An Introduction to Topos Theory

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June 26, 2011

Abstract

The purpose of this text is to equip the reader with an intuitive but precise understanding of elementary structures of category and topos theory. In order to achieve this goal, we provide a guided tour through category theory, leading to the definition of an elementary (Lawvere–Tierney) topos. Then we turn to the investigation of consequences of this definition. In particular, we analyse in detail the topos $\mathbf{Set}^{\mathbf{2}^{op}}$, the internal structure of its subobject classifier and its variation over stages. Next we turn to the discussion of the interpretation of a logic and language in topos, viewed as a model of higher order intuitionistic multisort type theory, as well as the geometric perspective on a topos, viewed as a category of set-valued sheaves over base category equipped with a Grothendieck topology. This text is designed as an elementary introduction, written in a self-contained way, with no previous knowledge required.

$$
\begin{tikzcd}
X \arrow[dr, dotted] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& B \arrow[r, dotted, "!"] \arrow[d, tail, "m"] & \mathbf{1} \arrow[d, tail, "\top"] \\
& A \arrow[r, "\chi(m)"'] & \Omega
\end{tikzcd}
$$

`[ draft version - under construction ]`

Motto:

« Thinking is necessary in order to understand the empirically given, and concepts and “categories” are necessary as indispensable elements of thinking »

A. Einstein, 1949, *Reply to criticism*,
in: P.A. Schilpp (ed.), 1949, *Albert Einstein: Philosopher–Scientist.*

Contents

Cross-dependence of sections

$$
\begin{tikzcd}
\text{categories:} & 1 \arrow[r] & 2 \arrow[r] \arrow[d] & 3 \arrow[r] & 4 \arrow[r] & 5 \arrow[d] & \\
\text{toposes:} & & 6 \arrow[rrr] \arrow[d] & & & 7 \arrow[d] \arrow[ddll, bend right] & \\
\text{logic:} & 8 \arrow[r] & 9.1 \arrow[rrr] & & & 9.2 \arrow[r] & 9.3 \\
\text{geometry:} & 10.2 \arrow[r] & 10.3 & 10.1 \arrow[l] & & &
\end{tikzcd}
$$

Introduction

Categories

Category theory may be understood as a general theory of structure. The main idea of the category-theoretic approach is to decribe the properties of structures in terms of morphisms between objects, instead of the description in terms of elements and membership relations. Hence, set-theoretic notions of 'sets' or 'spaces' are replaced by 'objects', while 'elements' are replaced by 'arrows' or 'morphisms'.

In this way, category theory may be viewed not as a generalisation of set theory, but as an alternative foundational language which allows to describe structure in a *relative* way, that is, defined in terms of relations with other structures. From this perspective, the structure of every object is specified by all morphisms between this object and other objects.

The basic correspondence between set-theoretic and category-theoretic notions can be presented in the following way:

set theory	**category theory**
set	object
subset	subobject
set of truth values $\{0,1\}$	subobject classifier Ω
power set $P(A) = 2^A$	power object $P(A) = \Omega^A$
bijection	isomorphism
one-to-one function	monic arrow
surjection	epic arrow
one-element set $\{*\}$	terminal object $\mathbf{1}$
empty set $\emptyset$	initial object $\mathbf{0}$
element of a set X	arrow $\mathbf{1} \to X$
—	non-global element $Y \to X$
—	functors
—	natural transformations
—	limits and colimits
—	adjunctions

The characteristic aspect of a category theory is that all constructions of this theory are provided in the language of diagrams, consisting of appropriate morphisms between given objects. In this sense, the concept of a mathematical structure as a 'set of elements equipped with some properties' is not fundamental. The category-theoretic proofs are provided by showing the commutativity of diagrams, and usually involve such structural concepts as functors between categories, natural transformations between functors, as well as limits and adjunctions of functors, what has to be contrasted with the structure-ignorant methods of set-theoretic formalism, based on proving the equality between the elements of sets.

However, on the other hand, category theory allows to consider the notion of an element of an object of a given mathematical theory (modelled in a particular category) in essentially more general way than this is possible within the frames of set theory. In principle, every arrow can be considered as a generalised element of its own codomain. Hence, any given object X can be considered as consisting of different collections of elements $Y \to X$, depending on which domain Y of arrows is chosen for consideration. This is usually called 'varying of elements of X over the stages Y of definition'. The set-theoretic idea of an 'absolute' element (point) of the set, $x \in X$, can be expressed in terms as a map $x : \{*\} \to X$ from one-element set $\{*\}$ to set X. The category-theoretic generalisation of this map is based on the categorical notion of terminal object $\mathbf{1}$, which allows to define the notion of a *global element* of an object X as a morphism $x : \mathbf{1} \to X$. However, not every category has a terminal object. Moreover, if a given category has a terminal object, there still can exist many different generalised elements $Y \to X$ which are not global elements, and are not expressible in terms of global elements.

Toposes

The notion of a category is very general. There are many different categories whose arrows substantially differ from functions between sets. This leads to the question whether there is a kind of category which could be regarded as a category-theoretic replacement and generalisation of (the category of) sets and functions? Lawvere (in collaboration with Tierney) found that the right kind of category for this purpose is an *elementary topos*, the direct categorical generalisation of a *Grothendieck topos*, a category invented by Grothendieck (in collaboration with Giraud) as a replacement for the notion of a space.

From the revolutionary perspective provided by Grothendieck's vision of geometry and Lawvere's vision of logic, a topos is considered as an arena for mathematical discourse. This means that topos has all the features of the set-theoretical universe that are necessary for construction of mathematical structures and their models, but is a strictly categorical notion. In particular, it generalises the notions of space and logic that are usually associated with the structures constructed in set-theoretical framework. In other words, topos can be thought of as a category theoretic 'generalisation' (abstraction) of the *structure* of universe of sets and functions that removes certain logical and geometric restrictions (constraints) of this structure while maintaining its virtues. So, while category theory replaces the framework (language) of set theory, topos theory provides a definite arena within this new framework which has all the structural advantages that were (implicitly) used while working in the previous framework.

By formulation of a mathematical theory in the internal terms of some topos $\mathcal{E}$, instead of formulating it in terms of a category **Set**, one obtains an additional flexibility of the mathematical structures of this theory. The important example of a topos is a category $\mathbf{Set}^{\mathcal{C}^{op}}$ of contravariant functors from some base category $\mathcal{C}$ to the category **Set**. The objects of some mathematical theory $\mathbb{T}$ modelled as objects in the topos $\mathbf{Set}^{\mathcal{C}^{op}}$ become representable in terms of set-valued functors over the base category $\mathcal{C}$. This enables evaluating the theory at different stages ('contexts', 'local places of view'), provided by the objects of the base category $\mathcal{C}$. In comparision, the same mathematical theory formulated in terms of the category **Set** has no variability of its internal structures—they have to be considered as context-independent, absolute. There is also a possibility of moving between different toposes, using special functors called *geometric morphisms*, and exploring what particular form (interpretation) the given theory obtains in one or another particular topos. This way the change from set-theoretic to topos-theoretic framework of construction of mathematical theories introduces two new levels of variability of structure of these theories: the context-dependence provided by variation of functors within one given universe of discourse (one given topos) and the variation of the method of contextualisation introduced by changes of universes of discourse along the geometric morphisms.

The 'variation over stages' perspective provides probably the quickest way to intuitively grasp the logical and geometrical aspects of topoi. **[***On the logical side,...***]**

[*On the geometrical side,...***]**

Purpose of this paper

This text is intended as a very basic and user-friendly introduction to topos theory, covering all categorical notions needed in order to understand the definition of an *elementary topos* (every topos is elementary), and then working out its properties in details on the simplest non-trivial example of a presheaf category $\mathbf{Set}^{\mathcal{C}^{op}}$, namely the category $\mathbf{Set}^{\mathbf{2}^{op}}$. We do not pretend here to any kind of originality: our purpose is only to provide a short but not handwavy introduction which presumably is readable also for people who do not know what a category is. We stress that there exist very good presentations of the theory which cover a lot more content, and discuss all notions in a deeper way. If the reader finds the area of category and topos theory interesting, we passionately recommend to him or her the following path of education:

- F. William Lawvere, Stephen H. Schanuel, 1997, *Conceptual mathematics: a first introduction to categories* [96] — a great introduction to category theory that presents category theory as a natural way of mathematical expression of concepts. Starting from the definition of a category, it leads gently and meaningfully to a definition and basic examples of topoi.

- Marie La Palme Reyes, Gonzalo E. Reyes, Houman Zolfaghari, 2004, *Generic figures and their glueings: a constructive approach to functor categories* [67] — this text can be considered as a continuation of the previous book. It is devoted to the systematic study of presheaf categories, considered as toposes. The large number of *precisely worked out* examples enable reader to grasp intutively and in detail different aspects and notions of topos theory.
- Peter Johnstone, 2002, *Sketches of an elephant: a topos theory compendium* [59] — the insights into category and topos theory, provided by the previous two books, would make reader ready to deal with the full scope of the theory, covered in this three-volume compendium in a pedagogical and readable way (the third volume is still in preparation).

There are also other great books on category and topos theory:

- Saunders Mac Lane, 1971, *Categories for working mathematician* [104],
- Alexander Grothendieck, Michael Artin, Jean-Louis Verdier, 1972, *Théorie des topos et cohomologie étale des schémas (SGA4)* [5],
- Mihály Makkai and Gonzalo E. Reyes, 1977, *First order categorical logic* [111],
- Peter Johnstone, 1977, *Topos theory* [57],
- Robert Goldblatt, 1979, *Topoi: the categorical analysis of logic* [47],
- Anders Kock, 1981, *Synthetic differential geometry* [62],
- Michael Barr and Charles Wells, 1985, *Toposes, triples and theories* [7],
- Jim Lambek and Peter J. Scott, 1986, *Introduction to higher order categorical logic* [78],
- John L. Bell, 1988, *Toposes and local set theories: an introduction* [9],
- Jiří Adamek, Horst Herrlich and George E. Strecker, 1990, *Abstract and concrete categories: the joy of cats* [2],
- Ieke Moerdijk and Gonzalo E. Reyes, 1991, *Models for smooth infinitesimal analysis* [121],
- Colin McLarty, 1992, *Elementary categories, elementary topoi* [115],
- Saunders Mac Lane and Ieke Moerdijk, 1992, *Sheaves in geometry and logic: a first introduction to topos theory* [107].

Acknowledgments

I would like to thank Andreas Döring, Cecilia Flori, Chris Isham, Gonzalo Reyes, Jamie Vicary and Marek Zawadowski for very valuable discussions, comments, corrections and suggestions.

1 Categories

Definition 1.1 *A **category** $\mathcal{C}$ consists of:*

- ***objects**: X, Y, ..., whose collection is denoted by* $\mathrm{Ob}(\mathcal{C})$,
- ***arrows** or **morphisms** f, g, ... whose collection is denoted by* $\mathrm{Arr}(\mathcal{C})$ *or* $\mathrm{Mor}(\mathcal{C})$,
- ***operations*** $\mathrm{dom} : \mathrm{Arr}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{C})$ *and* $\mathrm{cod} : \mathrm{Arr}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{C})$ *that assign a **codomain** $X = \mathrm{cod}(f)$ and a **domain** $Y = \mathrm{dom}(f)$ to each arrow f, what is denoted by $f : X \to Y$ or $X \xrightarrow{f} Y$,*

such that:

1. *(associativity of composition) for each pair of morphisms $A \xrightarrow{f} B$ and $B \xrightarrow{g} C$, there exists a morphism $g \circ f : A \to C$, called the **composite arrow** of f and g, denoted by the commutative diagram*

$$\begin{tikzcd} & Y \arrow[dr, "g"] & \\ X \arrow[ur, "f"] \arrow[rr, "g\circ f"'] & & Z, \end{tikzcd} \tag{1}$$

and such that for any arrows $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$, it holds that $h \circ (g \circ f) = (h \circ g) \circ f$, i.e., the diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "h\circ(g\circ f)=(h\circ g)\circ f"'] \arrow[dr, "g\circ f"] & B \arrow[d, "g"] \arrow[dl, "h\circ g"'] \\ D & C \arrow[l, "h"] \end{tikzcd} \tag{2}$$

commutes.

2. *(identity arrow) for each object Y, there exists a morphism $\mathrm{id}_Y \equiv 1_Y : Y \to Y$ called the **identity arrow** such that, given any other two morphisms $f : X \to Y$ and $g : Y \to Z$, it holds that $\mathrm{id}_Y \circ f = f$ and $g \circ \mathrm{id}_Y = g$, that is, the diagram*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "f"] & Y \arrow[d, "\mathrm{id}_Y"] \arrow[dr, "g"] & \\ & Y \arrow[r, "g"] & Z \end{tikzcd} \tag{3}$$

commutes.

Definition 1.2 *A **diagram** in a category $\mathcal{C}$ is defined as a collection of objects and arrows that belong to the category $\mathcal{C}$ with specified operations* dom *and* cod, *while a **commuting diagram** is defined as such diagram that any two arrows of this diagram which have the same domain and codomain are equal.*

Examples

1. The category **Set** consists of objects which are sets, and arrows which are functions between them. The axioms of composition, associativity and identity hold due to standard properties of sets and functions.

2. The category **Grp** consists of groups and group homomorphisms. The category **Mon** consists of monoids (semigroups with unit) and monoid homomorphisms (unit-preserving semigroup homomorphisms). The category **Top** consists of topological spaces and continuous functions between them. The category **Rng** consists of rings with (two-sided) identity and ring homomorphisms between them which send ring identity to ring identity. The category **Bool** of boolean algebras and boolean homomorphisms[1]. The category R-**Mod** consists of R-modules over ring R as objects, and R-linear maps as morphisms. If R is a field, then R-**Mod** is denoted as $\mathbf{Vect}_R$ and its objects are vector spaces over the field R. The category $\mathbf{Vect}_K^{fin}$ consist of finite-dimensional vector spaces over the field K and linear maps between them. In all these cases the composition, associativity and identity arrow follow from the standard properties of the respective morphisms.

3. The ***empty category***, denoted by $\mathbf{0}$, that contains no objects and no arrows, and the ***degenerate category***,

$$\begin{tikzcd} \mathbf{1} \equiv & 1 \arrow[loop above, "\mathrm{id}_1"] \end{tikzcd}, \tag{4}$$

that contains only one object and only one arrow which is an identity arrow of this object.

4. A ***discrete category***, defined as such category $\mathcal{C}$ that every arrow in $\mathcal{C}$ is an identity arrow. Every *set* is a discrete category.

