathrm{d}^3\mathbf{x}$, where $\tilde{\phi}(\mathbf{k})$ are new operators and $\phi(\mathbf{x})$ are the field operators already introduced. (This equation is different from the previous only because ϕ now denotes operators; one might also use the hat notation.) These new operators are local in the $\mathbf{k}$-space TPS.

In the new notation and associated TPS, the vacuum wavefunctional becomes

$$\Psi_0[\tilde{\phi}] \propto \exp\left(-\frac{1}{2}\int_{\mathbb{R}^3} \frac{1}{(2\pi)^3}\omega_{\mathbf{k}}|\tilde{\phi}(\mathbf{k})|^2\right)\, \mathrm{d}^3\mathbf{k}, \tag{44}$$

as in [6]. This state is evidently unentangled with respect to the **k**-space TPS. In particular, it is the product of Gaussian wavefunctions. To further simplify the Hamiltonian, we introduce ladder operators

$$a_{\mathbf{k}} \approx \textstyle\int_{\mathbb{R}^3} e^{-i\mathbf{k}\cdot\mathbf{x}}(\omega_{\mathbf{k}}\phi(\mathbf{x}) + i\pi(\mathbf{x}))\, \mathrm{d}^3\mathbf{x} \tag{45}$$

$$a^+_{\mathbf{k}} \approx \textstyle\int_{\mathbb{R}^3} e^{-i\mathbf{k}\cdot\mathbf{x}}(\omega_{\mathbf{k}}\phi(\mathbf{x}) - i\pi(\mathbf{x}))\, \mathrm{d}^3\mathbf{x} \tag{46}$$

$$[a_{\mathbf{k}}, a^+_{\mathbf{k}'}] \approx (2\pi)^3\delta^3(\mathbf{k}-\mathbf{k}').$$

We may then write the Hamiltonian as

$$H \approx \frac{1}{2}\int_{\mathbb{R}^3} \omega_{\mathbf{k}} a^+_{\mathbf{k}} a_{\mathbf{k}}\, \mathrm{d}^3\mathbf{k}. \tag{47}$$

When formally calculating this expression, a term $[a^+_{\mathbf{k}}, a_{\mathbf{k}}] \propto \delta^3(0) \to \infty$ arises, which we then ignore. This constant is sometimes considered the infinite zero-point energy of the vacuum. The infinite constant also disappears if one uses the ubiquitous normal ordering procedure, in which one moves operators $a^+_{\mathbf{k}}$ to the left of $a_{\mathbf{k}}$ in all expressions while simultaneously ignoring the commutator term. The need for a normal ordering prescription may be understood as a consequence of not using the smeared fields discussed in Section 3.3.

By considering the free Hamiltonian in the **k**-space TPS, we will transition to a particle or Fock space picture of quantum field theory. First note the free Hamiltonian is ultralocal with respect to the **k**-space TPS, and time-evolution of product states is therefore simple. Likewise, eigenstates written in this TPS are easy to find. In general, when a Hamiltonian is written as a sum of local terms, one may find eigenstates simply by diagonalizing each term individually. Here, the individual terms $\omega_{\mathbf{k}} a^+_{\mathbf{k}} a_{\mathbf{k}}$ are familiar as a description of a harmonic oscillator. The free Hamiltonian therefore describes harmonic oscillators at each point in **k**-space, and these oscillators are uncoupled, in contrast to the oscillators associated with the spatial TPS. One then finds a basis of eigenstates of the form $a^+_{\mathbf{k}_1}...a^+_{\mathbf{k}_n}|\Psi_0\rangle$, called Fock states, where $|\Omega\rangle$ is the vacuum state $|\Psi_0\rangle$. The Fock states may also be written as $a^+_{\mathbf{k}_1}...a^+_{\mathbf{k}_n}|\Omega\rangle = |\mathbf{k}_1...\mathbf{k}_n\rangle$. Because $a^+_{\mathbf{k}}$ and $a^+_{\mathbf{k}'}$ commute, the order of the $\mathbf{k}_i$ in the expression $|\mathbf{k}_1...\mathbf{k}_n\rangle$ does not matter: $|\mathbf{k}_1\mathbf{k}_2\rangle$ and $|\mathbf{k}_2\mathbf{k}_1\rangle$ denote the same state. So a Fock state is fully characterized by the number of times $n_{\mathbf{k}}$ that each value $\mathbf{k} \in \mathbb{R}^3$ appears among the particular $\mathbf{k}_i$ listed. In other words, we may specify a Fock state by a list of occupancy numbers $\{n_{\mathbf{k}}, \mathbf{k} \in \mathbb{R}^3\}$ with $n_{\mathbf{k}} \in \mathbb{N}$.

Under the standard interpretation, a Fock state describes a situation with $n_{\mathbf{k}}$ particles of three-momentum $\mathbf{k}$. For instance, $a^+_{\mathbf{k}}|\Omega\rangle = |\mathbf{k}\rangle$ is a single-particle state describing a particle with momentum $\mathbf{k}$, and a generic Fock state is a multi-particle state with $\sum_{\mathbf{k}\in\mathbb{R}^3} n_{\mathbf{k}}$ particles.

We might also write $|\mathbf{k}_1...\mathbf{k}_n\rangle$ as $|\mathbf{k}_1\rangle \otimes ... \otimes |\mathbf{k}_n\rangle$, but this notation is slightly misleading because $|\mathbf{k}_1\mathbf{k}_2\rangle = |\mathbf{k}_1\rangle \otimes |\mathbf{k}_2\rangle$ and $|\mathbf{k}_2\mathbf{k}\rangle = |\mathbf{k}_2\rangle \otimes |\mathbf{k}\rangle$ denote the same state. If we call $\mathcal{H}_1 \approx \mathrm{span}(\{|\mathbf{k}\rangle, \mathbf{k} \in \mathbb{R}^3\})$ the space of single-particle states, then the space $\mathcal{H}_2$ of two-particle states might be defined as $\mathrm{Sym}(\mathcal{H}_1 \otimes \mathcal{H}_1)$, where $\mathrm{Sym}(\cdot)$ indicates taking the quotient of the tensor product by the equivalence relation $a \otimes b \sim b \otimes a$. More generally, $\mathcal{H}_n \equiv \mathrm{Sym}(\otimes_{j=1}^n \mathcal{H}_1) = \mathrm{Sym}(\mathcal{H}_1^{\otimes n})$ is the space of n-particle states, where $\mathrm{Sym}(\cdot)$ indicates the

fully-symmetrized tensor product, i.e. a quotient by the appropriate equivalence relation.

With the above in mind, the Hilbert space for the theory may be written

$$\begin{gathered}\mathcal{H} = \mathcal{F}_{Sym}(\mathcal{H}_1) \equiv \mathbb{C} \oplus \bigoplus_{n=1}^{\infty} \mathcal{H}_n \\ \mathcal{H}_n \equiv \mathrm{Sym}\left(\bigotimes_{j=1}^{n} \mathcal{H}_1\right) \\ \mathcal{H}_1 \approx \mathrm{span}(\{|\mathbf{k}\rangle, \mathbf{k} \in \mathbb{R}^3\}).\end{gathered} \tag{48}$$

Again, the space $\mathcal{H}_1$ is called the single-particle Hilbert space, $\mathcal{H}_n$ is the space of n-particle states, and $\mathbb{C}$ is $\mathcal{H}_0$, the vacuum eigenspace. The construction $\mathcal{F}_{Sym}(\cdot)$ is called the Fock construction and $\mathcal{H} = \mathcal{F}_{Sym}(\mathcal{H}_1)$ is the Fock space.

The construction does not depend on the choice of basis for $\mathcal{H}_1$, but creation and annihilation operators must be defined with respect to single-particle states. The operators $a_{\mathbf{k}}^+$ produce single-particle states in momentum state $|\mathbf{k}\rangle \in \mathcal{H}_1$, but more generically we could define a_v^+ to create a single-particle state $v \in \mathcal{H}_1$ and thereby produce a set of ladder operators a_v, a_v^+. For a change of basis on $\mathcal{H}_1$ defined by $|v_i\rangle = \sum_j M_{ij} |w_j\rangle$, we have $a_v^+ = \sum_j M_{ij} a_w^+$. Applying this formula to the position basis $\{|\mathbf{x}\rangle, \mathbf{x} \in \mathbb{R}^3\}$ of $\mathcal{H}_1$, we have

$$\begin{aligned} a_{\mathbf{x}}^+ &= \textstyle\int_{\mathbb{R}^3} \frac{1}{(2\pi)^3} e^{-i\mathbf{k}\cdot\mathbf{x}} a_k^+ \, \mathrm{d}^3\mathbf{k} \\ a_{\mathbf{x}} &= \textstyle\int_{\mathbb{R}^3} \frac{1}{(2\pi)^3} e^{i\mathbf{k}\cdot\mathbf{x}} a_k \, \mathrm{d}^3\mathbf{k}. \end{aligned} \tag{49}$$

Note that these ladder are different from the ladder operators $\alpha_{\mathbf{x}}^+, \alpha_{\mathbf{x}}$ defined in Equation (14).

