Generalised Brègman relative entropies: a brief introduction

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Abstract

We present some basic elements of the theory of generalised Brègman relative entropies over non-reflexive Banach spaces. Using nonlinear embeddings of Banach spaces together with the Euler–Legendre functions, this approach unifies two former approaches to Brègman relative entropy: one based on reflexive Banach spaces, another based on differential geometry. This construction allows to extend Brègman relative entropies, and related geometric and operator structures, to arbitrary-dimensional state spaces of probability, quantum, and postquantum theory. We give several examples, not considered previously in the literature.

1 Introduction

For any set Z, $D: Z \times Z \to [0, \infty]$ will be called an *information* on Z (and -D will be called a *relative entropy* on Z)¹ iff (cf. [8, p. 1019] [17, p. 794] [14, p. 161]) $D(x,y) = 0 \iff x = y \ \forall x, y \in Z$. If $\varnothing \neq K \subseteq Z$, $x \in Z$, and $\arg\inf_{y \in K} \{D(y,x)\}$ (resp., $\arg\inf_{y \in K} \{D(x,y)\}$) is a singleton set, then we will denote the element of this set by $\overleftarrow{\mathfrak{P}}_K^D(x)$ (resp., $\overrightarrow{\mathfrak{P}}_K^D(x)$), while the map $x \mapsto \overleftarrow{\mathfrak{P}}_K^D(x)$ [51, p. 32] [33, Ch. 3.2] (resp., $x \mapsto \overrightarrow{\mathfrak{P}}_K^D(x)$ [13, Eqn. (16)]) will be called a *left* (resp., *right*) D-projection of x onto K.

Let M be a \mathbb{C}^3 -manifold with a tangent bundle $\mathbf{T}M$, a \mathbb{C}^3 riemannian metric tensor \mathbf{g} on $\mathbf{T}M$, and a pair $(\nabla, \widetilde{\nabla})$ of \mathbb{C}^3 affine connections on $\mathbf{T}M$ (with an arbitrary torsion). Let \mathbf{t}_c^{∇} denote a ∇ -parallel transport in $\mathbf{T}M$ along a curve c in M. Then the **Norden–Sen geometry** is defined as a quadruple $(M, \mathbf{g}, \nabla, \widetilde{\nabla})$ satisfying any of the equivalent conditions [42, pp. 205–206, §2, §4] [52, p. 46]:

$$\mathbf{g}(\mathbf{t}_{c}^{\nabla}(\cdot), \mathbf{t}_{c}^{\widetilde{\nabla}}(\cdot)) = \mathbf{g},\tag{1}$$

$$\mathbf{g}(\nabla_u v, w) + \mathbf{g}(v, \widetilde{\nabla}_u w) = u(\mathbf{g}(v, w)) \ \forall u, v, w \in \mathbf{T}M.$$

If Z is a finite dimensional C³-manifold and $D \in C^3(Z \times Z; \mathbb{R}^+)$ has a positive definite hessian matrix, then a third order Taylor expansion of D on Z induces [17, pp. 795–796] [18, p. 357] a riemannian metric \mathbf{g}^D on $\mathbf{T}Z$ and a pair $(\nabla^D, \widetilde{\nabla}^D)$ of torsion-free affine connections on $\mathbf{T}Z$, satisfying the characteristic property (2) of the Norden–Sen geometry. This way the global geometric properties of D can be analysed in local terms of its torsion-free Norden–Sen differential geometry.³

¹Cf. «information is the negative of the quantity (...) defined as entropy» [58, p. 76].

²In comparison, given (M, \mathbf{g}) , the Levi-Civita affine connection $\nabla^{\mathbf{g}}$ is characterised among all torsion-free affine connections on $\mathbf{T}M$ by $\mathbf{g}(\mathbf{t}_c^{\nabla^{\mathbf{g}}}(\cdot), \mathbf{t}_c^{\nabla^{\mathbf{g}}}(\cdot)) = \mathbf{g}$. Each torsion-free Norden–Sen geometry determines $\nabla^{\mathbf{g}}$ by $\nabla^{\mathbf{g}} = \frac{1}{2}(\nabla + \widetilde{\nabla})$ [42, p. 211].

³Following [34, §4], the torsion-free Norden–Sen geometries are sometimes called "statistical manifolds". Apart from not crediting the original authors, this terminology is misleading, since these geometries are independent of any notion of statistics.

2 D_{Ψ} : Brègman vs Brunk-Ewing-Utz

Given a strictly convex, differentiable function $\Psi : \mathbb{R}^n \to \mathbb{R}$ (or $\Psi : M \to \mathbb{R}$ with convex $M \subseteq \mathbb{R}^n$), there are two approaches to construction of a functional encoding the first order Taylor expansion of Ψ (together with its further use in optimisation problems): one going back to Brègman's [8, p. 1021]

$$D_{\Psi}(x,y) := \Psi(x) - \Psi(y) - \sum_{i=1}^{n} (x_i - y_i) (\operatorname{grad} \Psi(y_i)) \ \forall x, y \in \mathbb{R}^n$$
 (3)

(or $\forall x, y \in M$), another going back to the Brunk–Ewing–Utz [10, Eqn. (4.4)]

$$D_{\Psi}^{\mu}(x,y) := \int_{\mathcal{X} \subset \mathbb{R}^m} \mu(\chi) D_{\Psi}(x(\chi), y(\chi)), \tag{4}$$

for $x, y: \mathcal{X} \to \mathbb{R}$, n = 1, and a measure μ on the Borel subsets of \mathbb{R}^m .