5. The categories

$$\begin{tikzcd} \mathbf{2} \equiv & 0 \arrow[loop above, "\mathrm{id}_0"] \arrow[r, "2"] & 1, \arrow[loop above, "\mathrm{id}_1"] \end{tikzcd} \tag{5}$$

$$\begin{tikzcd} \mathbf{3} \equiv & A \arrow[loop above, "\mathrm{id}_A"] \arrow[r, "a"] \arrow[rr, bend right, "b\circ a"'] & B \arrow[loop above, "\mathrm{id}_B"] \arrow[r, "b"] & C, \arrow[loop above, "\mathrm{id}_C"] \end{tikzcd} \tag{6}$$

$$\begin{tikzcd} \mathbf{n} \equiv & 0 \arrow[loop below, "\mathrm{id}_0"] \arrow[r, "f_{0,1}"] \arrow[rr, bend left, "f_{0,2}"] \arrow[rrr, bend left, "f_{0,\ldots}"] \arrow[rrrr, bend left, "f_{0,n-1}"] & 1 \arrow[loop below, "\mathrm{id}_1"] \arrow[r, "f_{1,2}"] \arrow[rr, bend left, "f_{1,\ldots}"] \arrow[rrr, bend right, "f_{1,n-1}"'] & 2 \arrow[loop below, "\mathrm{id}_2"] \arrow[r, "f_{2,\ldots}"] \arrow[rr, bend right, "f_{2,n-1}"'] & \ldots \arrow[r, "f_{\ldots,n-1}"] & n-1. \arrow[loop below, "\mathrm{id}_{n-1}"] \end{tikzcd} \tag{7}$$

6. The category $\mathbf{N}$ of all natural numbers consists of one object N, the identity arrow id_N corresponding to $0 \in \mathbb{N}$, and a collection of arrows which correspond to subsequent natural numbers, and compose according to addition:

$$\begin{tikzcd} N \arrow[r, "5"] \arrow[rr, bend right, "7\circ5=7+5=12"'] & N \arrow[r, "7"] & N. \end{tikzcd} \tag{8}$$

This allows to write symbolically

$$\begin{tikzcd} \mathbf{N} \equiv & N \arrow[loop above, "\mathrm{id}_N\equiv0"] \arrow[loop left, "1"] \arrow[loop below, "2"] \arrow[loop right, "\cdots"] \end{tikzcd} \tag{9}$$

7. A ***monoid*** is defined as a category consisting of only one object and a collection of arrows. When viewed as a set M with elements $x \in M$ given by its arrows and a binary operator $\circ : M \times M \to M$ defined by composition of arrows, a monoid is a semigroup $(M, \circ)$ equipped with neutral element

[1] For a definition of a boolean algebra and boolean homomorphism, see Section 8.2.

$e \in M$ corresponding to the identity arrow. A group can be defined equationally as a monoid such that

$$\forall x \in M \ \ \exists y \in M \ \ x \circ y = e = y \circ x,$$

or categorically, as a category $\mathbf{G}$ with a single object X, and of collection of arrows which have this single object as domain and codomain and correspond to elements of the group, such that the identity arrow id_X corresponds to the neutral element of a group, and for every arrow $f : X \to X$ there exists an arrow $f^{-1} : X \to X$ such that $f \circ f^{-1} = \mathrm{id}_X = f^{-1} \circ f$.

8. A ***groupoid*** is defined as a category consisting of a collection of objects and arrows such that for every arrow $f : X \to Y$, there exists an arrow $f^{-1} : Y \to X$ such that $f \circ f^{-1} = \mathrm{id}_Y$ and $f^{-1} \circ f = \mathrm{id}_X$. Hence, one can equivalently define a group as a groupoid with a single object.

9. Let P be a set. The properties

   (a) $p \leq p \ \ \forall p \in P$, (*reflexivity*)

   (b) $(p \leq q \wedge q \leq r) \Rightarrow p \leq r \ \ \forall p, q, r \in P$, (*transitivity*)

   define a ***preorder*** $(P, \leq)$. A ***partially ordered set*** (***poset***) is defined as a preorder $(P, \leq)$ for which

   (c) $(p \leq q \wedge q \leq p) \Rightarrow p = q \ \ \forall p \in P$ (*antisymmetry*)

   holds. Any poset $(P, \leq)$ and any preorder $(P, \leq)$ can be considered as a category $\mathbf{P}$ consisting of objects which are elements of a set P and morphisms defined by $p \to q \iff p \leq q$. An example of a preorder category which is not poset is:

$$\begin{tikzcd} A \arrow[loop left] \arrow[r, shift left] & B \arrow[l, shift left] \arrow[loop right] \end{tikzcd} \tag{10}$$

   The category **Poset** consists of objects which are posets and of arrows which are order-preserving functions between posets, that is, the maps $T : P \to P'$ such that

$$p \leq q \ \ \Rightarrow \ \ T(p) \leq T(q). \tag{11}$$

   The category **Preord** consists of objects which are preorders and of arrows which are order-preserving functions.

10. The category **Rel** consists of objects which are pairs (X, R), where X is a set and R is a binary *relation* on X, defined as a subset $R \subseteq X \times X$ and denoted by $xRy := \langle x, y \rangle \in R$. The morphisms $f : (X, R) \to (X', R')$ in **Rel** are given by such functions $f : X \to X'$ which are relation-preserving, that is, if xRy then $f(x)Rf(y)$.

11. The ***subcategory*** of a given category $\mathcal{C}$ is defined as such collection of arrows and objects of $\mathcal{C}$ that is also a category.

12. The ***product category*** $\mathcal{C} \times \mathcal{D}$, where $\mathcal{C}$ and $\mathcal{D}$ are categories, has objects given by the pairs (C, D) such that $C \in \mathrm{Ob}(\mathcal{C})$, $D \in \mathrm{Ob}(\mathcal{D})$ and morphisms given by the pairs $(f, g) : (C, D) \to (C', D')$ such that $f : C \to C'$ is an arrow in $\mathcal{C}$ and $g : D \to D'$ is an arrows in $\mathcal{D}$. The composition is given by $(f_2, g_2) \circ (f_1, g_1) = (f_2 \circ f_1, g_2 \circ g_1)$, while the identity arrows are given by $\mathrm{id}_{(C,D)} = (\mathrm{id}_C, \mathrm{id}_D)$.

13. The ***comma category*** $\mathcal{C}{\downarrow}A$, called also ***slice category*** and denoted by $\mathcal{C}/A$. Its objects are given by arrows in $\mathcal{C}$ with fixed codomain A. If $f : B \to A$ and $g : C \to A$ are two objects, then an arrow between them is an arrow $k : B \to C$ in $\mathcal{C}$ (from the domain of f to the domain of g) such that the diagram

$$\begin{tikzcd} B \arrow[rr, "k"] \arrow[dr, "f"'] & & C \arrow[dl, "g"] \\ & A & \end{tikzcd} \tag{12}$$

   commutes. The composition between the two arrows $j : B \to C$ and $i : C \to D$ is given by the commutative diagram

$$\begin{tikzcd} B \arrow[rr, "i\circ j"] \arrow[dr, "j"] \arrow[ddr, "f"'] & & D \arrow[ddl, "h"] \\ & C \arrow[ur, "i"'] \arrow[d, "g"] & \\ & A. & \end{tikzcd} \tag{13}$$

The identity arrow id_f on $f : B \to A$ is given by $\mathrm{id}_B : B \to B$ and the commutative diagram

$$\begin{tikzcd} B \arrow[rr, "\mathrm{id}_B"] \arrow[dr, "f"'] & & B \arrow[dl, "f"] \\ & A. & \end{tikzcd} \tag{14}$$

The examples above exemplify in what sense each category can be considered as a collection of diagrams. The collections $\mathrm{Ob}(\mathcal{C})$ and $\mathrm{Arr}(\mathcal{C})$ in an arbitrary category $\mathcal{C}$ do not have necessarily to be *sets*, because they can be too large (for example, the collection of all sets is not a set). It is hence useful to define a notion for when they are.

Definition 1.3 *A category $\mathcal{C}$ is called* ***locally small*** *iff for any of its objects X, Y, the collection of arrows from X to Y is a set. This collection is denoted as $\mathcal{C}(X,Y)$, $\mathrm{Hom}_{\mathcal{C}}(X,Y)$, or just $\mathrm{Hom}(X,Y)$, and is called the* ***hom-set*** *of X, Y. A category is called* ***small*** *iff the collection of all its arrows is a set.*

It then follows that the collection of all objects in a small category must also be a set, since there is an identity arrow for every object, and the collection of identity arrows is a subcollection of the collection of all arrows. A category is called ***finite*** iff it is small and it has only a finite number of objects and arrows.

2 Arrows and elements

2.1 Basic types of arrows

Definition 2.1 *An arrow $f : X \to Y$ is called* ***monic*** *or* ***monomorphic*** *or a* ***monomorphism*** *iff $f \circ g = f \circ h \Rightarrow g = h$ for any two arrows $g, h : Z \to X$. Expressing this in category-theoretic style, we say that the diagram*

$$\begin{tikzcd} Z \arrow[r, shift left, "g"] \arrow[r, shift right, "h"'] & X \arrow[r, "f"] & Y \end{tikzcd} \tag{15}$$

should commute. If f is a monic arrow, we denote it by $f : X \rightarrowtail Y$.

Definition 2.2 *An arrow $f : Y \to X$ is called* ***epic*** *or* ***epimorphic*** *or a* ***epimorphism*** *iff for any two arrows $g, h : X \to Z$ the diagram*

$$\begin{tikzcd} Z & X \arrow[l, shift right, "g"'] \arrow[l, shift left, "h"] & Y \arrow[l, "f"] \end{tikzcd} \tag{16}$$

commutes. If f is an epic arrow, we denote it by $f : Y \twoheadrightarrow X$.

If there is only one unique morphism between two objects, we denote it using the exclamation sign ! and a dotted arrow. Hence, if an arrow $f : A \to B$ is a unique arrow between A and B, then one denotes it by $A \overset{!f}{\dashrightarrow} B$.

Definition 2.3 *An arrow $f : X \to Y$ is called* ***invertible****,* ***isomorphic****,* ***iso*** *or an* ***isomorphism*** *iff there exists a unique arrow g called the* ***inverse*** *of f such that $g \circ f = \mathrm{id}_X$ and $f \circ g = \mathrm{id}_Y$. In category-theoretic style, we say that both triangles of diagram*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\mathrm{id}_X"'] & Y \arrow[d, "\mathrm{id}_Y"] \arrow[dl, dotted, "!g"] \\ X \arrow[r, "f"'] & Y \end{tikzcd} \tag{17}$$

must commute. The unique inverse of f is usually denoted by f^{-1}. If there exists an invertible arrow between X and Y, then X and Y are called ***isomorphic*** *(in $\mathcal{C}$), which is denoted as $X \cong Y$.*

Proposition 2.4 *An iso arrow is always monic and epic.*

Proof. Consider an isomorphic arrow $f : A \to B$, two arrows $g, h : C \to A$ such that $f \circ g = f \circ h$, and two arrows $k, l : B \to D$ such that $k \circ f = l \circ f$. Then $g = \mathrm{id}_A \circ g = (f^{-1} \circ f) \circ g = f^{-1} \circ (f \circ g) = f^{-1} \circ (f \circ h) = (f^{-1} \circ f) \circ h = h$. So, the diagram

$$\begin{tikzcd} C \arrow[r, shift left, "g"] \arrow[r, shift right, "h"'] & A \arrow[r, "f"] & B \end{tikzcd} \tag{18}$$

commutes, and f is monic. Moreover, $k = k \circ \mathrm{id}_B = k \circ (f \circ f^{-1}) = (k \circ f) \circ f^{-1} = (l \circ f) \circ f^{-1} = l \circ (f \circ f^{-1}) = l$. So,

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, shift left, "k"] \arrow[r, shift right, "l"'] & D \end{tikzcd} \tag{19}$$

commutes, and f is epi. □

However, not every arrow which is monic and epic is also iso.

Examples

1. In **Set** monic arrows are injections (one-to-one functions), epic arrows are surjections (onto functions), while invertible arrows are bijections. Hence, each mono epic arrow in **Set** is iso. For monic arrows in **Set** the notation $f : X \hookrightarrow Y$ is often used.
2. In **Top** isomorphic arrows are homeomorphisms; monic and epic arrows are the same as in **Set**.
3. In **Grp** every mono epic arrow is iso. In **Mon** a mono epic arrow might not be iso.
4. In poset $(P, \leq)$, when viewed as a category **P**, every arrow is monic and epic, because there are no two different arrows between any two objects. However, the only invertible arrows in **P** are the identity arrows. If $f : p \to q$, given by $p \leq q$ is iso, then there exists $f^{-1} : q \to p$ given by $q \leq p$, hence, from the antisymmetry property of a poset, $p = q$. So, $f = \mathrm{id}_p = \mathrm{id}_q$.
5. Identity arrows are invertible in any category.
6. If $g \circ f$ is epic, then g is epic. If $g \circ f$ is mono, then f is mono.
7. A category in which all arrows are iso is a groupoid. A category with a single object in which all arrows are iso is a group.
8. In **Rng** epic arrows may be not surjective. For example, the embedding $f : \mathbb{Z} \to \mathbb{Q}$ is not surjective, but is epic, because for any $n/m \in \mathbb{Q}$ and any $h, k : \mathbb{Q} \to K$ in **Rng** such that $h \circ f = k \circ f$ one has $h(\frac{n}{m}) = h(n) \cdot h(\frac{1}{m}) \cdot h(1) = k(n) \cdot h(\frac{1}{m}) \cdot k(1) = k(n) \cdot h(\frac{1}{m}) \cdot k(m) \cdot k(\frac{1}{m}) = k(n) \cdot h(\frac{1}{m}) \cdot h(m) \cdot k(\frac{1}{m}) = k(n) \cdot h(1) \cdot k(\frac{1}{m}) = k(n) \cdot k(1) \cdot k(\frac{1}{m}) = k(\frac{n}{m})$.

Because all notions in category theory are defined in terms of arrows, they are specified only up to an isomorphism. Consider Definitions 2.1 and 2.2: they are very similar. When described in terms of diagrams, they differ from each other only by the direction of arrows. In fact, this is the simplest example of ***duality***. For a diagram Σ (which can be considered as a ***statement*** in category-theoretic language), the ***dual diagram*** (called also ***dual statement*** or ***opposite statement***) Σ^{op} has the same objects, but has all arrows inverted. That is, $A \xrightarrow{f} B$ in Σ iff $A \xleftarrow{f} B$ in Σ^{op}. In other words, the operation ${}^{op} : f \mapsto f^{op}$ interchanges the domain and codomain of the arrow. Following the line of dual statements we discover ***dual notions*** and ***dual categories***. It is often to use *conotion* as a name for a notion which is dual to a given categorical *notion*, and to use $\mathcal{C}^{op}$ to denote a category dual to category $\mathcal{C}$. More precisely:

Definition 2.5 *A category is called the* ***dual*** *or* ***opposite*** *or* ***mirror*** *category of $\mathcal{C}$ and is denoted by $\mathcal{C}^{op}$ iff*

1. *(reversion of arrows)* $\mathrm{Ob}(\mathcal{C}^{op}) = \mathrm{Ob}(\mathcal{C})$ *and* $\mathrm{Arr}(\mathcal{C}^{op}) \ni f : Y \to X \iff \mathrm{Arr}(\mathcal{C}) \ni f : X \to Y$*.*
2. *(reversion of composition) For every composition of arrows in $\mathcal{C}$ such that the diagram*

$$\begin{tikzcd} X \arrow[rr, "z \circ x"] \arrow[dr, "x"'] & & Y \\ & Z \arrow[ur, "z"'] & \end{tikzcd} \tag{20}$$

is commutative, there is a corresponding commutative diagram in $\mathcal{C}^{op}$:

$$\begin{tikzcd} X & & Y \arrow[ll, "x \circ z"'] \arrow[dl, "z"] \\ & Z. \arrow[ul, "x"] & \end{tikzcd} \tag{21}$$

An important example of pair of dual notions is given by an intial object and a terminal object.