2.7 Localized particles

The occupancy numbers suggest that the Fock space has an overall TPS. To be specific, given of a choice of basis $\{|a_i\rangle, i \in S\}$ of $\mathcal{H}_1$, the Fock space may be written with TPS

$$\mathcal{H} \approx \bigotimes_{i \in S} \mathrm{span}(\{|0\rangle_i, |1\rangle_i, |2\rangle_i \ldots\}). \tag{50}$$

Here, $|n\rangle_i$ represents the occupancy number of single-particle state $|a\rangle$. Evidently, this TPS depends on the choice of basis for $\mathcal{H}_1$. The creation and annihilation operators with respect to this single-particle basis will be local in the resulting TPS. If we choose the naive momentum basis $\{|\mathbf{k}\rangle, \mathbf{k} \in \mathbb{R}^3\}$, the resulting TPS is precisely the $\mathbf{k}$-space TPS already discussed. On the other hand, if we choose the position basis $\{|\mathbf{x}\rangle, \mathbf{x} \in \mathbb{R}^3\}$, we do *not* obtain the spatial TPS of Equation (2). We will call this other TPS the Fock position TPS. To see why the spatial TPS and Fock position TPS are inequivalent, recall $\phi(\mathbf{x})$ is local in the spatial TPS. Using Equation (45) to solve for $\phi(\mathbf{x})$ and then using Equation (49), one finds that $\phi(\mathbf{x})$ is of the form $\int_{\mathbb{R}^3} f(\mathbf{x}) a_{\mathbf{x}}^+ + g(\mathbf{x}) a_{\mathbf{x}} \, \mathrm{d}^3\mathbf{x}$ for functions f, g with non-local support. But $a_{\mathbf{x}}^+$ and $a_{\mathbf{x}}$ are local to $\mathbf{x}$ with respect to the Fock position TPS, so $\phi(\mathbf{x})$ cannot be, and the TPS must be different.

Which TPS should be considered significant for spatial locality? The spatial TPS was essentially postulated to have physical significance, e.g. self-adjoint local operators $O_x : \mathcal{H}_{\mathbf{x}} \to \mathcal{H}_{\mathbf{x}}$ were assumed to be the observables accessible to an observer located at position $\mathbf{x} \in \mathbb{R}^3$. Still, we can explore the question further by considering the states $a_{\mathbf{x}}^+ |\Omega\rangle$ and $\phi(\mathbf{x}) |\Omega\rangle$, created by acting on the vacuum with an operator local to

the Fock position TPS and spatial TPS, respectively. The state $a^+_{\mathbf{x}} |\Omega\rangle$ is the one-particle state associated with the state $|\mathbf{x}\rangle \in \mathcal{H}_1$, so one may be tempted to think it represents a single particle localized to $\mathbf{x} \in \mathbb{R}^3$. On the other hand, the state $\phi(\mathbf{x}) |\Omega\rangle$ also suggests an interpretation as a localized one-particle state. But $\phi(\mathbf{x}) |\Omega\rangle \neq a^+_{\mathbf{x}} |\Omega\rangle$, which is evident from the fact that the spatial TPS and Fock position TPS are different. Which of these two states, if any, should represent a localized particle?

If we accept the spatial TPS, the proposal that $\phi(\mathbf{x}) |\Omega\rangle$ represents a localized particle has the advantage that this state is created by operating locally on the vacuum. Recalling the discussion of microcausality, the state $\phi(\mathbf{x}) |\Omega\rangle$ has other nice properties: it evolves in time like $\phi(\mathbf{x}, t) |\Omega\rangle$, and $\phi(\mathbf{x}, t)$ is an operator whose support grows at the speed of light. Most importantly, measurements local to $y \in \mathbb{R}^4$ will not detect the presence of a particle described by $\phi(x) |\Omega\rangle$ when $(x - y)$ spacelike, i.e. measurements at y will not distinguish between $\phi(x) |\Omega\rangle$ and the vacuum.

One may object that even though $\phi(\mathbf{x}) |\Omega\rangle$ and $\phi(\mathbf{y}) |\Omega\rangle$ are supposed to describe particles local to different points, these states are not orthogonal; indeed, their inner product $\langle\Omega| \phi(\mathbf{x})\phi(\mathbf{y}) |\Omega\rangle$ is related to the Feynman propagator. The implication is that if we make a projective measurement on $\phi(\mathbf{y}) |\Omega\rangle$ that projects onto the ray $\phi(\mathbf{x}) |\Omega\rangle$, then the state $\phi(\mathbf{y}) |\Omega\rangle$ might "collapse" to $\phi(\mathbf{x}) |\Omega\rangle$ for some $\mathbf{x}$ arbitrarily far away. Alternatively, note that the $\phi(\mathbf{x}) |\Omega\rangle$ states are not analogous to the position states $|\mathbf{x}\rangle$ of quantum mechanics: there is no analogous position observable $\hat{\mathbf{X}}$ on the QFT Hilbert space for which $\phi(\mathbf{x}) |\Omega\rangle$ and $\phi(\mathbf{y}) |\Omega\rangle$ are orthogonal eigenstates.

On the other hand, the states $a^+_{\mathbf{x}} |\Omega\rangle$ *are* orthogonal for different $\mathbf{x} \in \mathbb{R}^3$, so they are good candidates for "position eigenstates" of some QFT position observable. This way of thinking is related to the Newton-Wigner localization scheme, in which one constructs such position observables. Using the position states $a^+_{\mathbf{x}} |\Omega\rangle$, any single-particle state may then be written as a superposition $\int_{\mathbb{R}^3} f(\mathbf{x}) a^+_{\mathbf{x}} |\Omega\rangle \ \mathrm{d}^3\mathbf{x}$ for some "wavefunction" $f(\mathbf{x})$. However, if $f(\mathbf{x})$ has compact support, then after any finite time, the support will become infinite under time-evolution by the free Hamiltonian, just as in the case of a free particle wavefunction in non-relativistic quantum mechanics. In addition, under a Lorentz transformation, the support of the "wavefunction" $f(\mathbf{x})$ may transform from compact to infinite, so this sense of spatial localization is not Lorentz-invariant. These results are troubling when thinking of $a^+_{\mathbf{x}} |\Omega\rangle$ as a localized particle. Perhaps more importantly, the notion of a global position observable may not even be desirable. As discussed in the beginning of Section 2, global spatial wavefunctions that collapse to points upon global measurement are antithetical to the general notion of spacetime locality. We expect that measurements performed in some region of space will be ignorant of the "wavefunction" arbitrarily far away. Based on this discussion, one might abandon the notion of particles as fundamental objects with position wavefunctions, instead imagining a Hilbert space like that of Equation (2), whose states encode the outcomes of local measurements associated with $\phi(\mathbf{x})$ and $\pi(\mathbf{x})$. Particles are then a useful formalism for calculating dynamics, but they are not objects whose positions are to be measured in the traditional quantum-mechanical sense.

For an independent discussion related to the one above, see [8, 9].

Experiments should serve as the guide for any interpretation we impose. Whether we interpret either $a^+_{\mathbf{x}} |\Omega\rangle$ or $\phi(\mathbf{x}) |\Omega\rangle$ as spatially localized, we demand consistency with the predictions discussed in Section 2.4. Of course, scattering experiments are not about localized particles, so there is no explicit constraint on our interpretation of locality due to experiment. But when translating correlation functions into scattering predictions, the standard derivation (as in [3]) includes several interpretive elements, including the

use of particle wavepackets distributed over space to represent incoming and outgoing particles. Are these wavepackets spread out over space in the sense of a superposition of states $a^+_{\mathbf{x}} |\Omega\rangle$, or are they a superposition of states $\phi(x) |\Omega\rangle$? (It seems that actually, for sufficiently spread out wavepackets, these choices are equivalent. Similarly, the spatial TPS and Fock position TPS coincide in the formal non-relativistic limit $c \to \infty$.)

We ignore this question of interpretation. For now, we seek to formalize the mathematical structure of quantum field theory, bearing in mind that our formalization should ultimately produce correlation functions or other outputs used to make predictions. The right mathematical structure for this task may even suggest interpretations not considered before.

3 Seeking rigor

After sketching heuristic pictures of quantum field theory, we are ready to begin various attempts to formalize the structure. In the following subsections, we explain some of the tools used in formalization. Most mathematical definitions will be introduced, but not all exposition will be fully self-contained.

3.1 Hilbert spaces

To many physicists, the ostensible setting of quantum field theory is the Hilbert space. Yet we are rarely working in a well-defined Hilbert space. In order to introduce the subtleties of working in infinite dimensions, I will briefly review Hilbert spaces and related concepts.