The former approach has been generalised and widely developed for \mathbb{R}^n replaced by a reflexive Banach space $(X, \|\cdot\|_X)$ (see Section 3). On the other hand, the latter approach was generalised and further developed for (\mathcal{X}, μ) given by any countably finite nonzero measure space (see [15] and references therein).

The passage from probabilistic to quantum theoretic setting corresponds to replacing $(L_1(\mathcal{X}, \mu), \|\cdot\|_1)$ by the Banach predual \mathcal{N}_{\star} of a W*-algebra \mathcal{N} (all of these spaces are nonreflexive). The noncommutative analogue $D_{\Psi}^{\text{tr}_{\mathcal{H}}}$ of D_{Ψ}^{μ} was introduced in [56, §2.2] for finite dimensional real Hilbert spaces, and in [43, pp. 127–129]⁴ for type I W*-algebras (see also [23, §V] for type I_n JBW-algebras). However, due to nonreflexivity of \mathcal{N}_{\star} , this definition is incapable of utilising the vast body of reflexive Banach space theoretic results obtained for D_{Ψ} , and it is also unclear how to extend the definition of $D_{\Psi}^{\text{tr}_{\mathcal{H}}}$ to arbitrary W*-algebras.

For a convex closed $C \subseteq M \subseteq \mathbb{R}^n$, D_{Ψ} given by (3) exhibits [8, Lemm. 1],

$$D_{\Psi}(x, \overleftarrow{\mathfrak{P}}_{C}^{D_{\Psi}}(y)) + D_{\Psi}(\overleftarrow{\mathfrak{P}}_{C}^{D_{\Psi}}(y), y) \ge D_{\Psi}(x, y) \ \forall (x, y) \in C \times M$$
 (5)

(and analogously for $\overrightarrow{\mathfrak{P}}_{C}^{D_{\Psi}}$ [37, Prop. 4.11]; cf. also [13, Thm. 1]), with \geq replaced by = for affine closed C. This property is a nonlinear generalisation of a pythagorean theorem, and is interpreted as an additive decomposition of an (information about) "data" into "signal" and "noise". It is a fundamental feature of D_{Ψ} , characterising $\overleftarrow{\mathfrak{P}}_{C}^{D_{\Psi}}$ [6, Cor. 3.35] and $\overrightarrow{\mathfrak{P}}_{C}^{D_{\Psi}}$ [37, Prop. 4.11].

3 D_{Ψ} : reflexive Banach space setting

 $(X, \|\cdot\|_X)$ will denote a Banach space over \mathbb{R} . A Banach space $(X^*, \|\cdot\|_{X^*})$, consisting of elements given by continuous linear maps $X \to \mathbb{R}$, with a norm

$$||y||_{X^{\star}} := \sup\{|y(x)| \mid x \in B(X, ||\cdot||_X) := \{x \in X \mid ||x||_X \le 1\}\} \ \forall y \in X^{\star}, \tag{6}$$

is called a **Banach dual** of $(X, \|\cdot\|_X)$, with respect to a bilinear duality

$$[x, y]_{X \times X^*} := y(x) \in \mathbb{R} \ \forall (x, y) \in X \times X^*. \tag{7}$$

If there exists $(Y, \|\cdot\|_Y)$ with $(Y^*, \|\cdot\|_{Y^*}) = (X, \|\cdot\|_X)$, then $Y =: X_*$ is called a **predual** of X. Symbol $\operatorname{int}(W)$ (resp., $\operatorname{cl}(W)$) will denote an interior (resp., closure) of $W \subseteq X$ with respect to a topology of $\|\cdot\|_X$.

Given a Banach space $(X, \|\cdot\|_X), \Psi: X \to]-\infty, \infty]$ is called: **proper** iff

$$\operatorname{efd}(\Psi) := \{ x \in X \mid \Psi(x) \neq \infty \} \neq \emptyset; \tag{8}$$

⁴More precisely, $D_{\Psi}^{\mathrm{tr}_{\mathcal{H}}}(x,y) := \mathrm{tr}_{\mathcal{H}}(D_{\Psi}(x,y))$ for a convex and Gateaux differentiable $\Psi : W \to \mathfrak{B}(\mathcal{H})$, where W is a convex subset of a Banach space, e.g. $W = (\mathfrak{B}(\mathcal{H}))_{\star}^+$. The evaluation of $D_{\Psi}^{\mathrm{tr}_{\mathcal{H}}}(x,y)$ is thus defined by spectral calculus applied to Ψ.

convex (resp., strictly convex) iff $\forall x, y \in \text{efd}(\Psi) \ \forall \lambda \in]0,1[$

$$x \neq y \Rightarrow \Psi(\lambda x + (1 - \lambda)y) \le (\text{resp.}, <) \lambda \Psi(x) + (1 - \lambda)\Psi(y).$$
 (9)

Let $\Gamma(X, \|\cdot\|_X)$ (resp., $\Gamma^{\mathrm{G}}(X, \|\cdot\|_X)$) be the set of all proper, convex, lower semicontinuous functions $\Psi: X \to]-\infty, \infty]$ (resp., that are also Gateaux differentiable on $\operatorname{int}(\operatorname{efd}(\Psi)) \neq \varnothing$, with $\mathfrak{D}^G \Psi$ denoting a Gateaux derivative of Ψ).