Definition 2.6 *An* ***initial object*** *in a category $\mathcal{C}$ is an object $\mathbf{0} \in \mathrm{Ob}(\mathcal{C})$ such that for any object X in $\mathcal{C}$, there exists a unique arrow $\mathbf{0} \to X$. A* ***terminal object*** *in a category $\mathcal{C}$ is an object $\mathbf{1} \in \mathrm{Ob}(\mathcal{C})$ such that for any object X in $\mathcal{C}$, there exists a unique arrow $X \to \mathbf{1}$. A* ***null object*** *is an object which is both terminal and initial.*

Proposition 2.7 *All initial objects in a category are isomorphic. All terminal objects in a category are isomorphic.*

Proof. Let A and B be two initial objects in $\mathcal{C}$. Then we have $f : A \to B$ and $g : B \to A$ such that $f \circ g = \mathrm{id}_B$ and $g \circ f = \mathrm{id}_A$, hence $A \cong B$. The same argument holds for terminal objects. □

2.2 Generalised elements

Category-theoretic notions are defined in terms of arrows, which in particular means that categorical definition and proofs do not require consideration of elements of objects. However, this does not mean that one cannot define what an element of an object in some category is. Such a term is useful, especially for transcribing the set-theoretical terms into categorical ones, although it is not fundamental.

Definition 2.8 *Let A be an object in $\mathcal{C}$. An* ***element*** *or a* ***generalised element*** *or a* ***variable element*** *of A is an arrow in $\mathcal{C}$ with codomain A. The domain of this arrow is called the* ***stage of definition*** *(or the* ***domain of variation****) of the element.*

This means that one can regard an arrow $x : B \to A$ as a generalised element of A *defined over* B, or a variable element of A at stage B. The object B is called a *stage* or a *domain of variation* in order to express the intuition that it is a ‘place of view’ on A. It is (naïve but) useful to think about x as a set of elements of A indexed by B, or as an element of A defined in terms of a parameter in B.

A generalised element of A at stage B is usually denoted as

$$x \in_B A, \tag{22}$$

which reads as '*at stage B, $x \in A$ is satisfied*'. One can change the domain of variation with $y : C \to B$, and concern $x \circ y : C \to A$, that is, $x \circ y \in_C A$. This notation is often abused by writing just $x \in_C A$.

We say that we use the ***naïve style*** if we speak about arrows as 'elements' and about objects as 'sets'. We say that a property or notion is ***purely categorical*** if it can be defined in terms of arrows and their compositions only, such that it does not depend in its essence on any set-theoretical background.

From the definition of the terminal object and its uniqueness up to an isomorphism follows that for any domain of variation in $\mathcal{C}$, the *terminal* object has always exactly one element. Hence, one can think about $\mathbf{1}$ as a categorical generalisation of the idea of a one-element set $\{*\}$. Note that every 'set-theoretic' element of a set A in the category **Set** is just (the codomain of) an arrow

$$\{*\} \xrightarrow{\ulcorner a \urcorner} A \tag{23}$$

from an arbitrary, but fixed one-element set $\{*\}$ (called the ***singleton***) to A. On the other hand, for every set X in **Set**, there exists a unique arrow $X \to \{*\}$ to the one-element set $\{*\}$. This means that $\mathbf{1} \cong \{*\}$ in **Set**, and the elements in the ordinary set-theoretical sense of any set A are in bijective correspondence to arrows from the one-element set to A:

$$a \in A \quad \Longleftrightarrow \quad a : \mathbf{1} \longrightarrow A. \tag{24}$$

One can use this observation to define a notion expressing the set-theoretical way of being an element of an object (in any category with a terminal object). But first let us check what the initial object in **Set** is. It is a set $\mathbf{0}$ such that for every set X in **Set**, there exists only one function $\mathbf{0} \to X$. There is only one set which has this property: the empty set $\emptyset$.

Note also that in any category $\mathcal{C}$ with a terminal object $\mathbf{1}$ we have $x \circ g : Y \to X$ for any arrow $g : Y \to \mathbf{1}$ and global element $x : \mathbf{1} \to X$. If two different elements of A are defined at the stage of terminal object $\mathbf{1}$ (i.e., the terminal object is the domain),

$$\begin{tikzcd} \mathbf{1} \arrow[r, shift left, "a"] \arrow[r, shift right, "b"'] & A, \end{tikzcd} \tag{25}$$

then for any X there exists a unique arrow $x : X \dashrightarrow \mathbf{1}$ such that $x \circ a$ will differ from $x \circ b$ also at stage X ('from the point of view of X'). Hence, the elements defined at the terminal stage can be uniquely translated to elements at any other stage. This means that they are 'globally observable', and this is the reason for calling them 'global elements'. Hence, there are two different ways of being an element of an object:

- A generalised element, which $x \in_B A$ is dependent on the reference (stage) B, and varies with the change of stage.
- A global element $x \in_{\mathbf{1}} A$, which does not depend on any object of reference (what is a reason to denote $x \in_{\mathbf{1}} A$ as $x \in A$).

Definition 2.9 *An arrow $\mathbf{1} \to X$ is called a **global element** or a **global section** or a **point** of X. An arrow $Y \to X$, if Y is not isomorphic to $\mathbf{1}$, is called the **local element** or a **local section** of X at **stage** Y. An arrow $\mathrm{id}_X : X \to X$ is called the **generic element** of X.*

In a category with a terminal object, some objects may have global elements, others may not. Moreover, not all elements of a given object have to be global. Indeed, $x \in_{\mathbf{1}} X$ is a global element, but $y \in_Y X$ is not. Even if one can take some $f : \mathbf{1} \to Y$ and obtain the commutative diagram:

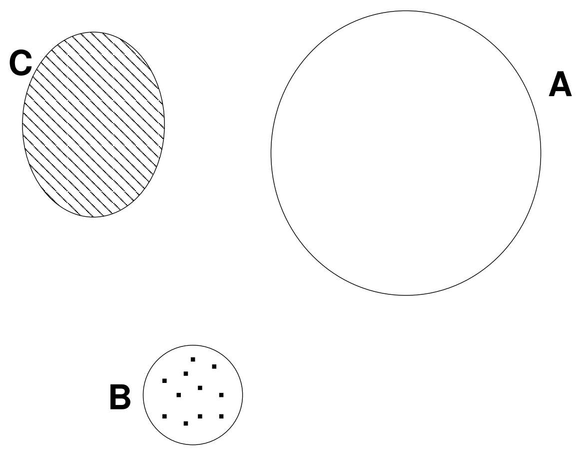
$$\begin{tikzcd} \mathbf{1} \arrow[rr, "f \circ y"] \arrow[dr, "f"'] & & X \\ & Y, \arrow[ur, "y"'] & \end{tikzcd} \tag{26}$$

there still may be other nonidentical arrows $\mathbf{1} \to Y$, which provide nonequivalent 'globalisations' of the local element $y \in_Y X$. Hence, X can have non-global elements which are seen only from some stages of definition ('contexts of observation'). So, while the global elements of an object correspond to the set-theoretic idea of *points*, there can be also such elements of an object, which are not points. The importance of this idea cannot be overestimated. It provides a fresh view on many structures. For example, let us quote [62]:

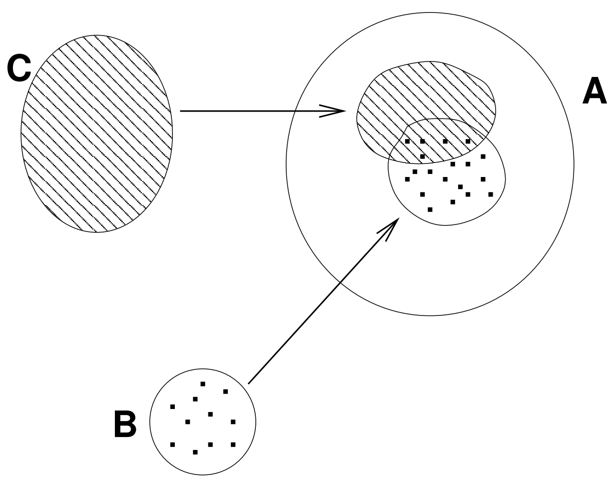
> *When thinking in terms of physics (of which geometry of space is a special case), the reason for the name "domain of variation" (instead of "stage of definition") becomes clear: for a nonatomistic point of view, a body B is not described just in terms of its "atoms" $b \in B$, that is, maps $\mathbf{1} \to B$, but in terms of "particles" of varying size X, or in terms of motions that take place in B and are parametrised by a temporal extent X; both of these situations being described by maps $X \to B$ for suitable domain of variation X.*

2.3 Basic types of diagrams

Let us now take a look at a little more complex structures, which are build from the arrows of a category. Consider three sets A, B, C.



The inclusion $B \subseteq A$ and $C \subseteq A$ may be represented as



or, categorically, by inclusions arrows

$$\begin{tikzcd}
C \arrow[dr, hook] & \\
& A. \\
B \arrow[ur, hook] &
\end{tikzcd} \tag{27}$$

If the set B is also a subset of C, then $C \subseteq A$, $B \subseteq A$ and $C \subseteq B$, hence the diagram

$$\begin{tikzcd}
C \arrow[dr, hook] \arrow[dd, hook, bend right] & \\
& A \\
B \arrow[ur, hook] &
\end{tikzcd} \tag{28}$$

commutes. Now one can ask what condition is necessary and sufficient for regarding C and B as "the same" (equivalent) subsets of A? The answer is that the inclusion $C \subseteq B$ should be replaced by a bijection $C \cong B$, what can be represented in terms of the commutative diagram

$$\begin{tikzcd}
C \arrow[dr, hook] \arrow[dd, "\cong"] & \\
& A. \\
B \arrow[ur, hook] &
\end{tikzcd} \tag{29}$$

This observation can be generalised to the categorical terms by using monic arrows instead of injections and isomorphisms instead of bijections. **[***Show that commutativity of mutual inclusion diagram implies isomorphism!***]**

Definition 2.10 *Two monic arrows x and y which satisfy*

$$\begin{tikzcd}
C \arrow[d, "\cong"'] \arrow[dr, tail, "y"] & \\
B \arrow[r, tail, "x"'] & A
\end{tikzcd} \tag{30}$$

are called ***equivalent****, which is denoted as $x \sim y$. The* ***equivalence class*** *of x is denoted as $[x]$, i.e., $[x] = \{y \mid x \sim y\}$. A* ***subobject*** *of any object is defined as an equivalence class of monomorphisms into it. The* ***class of subobjects*** *of an object A is denoted as*

$$\mathrm{Sub}(A) := \{[f] \mid \mathrm{cod}(f) = A \ \wedge \ f \text{ is monic}\}. \tag{31}$$

For a given class $[x]$ of equivalent monic arrows with codomain A, one can use any of its members to talk about this particular subobject of A, for example $x : B \rightarrowtail A$, writing x instead of $[x]$. Moreover, one can define a partial order on $\mathrm{Sub}(A)$, using the inclusion $[f] \subseteq [g]$ (denoted also by $f \subseteq g$), because for $[f] \subseteq [g]$ and $[g] \subseteq [f]$ one has $f \sim g$, hence $[f] = [g]$. The partially ordered set of subobjects of A is denoted by $(\mathrm{Sub}(A), \subseteq)$.

One can now consider the categorical analogue of a subset which consists of elements of a given set such that two given functions are equal on them. For any two arrows $f,\ g : A \to B$, their *equalising set* $E \subseteq A$ is defined as

$$E := \{e \mid e \in A \wedge f(e) = g(e)\}. \tag{32}$$

This definition may be naturally extended to categorical terms.

Definition 2.11 *An* ***equaliser*** *of two given morphisms $f, g : A \to B$ is an object E together with a morphism $e : E \to A$ such that $f \circ e = g \circ e$, and for any object D and morphism $h : D \to A$ there exists a unique morphism $k : D \to E$ such that the diagram*

$$\begin{tikzcd} E \arrow[r, "e"] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \\ & D \arrow[ul, dotted, "!k"] \arrow[u, "h"'] & \end{tikzcd} \tag{33}$$

commutes. We say that a category $\mathcal{C}$ ***has equalisers*** *iff every diagram $A \overset{f}{\underset{g}{\rightrightarrows}} B$ in $\mathcal{C}$ has an equaliser.*

The additional object D and the arrow $h : D \to A$ are considered in order to guarantee that the object E is indeed the unique solution (up to isomorphism) of the problem. This means that if there is some D which also has the properties that E has, then there must be a unique transformation (arrow) from D to E and, by symmetry, another arrow from E to D, making E and D related to each other by a unique isomorphism.

Consider now two morphisms $f : A \to C$ and $g : B \to C$ (between sets for the moment), that is, the diagram

$$\begin{tikzcd} & B \arrow[d, "g"] \\ A \arrow[r, "f"'] & C. \end{tikzcd} \tag{34}$$

One can ask about the object which will be the categorical analogue (generalisation) of the set

$$A \times_C B := \{(a, b) \in A \times B \mid f(a) = g(b)\}. \tag{35}$$

This is in fact a question about some object $A \times_C B$ together with arrows $p_A : A \times_C B \ni (a, b) \mapsto a \in A$ and $p_B : A \times_C B \ni (a, b) \mapsto b \in B$ making the diagram

$$\begin{tikzcd} A \times_C B \arrow[r, "p_B"] \arrow[d, "p_A"'] & B \arrow[d, "g"] \\ A \arrow[r, "f"'] & C \end{tikzcd} \tag{36}$$

commute. However, it is not a complete solution of the problem, because one has to make sure that the given construction of $A \times_C B$ is unique (up to isomorphism). That is, for a set $A \times_C B$, and maps p_A, p_B, making (36) commutative, one has to show that for any given D, equipped with maps $k : D \to B$ and $h : D \to A$ making the corresponding diagram commutative, there exists a unique map $\langle k, h\rangle : D \to A \times_C B$. This leads to an important definition:

Definition 2.12 *Let $f : A \to C$, $g : B \to C$ be a pair of morphisms in $\mathcal{C}$. A* ***pullback*** *or* ***fiber product*** *of f and g (or of A and B over C, if morphisms f, g between them are fixed) is an object $A \times_C B$ in $\mathcal{C}$ together with arrows $p_A : A \times_C B \to A$ and $p_B : A \times_C B \to B$, called* ***projections****, such that $f \circ p_A = g \circ p_B$, and for any object D in $\mathcal{C}$ and morphisms $h : D \to A$ and $k : D \to B$ such that $f \circ h = g \circ k$, there exists a unique morphism $\langle h, k\rangle : D \to A \times_C B$ such that the diagram*

$$\begin{tikzcd} D \arrow[dr, dotted, "!\langle h,k\rangle"] \arrow[drr, bend left, "k"] \arrow[ddr, bend right, "h"'] & & \\ & A \times_C B \arrow[r, "p_B"] \arrow[d, "p_A"'] & B \arrow[d, "g"] \\ & A \arrow[r, "f"'] & C \end{tikzcd} \tag{37}$$

commutes (hence, $p_A \circ \langle h, k\rangle = h$ and $p_B \circ \langle h, k\rangle = k$). We say that a category $\mathcal{C}$ ***has pullbacks*** *iff every diagram $A \xrightarrow{f} C \xleftarrow{g} B$ in $\mathcal{C}$ has a pullback.*

Proposition 2.13 *Pullbacks of monic arrows are monic. That is, if the arrow $f : A \to C$ in a pullback square*

$$\begin{tikzcd} D \arrow[r, "f'"] \arrow[d, "g'"'] & B \arrow[d, "g"] \\ A \arrow[r, tail, "f"'] & C \end{tikzcd} \tag{38}$$

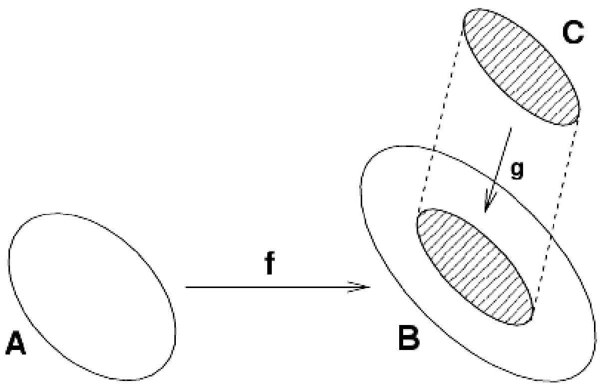
is monic, so is $f' : D \to B$.