The linear-algebraic structure of a vector space usually becomes inadequate when working in infinite dimensions. In a vector space equipped only with its linear-algebraic structure, there is no natural notion of adding infinitely many vectors. One therefore introduces topology in order to consider the convergence properties of sequences. The basic structure is a topological vector space, i.e. a vector space with topology such that addition and scaling are continuous. (The topological aspects of a topological vector space only become interesting in infinite dimensions – the only Hausdorff topology on $\mathbb{R}^n$ or $\mathbb{C}^n$ is the standard one.) A more rigid structure is the Banach space, a topological vector space in which the topology is induced by a norm such that the space is Cauchy complete in the norm. An even more rigid structure is the Hilbert space, a Banach space in which the norm is induced by a sesquilinear, conjugate-symmetric, positive-definite inner product. Here, Banach spaces and Hilbert spaces will be taken over the field $\mathbb{C}$.

The algebraic span of a set of vectors consists of all finite linear combinations, and likewise algebraic linear independence involves only finite combinations. These notions are used to define the algebraic basis, or Hamel basis. The cardinality of this basis is the algebraic dimension of the space. Meanwhile, in Hilbert spaces, one may have a set of orthonormal vectors whose algebraic span is merely dense in the whole space. This is the usual notion of an orthonormal basis in a Hilbert space, and the cardinality of the basis is the Hilbert dimension. It turns out that any Hilbert space with infinite algebraic dimension will actually have uncountable algebraic dimension. On the other hand, a Hilbert space may easily have countable Hilbert dimension, and these spaces are called separable. One example of a separable Hilbert spaces is $L^2(\mathbb{R})$, the

space of square-integrable functions $\mathbb{R} \to \mathbb{C}$.[3] More generally, for any set X with measure μ, the space $L^2(X, \mu)$ of functions square-integrable with respect to the measure μ is a Hilbert space. (See Section 3.5 as well.) Another example of a separable Hilbert space is ℓ^2, the space of square-summable sequences $\{f : \mathbb{N} \to \mathbb{C}\}$. In fact, all separable Hilbert spaces are isomorphic. Note finally that operators on Hilbert spaces are often defined only on a dense subspace. For instance, the momentum operator on $L^2(\mathbb{R})$ is only defined on differentiable functions, and the set of differentiable functions is only dense in $L^2(\mathbb{R})$. The definition of a self-adjoint operator on $\mathcal{H}$ only requires that the operator be defined on a dense subspace.

Naive constructions of Hilbert spaces such as $\mathcal{H} \approx \bigotimes_{\mathbf{x} \in \mathbb{R}^3} L^2(\mathbb{R})$ of Equation (2) do not correspond to actual Hilbert spaces, at least not obviously. Still, spaces like this are often easy to define as mere vector spaces, without the extra structure. Why do we need Hilbert spaces for quantum theory? Well-defined inner products are necessary for the standard probabilistic interpretation of the theory. Defining an inner product will often present us with the most difficulty when constructing a Hilbert space. On the other hand, completeness is a less difficult requirement: any incomplete space may simply be completed. That is, given an incomplete "pre-Hilbert space" with only an inner product, there exists a canonical embedding into a Hilbert space (called the completion) such that the image of the pre-Hilbert space is dense. Finally, the separability condition is not strictly necessary for quantum theory, but separable Hilbert spaces are often sufficient and more mathematically tractable. The Wightman axioms for QFT will take a separable Hilbert space as their starting point.

To see why Equation (2) does not obviously define a Hilbert space, first consider the tensor product of just two separable Hilbert spaces $\mathcal{H}_A \otimes \mathcal{H}_B$. As a vector space, the space is easy to define: for instance, the algebraic basis is just given by the Cartesian product of the bases of the individual spaces. This space becomes a pre-Hilbert space when equipped with an inner product $\langle a \otimes b, c \otimes d \rangle = \langle a, c \rangle \langle b, d \rangle$, and we take the completion to form a Hilbert space. Now, consider the tensor product of countably-many separable Hilbert spaces. For example, take the Hilbert space $\mathcal{H}_{sc}$ we might associate to an infinite spin chain,

$$\mathcal{H}_{sc} = \bigotimes_{i \in \mathbb{N}} \mathbb{C}^2. \tag{51}$$

If we only wanted to define $\mathcal{H}_{sc}$ as a vector space, we could use the algebraic tensor product, which may be defined as the set of multilinear maps $\{f : \prod_{i \in \mathbb{N}} \mathbb{C}^2 \to \mathbb{C}\}$ from the infinite Cartesian product. This vector space contains vectors of the sort $(a_1 \otimes a_2 \otimes a_3 \otimes ...)$ for $a_i \in \mathbb{C}^2$. However, now we have trouble defining an inner product on the space. For instance, if we try to mimic our construction for finite tensor products, then the ostensibly nonzero vector $a^{\otimes \infty} \equiv a \otimes a \otimes ...$ has zero norm when $|a| < 1$, because $|a^{\otimes \infty}|^2 = \langle a^{\otimes \infty}, a^{\otimes \infty} \rangle = \langle a, a \rangle^{\infty} \to 0$. Therefore, to define an infinite tensor product, we first choose a distinguished unit vector of each tensor factor. For instance, to define $\mathcal{H}_{sc}$, for each tensor factor we will choose the unit vector $|0\rangle \in \mathbb{C}^2$. Then, to define the tensor product Hilbert space, we define a basis of simple tensors (i.e. product states) where all but finitely many of the tensor factors are equal to the distinguished unit vector. In particular, for $\mathcal{H}_{sc}$ we define the basis vectors $(a_1 \otimes a_2 \otimes ...)$ where $a_i \in \mathbb{C}^2$ and $a_i = |0\rangle$ for all but finitely many i. An inner product on this space is now well-defined, and we take the completion to form a true Hilbert space. Thus $\mathcal{H}_{sc}$ will contain vectors like $(|0\rangle\,|0\rangle\,|0\rangle\,...)$, $(|1\rangle\,|0\rangle\,|0\rangle\,...)$, and $\bigotimes_{i \in \mathbb{N}} (|0\rangle + \frac{1}{n}\,|1\rangle)$,

[3] When working with $L^2(\mathbb{R})$, it is important to realize that the so-called position basis kets $|x\rangle$ are not actual vectors of the Hilbert space: the associated "δ-function" wavefunction is not a square-integrable function. To define an orthonormal basis for $L^2(\mathbb{R})$ requires other functions.

but not vectors like $(|1\rangle |1\rangle |1\rangle ...)$. In a sense, our Hilbert space is built around the "vacuum" state $|0\rangle^{\otimes\infty}$.

Finally we return to our naive Hilbert space $\mathcal{H} \approx \bigotimes_{\mathbf{x}\in\mathbb{R}^3} \mathcal{H}_{\mathbf{x}}$. What does this uncountably-indexed tensor product mean? It may make sense as a vector space, but there is no obvious inner product. It turns out that in this example, we are better off working with the formulation $\mathcal{H} \approx L^2(C(\mathbb{R}^3), \mu)$ and searching for a legitimate measure μ. This approach is discussed in Section 3.5.

3.2 Fock space

So far, none of our original expressions for a Hilbert space have found obvious definitions. The Fock space of Equation (48) will be an exception. There, we had

$$\mathcal{H} = \mathbb{C} \oplus \bigoplus_{n=1}^{\infty} \mathrm{Sym} \left(\bigotimes_{j=1}^{n} \mathcal{H}_1 \right). \tag{52}$$

The only part of the expression demanding further attention is the infinite direct sum. We may define a countably-infinite direct sum of separable Hilbert spaces as the vector space formed by finite linear combinations of vectors in each direct summand; we then define an inner product by considering vectors in different summands to be orthonormal, and we take the completion. The resulting space is then countable: we have a countable orthonormal basis formed from the countable union of the countable orthonormal bases for each summand.

The Fock space defined this way will contain states like $\sum_{n\in\mathbb{N}} \frac{1}{n} a^{\otimes n}$ for any $a \in \mathcal{H}_1$, which exhibit a superposition of multi-particle states with arbitrarily high particle number. However, the Fock space will contain no "infinite-particle state" $(a \otimes a \otimes a ...)$, nor even states like $\sum_{n\in\mathbb{N}} \frac{1}{\sqrt{n}} a^{\otimes n}$. In this sense, the Fock space is built around the zero-particle vacuum state.

The construction of Equation (50) also makes sense more rigorously. In that discussion, we considered the Fock space as an overall tensor product, where the TPS corresponded to the occupancy number notation. For each basis element $|a\rangle$ of the single-particle Hilbert space $\mathcal{H}_1$, there was a corresponding tensor factor with basis $\{|0\rangle, |1\rangle, |2\rangle, ...\}$ encoding the occupancy number of that single-particle basis state $|a\rangle$. For separable Hilbert space $\mathcal{H}_1$ with countable basis $\{|a_i\rangle, i \in \mathbb{N}\}$, we may indeed define the countably-infinite tensor product

$$\mathcal{H} = \bigotimes_{i\in\mathbb{N}} \mathrm{span}(\{|0\rangle_i, |1\rangle_i, |2\rangle_i, ...\}). \tag{53}$$

As before, we must specify a distinguished unit vector in each tensor factor for this expression to be well-defined. If we choose $|0\rangle_i$ as our distinguished vector, then we have again built our space around the zero-particle vacuum state, and this construction will correspond to the Fock space construction.