For $\Psi \in \Gamma^{G}(X, \|\cdot\|_{X})$ the **Brègman function** reads [1, Eqn. (1)] $\forall x \in X$

$$D_{\Psi}(x,y) := \Psi(x) - \Psi(y) - \left[\left[x - y, \mathfrak{D}^{G} \Psi(y) \right] \right]_{X \times X^{\star}} \forall y \in \operatorname{int}(\operatorname{efd}(\Psi)),$$
(10)

and $D_{\Psi}(x,y) := \infty \ \forall y \in X \setminus \operatorname{int}(\operatorname{efd}(\Psi))$. D_{Ψ} is an information on X iff Ψ is strictly convex on $int(efd(\Psi))$ [12, Prop. 1.1.9].

For a proper $\Psi: X \to]-\infty, \infty]$, a **Fenchel dual** map [21, p. 75] [39, p. 8]

$$X^{\star} \ni y \mapsto \Psi^{\mathbf{F}}(y) := \sup_{x \in X} \{ [x, y]_{X \times X^{\star}} - \Psi(x) \} \in] - \infty, \infty], \tag{11}$$

satisfies $\Psi^{\mathbf{F}} \in \Gamma(X^{\star}, \|\cdot\|_{X^{\star}})$ [9, Thm. 3.6]. If $(X, \|\cdot\|_{X})$ is reflexive and $\Psi \in \Gamma^{\mathbf{G}}(X, \|\cdot\|_{X})$, then Ψ will be called $\operatorname{{\it Euler-Legendre}}^5$ iff [5, Def. 5.2.(iii), Thm. 5.4, Thm. 5.6] [47, §2.1] $\Psi^{\mathbf{F}} \in \Gamma^{\mathbf{G}}(X^{\star}, \|\cdot\|_{X^{\star}})$ and

$$\begin{cases}
\operatorname{efd}(\mathfrak{D}^{G}\Psi) := \{x \in \operatorname{efd}(\Psi) \mid \exists \mathfrak{D}^{G}\Psi(x)\} = \operatorname{int}(\operatorname{efd}(\Psi)), \\
\operatorname{efd}(\mathfrak{D}^{G}\Psi^{F}) = \operatorname{int}(\operatorname{efd}(\Psi^{F})).
\end{cases}$$
(12)

For $X = \mathbb{R}^n$, the definition of Euler-Legendre functions goes back to Rockafellar, who showed [49, Thm. C-K] [50, Thm. 1] that if $\emptyset \neq U \subseteq \mathbb{R}^n$ is open and convex, while $\Psi: U \to]-\infty, \infty$] is strictly convex, differentiable on U, and

$$\lim_{t \to +0} \frac{\mathrm{d}}{\mathrm{d}t} \Psi(tx + (1-t)y) = -\infty \ \forall (x,y) \in U \times (\mathrm{cl}(U) \setminus U), \tag{13}$$

then grad Ψ is a bijection on U, grad $(\Psi^{\mathbf{F}}) = (\operatorname{grad} \Psi)^{-1}$ on $(\operatorname{grad} \Psi)(U)$, and $\Psi^{\mathbf{F}}$ on $(\operatorname{grad} \Psi)(U)$ satisfies the same conditions as Ψ on U.

4 D_{Ψ} : dually flat setting

The dually flat (a.k.a. hessian) geometry [53, Prop. (p. 213)] is characterised among all torsionfree Norden–Sen geometries by the flatness of ∇ and ∇ . This is equivalent with existence of two coordinate systems, $\{\theta_i \mid i \in \{1, \dots, n\}\} : M \to \mathbb{R}^n$ and $\{\eta_i \mid i \in \{1, \dots, n\}\} : M \to \mathbb{R}^n$, such that, $\forall \rho \in M$,

$$\eta_i(\rho) = \frac{\partial \Psi(\theta(\rho))}{\partial \theta^i}, \quad \theta_i(\rho) = \frac{\partial \Psi^{\mathbf{F}}(\eta(\rho))}{\partial \eta^i} \tag{14}$$

$$\begin{cases}
\eta_i(\rho) = \frac{\partial \Psi(\theta(\rho))}{\partial \theta^i}, & \theta_i(\rho) = \frac{\partial \Psi^{\mathbf{F}}(\eta(\rho))}{\partial \eta^i} \\
\Psi^{\mathbf{F}}(y) = \sup_{x \in \mathbb{R}^n} \left\{ \sum_{i=1}^n x_i y_i - \Psi(x) \right\} \quad \forall x \in \mathbb{R}^n,
\end{cases} \tag{14}$$

and, for $D_{\theta,\Psi}(\rho,\sigma) := D_{\Psi}(\theta(\rho),\theta(\sigma))$ with D_{Ψ} defined by (3),

$$\begin{cases}
\Gamma_{ijk}^{\nabla^{D_{\theta,\Psi}}}(\theta(\rho)) = 0, & \Gamma_{ijk}^{\widetilde{\nabla}^{D_{\eta,\Psi}}}(\eta(\rho)) = 0 \\
\mathbf{g}_{ij}^{D_{\theta,\Psi}}(\theta(\rho)) = \frac{\partial^{2}\Psi(\theta(\rho))}{\partial\theta^{i}\partial\theta^{j}},
\end{cases} (16)$$

$$\mathbf{g}_{ij}^{D_{\theta,\Psi}}(\theta(\rho)) = \frac{\partial^2 \Psi(\theta(\rho))}{\partial \theta^i \partial \theta^j},\tag{17}$$