Proof. Consider two arrows $p, q : D' \to D$ such that $f' \circ p = f' \circ q$. Then $g \circ f' \circ p = g \circ f' \circ q$ and $f \circ g' \circ p = g \circ f' \circ p = g \circ f' \circ q = f \circ g' \circ q$. Canceling f (because it is monic) we obtain $g' \circ p = g' \circ q$. The diagram (38) is a pullback, hence there is a unique $h : D' \to D$ such that $f' \circ h = f' \circ p$ and $g' \circ h = g' \circ q$. These two equations hold for $h = p$ and $h = q$, so $p = q = h$. Thus, f' is monic. □

For any pullback diagram

$$\begin{tikzcd} D \arrow[r, "f'"] \arrow[d, "g'"'] & B \arrow[d, "g"] \\ A \arrow[r, "f"'] & C \end{tikzcd} \tag{39}$$

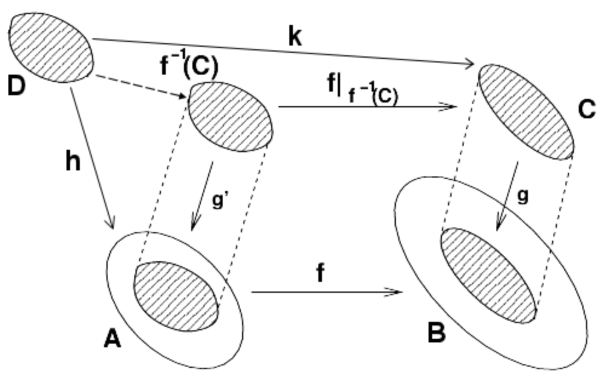
we say that f' is an ***inverse image*** of f with respect to g, or that f' is a ***result of raising*** f along g. For example, given objects $A, B, C \in \mathrm{Ob}(\mathbf{Set})$ such that $C \subset B$,



which is represented by the diagram

$$A \xrightarrow{f} B \xhookleftarrow{g} C, \tag{40}$$

one obtains as its pullback



where $f^{-1}(C)$ is such that $f \circ g' = g \circ f|_{f^{-1}(C)}$, i.e., there is a commutative diagram

$$\begin{tikzcd}
D \arrow[dr, dotted] \arrow[drr, bend left, "k"] \arrow[ddr, bend right, "h"'] & & \\
& f^{-1}(C) \arrow[r, "f|_{f^{-1}(C)}"] \arrow[d, hook, "g'"'] & C \arrow[d, hook, "g"] \\
& A \arrow[r, "f"'] & B.
\end{tikzcd} \tag{41}$$

Using duality, one can define the notions dual to the notions of equaliser and pullback, namely the coequaliser and the pushout.

Definition 2.14 *A* ***coequaliser*** *of two morphism $f,\ g : B \to A$ is an object E together with a morphism $e : A \to E$ such that $e \circ f = e \circ g$, and for any object D and morphism $h : A \to D$, there exists a unique morphism $k : E \to D$ such that the diagram*

$$\begin{tikzcd}
E \arrow[dr, dotted, "!k"'] & & A \arrow[ll, "e"'] \arrow[dl, "h"] & B \arrow[l, shift right, "f"'] \arrow[l, shift left, "g"] \\
& D & &
\end{tikzcd} \tag{42}$$

commutes. We say that a category $\mathcal{C}$ ***has coequalisers*** *iff every diagram $A \overset{f}{\underset{g}{\leftleftarrows}} B$ in $\mathcal{C}$ has a coequaliser.*

Proposition 2.15 *Every equaliser is mono. Every coequaliser is epic. Every epic coequaliser is iso. Every mono coequaliser is iso.*

Proof. Consider equaliser diagram (33). Let $a : D \to E$ and $b : D \to E$ be such arrows that $e \circ a = e \circ b$, and let $h = e \circ a$. Then $f \circ h = f \circ (e \circ a) = (f \circ e) \circ a = (g \circ e) \circ a = g \circ h$. Hence, there exists a unique arrows k such that $e \circ k = h$. So, $k = a$. On the other hand, $e \circ b = e \circ a = h$, so $k = b$. In consequence, $a = b$, hence every equaliser is mono. Now let us assume that e is epic. This implies $f = g$. Assuming also that $D = A$ and $h = \mathrm{id}_A$, we obtain $f \circ \mathrm{id}_A = g \circ \mathrm{id}_A$. Hence, there exists a unique arrow k such that $\mathrm{id}_A = e \circ k$. In consequence, $e \circ \mathrm{id}_A = \mathrm{id}_A = \mathrm{id}_A \circ e = e \circ k \circ e$. From the fact that e is an equaliser it follows that $\mathrm{id}_A = k \circ e$. Hence, k is an inverse of e, and e is iso. The statements for coequaliser follow by duality. □

Definition 2.16 *Let $f : C \to A$, $g : C \to B$ be a pair of morphisms in $\mathcal{C}$. A* ***pushout*** *or* ***fiber coproduct*** *of f and g (or of A and B over C, if morphisms f, g between them are fixed) is an object $A +_C B$ in $\mathcal{C}$ together with arrows $i_A : A \to A +_C B$ and $i_B : B \to A +_C B$, called* ***injections***[2]*, such that $i_A \circ f = i_B \circ g$, and for any object D in $\mathcal{C}$ and morphisms $h : A \to D$ and $k : B \to D$ such that $k \circ g = h \circ f$, there exists a unique morphism $[h, k] : A +_C B \to D$ such that the diagram*

$$\begin{tikzcd}
D & & \\
& A +_C B \arrow[ul, dotted, "{![h,k]}"] & B \arrow[l, "i_B"] \arrow[ull, bend right, "k"'] \\
& A \arrow[u, "i_A"] \arrow[uul, bend left, "h"] & C \arrow[l, "f"] \arrow[u, "g"']
\end{tikzcd} \tag{43}$$

commutes (thus, $[h, k] \circ i_A = h$ and $[h, k] \circ i_B = k$). We say that category $\mathcal{C}$ ***has pushouts*** *if every diagram $A \xleftarrow{f} C \xrightarrow{g} B$ in $\mathcal{C}$ has a pushout.*

[2] This naming convention is standard, but might be misleading, because these maps do not need to be neither set-theoretic inclusions, nor monomorphisms. We will try to avoid this name.

Proposition 2.17 *Pushouts of epic arrows are epic, so if the arrow $f : C \to A$ in a pushout square*

$$\begin{tikzcd} D & B \arrow[l, "f'"'] \\ A \arrow[u, "g'"] & C \arrow[l, two heads, "f"] \arrow[u, "g"'] \end{tikzcd} \tag{44}$$

is epic, then $f' : B \to D$ is epic too.

Proof. By Proposition 2.13 and duality. □

So far, we have defined categorical 'generalisations' of the sets

$$E = \{e \mid e \in A \wedge f(e) = g(e)\}, \tag{45}$$

$$A \times_C B = \{(x, y) \mid x \in A \wedge y \in B \wedge f(x) = g(y) \in C\}, \tag{46}$$

as well as their duals. This suggests to consider a categorical 'equivalent' of the set

$$A \times B = \{(x, y) \mid x \in A \wedge y \in B\}, \tag{47}$$

as well as the corresponding dual notion.

Definition 2.18 *Let A, B be objects in $\mathcal{C}$. A* ***(binary) product*** *of A and B is an object $A \times B$ together with arrows $p_A : A \times B \to A$ and $p_B : A \times B \to B$, called* ***projections****, such that for any object C of $\mathcal{C}$ and morphisms $f : C \to A$ and $g : C \to B$, there exists a unique morphism $\langle f, g\rangle : C \to A \times B$ such that the diagram*

$$\begin{tikzcd} & C \arrow[dl, "f"'] \arrow[d, dashed, "!\langle f,g\rangle"] \arrow[dr, "g"] & \\ A & A \times B \arrow[l, "p_A"] \arrow[r, "p_B"'] & B \end{tikzcd} \tag{48}$$

commutes. We say that a category $\mathcal{C}$ ***has (binary) products*** *if every pair A, B of objects in $\mathcal{C}$ has a product $A \times B$ in $\mathcal{C}$.*

Definition 2.19 *Let A, B be the objects in $\mathcal{C}$. A* ***(binary) coproduct*** *of A and B is an object $A + B$ together with arrows $i_A : A \to A + B$ and $i_B : B \to A + B$, called* ***injections****, such that for any object C of $\mathcal{C}$ and morphisms $f : A \to C$ and $g : B \to C$, there exists a unique morphism $[f, g] : A + B \to C$ such that the diagram*

$$\begin{tikzcd} & C & \\ A \arrow[ur, "f"] \arrow[r, "i_A"'] & A + B \arrow[u, dashed, "!{[f,g]}"] & B \arrow[ul, "g"'] \arrow[l, "i_B"] \end{tikzcd} \tag{49}$$

commutes. We say that a category $\mathcal{C}$ ***has (binary) coproducts*** *if every pair A, B of objects in $\mathcal{C}$ has a coproduct $A + B$ in $\mathcal{C}$.*

Examples

1. In **Set**, the terminal object **1** is a single element (singleton) set $\{*\}$, the initial object **0** is an empty set , the product $\times$ is a cartesian product of sets, the coproduct $+$ is a disjoint union of sets, given by

   $$A + B := (\{0\} \times A) \cup (\{1\} \times B) := \{\langle 0, a\rangle \mid a \in A\} \cup \{\langle 1, b\rangle \mid b \in B\},$$

   with $i_A(a) = \langle 0, a\rangle$ and $i_B(b) = \langle 1, b\rangle$, the pullback is given by

   $$\{\langle a, b\rangle \in A \times B \mid f(a) = g(b)\} = \coprod_{c \in C} f^{-1}(c) \times g^{-1}(c),$$

   while the pushout is isomorphic to the coequaliser of the pair of compositions $A \xrightarrow{f} B \hookrightarrow B + C$ and $A \xrightarrow{g} C \hookrightarrow B + C$ (for an explicit construction, see Example 15).

2. In **Top**, $\mathbf{1}$ is a one-element topological space, $\mathbf{0}$ is an empty space, $\times$ is a topological product, $+$ is a disjoint topological sum, while pushout is given by the disjoint topological sum equipped with a maximal (final) topology among those for which both injections are continuous.
3. In a single poset $(P, \leq)$, when considered as a category $\mathbf{P}$, the product $\times$ is given by an infimum $\vee$, defined by the conditions
   (a) $p \vee q \leq p$,
   (b) $p \vee q \leq q$,
   (c) if $r \leq p$ and $r \leq q$ then $r \leq p \vee q$,

   the coproduct $+$ is given by a supremum $\wedge$, defined by the conditions
   (a) $p \leq p \wedge q$,
   (b) $q \leq p \wedge q$,
   (c) if $p \leq r$ and $q \leq r$ then $p \wedge q \leq r$,

   the terminal object $\mathbf{1}$, if it exists, is given by a maximal element $1 := \max(P)$, defined as such $p \in P$ that $q \leq p \; \forall q \in P$, while the initial object $\mathbf{0}$, if it exists, is given by a minimal element $0 := \min(P)$, defined as such $p \in P$ that $p \leq q \; \forall q \in P$.
4. In a preorder category given by diagram (10) both objects are initial.
5. In **Rng**, the terminal object is the zero ring, while the ring $\mathbb{Z}$ of integers is an initial object, with the unique arrow $\mathbb{Z} \to K$ provided by the map $\mathbb{Z} \ni 1 \mapsto 1 \in K$, where 1 is the unit of the corresponding ring.
6. In **Ab**, the null object is given by the one-element group, while the equaliser is given by the kernel of homomorphism $(f - g)$ of abelian groups.
7. The object 1 in the degenerate category $\mathbf{1}$ is a null object. The object 0 in the category $\mathbf{n}$ is an initial object.
8. In **Bool**, the initial object is a two-element boolean algebra.
9. In the category of fields **Fields** there are no products, and no coproducts.
10. The terminal object in $\mathbf{Vect}_K^{fin}$ is given by the vector space $\{0\}$, while the initial object is given by the one-element vector space.
11. In **CommRng**, the coproduct $A +_C B$ is given by the tensor product $A \otimes_C B$ of A and B considered as C-modules, with $i_A(a) = a \otimes 1$ and $i_B(b) = 1 \otimes b$.
12. The null object in **Grp** and in **Mon** is given by the group with one element.
13. Every morphism e of an equaliser is a monomorphism.
14. Every non-trivial group G, when considered as a category $\mathbf{G}$, has pullbacks and pushouts, but does not have: equalisers, products, terminal object, coequalisers, coproducts, initial object.
15. **Definition** 2.20 *An **equivalence relation** in* **Set** *is defined as a subset (relation) $R \subseteq A \times A$ in* **Set** *satisfying, for every $a, b, c \in A$,*
   (a) aRa (reflexivity),
   (b) if aRb then bRa (symmetry),
   (c) if aRb and bRc then aRc (transitivity).

   *The **equivalence class** of equivalence relation R for a given $a \in A$ is defined as a set*
   $$[a] := \{b \mid aRb\} \subseteq A. \tag{50}$$
   *The **set of equivalence classes** is defined as*
   $$A/R := \{[a] \mid a \in A\}. \tag{51}$$

   Proposition 2.21 *The function $f_R : A \ni a \mapsto [a] \in A/R$ is a coequaliser of a pair of functions $f : R \ni \langle a, b \rangle \mapsto a \in A$ and $g : R \ni \langle a, b \rangle \mapsto b \in A$.*

Proof. One has $f_R \circ f = f_R \circ g$, because $f_R(a) = f_R(b)$ for any aRb. It remains to prove that there exists a unique arrow k such that (42) commutes. Let $k : A/R \to D$ and $h : A \to D$ be such arrows that $h = k \circ f_R$. Then for each $[a] \in A/R$ we have $k([a]) = k \circ f_R(a) = h(a)$. This determines the value of k at each $[a]$ by the values of h at each a. However, it still remains to prove that if $[a] = [b]$ then $k([a]) = k([b])$. This follows from $[a] = [b] \iff aRb \iff \langle a, b\rangle \in R$ and $h \circ f = h \circ g \Rightarrow h \circ f(\langle a, b\rangle) = h \circ g(\langle a, b\rangle) \Rightarrow h(a) = h(b) \Rightarrow k([a]) = k([b])$. □

Hence, the construction of sets of equivalence classes in **Set** can be considered as a construction of a particular coequaliser. Moreover, every coequaliser in **Set** can be described as a particular example of an equivalence class. If $f : B \to A$ and $g : B \to A$ are functions in **Set**, then their coequaliser is given by the set A/R, where R is the **[***smallest?***]** equivalence relation on A that contains the equivalence relation $R' := \{\langle f(b), g(b)\rangle \mid b \in B\}$.