For a more extensive discussion of the rigorous Fock space construction, including ladder operators and occupancy number notation, see for instance [10].

3.3 Smeared fields

Though the Fock space has been made rigorous, we are still unequipped to formalize even the free theory for a scalar field. The remaining task is to specify field operators $\phi(\mathbf{x}), \pi(\mathbf{x})$ and Hamiltonian H along with

transformations under the Poincaré group G. Unfortunately, the operators $\phi(\mathbf{x})$ are problematic. One may be suspicious of them first on physical grounds: are observables truly localizable to a single point $\mathbf{x} \in \mathbb{R}^3$? We might also be suspicious on mathematical grounds. The Fock space construction is rigorous, but the idea of the exactly localized $\phi(\mathbf{x})$ operators traces to Equation (2), with its naive uncountably-indexed TPS. If we use the Fock space and attempt to calculate $\langle\Omega|\,\phi(\mathbf{x})\phi(\mathbf{x})\,|\Omega\rangle = |\phi(\mathbf{x})\,|\Omega\rangle\,|^2$ for the naively-defined operator $\phi(\mathbf{x})$, we obtain $|\phi(\mathbf{x})\,|\Omega\rangle\,|^2 \approx \infty$ as in [5], another indication that the $\phi(\mathbf{x})$ operators are problematic.

The exactly localized operator $\phi(\mathbf{x})$ is loosely analogous to the state $|\mathbf{x}\rangle \in L^2(\mathbb{R})$. This analogy will suggest an avenue for formalization. While $|\mathbf{x}\rangle$ does not represent a true element of $L^2(\mathbb{R})$, the function $f : \mathbb{R} \to \mathbb{R} \subset \mathbb{C}$ does. We might write

$$|f\rangle = \int_{\mathbb{R}} f(\mathbf{x})\,|\mathbf{x}\rangle\ \mathrm{d}\mathbf{x} \tag{54}$$

where the LHS is well-defined as an element $f \in L^2(\mathbb{R})$, while the RHS is just a formal expression. Likewise, for functions $f : \mathbb{R}^3 \to \mathbb{R}$, we might write

$$\phi(f) = \int_{\mathbb{R}^3} |\mathbf{x}\rangle\,\phi(\mathbf{x})\ \mathrm{d}^3\mathbf{x} \tag{55}$$

where the LHS is an operator for which we hope to provide a rigorous definition, while the RHS is a formal expression indicating our intention.

Therefore, rather than define a set of operators $\{\phi(\mathbf{x}) : \mathbf{x} \in \mathbb{R}^3\}$, we define the map $\phi : f \mapsto \phi(f)$, where f is some well-behaved function $f : \mathbb{R}^3 \to \mathbb{R}$ and $\phi(f)$ is a densely-defined operator on $\mathcal{H}$. We also require that the map ϕ is linear in f. Hence ϕ is called an operator-valued distribution. (Recall that a distribution is usually understood to be a real- or complex-valued continuous linear functional acting on a function space. An operator-valued distribution merely has different codomain.) The domain of ϕ is often restricted to be a sufficiently well-behaved class of functions, such as the Schwartz space $\mathscr{S}(\mathbb{R}^3) \subset C(\mathbb{R}^3)$ of rapidly decaying functions. The field operator $\phi(f)$ is sometimes called a smeared field, smeared by the function f. For a function f with support localized in a small neighborhood of $\mathbf{x}$, $\phi(f)$ approximates the original expression $\phi(\mathbf{x})$. In general, $\phi(f)$ represents an observable local to the support of f.

One may define both ϕ and π as operator-valued distributions and request that they satisfy the CCR given in Equation (13), adapted for smeared fields. One may arrive at the appropriate form of the smeared-field CCR by formally manipulating the expressions of Equation (13). It is then possible to rigorously define operator-valued distributions ϕ and π acting on the Fock space that satisfy the informal definition of scalar quantum field theory given in Section 2.3, appropriately modified for smeared fields. The details of this construction are clearly explained in [10].

Although a free field theory may be constructed with these spatially-smeared fields, in general interacting theories it becomes necessary to consider fields smeared over spacetime. Much of the reasoning that motivates spatial smearing similarly motivates temporal smearing. Thus we define ϕ as an operator-valued distribution acting on the space $\mathscr{S}(\mathbb{R}^4) \subset C(\mathbb{R}^4)$.

Using spacetime-smeared fields allows one to shift attention from ϕ, π to consider only ϕ. In Section 2.3, we discussed that $\phi(\mathbf{x})$ and $\pi(\mathbf{x})$ should together generate the algebra of all observables local to $\mathbf{x}$. Meanwhile, at least naively, the algebras of operators on $\mathcal{H}_\mathbf{x}$ should together generate the algebra of operators on $\mathcal{H}$, so the algebra on $\mathcal{H}$ is in turn generated by the fields $\pi(\mathbf{x})$ and $\phi(\mathbf{x})$. Thus, naively, any operator may be written

as an algebraic combination of field operators $\phi(\mathbf{x}), \pi(\mathbf{x})$. We also have the relation $\partial_t \phi(\mathbf{x}) = \frac{\partial \mathcal{H}}{\partial \pi(\mathbf{x})} = \pi(\mathbf{x})$, so we imagine any operator may be written in terms of operators $\phi(\mathbf{x})$ and $\partial_t \phi(\mathbf{x}, t)|_{t=0}$. Note that $\partial_t \phi(\mathbf{x})$ may be approximated by a spacetime-smeared field ϕ for an appropriate smearing function (i.e. a smearing function that looks like $\delta'(t)$). Then we conclude that the algebra of operators on $\mathcal{H}$ should be generated by the spacetime-smeared fields ϕ. If this conclusion appears haphazard, it may be taken as definitional, as in the axioms soon discussed. Either way, are fortunate to no longer have a need for the operator $\pi(\mathbf{x})$ when considering spacetime-smeared fields, because the CCR that define the relationship between $\pi(\mathbf{x})$ and $\phi(\mathbf{x})$ have no obvious spacetime-smeared counterpart.

With spacetime-smeared fields, the correlation functions of Equation (27) become distributional. That is, we have a multilinear map

$$\begin{aligned} \mathscr{S}(\mathbb{R}^4) \times ... \times \mathscr{S}(\mathbb{R}^4) &\to \mathbb{C} \\ (f_n, ..., f_1) &\mapsto \langle \Omega | \, \mathcal{T}(\phi(f_n)...\phi(f_1)) \, | \Omega \rangle \, . \end{aligned} \tag{56}$$

Given certain continuity conditions, the multilinear n-point correlation function may be understood as as a single distribution $\mathcal{W}_n$ acting on $\mathscr{S}(\mathbb{R}^{4n}) \approx \mathscr{S}(\mathbb{R}^4) \otimes ... \otimes \mathscr{S}(\mathbb{R}^4)$. (To see why, recall the definition of tensor products involving multilinear maps.) The distributions $\{\mathcal{W}_1, \mathcal{W}_2, ...\}$ are called the Wightman distributions.

Though operator-valued distributions are more amenable to rigorous construction, they present a new difficulty: how can we multiply them? In general, multiplication is not defined for distributions. For non-smeared fields, we imagine simply using operator multiplication to define expressions like $\phi(x)^4$. But with operator-valued distributions, we have no corresponding expression. (One might mistakenly imagine $\phi(f)^4$ is the right expression, but quickly examining Equation (55) shows this is not the case.) The inability to multiply distributions is a serious obstacle to rigorously constructing interacting theories – if we knew how to define $\phi(\mathbf{x})^4$, we might just take a rigorously-constructed free theory and define an interaction Hamiltonian by adding this term.

It turns out that the above difficulty is closely related to the need for normal ordering in conventional QFT calculations. In some cases, multiplying field operators may be defined by a formalization of the normal ordering procedure [11].

3.4 From Hilbert space to algebra

We have seen that the Fock space is built around the vacuum of the free theory. We encounter difficulties when we try to define interacting theories from within the Fock space. In a sense that will become clear, the Fock space restricts us to perturbative or non-rigorous calculations.

Yet if we are looking for a separable Hilbert space, and all of these spaces are isomorphic, in which sense can we distinguish the Fock space as a poor choice? The answer is that when we build an interacting theory from the Fock space, we are not just working with the Fock space as an abstract Hilbert space. We actually begin with the Fock space, the free field operators on the space, and the transformation properties of these fields under the Lorentz group. That is, we use the kinematical setup of the Fock space in order to write down the interacting theory.

To be more specific, consider the following ill-fated attempt to define an interacting theory. We begin with the Fock space. In this setting, we may rigorously define the field operators as operator-valued distributions.