These functions are usually called "Legendre" (for $X = \mathbb{R}^n$ they were introduced namelessly in [49, Thm. C-K]). Yet, the transformation $\mathrm{d}(z(x,y)-px-qy)=-x\mathrm{d}p-y\mathrm{d}q$, with $p=\frac{\partial z(x,y)}{\partial x}$ and $q=\frac{\partial z(x,y)}{\partial y}$, was introduced first by Euler [19, Part I, Probl. 11], and only 17 years later by Legendre [35, p. 347].

where $\Gamma^{\nabla}(u,v,w) := \mathbf{g}(\nabla_u v,w) \ \forall u,v,w \in \mathbf{T}M$, while the subscript i denotes evaluation at the i-th component of a basis in $\mathbf{T}M$ given by coordinate system differentials (i.e., setting $u = \frac{\partial}{\partial \theta^i}$, etc., in (16)). (Also, $\mathbf{g}_{ij}^{D_{\eta,\Psi}}(\eta(\rho)) = \frac{\partial^2 \Psi^{\mathbf{F}}(\eta(\rho))}{\partial \eta^i \partial \eta^j}$.) When reconsidered in this setting, the left (resp., right) generalised pythagorean theorem is equivalent with: a projection of $y \in M$ onto C along $\widetilde{\nabla}^{D_{\eta,\Psi}}$ -(resp., $\nabla^{D_{\theta,\Psi}}$ -)geodesics is $\mathbf{g}^{D_{\theta,\Psi}}$ -orthogonal (= $\mathbf{g}^{D_{\eta,\Psi}}$ -orthogonal) to C [3, Thm. 3.4].

Equation (15) is a special case of (11). Furthermore, (14) require only C¹-differentiability. The approach presented in Section 5 is rooted in an observation that the correct generalisation of (14) requires two components: Euler-Legendre Ψ on a reflexive Banach space $(X, \|\cdot\|_X)$, and nonlinear embeddings into $(X, \|\cdot\|_X)$ and $(X^*, \|\cdot\|_{X^*})$, replacing, respectively, θ and η .

5 $D_{\ell,\Psi}$

In [31, §3] we introduced a generalisation, $D_{\ell,\Psi}$, of a family of Brègman informations D_{Ψ} on reflexive Banach spaces $(X, \|\cdot\|_X)$, applicable to a wide range of nonreflexive Banach spaces $(Y, \|\cdot\|_Y)$. (E.g., to postquantum state spaces, given by bases $Z \subseteq V^+$ of positive cones V^+ of radially compact base normed spaces in spectral duality, $(V, \|\cdot\|_V) = (Y, \|\cdot\|_Y)$.) The main idea is to pull back the properties exhibited by D_{Ψ} with Euler–Legendre Ψ acting on $(X, \|\cdot\|_X)$ into the properties exhibited by $D_{\ell,\Psi}(\cdot,\cdot) := D_{\Psi}(\ell(\cdot),\ell(\cdot))$, where $\ell: Z \to X$ and $Z \subseteq Y$.

Definition 5.1. [31, Def. 3.1] Let $(Y, \|\cdot\|_Y)$ be a Banach space, let $(X, \|\cdot\|_X)$ be a reflexive Banach space, let $\Psi \in \Gamma^{\mathrm{G}}(X, \|\cdot\|_X)$ be strictly convex on $\operatorname{int}(\operatorname{efd}(\Psi))$ and Euler-Legendre, let $\varnothing \neq Z \subseteq Y$, and let $\ell: Z \to \ell(Z) \subseteq X$ be a bijection such that $\ell(Z) \cap \operatorname{int}(\operatorname{efd}(\Psi)) \neq \varnothing$. Then:

- (i) if $\varnothing \neq C \subseteq Y$, and $\ell(C)$ is convex (resp., closed; affine), then C will be called ℓ -convex (resp., ℓ -closed; ℓ -affine);
- (ii) a triple (Z, ℓ, Ψ) will be called a generalised pythagorean geometry;
- (iii) an (ℓ, Ψ) -information (a generalised Brègman information) on Z is

$$D_{\ell,\Psi}(\phi,\psi) := D_{\Psi}(\ell(\phi),\ell(\psi)) \ \forall (\phi,\psi) \in Z \times \ell^{-1}(\ell(Z) \cap \operatorname{int}(\operatorname{efd}(\Psi))).$$
 (18)