Corollary 2.22 *In any category with binary products the objects $A \times (B \times C)$ and $(A \times B) \times C$ are isomorphic. In any category with binary products the objects $A+(B+C)$ and $(A+B)+C$ are isomorphic.*

This allows to consider n-fold products $A_1 \times \ldots \times A_n$ and n-fold coproducts of objects of a given category.

Definition 2.23 *A category which has n-fold products (respectively, coproducts) for any $n \in \mathbb{N}$ is said to* ***have finite products*** *(respectively,* ***have finite coproducts****).*

One can use the notation A^n for n-fold product of the same object A. The terminal object $\mathbf{1}$ can be considered as a 0-ary product of any object A (i.e., A^n for $n = 0$). By duality, the initial object $\mathbf{0}$ can be considered as a 0-ary coproduct of any object A.

Note that the definitions of equaliser, pullback, product and their duals have similar properties: all are given by a commutative diagram with projective (or injective) arrows, and one unique arrow. In fact all these notions are examples of a notion of ***limit***, which will be defined later. As suggested by the observation made above, also the initial and terminal objects can be considered as examples of limits.

3 Functors

Now we turn to the second fundamental concept in category theory, namely that of a *functor*. Roughly speaking, a functor is a map between two categories which preserves the structure of compositions and identities.

Definition 3.1 *A* ***(covariant) functor*** *$F : \mathcal{C} \to \mathcal{D}$ from category $\mathcal{C}$ to $\mathcal{D}$ is an assignment of an object $F(C)$ in $\mathcal{D}$ to each object C in $\mathcal{C}$ and an arrow $F(f) : F(A) \to F(B)$ in $\mathcal{D}$ to each arrow $f : A \to B$ in $\mathcal{C}$ in such a way that compositions and identities are preserved, i.e.*

1. *if f and g are arrows in $\mathcal{C}$ with $\mathrm{cod}(f) = \mathrm{dom}(g)$, then $F(g \circ f) = F(g) \circ F(f)$ in $\mathcal{D}$,*
2. *$F(\mathrm{id}_A) = \mathrm{id}_{F(A)}$ for any object A of $\mathcal{C}$.*

A functor F is called a ***contravariant functor*** *from $\mathcal{C}$ to $\mathcal{D}$, and denoted $F : \mathcal{C}^{op} \to \mathcal{D}$, if it obeys the definition given above for $\mathcal{C}$ replaced by $\mathcal{C}^{op}$. A functor $F : \mathcal{C} \to \mathcal{C}$ is called an* ***endofunctor****.*

In other words, a contravariant functor $F : \mathcal{C} \to \mathcal{D}$ is by definition equal to the covariant functor $F : \mathcal{C}^{op} \to \mathcal{D}$. Hence, a covariant functor is given by

$$\begin{tikzcd}
A & & F(A) \\
B \arrow[u, "f"] \arrow[loop right, "\mathrm{id}_B"] & \arrow[r, Mapsto, "F"] & F(B) \arrow[u, "F(f)"'] \arrow[loop left, "\mathrm{id}_{F(B)}"] \\
C \arrow[u, "g"] \arrow[uu, bend left, "f \circ g"] & & F(C) \arrow[u, "F(g)"'] \arrow[uu, bend right, "F(f) \circ F(g)"']
\end{tikzcd} \tag{52}$$

$$\mathcal{C} \overset{F}{\Longrightarrow} \mathcal{D}$$

while a contravariant functor is given by

$$\begin{tikzcd}
A & & F(A) \arrow[d, "F(f)"'] \arrow[dd, bend left, "F(g) \circ F(f)"] \\
B \arrow[u, "f"] \arrow[loop right, "\mathrm{id}_B"] & \arrow[r, Mapsto, "F"] & F(B) \arrow[d, "F(g)"'] \arrow[loop left, "\mathrm{id}_{F(B)}"] \\
C \arrow[u, "g"] \arrow[uu, bend left, "f \circ g"] & & F(C)
\end{tikzcd} \tag{53}$$

$$\mathcal{C}^{op} \overset{F}{\Longrightarrow} \mathcal{D}$$

This allows to consider the dualisation procedure of reversing of all arrows in any category $\mathcal{C}$ as an example of a contravariant functor.

Examples

1. The ***identity functor*** $\mathrm{id}_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ (denoted also by $1_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$), defined by $\mathrm{id}_C(A) = A$ and $\mathrm{id}_{\mathcal{C}}(f) = f$ for every $A \in \mathrm{Ob}(\mathcal{C})$ and every $f \in \mathrm{Arr}(\mathcal{C})$.
2. The ***constant functor*** $\Delta_A : \mathcal{C} \to \mathcal{D}$ which assigns a fixed $A \in \mathrm{Ob}(\mathcal{D})$ to any object of $\mathcal{C}$ and id_A, the identity arrow on A, to any morphism from $\mathcal{C}$:

$$\mathcal{C} \overset{\Delta_A}{\Longrightarrow} \mathcal{D} \tag{54}$$

$$\begin{tikzcd}
C \arrow[d, "f"'] & & A \arrow[d, "\mathrm{id}_A"] \\
C' \arrow[r, phantom, ""] & \arrow[u, phantom, ""] & A.
\end{tikzcd}
\quad \text{with } \xmapsto{\Delta_A}$$

   with compositions and identities preserved in a trivial way.
3. The ***forgetful functor***, which 'forgets' some part of structure, such as $U : \mathbf{Top} \to \mathbf{Set}$, $U' : \mathbf{Grp} \to \mathbf{Set}$, $U'' : \mathbf{Rng} \to \mathbf{Ab}$, where $\mathbf{Ab}$ is a category of abelian groups with group homomorphisms. For example, the forgetful functor $U : \mathbf{Top} \to \mathbf{Set}$ gives:

$$\begin{tikzcd}
(X, \mathcal{O}_X) \arrow[d, "f"'] & & X \arrow[d, "f"] \\
(Y, \mathcal{O}_Y) & & Y,
\end{tikzcd} \tag{55}$$

   where $\mathcal{O}_X$ and $\mathcal{O}_Y$ are the collections of open sets in X and Y, respectively. Hence, the functor U just 'forgets' about topology, however functions between sets are preserved. Again, preservation of compositions and identities is trivial.

4. Let $\mathcal{C}$ be a subcategory of $\mathcal{D}$. The ***inclusion functor***, sends objects and arrows of $\mathcal{D}$ into themselves in category $\mathcal{D}$. For example $\mathbf{Set}_{fin} \hookrightarrow \mathbf{Set}$, where $\mathbf{Set}_{fin}$ is the category of finite sets.
5. The composition $F \circ G : \mathcal{C} \to \mathcal{J}$ of any two functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{J}$ is a functor.
6. The ***diagonal functor*** $\Delta : \mathcal{C} \to \mathcal{C} \times \mathcal{C}$, $\Delta(A) = (A, A)$ and $\Delta(f) = (f, f)$ for $f : A \to A'$.
7. If G and H are two groups, considered as categories $\mathbf{G}$ and $\mathbf{H}$, then any functor $\mathbf{G} \to \mathbf{H}$ is a group homomorphism.
8. If two posets $(P, \leq)$ and $(Q, \leq')$ are considered as categories $\mathbf{P}$ and $\mathbf{Q}$, then any order-preserving map $T : P \to Q$ can be considered as a functor $T : \mathbf{P} \to \mathbf{Q}$. Hence, the arrows in **Poset** can be considered as functors.
9. The dual space functor $(-)^* : \mathbf{Vect}_K^{fin^{op}} \to \mathbf{Vect}_K^{fin}$, which assigns to each finite dimensional vector space V over the field K the linear space K^V of linear functionals from V to K and to each linear map $f : V \to W$ the linear map $f^* : K^W \to K^V$. This is a contravariant functor which diagrammatically can be represented as:

$$
\begin{tikzcd}
v \in \arrow[d, mapsto, "f"] & V \arrow[d, "f"] \arrow[dd, bend right, "g \circ f"'] & & K^V = V^* & \ni f^*(g^*(z^*)) = v^* = (f^* \circ g^*)(z^*) \\
w = f(v) \in \arrow[d, mapsto, "g"] & W \arrow[d, "g"] \arrow[r, Rightarrow, mapsto, "(-)^*"] & K^W = W^* \arrow[ur, "f^*"] & & \ni g^*(z^*) = w^* \arrow[u, mapsto, "f^*"] \\
(g \circ f)(v) = z = g(f(v)) \in & Z & K^Z = Z^* \arrow[u, "g^*"] \arrow[uur, bend right, "f^* \circ g^*"'] & & \ni z^* \arrow[u, mapsto, "g^*"]
\end{tikzcd}
\tag{56}
$$

10. For every locally small category $\mathcal{C}$, i.e., for every $\mathcal{C}$ such that, for any two $X, Y \in \mathrm{Ob}(\mathcal{C})$ the collection $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ of all arrows from X to Y is a *set*, there exist two functors:
    - The ***covariant hom-functor*** $\mathrm{Hom}_{\mathcal{C}}(X, -) : \mathcal{C} \to \mathbf{Set}$ such that $\mathrm{Hom}_{\mathcal{C}}(X, -)(Y) = \mathrm{Hom}_{\mathcal{C}}(X, Y)$, and for $f : Y \to Z$ in $\mathcal{C}$, $\mathrm{Hom}_{\mathcal{C}}(X, -)(f) : \mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{C}}(X, Z)$ is given by the map $g \mapsto f \circ g$.
    - The ***contravariant hom-functor*** $\mathrm{Hom}_{\mathcal{C}}(-, X) : \mathcal{C}^{op} \to \mathbf{Set}$ such that $\mathrm{Hom}_{\mathcal{C}}(-, X)(Y) = \mathrm{Hom}_{\mathcal{C}}(Y, X)$, and for $f : Y \to Z$ in $\mathcal{C}$, $\mathrm{Hom}_{\mathcal{C}}(-, X)(f) : \mathrm{Hom}_{\mathcal{C}}(Z, X) \to \mathrm{Hom}_{\mathcal{C}}(Y, X)$ is given by the map $g \mapsto g \circ f$.

    A diagrammatic presentation of these two functors is given as follows:

$$
\begin{tikzcd}
B \arrow[d, "f"] \arrow[dd, bend right, "g \circ f"'] & \mathrm{Hom}_{\mathcal{C}}(A, B) \ni b \arrow[d, "f \circ \ldots"'] \arrow[dd, bend left, "g \circ f \circ \ldots"] \\
C \arrow[d, "g"] \arrow[r, Rightarrow, mapsto, "\mathrm{Hom}_{\mathcal{C}}(A,-)"] & \mathrm{Hom}_{\mathcal{C}}(A, C) \ni f \circ b \arrow[d, "g \circ \ldots"'] \\
D & \mathrm{Hom}_{\mathcal{C}}(A, D) \ni g \circ f \circ b
\end{tikzcd}
\tag{57}
$$

$$
\mathcal{C} \xRightarrow{\mathrm{Hom}_{\mathcal{C}}(A,-)} \mathbf{Set}
$$

and

$$
\begin{tikzcd}
B & \mathrm{Hom}_{\mathcal{C}}(B, A) \ni a \arrow[d, "\ldots \circ f"'] \arrow[dd, bend left, "\ldots \circ (f \circ g)"] \\
C \arrow[u, "f"'] \arrow[r, Rightarrow, mapsto, "\mathrm{Hom}_{\mathcal{C}}(-,A)"] & \mathrm{Hom}_{\mathcal{C}}(C, A) \ni a \circ f \arrow[d, "\ldots \circ g"'] \\
D \arrow[u, "g"'] \arrow[uu, bend left, "f \circ g"] & \mathrm{Hom}_{\mathcal{C}}(D, A) \ni a \circ f \circ g
\end{tikzcd}
\tag{58}
$$

$$
\mathcal{C}^{op} \xRightarrow{\mathrm{Hom}_{\mathcal{C}}(-,A)} \mathbf{Set}
$$

The inverted direction of arrow(s) in the second diagram is typical for all contravariant functors. That diagram can be understood in the following way:

$$\left(A \underset{\mathrm{Hom}_{\mathcal{C}}(B,A)}{\overset{\longleftarrow}{\longleftarrow}} B \quad \forall f : C \to B \right) \iff \left(A \underset{\mathrm{Hom}_{\mathcal{C}}(B,A)}{\overset{\longleftarrow}{\longleftarrow}} B \xleftarrow[f]{} C \quad \forall f : C \to B \right) \iff \left(A \underset{\mathrm{Hom}_{\mathcal{C}}(C,A)}{\overset{\longleftarrow}{\longleftarrow}} C \right). \tag{59}$$

Hence, a morphism from C to B in $\mathcal{C}$ is mapped by the contravariant hom-functor into a morphism in **Set** from $\mathrm{Hom}_{\mathcal{C}}(B, A)$ to $\mathrm{Hom}_{\mathcal{C}}(C, A)$ *and not* to the morphism from $\mathrm{Hom}_{\mathcal{C}}(C, A)$ to $\mathrm{Hom}_{\mathcal{C}}(B, A)$. The reversion of direction of arrows corresponds to a functor not from $\mathcal{C}$ but from $\mathcal{C}^{op}$. For covariant hom-functors $\mathrm{Hom}_{\mathcal{C}}(A, -)$ the notation $\mathcal{C}(A, -)$, $\mathrm{Hom}(A, -)$, or just H_A is also used. Respectively, the contravariant hom-functors $\mathrm{Hom}_{\mathcal{C}}(-, A)$ are sometimes denoted by $\mathcal{C}(-, A)$, $\mathrm{Hom}(-, A)$, or simply H^A.

11. The ***hom-bifunctor*** $\mathrm{Hom}_{\mathcal{C}}(-, -) : \mathcal{C}^{op} \times \mathcal{C} \to \mathbf{Set}$, defined as a contravariant hom-functor at first argument and as a covariant hom-functor at second argument.
12. The category **Cat** consists of objects given by all small categories and morphisms given by functors all functors between them. The dualisation procedure that assigns to each category $\mathcal{C}$ its dual category $\mathcal{C}^{op}$ and to each functor $T : \mathcal{C} \to \mathcal{D}$ the functor $T^{op} : \mathcal{C}^{op} \to \mathcal{D}^{op}$, defined by $\mathrm{Ob}(\mathcal{C}^{op}) \ni C \mapsto TC \in \mathrm{Ob}(\mathcal{D}^{op})$, $\mathrm{Arr}(\mathcal{C}^{op}) \ni f^{op} \mapsto (Tf)^{op} \in \mathrm{Arr}(\mathcal{D}^{op})$ for every $f \in \mathrm{Arr}(\mathcal{C})$ defines a covariant functor $D : \mathbf{Cat} \to \mathbf{Cat}$.
13. The covariant ***power set functor*** $P : \mathbf{Set} \to \mathbf{Set}$ that assigns to each set X a ***power set*** $P(X)$ of X, with elements given by all subsets of X, and assigns to each $f : X \to Y$ the function $P(f) : P(X) \ni S \mapsto f(S) \in P(Y)$. The contravariant ***power set functor*** $P : \mathbf{Set}^{op} \to \mathbf{Set}$ that assigns to each set X the power set $P(X)$ of X and to each $f : X \to Y$ the function $P(f) : P(Y) \ni T \mapsto f^{-1}(T) \in P(Y)$.
14. The ***free group functor*** $F : \mathbf{Set} \to \mathbf{Grp}$ that assigns to each $A \in \mathrm{Ob}(\mathbf{Set})$ the free group $G \in \mathrm{Ob}(\mathbf{Grp})$ which generators are the elements of the set A, and assigns a free group[3] homomorphism to every function in **Set**.
15. The functor $GL_n : \mathbf{CommRng} \to \mathbf{Grp}$ that assigns to each commutative ring $K \in \mathrm{Ob}(\mathbf{CommRng})$ the set of all non-singular $n \times n$ matrices with entries in K (which is equal to the general linear group $GL_n(K)$) and assigns to each ring homomorphism $f : K \to K'$ the group homomorphism $GL_n(f) : GL_n(K) \to GL_n(K')$.
16. The ***Stone functor*** $S : \mathbf{Top}^{op} \to \mathbf{Bool}$ assigns to each topological space the boolean algebra of its closed-and-open subsets, and to each continuous function $f : X \to Y$ the morphism of algebras $S(f) : S(Y) \ni T \mapsto f^{-1}(T) \in S(X)$.