We prescribe a representation of the Lorentz group such that these fields transform like $\phi(x) \mapsto \phi(\Lambda x)$, $\pi(x) \mapsto \phi(\Lambda x)$. We also define a free Hamiltonian operator and compatible Poincaré representation.

Now we attempt to write down the interacting theory. In the Schrödinger picture, there is no distinction between an "interacting" field and a "free field" – this distinction is only relevant when considering time-evolution in the Heisenberg picture. Thus we use the "free" field operators to write down our interaction Hamiltonian in the Schrödinger picture. In particular, we use an operator such as ϕ^4, where ϕ denotes the free field. Furthermore, we these Schrödinger-picture field operators should transform as $U^{-1}(a)\phi(x)U(a) = \phi(x+a)$ under spatial translations, for unitary representation U of the Poincaré group with translations a.

From this perspective, it seems the very notion of what we *mean* by an interacting ϕ^4-theory relies on the setup of free fields in the Fock space. Yet the Fock space is not the appropriate setting for the interacting theory. Even if one could manage to write down an interacting Hamiltonian on the Fock space that resembled the ϕ^4 Hamiltonian, it appears unlikely that the ground state would exist in the Fock space. To see why, note that the interaction Hamiltonian does not preserve any subspace with finite particle number. Hence the vacuum will be a superposition of multi-particle states with arbitrarily large numbers of particles. As discussed in Section 3.2, we cannot expect such "states" to be true states of the Fock space, although some such states exist.

A variety of results under the name "Haag's theorem" state that the Fock space in fact cannot contain the interacting vacuum. Results of this sort may be formulated with varying levels of rigor, and they may be phrased in the language of the Wightman or Haag-Kastler axioms [12]. Here, the result is presented loosely, as partial motivation for the transition to a more algebraic framework. The formulation follows [13], where it is stated additionally in the language of C^*-algebras.

At the heart of this variant of Haag's theorem is the following fact. We already have a very rigid structure when we posit the free particle Fock space with the associated field operators and representation of the Poincaré group. We would like to introduce an interacting Hamiltonian along with a new representation of the Poincaré group, whereby the interacting Hamiltonian is the generator of time-translation. However, we want to introduce the interacting Hamiltonian on this same Fock space, and we would like the Hamiltonian to "use the same fields," which is to say that the new representation of the Poincaré group should translate the fields in the same way. This turns out to be impossible.

Informal theorem (Haag's theorem)**.** *Assume one is given the following:*

1. *Fock space $\mathcal{H}$ with canonical fields $\phi(x)$ and $\pi(x)$,*
2. *Free Hamiltonian H_F and interacting Hamiltonian $H_I^{(\lambda)}$ parameterized by coupling strength λ, with unique respective vacuum states $|\Omega\rangle_0$ and $|\Omega\rangle_\lambda$,*
3. *Unitary representations U_F and $U_I^{(\lambda)}$ of the Poincaré group, such that H_F and H_I are the respective time-translation generators for each representation.*

Assume also that for spatial translations $\boldsymbol{a}$ in the Poincaré group,

$$\begin{aligned} U(\boldsymbol{a})^{-1}\phi(t,\boldsymbol{x})U(\boldsymbol{a}) &= \phi(t,\boldsymbol{x}+\boldsymbol{a}) \\ U(\boldsymbol{a})^{-1}\pi(t,\boldsymbol{x})U(\boldsymbol{a}) &= \pi(t,\boldsymbol{x}+\boldsymbol{a}) \end{aligned} \tag{57}$$

holds for both $U = U_F$ *and* $U = U_I^{(\lambda)}$*. Then*

$$|\Omega\rangle_0 = |\Omega\rangle_\lambda \,. \tag{58}$$

The setup of the theorem directly formalizes the above discussion.

Briefly, the argument for Haag's theorem proceeds as follows. Condition (57) for U_F and $U_I^{(\lambda)}$ implies that $U_F = U_I^{(\lambda)}$ up to a phase – this fact is not shown, but see [13]. Thus let $U \equiv U_F = U_I^{(\lambda)}$. Note that both vacua are invariant under spatial translations $U(a)$, for instance because spacetime translations commute in the Poincaré group. However, in the free Fock space, there is only one vector (or ray, rather) invariant under spatial translations. Equation (58) follows, and this conclusion contradicts our intention that $\mathcal{H}_I$ represent the interaction Hamiltonian. In particular, it is clear that an interaction term like ϕ^4 will not have the free vacuum as an eigenstate, and this holds for most desirable interaction terms.

To better understand the general phenomenon behind Haag's theorem, let us consider a simpler system. Recall the spin chain Hilbert space $\mathcal{H}_{sc}$ of Equation (51). Like the Fock space, this space was built around a particular state $|0\rangle^{\otimes\infty}$. Now consider the simple Hamiltonian

$$H = \sum_{i\in\mathbb{N}} \sigma_i^z \tag{59}$$

where σ_i^z is the Pauli spin operator at site i. Let $|0\rangle$ be the eigenvector of σ_z with eigenvalue -1. The first problem we encounter is that H is not well-defined: the eigenstate $|0\rangle^{\otimes\infty}$ seems to have eigenvalue $-\infty$, for instance. We might resolve this by "adding an infinite constant." More mundanely, we may simply redefine σ_z to have eigenvalues 0 and 1. Then H will be a well-defined operator, and $|0\rangle^{\otimes\infty}$ will indeed be the groundstate. However, our formalism breaks when we consider a small perturbation

$$H = \sum_{i\in\mathbb{N}} \sigma_i^z + \lambda\sigma_i^x. \tag{60}$$

For nonzero λ, the lower eigenstate of $(\sigma^z + \lambda\sigma^x)$ is some vector $|a\rangle \in \mathbb{C}^2$, and the new vacuum should be $|a\rangle^{\otimes\infty}$. However, this "state" is not a part of our Fock space, nor is the perturbed Hamiltonian even well-defined on the space. What is going on here? Though we are forced to put "state" in quotation marks, it seems obvious that the "state" $|a\rangle^{\otimes\infty}$ is the correct vacuum. On one hand, we might just re-define the Hilbert space $\mathcal{H}_{sc}$, choosing $|a\rangle$ as our distinguished vector when we construct the infinite tensor product. In that case, the perturbed Hamiltonian will be a legitimate operator and $|a\rangle^{\otimes\infty}$ will be the legitimate vacuum. Likewise, in the case of the Fock space for QFT, we could again choose to work in a new space built around the "right" state. For instance, we could define a Hilbert space using the TPS of the Fock space given in Equation (60), but choosing a new distinguished vector with which to construct the tensor product. Of course, this would still be a difficult task: which vector would we choose? And in general, we may want to work with states like $|0\rangle^{\otimes\infty}$ and $|1\rangle^{\otimes\infty}$ both at once. Consider an extreme example given by the doubly-infinite spin chain with ferromagnetic Heisenberg XXY Hamiltonian. There we have a continuum of vacua of the form $|b\rangle^{\otimes\infty}$ for any $|b\rangle \in \mathbb{C}^2$ with $\langle b|b\rangle = 1$. We need some formalism to handle all of these states at once, especially in order to discuss phenomena like phase transitions.

One solution to is to somehow add all of the separable Hilbert spaces built around each state of interest.

For instance, considering the unperturbed and perturbed Hamiltonian of Equation (60), we may take the direct sum of Hilbert spaces built around the $|0\rangle^{\otimes\infty}$ and $|a\rangle^{\otimes\infty}$ states. Then we could consider both vacua as part of the same space. Even in the case of a continuum of vacua, we could formalize the notion of taking the direct sum of all of these spaces, and we would obtain an inseparable Hilbert space.

But there is a better solution, from both a mathematical and physical perspective. As motivation, first consider the case of the spin chain. One way to manage the various the various "states" is to instead consider the objects $\sigma_i^{x,y,z}$. Taking the operators as primary, the states become objects that assign operators to expectation values. Indeed, the states $|0\rangle^{\otimes\infty}$ and $|1\rangle^{\otimes\infty}$ are both well-defined as maps from local observables to expectation values. That is, for a state $|\psi\rangle$ we have $|\psi\rangle : \sigma_i^{x,y,z} \to \mathbb{R}$, $\sigma_i^{x,y,z} \mapsto \langle\psi| \sigma_i^{x,y,z} |\psi\rangle$. The algebra of operators may then be the scaffolding around which we build our states, without worrying about building Hilbert spaces to house them. From the physical perspective, too, it makes sense to consider observables as primary. Observables correspond to operations one may perform in the lab. The state then describes a system (or, alternatively, one's knowledge about a system) by assigning expectation values to various observables. Whether or not these states live in the same Hilbert space is immaterial.