Proposition 5.2. [31, Prop. 3.2] Under assumptions of Definition 5.1, let $\emptyset \neq C \subseteq Z$ be ℓ -convex and ℓ -closed, and let $\psi \in \ell^{-1}(\ell(Z) \cap \operatorname{int}(\operatorname{efd}(\Psi)))$. Then:

- (i) $D_{\ell,\Psi}$ is an information on Z;
- (ii) $\arg\inf_{\phi\in C} \{D_{\ell,\Psi}(\phi,\psi)\}\ is\ a\ singleton\ set,\ denoted\ \{\overleftarrow{\mathfrak{P}}_C^{D_{\ell,\Psi}}(\psi)\};$
- (iii) $\omega \in C$ is the unique solution of $D_{\ell,\Psi}(\phi,\omega) + D_{\ell,\Psi}(\omega,\psi) \leq D_{\ell,\Psi}(\phi,\psi) \ \forall \phi \in C$ iff $\omega = \overleftarrow{\mathfrak{P}}_C^{D_{\ell,\Psi}}(\psi)$ (in 'then' case, if C is ℓ -affine, then = replaces \leq);
- (v) if ℓ is norm-to-norm continuous and $\overleftarrow{\mathfrak{P}}_K^{D_{\Psi}}$ is norm-to-norm continuous for any convex closed $\varnothing \neq K \subseteq \ell(Z) \cap \operatorname{int}(\operatorname{efd}(\Psi))$, then $\overleftarrow{\mathfrak{P}}_C^{D_{\ell,\Psi}}$ is norm-to-norm continuous for any ℓ -convex and closed $\varnothing \neq C \subseteq \ell^{-1}(\ell(Z) \cap \operatorname{int}(\operatorname{efd}(\Psi)))$.

An analogous result for $\overrightarrow{\mathfrak{P}}^{D_{\ell,\Psi}}$ also holds [32, Part I] (cf. also [13, Thm. 1]).

For $X = \mathbb{R}^n$, $D_{\ell,\Psi}$ recovers the setting of Brègman information $D_{\theta,\Psi}$ on an n-dimensional C^1 -manifold (hence, in particular, C^{∞} -manifold) M, with the map $\ell: M \to \mathbb{R}^n$ (resp., $\mathfrak{D}^G \Psi \circ \ell: M \to \mathbb{R}^n$) given by the coordinate system $\{\theta_i\}$ (resp., $\{\eta_i\}$). More specifically, a domain M of a dually flat geometry is assumed to be a (suitably differentiable) manifold, covered by two global maps $\{\theta_i\}$ and $\{\eta_i\}$, without assuming $M \subseteq \mathbb{R}^n$, cf. [3, 54]. This is not addressed by (3), and is addressed (up to a weaker assumption on the order of differentiability) by (18).

This way the framework of generalised Brègman information $D_{\ell,\Psi}$ unifies reflexive Banach space theoretic and finite dimensional smooth information geometric approaches to Brègman information. If ℓ is a norm-to-norm continuous homeomorphism, then the ℓ -closed sets in Z are closed in terms of topology of $\|\cdot\|_Y$. This fragment of a theory provides a fusion of nonlinear convex analysis with nonlinear homeomorphic theory of Banach spaces. In particular, if ℓ is Hölder continuous, then it allows to pull back the conditions on Hölder continuity of $\overset{\circ}{\mathcal{P}}_K^{D_{\Psi}}$ and $\overset{\circ}{\mathcal{P}}_K^{D_{\Psi}}$ into results on Hölder continuity of $\overset{\circ}{\mathcal{P}}_C^{D_{\ell,\Psi}}$ and $\overset{\circ}{\mathcal{P}}_C^{D_{\ell,\Psi}}$. Generalised pythagorean geometry (Z,ℓ,Ψ) is a more general object than $D_{\ell,\Psi}$, and allows to suitably generalise also the affine connections (16) [32, Part IV].

In this context, our approach arises partially from an observation that the ℓ_{γ} (resp., ℓ_{Υ}) embeddings, cf. Example 6.1.(a) (resp., 6.1.(c)) below, used in [40, Eqn. (2.7)] (resp., [22, §7.2]), are finite dimensional Mazur (resp., Kaczmarz) maps [38, p. 83] (resp., [28, p.148]) on $(L_1(\mathcal{X}, \mu))^+$. Drawing from an important example in [27, §6–§8] (equal to Example 6.1.(a) with $\alpha = \gamma(1 - \gamma)$ and $\beta = \gamma$), an abstract framework aiming at this unification was proposed in [30, Eqns. (24), (31)], while its implementation, based on the use of Euler–Legendre Ψ , was given in [31, §3–§4]. The resulting theory is developed in details in [32].

6 Examples of (ℓ, Ψ) with $Z \subseteq V^+$ (for Proposition 5.2)

If $(Y, \|\cdot\|_Y)$ is partially ordered by \geq , then $Y^+ := \{x \in Y \mid x \geq 0\}$. All examples below feature $(Y, \|\cdot\|_Y)$ given by some kind of a radially compact base normed space $(V, \|\cdot\|_V)$. Such spaces provide the setting for the (linear) convex operational generalisation of quantum theory (a.k.a. "generalised probability theory" or "postquantum theory"), with state space given by $V_1^+ := \{\phi \in V^+ \mid \|x\|_V = 1\}$.