Definition 3.2 *A functor $F : \mathcal{C} \to \mathcal{D}$ is **full** iff for any pair of objects A, B in $\mathcal{C}$ the induced map $F_{A,B} : \mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Hom}_{\mathcal{D}}(F(A), F(B))$ is surjective. F is **faithful** if this map is injective.*

Definition 3.3 *A functor $F : \mathcal{C} \to \mathcal{D}$ is called to **preserve** a property $\wp$ of an arrow iff for every $f \in \mathrm{Arr}(\mathcal{C})$ that has a property $\wp$ it follows that $F(f) \in \mathrm{Arr}(\mathcal{D})$ has this property. A functor $F : \mathcal{C} \to \mathcal{D}$ is called to **reflect** a property $\wp$ of an arrow iff for every $F(f) \in \mathrm{Arr}(\mathcal{D})$ that has a property $\wp$ it follows that $f \in \mathrm{Arr}(\mathcal{C})$ has this property.*

Examples

1. Every inclusion functor is faithful. If inclusion functor $\mathcal{C} \to \mathcal{D}$ is also full, the category $\mathcal{C}$ is called a ***full subcategory*** of $\mathcal{D}$.
2. The forgetful functor $U : \mathbf{Top} \to \mathbf{Set}$ is full but not faithful. The discrete space functor $D : \mathbf{Set} \to \mathbf{Top}$ that assigns a discrete topology to each set is full.

[3] A group is called *free* iff there exists such subset of elements of this group that any element of this group can be written in one and only one way as a finite product of the elements of this subset and its inverses, with an assumption that the elements $f \circ f^{-1} = e = f^{-1} \circ f$ are identified as the same.

3. The free group functor $F : \mathbf{Set} \to \mathbf{Grp}$ is faithful because different functions $f : X \to Y$ of sets induce different free group homomorphisms. It is not full, because there exist group homomorphisms which do not transform the set of selected generators to the set of selected generators.
4. The covariant and contravariant power set functors are not full.
5. Every functor preserves isomorphisms. ($F(f) \circ F(f^{-1}) = F(f \circ f^{-1}) = F(\mathrm{id}_B) = \mathrm{id}_{F(B)}$, and $F(f^{-1}) \circ F(f) = \mathrm{id}_{F(A)}$.)
6. Every full and faithful functor reflects isomorphisms. (Let $f : F(A) \to F(B)$ be iso in $\mathcal{D}$, let $F : \mathcal{C} \to \mathcal{D}$ be a functor, let $g : A \to B$ be an arrow in $\mathcal{C}$ such that $F(g) = h$ and let $h : B \to A$ be the unique morphism in $\mathcal{C}$ such that $F(h) = f^{-1}$. The morphisms g and h exist, because F is full. Then $F(h \circ h) = F(h) \circ F(g) = f^{-1} \circ f = \mathrm{id}_{F(A)} = F(\mathrm{id}_A)$. Hence, $f \circ g = \mathrm{id}_A$, because F is faithful. In the same way, we obtain $g \circ h = \mathrm{id}_B$. So, g is iso.)
7. Every faithful functor reflects monomorphisms and epimorphisms.
8. $\mathbf{Grp}$ is a full subcategory of $\mathbf{Mon}$.
9. Each of inclusion functors
$$\mathbf{Poset} \to \mathbf{Preord} \to \mathbf{Rel} \tag{60}$$
is an inclusion of full subcategory.
10. Let $\mathbf{Tych}$ denote the category of Tychonoff spaces (completely regular T_1 topological spaces) and continuous maps between them, and let $\mathbf{Haus}$ denote the category of all Hausdorff topological spaces and continuous maps between them. Then each of inclusion functors
$$\mathbf{CompHaus} \to \mathbf{Tych} \to \mathbf{Haus} \to \mathbf{Top} \tag{61}$$
is an inclusion of full subcategory.
11. A *sequential space* is defined as such set X that every it sequentially closed subset $A \subseteq X$ is topologically closed. A subset A of a set X is called *sequentially closed* iff for every sequence $\{x_n\} \in A$ such that $\inf x_n = x \in X$ the conditions $x \in A$ holds. A function $Y : X \to Y$ is called *sequentially continuous* iff (if $\forall \{x_n\} \in X \quad \inf x_n = x \in X$ then $\inf f(x_n) = f(x)$). Let $\mathbf{Seq}$ denote the category of sequential spaces and sequentially continuous maps. The category $\mathbf{Seq}$ is full subcategory of $\mathbf{Top}$.

Definition 3.4 *Let $A \in \mathrm{Ob}(\mathcal{C})$, $B \in \mathrm{Ob}(\mathcal{D})$, and let $G : \mathcal{D} \to \mathcal{C}$ be a functor. Then a pair (B, u), where $u : A \to G(B)$, is called an* ***universal arrow*** *from A to G iff for each $C \in \mathrm{Ob}(\mathcal{D})$ and each $g : A \to G(C)$ there exists a unique $f : B \to C$ such that the diagram*

$$\begin{tikzcd} G(A) \arrow[r, "u"] \arrow[dr, "g"'] & G(B) \arrow[d, dotted, "G(f)"] & B \arrow[d, dotted, "f"] \\ & G(C) & C \end{tikzcd} \tag{62}$$

commutes.

Definition 3.5 *Let $A \in \mathrm{Ob}(\mathcal{C})$, $B \in \mathrm{Ob}(D)$ and let $F : \mathcal{C} \to \mathcal{D}$ be a functor. Then a pair (A, u), where $u : F(A) \to B$, is called an* ***universal arrow*** *from F to B iff for each $C \in \mathrm{Ob}(\mathcal{D})$ and each $f : F(C) \to B$ there exists a unique $g : C \to A$ such that the diagram*

$$\begin{tikzcd} F(A) \arrow[r, "u"] & F(B) & A \\ F(C) \arrow[u, dotted, "F(g)"] \arrow[ur, "f"'] & & C \arrow[u, dotted, "g"'] \end{tikzcd} \tag{63}$$

commutes.

The name ***couniversal arrow*** is not used, because ‘dual to universal’ could be misunderstood as ‘not universal at all’.

Proposition 3.6 *An universal arrow from A to G is an initial object in the category of pairs (D, f), where $D \in \mathrm{Ob}(\mathcal{D})$ and $f : A \to G(D)$, with morphisms $K : (D, f) \to (D', f')$ given by morphisms $g : D \to D'$ such that $f' = G(g) \circ f$.*

Proof. Follows directly from definition. □

4 Natural transformations and functor categories

A functor may be considered as an *arrow* between categories. One can also consider arrows between functors, called *natural transformations*. This is very powerful idea. In particular, it allows to form categories of functors and their morphisms, called *functor categories*. It is worth to note that the historical development of category theory had begun not with the notion of a category, but with notions of natural transformations of the functors [35].

4.1 Functor categories

Definition 4.1 *A **natural transformation** $\sigma : F \to G$ between functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{C} \to \mathcal{D}$ is a collection of morphisms $\sigma = \{\sigma_A\}_{A \in \mathrm{Ob}(\mathcal{C})} \in \mathrm{Arr}(\mathcal{D})$ such that for any morphism $f : A \to B$ in $\mathcal{C}$ the following diagram*

$$\begin{tikzcd} F(A) \arrow[r, "\sigma_A"] \arrow[d, "F(f)"'] & G(A) \arrow[d, "G(f)"] \\ F(B) \arrow[r, "\sigma_B"'] & G(B) \end{tikzcd} \tag{64}$$

*commutes. F and G are called the **domain** and **codomain** of σ, respectively. The maps $\{\sigma_A\}_{A \in \mathrm{Ob}(\mathcal{C})}$ are called the **components** of σ.*

It follows from the definition above that a natural transformation $\sigma : F \to G$ can also be considered as a map from objects in $\mathcal{C}$ to arrows in $\mathcal{D}$, satisfying the condition that for each $A \in \mathrm{Ob}(\mathcal{C})$, $\sigma(A) := \sigma_A$ is an arrow $\sigma_A : F(A) \to G(A)$ such that all diagrams of the form (64) commute. However, it is more useful to think of natural transformations as morphisms between functors.

Examples

1. For $f : X \to Y$ in $\mathcal{C}$, the natural transformations $\mathrm{Hom}_{\mathcal{C}}(-, f)$ and $\mathrm{Hom}_{\mathcal{C}}(f, -)$ are given by

$$\mathrm{Hom}_{\mathcal{C}}(-, f) : \mathrm{Hom}_{\mathcal{C}}(-, X) \to \mathrm{Hom}_{\mathcal{C}}(-, Y), \tag{65}$$

and

$$\mathrm{Hom}_{\mathcal{C}}(f, -) : \mathrm{Hom}_{\mathcal{C}}(Y, -) \to \mathrm{Hom}_{\mathcal{C}}(X, -), \tag{66}$$

where

$$\begin{tikzcd} \mathrm{Hom}_{\mathcal{C}}(A, X) \arrow[r, "\mathrm{Hom}_{\mathcal{C}}(-,f)(A)"] \arrow[d, "\mathrm{Hom}_{\mathcal{C}}(-,X)(f)"'] & \mathrm{Hom}_{\mathcal{C}}(A, Y) \arrow[d, "\mathrm{Hom}_{\mathcal{C}}(-,Y)(f)"] \\ \mathrm{Hom}_{\mathcal{C}}(B, X) \arrow[r, "\mathrm{Hom}_{\mathcal{C}}(-,f)(B)"'] & \mathrm{Hom}_{\mathcal{C}}(B, Y) \end{tikzcd} \tag{67}$$

and

$$\begin{tikzcd} \mathrm{Hom}_{\mathcal{C}}(Y,A) \arrow[r, "\mathrm{Hom}_{\mathcal{C}}(f,-)(A)"] \arrow[d, "\mathrm{Hom}_{\mathcal{C}}(Y,-)(f)"'] & \mathrm{Hom}_{\mathcal{C}}(X,A) \arrow[d, "\mathrm{Hom}_{\mathcal{C}}(X,-)(f)"] \\ \mathrm{Hom}_{\mathcal{C}}(Y,B) \arrow[r, "\mathrm{Hom}_{\mathcal{C}}(f,-)(B)"'] & \mathrm{Hom}_{\mathcal{C}}(X,B). \end{tikzcd} \tag{68}$$

The arrows $\mathrm{Hom}_{\mathcal{C}}(-,X)(f)$ and $\mathrm{Hom}_{\mathcal{C}}(X,-)(f)$ are sometimes denoted, respectively, as $\mathrm{Hom}_{\mathcal{C}}(X,f)$ and $\mathrm{Hom}_{\mathcal{C}}(f,X)$.

2. An insertion of a finite dimensional vector space over the field K into its double dual space is a natural transformation $\tau : \mathrm{id}_{\mathbf{Vect}_K^{fin}} \to (-)^{**}$, where $(-)^{**} : \mathbf{Vect}_K^{fin} \to \mathbf{Vect}_K^{fin}$ is a double dual functor, defined as $(-)^{**} := ((-)^*)^*$. A particular component of this natural transformation is $\tau_V : V \to V^{**} = \mathrm{Hom}_{\mathbf{Vect}_K^{fin}}(\mathrm{Hom}_{\mathbf{Vect}_K^{fin}}(V,K),K)$, with $\tau_V(x)(f) = f(x)$ for $x \in V$ and $f \in \mathrm{Hom}_{\mathbf{Vect}_K^{fin}}(V,K)$. Thus it is sufficient to say that the following diagram commutes:

$$\begin{tikzcd} 1_{\mathbf{Vect}_K^{fin}}(V) = V \arrow[r, "\tau_V"] \arrow[d, "1_{\mathbf{Vect}_K^{fin}}(f)=f"'] & (-)^{**}(V) = V^{**} \arrow[d, "f^{**}=(-)^{**}(f)"] \\ 1_{\mathbf{Vect}_K^{fin}}(W) = W \arrow[r, "\tau_W"] & (-)^{**}(W) = W^{**}. \end{tikzcd} \tag{69}$$

3. The determinant of the $n \times n$ matrix M with entries in the commutative ring K is the natural transformation $\det : GL_n \to \mathrm{Inv}$, where $\mathrm{Inv} : \mathbf{CommRng} \to \mathbf{Grp}$ assigns to each commutative ring K a group of its invertible elements, and to each ring homomorphism a corresponding group homomorphism. If $f : K \to K'$ in **CommRng**, then the natural transformation det can be described in term of the commutative diagram:

$$\begin{tikzcd} GL_n(K) \arrow[r, "\det_K"] \arrow[d, "GL_n(f)"'] & \mathrm{Inv}(K) \arrow[d, "\mathrm{Inv}(f)"] \\ GL_n(K') \arrow[r, "\det_{K'}"'] & \mathrm{Inv}(K'). \end{tikzcd} \tag{70}$$

Definition 4.2 *A natural transformation $\sigma : F \to G$ between functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{C} \to \mathcal{D}$ is called a **natural isomorphism** or a **natural equivalence** if each component σ_A is an isomorphism in $\mathcal{D}$. We denote the natural isomorphism of functors as $\sigma : F \cong G$, and call F and G **naturally isomorphic** to each other.*

Definition 4.3 *The cateories $\mathcal{C}$ and $\mathcal{D}$ are called **isomorphic** iff there exists such functor $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ that $\mathrm{id}_{\mathcal{C}} = G \circ F$ and $\mathrm{id}_{\mathcal{D}} = F \circ G$. The categories $\mathcal{C}$ and $\mathcal{D}$ are called **equivalent** iff there exist functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ such that there exist natural isomorphisms $\sigma : \mathrm{id}_{\mathcal{C}} \to G \circ F$ and $\tau : \mathrm{id}_{\mathcal{D}} \to F \circ G$.*

Definition 4.4 *A functor $F : \mathcal{C} \to \mathbf{Set}$ is called **set-valued**. Let $G : \mathcal{C} \to \mathbf{Set}$. The collection of natural transformations from F to G is denoted $\mathrm{Nat}(F,G)$. A set-valued functor F on $\mathcal{C}$ is called **covariantly representable** if it is naturally isomorphic to a functor $\mathrm{Hom}(A,-) : \mathcal{C} \to \mathbf{Set}$ for some A. A functor $G : \mathcal{C}^{op} \to \mathbf{Set}$ is called **contravariantly representable** if it is naturally isomorphic to a functor $\mathrm{Hom}(-,A) : \mathcal{C}^{op} \to \mathbf{Set}$ for some A. Such an object A is called a **representing object** for the functor F or G, respectively. The pair (τ, A), where τ is a natural isomorphism $\tau : \mathrm{Hom}(A,-) \to F$ (respectively, $\tau : \mathrm{Hom}(-,A) \to G$), is called a **representation of the functor** F (respectively, G).*