In our particular case of QFT and the free particle Fock space, there is further reason to consider the algebra of observables as primary. (One often says "algebra of observables" when referring to the algebra of operators, which are actually not all observable, i.e. self-adjoint.) When we attempt to write down an interacting theory following the informal rules of Section 2.3, in a sense we do not care "which" Hilbert space $\mathcal{H}$ we use so long as the conditions of the informal axioms hold. We may then ask, which Hilbert space should we use? The following discussion will be somewhat informal precisely because we are developing notions to motivate the algebraic formalism before using that formalism. The argument may be re-read in a more rigorous light after discussing C^*-algebras and their representations.

Consider two isomorphic choices of Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ upon which the operators $\phi_1(\mathbf{x}), \pi_1(\mathbf{x})$ and $\phi_2(\mathbf{x}), \pi_2(\mathbf{x})$ are defined, respectively. Informal reasoning led us to believe that the algebras $\mathcal{A}_1$ and $\mathcal{A}_2$ of operators on $\mathcal{H}_1$ and $\mathcal{H}_2$ are generated by the field operators $\phi_1(\mathbf{x}), \pi_1(\mathbf{x})$ and $\phi_2(\mathbf{x}), \pi_2(\mathbf{x})$. Meanwhile, considered as abstract algebras, $\mathcal{A}_1$ and $\mathcal{A}_2$ are completely specified by the CCR. Hence the algebras are isomorphic via an isomorphism $F : \mathcal{A}_1 \mapsto \mathcal{A}_2, F(\phi_1) = \phi_2, F(\pi_1) = \pi_2$. If there additionally exists a unitary isomorphism $U : \mathcal{H}_1 \to \mathcal{H}_2$ such that $F(\cdot) = U^{-1}(\cdot)U$, then we will call U an equivalence of the Hilbert spaces and associated operators, where the operators are understood to come with an identification F between operators on $\mathcal{H}_1$ and $\mathcal{H}_2$. Otherwise, the Hilbert spaces and associated operators are called inequivalent.

In the case of an equivalence, there is no meaningful difference between working with $(\mathcal{H}_1, \phi_1(\mathbf{x}), \pi_1(\mathbf{x}))$ or $(\mathcal{H}_2, \phi_2(\mathbf{x}), \pi_2(\mathbf{x}))$. For instance, consider Hamiltonians H_1 and H_2 expressed in terms of the field operators on $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. Assume also that these Hamiltonians are the "same," in the sense that the expression for $H_2 \mapsto H_1$ when we replace $\phi_1, \pi_1 \mapsto \phi_2, \pi_2$ in the expressions for the Hamiltonians. Then a unitary equivalence implies $H_2 = UH_1U^{-1}$, which in turn means that properties of the Hamiltonian like the spectrum or the existence of a vacuum state will be identical for $\mathcal{H}_1$ and $\mathcal{H}_2$. On the other hand, if the choices $(\mathcal{H}_1, \phi_1(\mathbf{x}), \pi_1(\mathbf{x}))$ or $(\mathcal{H}_2, \phi_2(\mathbf{x}), \pi_2(\mathbf{x}))$ were inequivalent, we would not be able to make these statements about $\mathcal{H}_1$ and $\mathcal{H}_2$, and we would generically be working with a different structure.

Is it possible to have such an inequivalence when the Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ are isomorphic? In finite dimensions, the answer is no:

Proposition. *Consider two isomorphic finite-dimensional Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2$. Let $\mathcal{A}_1 = End(\mathcal{H}_1)$ and $\mathcal{A}_2 = End(\mathcal{H}_2)$ be the algebra of operators on $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. Assume $\mathcal{A}_1$ and $\mathcal{A}_2$ are isomorphic as algebras via the isomorphism $F : \mathcal{A}_1 \to \mathcal{A}_2$, i.e. F preserves multiplication, addition, scaling, and taking adjoints. Then there exists a unitary isomorphism $U : \mathcal{H}_1 \to \mathcal{H}_2$ such that the associated isomorphism $\mathcal{A}_1 \to \mathcal{A}_2,\ A_1 \mapsto U^{-1}A_1U$ is given by F.*

If the above were true in infinite dimensions as well, it would tell us that for any choice of Hilbert space and field operators that have the prescribed algebra, we would be working with isomorphic structures. In particular, it would tell us that if we had a Hilbert space with the algebra of fields prescribed by the CCR, then the questions like "Does the Hamiltonian with interaction term ϕ^4 have a vacuum in the Hilbert space?" would not depend on our choice of Hilbert space. This would be bad news in light of Haag's theorem, because then no choice of Hilbert space (and associated fields) would be different from the Fock space, which is inadequate for interacting theories. Fortunately, the above proposition is false in infinite dimensions; the existence of "inequivalent" choices saves us.

These arguments are sufficient motivation to formalize the idea of taking observables as primary. We want a mathematical structure that abstractly captures the idea of an algebra of bounded operators *without* making reference to Hilbert spaces. This is the theory of abstract operator algebras, and in particular we consider C^*-algebras. This abstract algebra, in addition to having the usual addition, multiplication, and scaling, should have an operation $(\cdot)^* : A \mapsto A^*$ that abstracts the notion of adjoints. With such an operation, one may distinguish between ordinary elements of the algebra and observables, i.e. the elements that satisfy $A = A^*$, which we would call self-adjoint operators in the Hilbert space setting. Our new notion of state is that of a complex-valued linear functional acting on the the algebra. These abstract states acting on the algebra are analogous to (possibly mixed) states on a Hilbert space, where the state ρ on the Hilbert space is considered as a map $\rho : O \mapsto \mathrm{Tr}(\rho O)$. The state should take values in $\mathbb{R}^+ \subset \mathbb{C}$ when it acts on observables.

These notions are directly formalized in the C^*-algebraic structure. A C^*-algebra is a Banach algebra with extra structure given by the $(\cdot)^*$ operation. A Banach algebra $\mathcal{A}$ is a Banach space which is also an algebra, obeying the additional condition that $|xy| \leq |x||y|\ \forall x, y \in A$, which ensures that multiplication in the algebra is continuous. (I will assume all algebras have unit.) A C^*-algebra is then defined as a Banach algebra $\mathcal{A}$ with an operation $* : \mathcal{A} \mapsto \mathcal{A}$ such that $\forall x, y \in \mathcal{A}$ and $\lambda \in \mathbb{C}$, we have $x^{**} = x$, $(x+y)^* = x^* + y^*$, $(xy)^* = y^*x^*$, $(\lambda x)^* = \lambda^* x$, and $|x^*x| = |x||x^*|$

For a standard example of a C^*-algebra – in particular, the example which motivated our definition – consider the algebra $B(\mathcal{H})$ of bounded operators on a Hilbert space $\mathcal{H}$. Here, bounded refers to boundedness in the operator norm $|O| \equiv \sup_{v \in \mathcal{H} s.t. |v|=1} |Ov|$ for operator $O : \mathcal{H} \to \mathcal{H}$. Using this norm, $B(\mathcal{H})$ is already a Banach algebra. It may then be considered as a C^*-algebra when the action of the $(\cdot)^*$ operator is defined to coincide with taking the adjoint of the operator. More generally, any C^*-algebra is isomorphic to a sub-algebra $\mathcal{A} \subset B(\mathcal{H})$, where the latter is again considered as a C^*-algebra with the operator norm.

Given a C^*-algebra $\mathcal{A}$ and Hilbert space $\mathcal{H}$, a map $\pi : \mathcal{A} \mapsto B(\mathcal{H})$ is called a C^*-algebra representation or $*$-representation (here, just "representation") if it is additive, multiplicative, and preserves the unit and $(\cdot)^*$ operation. Two representations $\pi_1 : \mathcal{A} \mapsto B(\mathcal{H}_2)$ and $\pi_2 : \mathcal{A} \mapsto B(\mathcal{H}_2)$ of the same algebra are called unitarily equivalent if there exists a unitary isomorphism $U : \mathcal{H}_1 \mapsto \mathcal{H}_2$ such that $\pi_2(A) = U\pi_1(A)U^{-1}\ \forall A \in \mathcal{A}$. A representation π is called faithful if it is injective, and a faithful representation therefore defines a C^*-

algebraic isomorphism onto its image. A representation is called irreducible if its image acts transitively on the associated Hilbert space. For instance, the identity representation $\pi : B(\mathcal{H}) \to B(\mathcal{H})$ of the C^*-algebra $B(\mathcal{H})$ on $\mathcal{H}$ is both faithful and irreducible.

The proposition discussed earlier in the section then becomes

Proposition. *All faithful, irreducible representations of a finite-dimensional C^*-algebra are unitarily equivalent.*

On the other hand, this proposition is false for arbitrary C^*-algebras. The discussion following Haag's theorem may also be reformulated in the language of inequivalent representations. If the field operators are formalized as part of an abstract C^*-algebra (whose algebraic structure is determined by the CCR), it turns out there will be inequivalent representations of the field operators as operators on a Hilbert space. The choice of the Fock space and associated field operators is a specific choice among these inequivalent representations. In our previous discussion, using "the same field operators" but on a "different Hilbert space" then refers to using an inequivalent representation of the algebra. Similarly, the different constructions of the Hilbert space for the spin chain discussed earlier may be understood as inequivalent representations of the Pauli spin algebra.