Example 6.1.

(a). (=[31, Prop. 4.2].) If \mathcal{N} is a W*-algebra, $\alpha \in]0, \infty[, \beta, \gamma \in]0, 1[, (X, \|\cdot\|_X) = (L_{1/\gamma}(\mathcal{N}), \|\cdot\|_{1/\gamma}),$ then the Mazur map

$$\ell = \ell_{\gamma} : Z = \mathcal{N}_{\star}^{+} \ni \phi \mapsto \phi^{\gamma} \in (L_{1/\gamma}(\mathcal{N}))^{+}$$
(19)

is Hölder continuous [48, Thm. (p. 37)]. If $\Psi = \Psi_{\alpha,\beta} := \frac{\beta}{\alpha} \| \cdot \|_X^{1/\beta}$, then

$$D_{\ell_{\gamma},\Psi_{\alpha,\beta}}(\phi,\psi) = \alpha^{-1}(\beta \|\phi\|_{1}^{\gamma/\beta} + (1-\beta)\|\psi\|_{1}^{\gamma/\beta} - \|\psi\|_{1}^{\gamma/\beta-1} \int (\phi^{\gamma}\psi^{1-\gamma}))$$
(20)

 $\forall \phi, \psi \in \mathcal{N}_{\star}^{+}$, where \int is understood as in [20, Eqn. (3.12')]; if $\mathcal{N} = \mathfrak{B}(\mathcal{H}) := \{\text{bounded operators on a Hilbert space } \mathcal{H}\}$, then $\mathcal{N}_{\star} = \mathfrak{G}_{1}(\mathcal{H}) \equiv \{\text{trace class operators on } \mathcal{H}\}$, $L_{1/\gamma}(\mathcal{N}) =: \mathfrak{G}_{1/\gamma}(\mathcal{H})$, and $\int \cdot = \text{tr}_{\mathcal{H}}(\cdot) = \|\cdot\|_{1}$.

(b). (=[31, Prop. 4.7].) Let A be a semifinite JBW-algebra with a Jordan product \bullet , a faithful normal semifinite trace τ , $\alpha \in]0, \infty[$, $\beta, \gamma \in]0, 1[$, $(X, \|\cdot\|_X) = (L_{1/\gamma}(A, \tau), \|\cdot\|_{1/\gamma})$, $\Psi = \Psi_{\alpha,\beta}$. Then $\ell = \ell_{\gamma} : A_{\star}^+ \ni \phi \mapsto \phi^{\gamma} \in (L_{1/\gamma}(A, \tau))^+$ is Hölder continuous [31, Prop. 4.6], and $\forall \omega, \phi \in Z = A_{\star}^+ D_{\ell_{\gamma},\Psi_{\alpha,\beta}}(\omega,\phi) =$

$$\alpha^{-1}(\beta(\tau(\omega))^{\gamma/\beta} + (1-\beta)(\tau(\phi))^{\gamma/\beta} - (\tau(\phi))^{\gamma/\beta - 1}\tau(\omega^{\gamma} \bullet \phi^{1-\gamma})). \tag{21}$$

(c). (=[31, Cor. 4.12].) If (\mathcal{X}, μ) is a nonatomic measure space, $\mu(\mathcal{X}) < \infty$, $\Upsilon : \mathbb{R} \to \mathbb{R}^+$ is even, strictly convex, continuously differentiable, with $\Upsilon(1) = 1$, $\Upsilon(u) = 0$ iff u = 0, $\limsup_{u \to \infty} \frac{\Upsilon(2u)}{\Upsilon(u)} < \infty$, $\liminf_{u \to \infty} \frac{\Upsilon(2u)}{\Upsilon(u)} > 2$, $\lim_{u \to +0} \frac{\Upsilon(u)}{u} = 0$, $\lim_{u \to \infty} \frac{\Upsilon(u)}{u} = \infty$, $t, s \in \mathbb{R}^+$, t < s, $u \mapsto \frac{\Upsilon^{-1}(u)}{u^t}$ is nondecreasing, and $u \mapsto \frac{\Upsilon^{-1}(u)}{u^s}$ is nonincreasing, then the Kaczmarz map

$$\ell = \ell_{\Upsilon} : Z = (L_1(\mathcal{X}, \mu))_1^+ \ni \phi \mapsto \Upsilon^{-1}(\phi) \in (L_{\Upsilon}(\mathcal{X}, \mu))_1^+$$
 (22)

is Hölder continuous for the Morse–Transue–Nakano–Luxemburg norm $\|\cdot\|_{\Upsilon}$ on Orlicz space $L_{\Upsilon}(\mathcal{X}, \mu)$ [16, Cor. 2.5]. For $\Psi = \Psi_{\beta,\beta}$, $\beta \in]0,1[$, this gives

$$D_{\ell_{\Upsilon},\Psi_{\beta,\beta}}(\omega,\phi) = \beta^{-1}(1-\bar{\Upsilon}(\omega,\phi)/\bar{\Upsilon}(\phi,\phi)), \tag{23}$$

where $\bar{\Upsilon}(\omega,\phi) := \int \mu \Upsilon^{-1}(\omega) \Upsilon'(\Upsilon^{-1}(\phi))$, and $(\cdot)'$ denotes a derivative.