Now it is possible to abstract further and define a *category of functors*, in which natural transformations will play the role of morphisms. The composition $\tau \circ \sigma$ of natural transformations $\sigma : F \to G$ and

$\tau : G \to H$ is defined by $(\tau \circ \sigma)_A := \tau_A \circ \sigma_A$. It is also a natural transformation, what follows from the commutativity of the diagram

$$\begin{tikzcd} F(A) \arrow[r, "\sigma_A"] \arrow[d, "F(f)"'] \arrow[rr, bend left, "(\tau\circ\sigma)_A"] & G(A) \arrow[r, "\tau_A"] \arrow[d, "G(f)"'] & H(A) \arrow[d, "H(f)"'] \\ F(B) \arrow[r, "\sigma_B"'] \arrow[rr, bend right, "(\tau\circ\sigma)_B"'] & G(A) \arrow[r, "\tau_B"'] & H(B). \end{tikzcd} \tag{71}$$

This composition is sometimes denoted by the commutative diagram

$$\begin{tikzcd} \mathcal{C} \arrow[rr, bend left=60, "F"] \arrow[rr, "G"] \arrow[rr, bend right=60, "H"'] \arrow[rr, phantom, shift left=3ex, "\Downarrow\sigma"] \arrow[rr, phantom, shift right=3ex, "\Downarrow\tau"] & & \mathcal{D} \end{tikzcd} \tag{72}$$

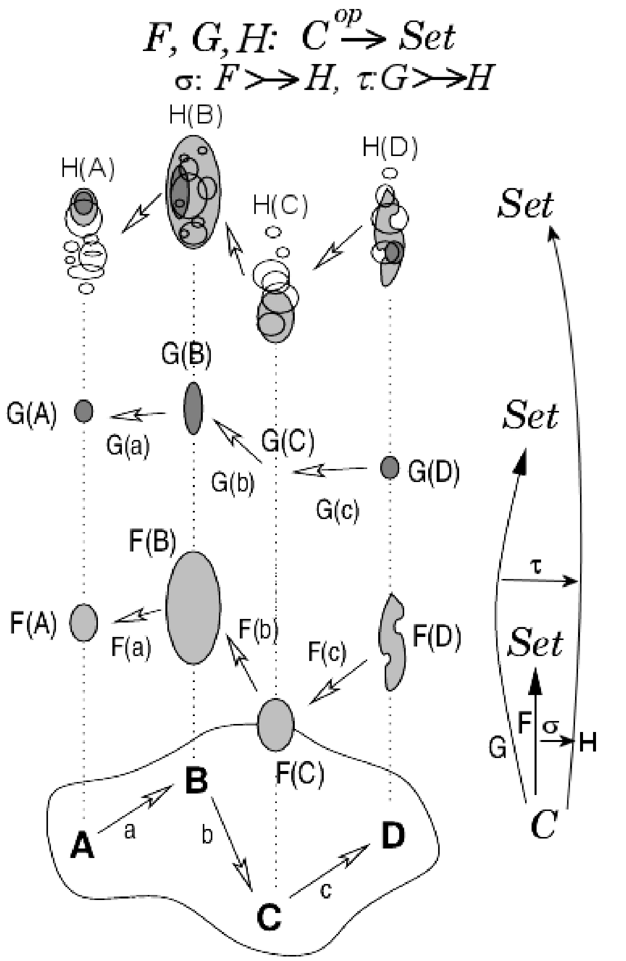
Definition 4.5 *Let $\mathcal{C}$ be a small category, and let $\mathcal{D}$ be some category. The* ***functor category*** $\mathcal{D}^{\mathcal{C}}$ *is a category whose objects are functors from $\mathcal{C}$ to $\mathcal{D}$, and whose arrows are natural transformations between them. The identity arrows are defined by natural transformations* $\mathrm{id}_F : F \to F$*, with components* $(\mathrm{id}_F)_A := \mathrm{id}_{F(A)}$.

From now on we will assume $\mathcal{C}$ to be a small category. Because any category may be thought of as a collection of diagrams one can think of a functor category $\mathcal{D}^{\mathcal{C}}$ as a category of diagrams in $\mathcal{D}$ *indexed* (or labelled) by the objects from $\mathcal{C}$.

Examples

1. The functor category of contravariant set-valued functors $\mathbf{Set}^{\mathcal{C}^{op}}$, called the category of ***presheaves*** or ***varying sets***[4], the objects of which are contravariant functors $\mathcal{C}^{op} \to \mathbf{Set}$. It may be regarded as a category of diagrams in **Set** indexed contravariantly by the objects of $\mathcal{C}$. By definition, objects of $\mathcal{C}$ play the role of ***stages***, marking the 'positions' (in passive view) or 'movements' (in active view) of the varying set $F : \mathcal{C}^{op} \to \mathbf{Set}$. For every A in $\mathcal{C}^{op}$, the set $F(A)$ is a set of elements of F *at stage* A. An arrow $f : B \to A$ between two objects in $\mathcal{C}^{op}$ induces a *transition map* $F(f) : F(A) \to F(B)$ between the varying set F at stage A and the varying set F at stage B.

[4]The name 'varying set' describes also the topos $\mathbf{Set}^{\mathcal{C}}$. The only difference between $\mathbf{Set}^{\mathcal{C}}$ and $\mathbf{Set}^{\mathcal{C}^{op}}$ is the direction of arrows in $\mathcal{C}^{op}$ which is opposite to the direction of arrows in $\mathcal{C}$.



2. The functor category $(K\text{-}\mathbf{Mod})^{\mathbf{G}}$ of K-linear representations of group G, where K is a commutative ring and $\mathbf{G}$ is a group G considered as a category. The objects of $(K\text{-}\mathbf{Mod})^{\mathbf{G}}$ are the functors $T : \mathbf{G} \to K\text{-}\mathbf{Mod}$, which assign to the single object G the K-module V and assign morphisms $V \to V$ to the morphisms $G \to G$ accordingly to the representation $T : G \to \mathrm{Aut}(G) := \mathrm{Hom}_{K\text{-}\mathbf{Mod}}(V, V)$. The natural transformation $\sigma : T \to T'$ to another representation $T' : \mathbf{G} \to K\text{-}\mathbf{Mod}$ is given for every $g : G \to G$ by the commutative diagram

$$\begin{array}{ccc} T(G) = V & \xrightarrow{\sigma_G} & V' = T'(G) \\ {\scriptstyle T(g)}\downarrow & & \downarrow{\scriptstyle T'(g)} \\ T(G) = V & \xrightarrow[\sigma_G]{} & V' = T'(G). \end{array} \tag{73}$$

The map σ_G is called an ***intertwinner*** of representations or an ***intertwinning operator***. Hence, the functor category $(K\text{-}\mathbf{Mod})^{\mathbf{G}}$ has objects given by the K-linear representations of G and morphisms given by intertwinners of representations.

3. The forgetful functor $U : \mathbf{Grp} \to \mathbf{Set}$ is represented by the group $\mathbb{Z}$, considered as a free group with one generator. The forgetful functor $U : \mathbf{Vect}_K \to \mathbf{Set}$ is represented by the vector space $\mathbb{R}$, while the forgetful functor $U : \mathbf{Top} \to \mathbf{Set}$ is represented by one-point topological space.

4. The contravariant power set functor $P : \mathbf{Set}^{op} \to \mathbf{Set}$ is represented by any two-element set (see Proposition 6.2), while the covariant power set functor $P : \mathbf{Set} \to \mathbf{Set}$ is not representable.

4.2 Yoneda's lemma

Because every contravariant functor $F : \mathcal{C} \to \mathbf{Set}$ is an object in $\mathbf{Set}^{\mathcal{C}^{op}}$, it is valid to consider the natural transformations into F as generalised elements of F. On the other hand, one can identify sets in $\mathbf{Set}$

indexed by the objects in $\mathcal{C}$ with the corresponding hom-sets of arrows between objects in $\mathcal{C}$, using the relationship between the *functor* $\mathrm{Hom}_{\mathcal{C}}(-,A) \in \mathrm{Ob}(\mathbf{Set}^{\mathcal{C}^{op}})$ and the *sets* $\mathrm{Hom}_{\mathcal{C}}(B,A)$, $\mathrm{Hom}_{\mathcal{C}}(C,A)$, ... of all arrows in $\mathcal{C}$ with codomain A and a given domain. The Yoneda lemma takes advantage of both perspectives, stating that there exists a natural bijection between the set of generalised elements of the functor F at stage $\mathrm{Hom}_{\mathcal{C}}(-,A)$ and the 'ordinary' set $F(A)$. This allows to describe the representations (τ, A) of the functor F in terms of pairs (x, A), where $x \in F(A)$.

Definition 4.6 *The* ***Yoneda embedding*** *is defined as a functor $Y : \mathcal{C} \to \mathbf{Set}^{\mathcal{C}^{op}}$ such that for $f : C \to C'$ in $\mathcal{C}$:*

1. $Y : \mathrm{Ob}(\mathcal{C}) \ni C \mapsto Y_C : \mathrm{Hom}_{\mathcal{C}}(-,C) \in \mathrm{Ob}(\mathbf{Set})$,

2. $Y : \mathrm{Arr}(\mathcal{C}) \ni f \mapsto Y_f \in \mathrm{Arr}(\mathbf{Set}^{\mathcal{C}^{op}})$ *is a natural transformation* $Y_f : Y_C \to Y_{C'}$ *such that*

$$(Y_f)_D : Y_C(D) \ni g \mapsto (Y_f)_D(g) = f \circ g \in Y_{C'}(D). \tag{74}$$

The representable Yoneda functor $Y_C := \mathrm{Hom}_{\mathcal{C}}(-,C) : \mathcal{C}^{op} \to \mathbf{Set}$ is the 'varying set' of all generalised elements of an object C. On the other hand, by natural transformation $\sigma : Y_C \to F$, it is a generalised element of a functor $F : \mathcal{C}^{op} \to \mathbf{Set}$. The Yoneda lemma states that there exists a natural bijection between the sets of generalised elements of F at stage Y_C and the sets $F(C)$. Hence, the set $F(C)$ can be considered as a representation of generalised elements of the functor F at stage $\mathrm{Hom}_{\mathcal{C}}(-,C)$.

Lemma 4.7 (Yoneda) *For any functor $F : \mathcal{C}^{op} \to \mathbf{Set}$ and any object C in $\mathcal{C}$ there exists a bijection $\sigma_{C,F} : \mathrm{Hom}_{\mathbf{Set}^{\mathcal{C}^{op}}}(Y_C, F) \to F(C)$. Moreover, this bijection is natural in the sense that for any $f : C \to C'$ in $\mathcal{C}$ and any $\mu : F \to F'$ the diagram*

$$\begin{tikzcd}
\mathrm{Hom}_{\mathbf{Set}^{\mathcal{C}^{op}}}(Y_{C'}, F) \arrow[r, "\sigma_{C',F}"] \arrow[d, "\mathrm{Hom}_{\mathbf{Set}^{\mathcal{C}^{op}}}(Y_f,\mu)"'] & F(C') \arrow[d, "\mu_C \circ F(f) = F'(f) \circ \mu_{C'}"] \\
\mathrm{Hom}_{\mathbf{Set}^{\mathcal{C}^{op}}}(Y_C, F') \arrow[r, "\sigma_{C,F'}"'] & F'(C)
\end{tikzcd} \tag{75}$$

commutes.

Proof. For $\alpha : Y_C \to F$, $D \in \mathrm{Ob}(\mathcal{C})$, $g : D \to C$ in $\mathcal{C}$, and $x \in F(C)$ let us define

$$\sigma_{C,F} : \mathrm{Hom}_{\mathbf{Set}^{\mathcal{C}^{op}}}(Y_C, F) \ni \alpha \mapsto \sigma_{C,F}(\alpha) := \alpha_C(\mathrm{id}_C) \in F(C), \tag{76}$$

and

$$\tau_{C,F} : F(C) \ni x \mapsto \tau_{C,F}(x) := \{(\tau_{C,F}(x))_D(g) = F(g)(x)\}_{D \in \mathrm{Ob}(\mathcal{C})} \in \mathrm{Hom}_{\mathbf{Set}^{\mathcal{C}^{op}}}(Y_C, F). \tag{77}$$

The former definition can be understood by noticing that $\alpha : Y_C \to F$ implies $\alpha_C : Y_C(C) \to F(C)$, hence $\alpha_C(\mathrm{id}_C) \in F(C)$. The latter definition can be understood by consideration of the commutative diagram of the natural transformation $\tau_{C,F}(x)$:

$$\begin{tikzcd}
Y_C(D') \arrow[rrrr, "(\tau_{C,F}(x))_{D'}"] &&&& F(D') \\
Y_C(D) \arrow[u, "Y_C(h)"] \arrow[rrrr, "(\tau_{C,F}(x))_D"] \arrow[dr, phantom, "\ni"] &&&& F(D) \arrow[u, "F(h)"'] \arrow[dl, phantom, "\in"] \\
& g \arrow[r, mapsto] & (\tau_{C,F}(x))_D(g) \arrow[r, equal] & F(g)(x) & \\
& \mathrm{id}_C \arrow[u, mapsto] \arrow[r, mapsto] & (\tau_{C,F}(x))_C(\mathrm{id}_C) = F(\mathrm{id}_C)(x) \arrow[r, equal] & x \arrow[u, mapsto] & \\
Y_C(C) \arrow[uuu, "Y_C(g)"] \arrow[rrrr, "(\tau_{C,F}(x))_C"'] \arrow[ur, phantom, "\ni"] &&&& F(C). \arrow[uuu, "F(g)"'] \arrow[ul, phantom, "\in"]
\end{tikzcd} \tag{78}$$

Consider first the lower square. The equality $(\tau_{C,F}(x))_C(\mathrm{id}_C) = F(\mathrm{id}_C)(x)$ follows from the definition of $\tau_{C,F}(x)$, while the equality $F(\mathrm{id}_C)(x) = x$ follows from the fact that F is a functor. In order to check that $\tau_{C,F}$ is a natural transformation, we have to check the commutativity of the upper square. We have

$$F(h) \circ (\tau_{C,F}(x))_D(g) = F(h) \circ F(g)(x) = F(g \circ h)(x) = (\tau_{C,F}(x))_{D'}(g \circ h) = (\tau_{C,F}(x))_D \circ Y_C(g), \tag{79}$$

where the first and third equality follow from the definition of $\tau_{C,F}(x)$, the second equality follows from the definition of the functor, while the last equality follows from the defition of Y_C.