3.5 Euclidean path integral

We have motivated a change in focus from the Hilbert space to the algebra of observables, the route taken by the AQFT program. We have both mathematical and physical motivations for such a transition. However, to date, another approach has been more successful in producing models of simple interacting theories. This is approach is inspired by the path integral, and it is usually referred to as constructive quantum field theory.

At first, we might attempt to formalize the Hilbert space of Equation (5),

$$\mathcal{H} \approx L^2(C(\mathbb{R}^3), \mu). \tag{61}$$

To briefly review, recall that a measure is a map that assigns sizes to subsets of a space; generally, the measure only assigns sizes to a certain collection of subsets called the measurable sets. For a set X, the collection $\mathcal{F} \subset \{U : U \subset X\}$ of measurable subsets is called a σ-algebra. A σ-algebra must in particular contain $X \subset P(X)$ and be closed under countable unions, countable intersections, and complements. A measure μ is then a map $\mu : \mathcal{F} \to \mathbb{R}^+ \cup \{\infty\}$ such that μ is additive on countable collections of disjoint sets and $\mu(\emptyset) = 0$. Most importantly for our purposes, when a space X is equipped with a measure, there is a general theory of integration that allows one to define integrals $\int_X f(x)\ \mathrm{d}\mu$ for for suitable functions $f : X \to \mathbb{R}, \mathbb{C}$. With this definition of integration, $L^2(X, \mu)$ becomes a Hilbert space with inner product $\langle f, g\rangle = \int_{x\in X} f^*(x)g(x)\ \mathrm{d}\mu$.

We need a measure μ on the space $C(\mathbb{R}^3)$ of field configurations in order to define integration on the space and thus define the space of square-integrable functions. We would then have the Hilbert space $\mathcal{H}$ for the wavefunctional with inner product

$$\langle \Psi_2, \Psi_1\rangle = \int_{\phi\in C(\mathbb{R}^3)} \Psi_1[\phi]^* \Psi_2[\phi]\ \mathrm{d}\mu, \tag{62}$$

where the measure μ replaces our informal $\mathcal{D}\phi$.

Because $C(\mathbb{R}^3)$ is a vector space, our first instinct might be for μ to emulate the Lebesgue measure, a canonical measure on $\mathbb{R}^n$. The Lebesgue measure assigns the standard volume to boxes in $\mathbb{R}^n$, and it further satisfies several intuitive properties. These properties include: (1) every open set is measurable and (2) the measure of a subset is invariant under translations $T_x : y \mapsto x + y$. We would like these properties for μ. To formulate (1) and (2) for a general space X with measure μ, X must have the notion of open sets and translations, hence we need a topological vector space. If we restrict our attention to bounded functions, the function space $C_b(\mathbb{R}^3)$ may indeed be considered as a separable Banach space with appropriate norm. We then ask, is there a measure μ on a separable Banach space that satisfies the same properties of the Lebesgue measure?

Theorem (No infinite-dimensional "Lebesgue" measure)**.** *A separable Banach space with a nonzero measure μ satisfying properties (1) and (2) must assign infinity to all open sets.*

For a quick proof of this common statement, see [14]. The result means we cannot find a good formalization of the desired measure on field configurations to produce $L^2(C(\mathbb{R}^3)$. To solve the problem, we shift focus to Equation (42). To define this integral, we would need a measure on $C(\mathbb{R}^4)$, the space of field configurations over space*time*. The integral would lend meaning to correlation functions. With this data, we might be satisfied even without an explicit construction of a Hilbert space. Of course, we cannot directly formalize the measure $\mathcal{D}\varphi$ in Equation (42) for the same reasons discussed. The solution is to not formalize $\mathcal{D}\varphi$ as a measure but to instead formalize $[e^{S_E}\mathcal{D}\varphi]$ of Equation (42). In other words, we want a measure such that $\mathrm{d}\mu \approx [e^{-S_E[\varphi]}\mathcal{D}\varphi]$. The desired maneuver is similar in spirit to a change of variables in ordinary calculus. We have also chosen to focus on the Euclidean formulation of the path integral for the reasons already discussed: even after discretization, the Minkowski path integral expression is ill-defined, whereas the Euclidean expression is not. The no-go theorem stated above will not apply to our hypothetical measure $\mathrm{d}\mu \approx [e^{-S_E[\varphi]}\mathcal{D}\varphi]$ because the measure we want is no longer translation-invariant (due to terms in the Lagrangian that are nonlinear in ϕ).

With our desired measure μ, we could define vacuum expectation values by

$$\langle\Omega|\,\mathcal{T}(\phi(\mathbf{x}_n,-it_n)...\phi(\mathbf{x}_1,-it_1))\,|\Omega\rangle = \int_{\varphi\in C(\mathbb{R}^4)} \varphi(\mathbf{x}_n,t_n)...\varphi(\mathbf{x}_1,t_1)d\mu \tag{63}$$

The denominator of Equation (42), which provided normalization, has disappeared. The normalization is unnecessary if we demand a probability measure, i.e. a measure that assigns measure 1 to the entire space. To understand the integrand on the RHS as a function on the domain of integration, it may help to think of $\varphi(x)$ as a map $\mathrm{Ev}_x : C(\mathbb{R}^4) \mapsto \mathbb{C}$, $\varphi \mapsto \varphi(x)$, where Ev indicates "evaluation at."

This route to formalization is the focus of the Osterwalder-Schrader axioms described in Section 4.3.

4 Axioms for quantum field theory

We have finally developed the tools to motivation to state various systems of axioms for quantum field theory.

4.1 Wightman axioms

The Wightman axioms focus on the Hilbert space, and they are similar to those discussed in section 2.3. There are several variants of these axioms, which were originally posed by Wightman and Gårding. The relatively weak version of the axioms used here follows Glimm and Jafffe [15] because the authors have proven several useful theorems with this version. By these axioms, a scalar quantum field theory is

Definition (Wightman axioms for real scalar quantum field theory)**.** *A quantum field theory is given by a separable Hilbert space $\mathcal{H}$, a unitary representation $U : G \mapsto B(\mathcal{H})$ of the Poincaré group G, and an operator-valued distribution ϕ with domain $\mathscr{S}(\mathbb{R}^4)$ such that $\phi(f)$ is a self-adjoint operator on $\mathcal{H}$, as described in Section 3.3. The theory must additionally satisfy*

1. *(Positive energy) The generators $\{P^\mu\}$ on $\mathcal{H}$ of the Poincaré group representation have joint spectrum $S \subset \mathbb{R}^4$ such that S lies in the forward lightcone: $S \subset \{p^\mu : (p^0)^2 - \sum_i (p^i)^2 \geq 0\} \subset \mathbb{R}^4$.*

2. *(Vacuum) There is a vector $|\Omega\rangle \in \mathcal{H}$ invariant under G. The vacuum $|\Omega\rangle$ lies in the domain of any polynomial of the field operators, and the subspace algebraically spanned by polynomial combinations of field operators acting on $|\Omega\rangle$ is dense in $\mathcal{H}$. The vacuum is also the unique vector invariant under time-translation, i.e. the unique eigenstate of the Hamiltonian.*

3. *(Microcausality) If $f, g \in \mathscr{S}(\mathbb{R}^4)$ such that their supports $supp(f), supp(g) \subset \mathbb{R}^4$ are spacelike separated, then $[\phi(f), \phi(g)] = 0$. (Regions $B_1, B_2 \subset \mathbb{R}^4$ are called spacelike separated if $(x_1 - x_2)$ spacelike for all $x_1 \in B_1, x_2 \in B_2$.)*

4. *(Poincaré covariance) For $f \in \mathscr{S}(\mathbb{R}^4)$, $U^{-1}(g)\phi(f)U(g) = \phi(fg)$ $\forall g \in G$.*

These axioms are essentially the informal axioms given in Section 2.3, modified as suggested by the discussion of smeared fields in Section 3.3. Given a model of the axioms, one may define the Wightman distributions of Equation (56).

4.2 Haag-Kastler axioms

The Haag-Kastler axioms focus on the algebra of observables, as motivated in Section 3.4. There are several variants of the AQFT axioms originally posed by Haag and Kastler, but the weaker version used here again follows Glimm and Jafffe [15]. Various strengthenings of the axioms are discussed in [13].

We begin by defining a net of observables. This structure consists of an abstract algebra of observables along with a specification of which observables are local to which regions in spacetime. The notion of an operator local to a region in spacetime is the same as that discussed in Section (2.3). Recall that in the Wightman axioms, the algebra of observables local to a region $B \subset \mathbb{R}^4$ is generated by smeared field operators $\phi(f)$ with $\text{supp}(f) \subset B$. In AQFT, there is no such operator-valued distribution or Hilbert space. Instead, all observables belong to the net of observables, which has additional structure to encode the locality of operators.