All these cases have norm-to-norm continuous $\overleftarrow{\mathfrak{P}}_{C}^{D_{\ell,\Psi}}$. In [32] we prove this also for $\overrightarrow{\mathfrak{P}}_{C}^{D_{\ell,\Psi}}$, and establish conditions for Hölder continuity of $\overleftarrow{\mathfrak{P}}_{C}^{D_{\ell,\Psi}}$ and $\overrightarrow{\mathfrak{P}}_{C}^{D_{\ell,\Psi}}$.

Example 6.2.

(= [31, Prop. 4.14] for $\varphi(t) = \varphi_{\alpha,\beta}(t) = \frac{1}{\alpha}t^{1/\beta-1}$, i.e. $\Psi = \Psi_{\alpha,\beta} = \Psi_{\varphi_{\alpha,\beta}}$; [32, Part I] for $\Psi = \Psi_{\varphi}$). Let $(V, \|\cdot\|_V)$ be a generalised spin factor [7, Def. 4], i.e. $V = \mathbb{R} \oplus X$, where $(X, \|\cdot\|_X)$ is a reflexive Banach space, and

$$\forall v = (\lambda, x) \in V \quad \left\{ \begin{array}{l} v \ge 0 : \iff \lambda \ge \|x\|_X \\ \|v\|_V := \max\{|\lambda|, \|x\|_X\}. \end{array} \right. \tag{24}$$

Let $\Psi(x) = \Psi_{\varphi}(x) := \int_0^{\|x\|_X} dt \, \varphi(t)$, where $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is positive, strictly increasing, continuous, $\varphi(0) = 0$, and $\lim_{t \to \infty} \varphi(t) = \infty$. Then Ψ_{φ} (and, in particular, $\Psi_{\alpha,\beta}$) is Euler–Legendre iff $(V, \|\cdot\|_V)$ satisfies spectral duality condition [2, Def. (p. 55)]. This gives a family $D_{\ell_X, \Psi_{\varphi}}$ on $Z = \{w \in V^+ \mid \|w\|_V = 1\}$, where $\ell = \ell_X : Z \ni v =: (1, x) \mapsto x \in B(X, \|\cdot\|_X)$. (25)

Example 6.3.

Let \mathcal{H} be a Hilbert space over \mathbb{C} with $n := (\dim \mathcal{H})^2 \in \mathbb{N}$ (hence, $\mathfrak{G}_{1/\widetilde{\gamma}}(\mathcal{H}) = \mathfrak{G}_{1/\gamma}(\mathcal{H}) \ \forall \gamma, \widetilde{\gamma} \in]0,1[$). Let $(\cdot)^{\mathrm{sa}} := \mathrm{self}$ -adjoint part of (\cdot) . Let $\lambda(x)$, with

$$\mathcal{K} := (\mathfrak{G}_2(\mathcal{H}))^{\text{sa}} = \{\text{hermitean } n \times n \text{ matrices}\} \ni x \mapsto \lambda(x) \in \mathbb{R}^n, \tag{26}$$

be a vector of eigenvalues of x ordered nonincreasingly. For $\Phi : \mathbb{R}^n \to]-\infty, \infty]$, let $\Phi(s(x)) = \Phi(x)$ \forall permutation matrices $s : \mathbb{R}^n \to \mathbb{R}^n$. Then $\Psi = \Phi \circ \lambda$ is Euler-Legendre iff Φ is Euler-Legendre [36, Cor. 3.2, Cor. 3.3]. E.g., if: $\Phi(x) = \Phi(x)$

- (a). [4, Ex. 6.5, Cor. 5.13] $\sum_{i=1}^{n} (x_i \log(x_i) x_i)$ if $x \ge 0$, and ∞ otherwise;
- (b). [11] [4, Ex. 6.7, Cor. 5.13] $-\sum_{i=1}^{n} \log(x_i)$ on $]0, \infty[^n]$, and ∞ otherwise;
- (c). [29, Eqn. (60)] [4, Ex. 6.6, Cor. 5.13] $\sum_{i=1}^{n} (x_i \log(x_i) + (1-x_i) \log(1-x_i))$ on $[0,1]^n$, and ∞ otherwise;
- (d). [4, Ex. 6.1, Cor. 5.13] $\sum_{i=1}^{n} \gamma |x_i|^{1/\gamma}$ on \mathbb{R}^n with $\gamma \in]0,1[$;
- (e). [46, Eqn. (37)] [46, §7.2] $\Phi_{\alpha}(x) := \frac{1}{\alpha 1} \sum_{i=1}^{n} (x_i^{\alpha} 1)$ for $(x, \alpha) \in [0, \infty[^n \times]0, 1[, -\Phi_{\alpha}(x)])$ for $(x, \alpha) \in [0, \infty[^n \times]0, \infty[^n \times]-\infty, 0[, \text{ and } \infty \text{ otherwise;}^8$

and $\mathcal{K}_0^+ := (\mathfrak{G}_2(\mathcal{H}))_0^+ = \{\text{strictly positive definite } n \times n \text{ matrices}\}, \text{ then: } D_{\Phi \circ \lambda}(\xi, \zeta) = 0$