Definition (Net of observables)**.** *A net of observables is a C^*-algebra $\mathcal{A}$ along with a map denoted $B \mapsto \mathcal{A}(B) \subset \mathcal{A}$ for bounded open regions $B \subset \mathbb{R}^4$. The map must satisfy*

1. *$\mathcal{A}(B_1) \subset \mathcal{A}(B_2)$ for all regions $B_1 \subset B_2$, and*

2. *$\mathcal{A} = \cup_B \mathcal{A}(B)$ where the union is taken over all regions $B \subset \mathbb{R}^4$.*

The abstract algebras $\mathcal{A}(B)$ are the algebras of observables local to B. (Again, the term "algebra of observables" is used, but only the self-adjoint elements represent observables.) Under the interpretation that the observables in $\mathcal{A}(B)$ are those accessible to an observer local to B, condition (1) demands that an observer local to B_1 is also local to $B_2 \supset B_1$. Condition (2) demands that the algebra of observables consists entirely of observables local to some bounded region; that is, all observables are hypothetically accessible to some local observer. Note that the net of observables assigns algebras $\mathcal{A}(B)$ only to *open* bounded regions of spacetime. The structure thereby incorporates the discussion of smeared fields in Section 3.3; there are no observables associated to single points in spacetime.

With this definition, we may state a weak version of the Haag-Kastler axioms:

Definition (Haag-Kastler axioms for real scalar quantum field theory)**.** *A quantum field theory is given by a net of observables $\mathcal{A}$ and a representation $\alpha : G \to Aut(\mathcal{A})$, $g \mapsto \alpha_g$ of the Poincaré group G. The net of observables and Poincaré group representation must have the following properties:*

1. *(Primitivity) There exists a faithful, unitary representation of $\mathcal{A}$. (Such an algebra is called primitive.)*
2. *(Microcausality) For regions B_1 and B_2 spacelike separated, the algebras $\mathcal{A}(B_1)$ and $\mathcal{A}(B_2)$ commute: $[A_1, A_2] = 0$ for all $A_1 \in \mathcal{A}(B_1), A_2 \in \mathcal{A}(B_2)$.*
3. *(Poincaré covariance) The representation of the Poincaré group satisfies $\alpha_g(\mathcal{A}(B)) = \mathcal{A}(\alpha_g(B))$ for all $g \in G$.*

One possible strengthening of these axioms is the following. Define the causal shadow $S(B)$ of a region $B \subset \mathbb{R}^4$ to be the set of points $x \in \mathbb{R}^4$ for which every timelike curve through x intersects B exactly once. From a classical special-relativistic perspective, the causal shadow is precisely the set of events completely determined by events in B. Meanwhile, the discussion of Heisenberg-picture operators and microcausality in Section 2.3 indicates that we should have

$$\mathcal{A}(S(B)) \subset \mathcal{A}(B), \tag{64}$$

in keeping with the special-relativistic notion. This statement may be promoted to an axiom and added to the previous axioms. Related suggestions are discussed in [13].[4]

4.3 Osterwalder-Schrader axioms

The Osterwalder-Schrader axioms formalize the approach outlined in Section 3.5. We seek a probability measure μ on the space of fields configurations over spacetime. However, because the fields themselves are best considered as distributional, we will ask for a measure μ on the space $\mathscr{D}(\mathbb{R}^4)$. Here, $\mathscr{D}(\mathbb{R}^4)$ is the space of distributions acting on $C_0^\infty(\mathbb{R}^4)$, the space of compactly supported smooth functions. For a distribution field $\phi \in \mathscr{D}(\mathbb{R}^4)$ and smearing function $f \in C_0^\infty(\mathbb{R}^4)$, there is a canonical pairing $\phi(f)$. This setup relates to

[4]Note that a net of observables may be reformulated as a precosheaf, i.e. as a functor from the category of regions in spacetime (with morphisms given by inclusion) to the category of C^*-algebras. The causal shadow axiom then has a clean categorical formulation as well, and this categorical thinking may yield structural insights about spacetime.

the formal path integrals of Section 2.5 via the correspondence

$$\int_{\varphi\in\mathscr{D}(\mathbb{R}^4)} \varphi(f_1)...\varphi(f_n)\ \mathrm{d}\mu \approx \int_{\mathbb{R}^{4n}} f(x_1)...f(x_n)\,\langle\Omega|\,\mathcal{T}(\phi(\mathbf{x}_1,-it_1)...\phi(\mathbf{x}_n,-it_n))\,|\Omega\rangle\ \mathrm{d}^4x_1...d^4x_n \tag{65}$$
$$\approx \frac{\int_{\varphi\in C(\mathbb{R}^4)} e^{-S_E[\varphi]} \int_{\mathbb{R}^{4n}} f(x_1)...f(x_n)\varphi(x_n)...\varphi(x_1)\ \mathrm{d}^4x_1...d^4x_n\mathcal{D}\varphi}{\int_{\varphi\in C(\mathbb{R}^4)} e^{-S_E[\varphi]}\mathcal{D}\varphi}.$$

Again, to understand the integrand on the LHS as a function of the domain of integration, it may help to rewrite $\phi(f_i)$ as $f_i(\phi)$. Note that this Euclidean path integral is providing the analytic extension of the Wightman distributions onto a complex domain, i.e. note the $-it$ of the RHS.

Some of the conditions on μ will be more easily specified in terms of conditions on its generating functional $S[\phi]$ obtained by inverse Fourier transform,

$$S[f] = \int e^{i\phi(f)} d\mu. \tag{66}$$

The Osterwalder-Schrader axioms are not transcribed here in their entirety, but the remaining details may be found in [15].

Definition (Osterwalder-Schrader axioms for a real scalar quantum field theory)**.** *A quantum field theory is given by a probability measure μ on $\mathscr{D}(\mathbb{R}^4)$ such that the following hold:*

1. *(Analyticity) The generating functional S is entire analytic, in the sense specified by [15].*
2. *(Regularity) S is bounded, in the sense specified by [15].*
3. *(Euclidean invariance) S is invariant under Euclidean transformations $E : \mathbb{R}^4 \to \mathbb{R}^4$, i.e. $S[f] = S[fE]$. Equivalently, μ is invariant under transformations induced by Euclidean transformations of the underlying space $\mathbb{R}^4$.*
4. *(Reflection positivitiy) See note below.*
5. *(Ergodicity) The time-translation group acts ergodically on the measure space $(\mathscr{D}(\mathbb{R}^4), \mu)$, in the sense specified by [15].*

The reflection positivity condition is not described in detail here. Briefly, this condition allows one to rigorously construct a Hilbert space heuristically equivalent to $L^2(C(\mathbb{R}^3), \nu)$, where we imagine obtaining a measure ν on spatial field configurations from a measure μ on spacetime field configurations. The condition may be shown to hold formally for the Euclidean path integral expressions discussed in Section 2.5, which justifies its use. The uniqueness of the vacuum in the constructed Hilbert space is ensured by the ergodicity condition. Meanwhile, Euclidean invariance ensures the Lorentz invariance of vacuum expectation values, as motivated in Section 2.5. Finally, the analyticity condition validates the analytic continuation necessary for obtaining the real-time vacuum expectation values from the Euclidean (imaginary time) values.

4.4 Consequences of the axioms

Glimm and Jaffe [15] prove that given a model of the Osterwalder-Schrader axioms, one may define a corresponding model satisfying the Wightman axioms. The correspondence between the two models is essentially

the first equality of Equation (63), where the RHS indicates the analytic extension of the Wightman distributions given by the Wightman axioms. In turn, the Wightman axioms allow one to define a model of the Haag-Kastler axioms, where the algebras $\mathcal{A}(B)$ (of the Haag-Kastler axioms) are generated by the smeared fields operators (of the Wightman axioms) with support on B.

Given two different sets A and B of axioms (such as those of Wightman and Haag-Kastler), then if one chooses a slightly weaker version of axioms B or a stronger version of axioms A, it becomes easier to prove results such as: "For any model of axioms A, there exists a corresponding a model of axioms B." Of course, this "technique" of tweaking the axioms to be alternately stronger and weaker does not work when we want to prove a two-way equivalence. In other words, it is much harder to prove the equivalence of the Wightman and Haag-Kastler axioms than it is to find versions of these axioms such that one implies the other. Nonetheless, one may find a discussion of different attempted formulations and equivalences in [13].

Several results about quantum field theory may be derived from an axiomatic setting. These include desired results like the spin-statistics theorem and the CPT theorem. However, there are also interesting or surprising results, such as the the Reeh-Schlieder theorem or the theorem on Borchers classes. (See the discussions in [9] and [13], respectively.)

5 Outlook

Although interesting models of the axioms are scarce, new mathematical tools may prove fruitful in the future. Success may also be found in other approaches not mentioned here, such as geometric quantization or functorial axioms. Regardless, the search for rigorous formalism may help one grasp the structure of quantum field theory.

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