(a). [57, Def.1] $\operatorname{tr}_{\mathcal{H}}(\xi(\log \xi - \log \zeta) - \xi - \zeta) \ \forall (\xi, \zeta) \in \mathcal{K}^+ \times \mathcal{K}_0^+;$

 $^{^6}$ Cf. [31, Rem. 4.15]. In [32] we also extend Example 6.1 to $\Psi=\Psi_{\varphi}.$

 $^{^7}D_{\Phi}(x,y) = \sum_{i=1}^n (-\log \frac{x_i}{y_i} + \frac{x_i}{y_i} - 1) \ \forall (x,y) \in (\mathbb{R}^n)_0^+ \times (\mathbb{R}^n)_0^+,$ corresponding to Φ in (b), was introduced by Pinsker in [44, Eqn. (4)] [45, Eqn. (10.5.4)]. The result by Itakura–Saito [25, Eqn. (7)], usually cited as a reference for this D_{Φ} , has appeared 8 years later, and contains only a formula $2\log(2\pi) + \frac{1}{2\pi} \int_{-\pi}^{\pi} dt (\log(y(t)) + \frac{x(t)}{y(t)})$.

⁸Cf.: $-\frac{2^{\alpha-1}(\alpha-1)}{2^{\alpha-1}-1}(\Phi_{\alpha}+\frac{n-1}{\alpha-1}) \ \forall \alpha > 0 \text{ in } [24, \text{ Thm. } 1]; \ -\Phi_{\alpha}-\frac{n-1}{\alpha-1} \ \forall \alpha \in \mathbb{R} \text{ in } [55, \text{ Eqn. } (1)]; \ a \ detailed analysis when } \frac{1}{\alpha}(-\Phi_{\alpha}-\frac{n}{\alpha-1}) \text{ is Euler-Legendre in } [59, \text{ Thm. } 5].$

- (b). [26, §5] $\langle \xi, \zeta^{-1} \rangle_{\mathcal{K}} \log \det(\xi \zeta^{-1}) n = h(\zeta^{-1/2} \xi \zeta^{-1/2}) n \ \forall (\xi, \zeta) \in \mathcal{K}_0^+ \times \mathcal{K}_0^+, \text{ for } h(\xi) := \operatorname{tr}_{\mathcal{K}}(\xi) \log \det(\xi);$
- (c). [41, p. 376] $\operatorname{tr}_{\mathcal{H}}(\xi(\log \xi \log \zeta) + (\mathbb{I} \xi)(\log(\mathbb{I} \xi) \log(\mathbb{I} \zeta))) \ \forall (\xi, \zeta) \in B^+ \times \operatorname{int}(B^+), \text{ where } B^+ := \mathcal{K}^+ \cap B(\mathcal{K}, \|\cdot\|_2);$
- (d). [31, Cor. 4.18.(ii)] $\operatorname{tr}_{\mathcal{H}}(\gamma|\xi|^{1/\gamma} + (1-\gamma)\zeta^{1/\gamma} \xi\zeta^{1/\gamma-1}) \ \forall (\xi,\zeta) \in \mathcal{K} \times \mathcal{K}_0^+$ (under restriction of a domain of ζ to \mathcal{K}_0^+);
- (e). [31, Cor. 4.18.(iii)] $D_{\alpha}(\xi,\zeta) := \operatorname{tr}_{\mathcal{H}}(\zeta^{\alpha} \frac{1}{1-\alpha}\xi^{\alpha} + \frac{\alpha}{1-\alpha}\zeta^{\alpha-1}\xi) \ \forall (\xi,\zeta,\alpha) \in \mathcal{K}^{+} \times \mathcal{K}_{0}^{+} \times]0,1[, -D_{\alpha}(\xi,\zeta) \ \forall (\xi,\zeta,\alpha) \in \mathcal{K}_{0}^{+} \times \mathcal{K}_{0}^{+} \times] \infty,0[;$

with " $D_{\Phi \circ \lambda}(\xi, \zeta) := \infty$ otherwise" in all cases, and $\langle \xi, \zeta \rangle_{\mathcal{K}} := \operatorname{tr}_{(\mathfrak{G}_2(\mathcal{H}))^{\operatorname{sa}}}(\xi \zeta)$. All cases (a)–(e) of $D_{\Phi \circ \lambda}$ are also the special cases of $D_{\Psi}^{\operatorname{tr}_{\mathcal{H}}}$, with a range of good optimisation theoretic properties implied by the fact that $\Phi \circ \lambda$ is Euler–Legendre. ℓ can be set to be any automorphism of $(\mathfrak{G}_2(\mathcal{H}))^{\operatorname{sa}}$ preserving $\operatorname{int}(\operatorname{efd}(\Phi \circ \lambda))$, e.g. a restriction of $\ell_{1/2}$ to a subset of $(\mathfrak{G}_1(\mathcal{H}))^{\operatorname{sa}}$, corresponding to $\operatorname{int}(\operatorname{efd}(\Phi \circ \lambda))$.

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Remark. Cyrillic names and titles were bijectively transliterated from the original, using the system: $\pi = c$, $\pi = c$, and analogously for capitalised letters. Symbol * in front of a bibliographic item indicates that I have not seen this work.

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