Note that the natural transformation $\tau_{C,F}$ is determined by evaluation on a single element, that is, $(\tau_{C,F}(x))_C(\mathrm{id}_C) = x$. This condition is designed to restrict the class of all possible natural transformations to those that are natural. We will first prove that $\sigma_{C,F}$ and $\tau_{C,F}$ form a bijection. Composing them in one direction, we obtain

$$(\sigma_{C,F} \circ \tau_{C,F})(x) = (\tau_{C,F}(x))_C(\mathrm{id}_C) = F(\mathrm{id}_C)(x) = x, \tag{80}$$

where the first equality follows from the definition of $\sigma_{C,F}$, while the latter equalities were already discussed. In order to show the bijection in an opposite direction, that is

$$(\tau_{C,F} \circ \sigma_{C,F})(\alpha) = \alpha, \tag{81}$$

it is sufficient to show that

$$((\tau_{C,F} \circ \sigma_{C,F})(\alpha))_D(g) = \alpha_D(g) \tag{82}$$

for every $D \in \mathrm{Ob}(\mathcal{C})$ and every $g : D \to C$. Starting from the left hand side of the last equation, we obtain

$$\begin{aligned} ((\tau_{C,F} \circ \sigma_{C,F})(\alpha))_D(g) = (\tau_{C,F})\sigma_{C,F}(\alpha)))_D(g) = (\tau_{C,F})\alpha_C(\mathrm{id}_C)))_D(g) = \\ = F(g)(\alpha_C(\mathrm{id}_C)) = \alpha_D(Y_C(g)(\mathrm{id}_C)) = \alpha_D(g), \end{aligned} \tag{83}$$

where the second equality follows from the definition of $\sigma_{C,F}$, the third equality follows from the definition of $\tau_{C,F}$, the fourth equality follows from the following natural transformation commutative diagram

$$\begin{tikzcd}
Y_C(C) \arrow[r, "\alpha_C"] \arrow[d, "Y_C(g)"'] & F(C) \arrow[d, "F(g)"] \\
Y_C(D) \arrow[r, "\alpha_D"'] & F(D),
\end{tikzcd} \tag{84}$$

while the last equality follows from the definition of Y_C.

Now we can prove that this bijection is natural, that is, that the diagram

$$
\begin{tikzcd}
\mathrm{Hom}_{\mathbf{Set}^{\mathcal{C}^{op}}}(Y_{C'},F) \arrow[rrr, "\sigma_{C',F}"] \arrow[dddddd, "\mathrm{Hom}_{\mathbf{Set}^{\mathcal{C}^{op}}}(Y_f,\mu)"'] & & & F(C') \arrow[dddddd, "\mu_C \circ F(f) = F'(f) \circ \mu_{C'}"] \\
& \ni & \in & \\
& \alpha \arrow[r, mapsto] \arrow[dd] & \alpha_{C'}(\mathrm{id}_{C'}) \arrow[d, mapsto] & \\
& & (\mu_C \circ F(f))(\alpha_{C'}(\mathrm{id}_{C'})) \arrow[d, equal] & \\
& \mu \circ \alpha \circ Y_f \arrow[r, mapsto] & (\mu \circ \alpha \circ Y_f)_C(\mathrm{id}_C) & \\
& \ni & \in & \\
\mathrm{Hom}_{\mathbf{Set}^{\mathcal{C}^{op}}}(Y_C,F') \arrow[rrr, "\sigma_{C,F'}"'] & & & F'(C)
\end{tikzcd}
\tag{85}
$$

commutes. In order to show the commutativity of this diagram, it is sufficient and necessary to show that

$$(\mu_C \circ F(f))(\alpha_{C'}(\mathrm{id}_{C'})) = (\mu \circ \alpha \circ Y_f)_C(\mathrm{id}_C). \tag{86}$$

Starting from the right hand side of this equation, we obtain

$$
\begin{aligned}
(\mu \circ \alpha \circ Y_f)_C(\mathrm{id}_C) &= (\mu \circ \alpha)_C((Y_f)_C(\mathrm{id}_C)) = (\mu \circ \alpha)_C(f) = \\
&= \mu_C(\alpha_C \circ Y_{C'}(f)(\mathrm{id}_{C'})) = \mu_C \circ F(f)(\alpha_{C'}(\mathrm{id}_{C'})),
\end{aligned}
\tag{87}
$$

where the second equality follows from the definition of Y_f, the third equality follows from the definition of $Y_{C'}(f)$, while the last equality follows from the commutativity of the natural transformation diagram

$$
\begin{tikzcd}
Y_C(C') \arrow[r, "(Y_f)_{C'}"] \arrow[d, "Y_C(f)"'] & Y_{C'}(C') \arrow[r, "\alpha_{C'}"] \arrow[d, "Y_{C'}(f)"'] & F(C') \arrow[r, "\mu_{C'}"] \arrow[d, "F(f)"'] & F'(C') \arrow[d, "F'(f)"] \\
Y_C(C) \arrow[r, "(Y_f)_{C'}"'] & Y_{C'}(C) \arrow[r, "\alpha_C"'] & F(C) \arrow[r, "\mu_C"'] & F'(C).
\end{tikzcd}
\tag{88}
$$

□

Proposition 4.8 *The Yoneda embedding functor $Y : \mathcal{C} \to \mathbf{Set}^{\mathcal{C}^{op}}$ is full and faithful.*

Proof. Let $f : C \to C'$ in $\mathcal{C}$. By Yoneda lemma

$$\mathrm{Hom}_{\mathbf{Set}^{\mathcal{C}^{op}}}(Y_C, Y_{C'}) \equiv \mathrm{Nat}(\mathrm{Hom}_{\mathcal{C}}(-,C), \mathrm{Hom}_{\mathcal{C}}(-,C')) \cong \mathrm{Hom}_{\mathcal{C}}(C,C') \equiv Y_{C'}(C). \tag{89}$$

It remains to prove that there exists a natural transformation $\tau_{C,Y_{C'}} : f \mapsto \tau_{C,Y_{C'}}(f) = Y_f$. Let $g : D \to C$, then

$$(\tau_{C,Y_{C'}}(f))_D(g) = Y_{C'}(f)(g) = f \circ g = (Y_f)_D(g), \tag{90}$$

so $\tau_{C,Y_{C'}}(f) = Y_f$, what finishes the proof. Alternatively, one can prove step by step that the map $f \mapsto Y_f$ is injective and surjective. For $Y_f : Y_C \to Y_{C'}$ one has $(Y_f)_C(\mathrm{id}_C) \in Y_{C'}(C)$. Hence, by Yoneda lemma, $f \circ \mathrm{id}_C \in Y_{C'}(C)$, so $f \in Y_{C'}(C)$, what proves injectivity. On the other hand, for any natural transformation $\alpha : Y_C \to Y_{C'}$ let $f := \alpha_C(\mathrm{id}_C) : C \to C'$. Then, by Yoneda lemma, $\alpha_C(\mathrm{id}_C) = f = f \circ \mathrm{id}_C = (Y_f)_C(\mathrm{id}_C)$, what proves surjectivity. □

So, every natural transformation $Y_f : \mathrm{Hom}_{\mathcal{C}}(-,A) \to \mathrm{Hom}_{\mathcal{C}}(-,B)$ in $\mathbf{Set}^{\mathcal{C}^{op}}$ determines a morphism $f : A \to B$ (because Y is full), and every morphism f in $\mathcal{C}$ determines a natural transformation Y_f in $\mathbf{Set}^{\mathcal{C}^{op}}$ (because Y is faithful). The existence of a natural bijection $\sigma_{C,F} : \mathrm{Hom}_{\mathbf{Set}^{\mathcal{C}^{op}}}(Y_C, F) \to F(C)$ implies that there exists a bijection between the set of all natural transformations from $\mathrm{Hom}_{\mathcal{C}}(-,C)$ to any set-valued functor F and the *set* $F(C)$, i.e.

$$\mathrm{Nat}(\mathrm{Hom}_{\mathcal{C}}(-,C), F) \cong F(C). \tag{91}$$

Because Y is full and faithful, it reflects isomorphisms, hence for any $C, C' \in \mathrm{Ob}(\mathcal{C})$, if $Y(C) \cong Y(C')$ then $C \cong C'$.

As noticed in the proof, the Yoneda lemma implies that the representation $\tau : \mathrm{Hom}(-, C) \to F$ of the functor $F : \mathcal{C}^{op} \to \mathbf{Set}$ is uniquely determined by the pair (C, x), where $x := \tau_C(\mathrm{id}_C) \in F(C)$.

[*Add some propositions on representations of functors*]

4.3 Limits and colimits

Now we will define the notion of a *limit* of a diagram. For this purpose we must consider the category of diagrams in category $\mathcal{C}$ which are indexed by the objects of a category $\mathcal{J}$, in such way that the objects of diagrams in $\mathcal{C}$ are indexed by the objects of $\mathcal{J}$, while the arrows of diagrams in $\mathcal{C}$ are indexed by the arrows in $\mathcal{J}$. Such category of diagrams is just a functor category $\mathcal{C}^{\mathcal{J}}$. so, a particular diagram in $\mathcal{C}$ can be considered as an object in $\mathcal{C}^{\mathcal{J}}$ or as a functor $\mathcal{J} \to \mathcal{C}$.

Definition 4.9 *Let $\mathcal{C}^{\mathcal{J}}$ be a functor category, where $\mathcal{J}$ is a small category. Let Δ_C be a constant functor, which assigns the same object C in $\mathcal{C}$ to any object J in $\mathcal{J}$. Let $K \in \mathrm{Ob}(\mathcal{J})$ and let $j \in \mathrm{Arr}(\mathcal{J})$ such that $j : J \to K$. Let F be any functor in $\mathcal{C}^{\mathcal{J}}$. A natural transformation $\pi : \Delta_C \to F$, defined as a family of morphisms $\pi_J : C \to F(J)$, such that the triangle*

$$\begin{tikzcd} & C \arrow[dl, "\pi_J"'] \arrow[dr, "\pi_K"] & \\ F(J) \arrow[rr, "F(j)"'] & & F(K) \end{tikzcd} \tag{92}$$

commutes, is called a ***cone*** *on the functor (diagram) F with* ***vertex*** *C.*

Note that $F(J)$, $F(K)$ and C are the objects in $\mathcal{C}$, while π_J, π_K and $F(j)$ are arrows in $\mathcal{C}$. For an example of a cone, consider the situation when the category $\mathcal{J}$ is a diagram

$$\begin{tikzcd} P \arrow[r, "q"] & Q \arrow[d, "r"] \\ S \arrow[u, "p"] & R. \arrow[l, "s"] \end{tikzcd} \tag{93}$$

Then a cone on the functor $F : \mathcal{J} \to \mathcal{C}$ is

$$\begin{tikzcd} & & C \arrow[dl, "\pi_P"] \arrow[dr, "\pi_Q"] \arrow[dd, "\pi_R"] \arrow[ddll, "\pi_S"'] & \\ & F(P) \arrow[rr, "F(q)"] & & F(Q) \arrow[dl, "F(r)"] \\ F(S) \arrow[ur, "F(p)"] & & F(R). \arrow[ll, "F(s)"] & \end{tikzcd} \tag{94}$$

Definition 4.10 *A cone on F with vertex C is called a* ***limiting cone*** *on F or a* ***universal cone*** *on F or a* ***limit*** *of F, and is denoted as* $\mathrm{Lim}F$ *or* $(\mathrm{Lim}F, \pi)$*, if for every cone on F with vertex D there* *exists a unique arrow $f : D \to C$ such that the following diagram commutes:*

$$
\begin{tikzcd}
 & D \arrow[d, dotted, "!f"] \arrow[ddl, "\sigma_J"'] \arrow[ddr, "\sigma_K"] & \\
 & C \arrow[dl, "\pi_J"] \arrow[dr, "\pi_K"'] & \\
F(J) \arrow[rr, "F(j)"'] & & F(K).
\end{tikzcd}
\tag{95}
$$

*We say that a category $\mathcal{C}$ **has (finite) limits** or is (**finitely**) **complete** if every diagram $D : \mathcal{J} \to \mathcal{C}$, where $\mathcal{J}$ is a (finite) category, has a limit. A category $\mathcal{C}$ is called **left exact** iff it is finitely complete.*

Thus, a limit Lim$F$ is just a terminal object in a category **Cone**(F), which consist of objects given by cones (C, π) on F and arrows $(C, \pi) \to (C', \pi')$ given by the maps $j : J \to K$ in $\mathcal{J}$ such that $\pi_J = \pi'_K \circ j$. The important aspect of the above definitions is that they are intended for viewing the category $\mathcal{J}$ as an indexing graph, that is, without taking into consideration the compositions of arrows and identity arrows in $\mathcal{J}$. This can be formalised rigorously (see e.g. [7]) by defining the category **Cone**(F) using the category $\mathcal{C}$ and the graph (but not a category) $\mathcal{J}$, but we do not need it here.

Definition 4.11 *Let $\mathcal{J}$ be a directed poset (considered as a small category with arrows $i \to j$ defined by $i \leq j$) and $F : \mathcal{J} \to \mathcal{C}$ be a contravariant functor. The limit of F is called an **inverse limit** or **projective limit**, and is denoted $\varprojlim_{x_i} F$.*

Examples

1. The limit of $F : \{0, 1\} \to \mathcal{C}$ is a *product*:

diagram: cone: limit:

$$
\begin{tikzcd}
0 & 1
\end{tikzcd}
\qquad
\begin{tikzcd}
 & C \arrow[dl, "\pi_0"'] \arrow[dr, "\pi_1"] & \\
F(0) & & F(1)
\end{tikzcd}
\qquad
\begin{tikzcd}
 & X \arrow[d, dotted, "!\langle f,g \rangle"] \arrow[ddl, bend right, "f"'] \arrow[ddr, bend left, "g"] & \\
 & F(0) \times F(1) \arrow[dl, "\pi_0"] \arrow[dr, "\pi_1"'] & \\
F(0) & & F(1).
\end{tikzcd}
\tag{96}
$$

2. The limit of $F : \{0 \xrightarrow{3} 1 \xleftarrow{4} 2\} \to \mathcal{C}$ is a *pullback* (fiber product):

diagram: cone: limit:

$$
\begin{tikzcd}
 & 2 \arrow[d, "4"] \\
0 \arrow[r, "3"'] & 1
\end{tikzcd}
\qquad
\begin{tikzcd}
C \arrow[r, "\pi_2"'] \arrow[d, "\pi_0"] & F(2) \arrow[d, "F(4)"] \\
F(0) \arrow[r, "F(3)"'] & F(1)
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[dr, dotted, "!\langle f,g \rangle"] \arrow[drr, bend left, "g"] \arrow[ddr, bend right, "f"'] & & \\
 & P \arrow[r, "\pi_2"'] \arrow[d, "\pi_0"] & F(2) \arrow[d, "F(4)"] \\
 & F(0) \arrow[r, "F(3)"'] & F(1),
\end{tikzcd}
\tag{97}
$$

where $P = F(0) \times_{F(1)} F(2) = \{(\pi_0, \pi_2) \in F(0) \times F(2) \mid F(3)(\pi_0) = F(4)(\pi_2)\}$.

3. The limit of $F : \{\, 0 \underset{3}{\overset{2}{\rightrightarrows}} 1 \,\} \to \mathcal{C}$ is an *equaliser*:

diagram: cone: limit:

$$
0 \underset{3}{\overset{2}{\rightrightarrows}} 1
\qquad
\begin{tikzcd}
 & C \arrow[dl, "\pi_0"'] \arrow[dr, "\pi_1"] & \\
F(0) \arrow[rr, shift left, "F(2)"] \arrow[rr, shift right, "F(3)"'] & & F(1)
\end{tikzcd}
\qquad
\begin{tikzcd}
 & X \arrow[d, dotted, "!k"] \arrow[ddl, "f"'] \arrow[ddr, "g"] & \\
 & E \arrow[dl, "\pi_0"'] \arrow[dr, "\pi_1"] & \\
F(0) \arrow[rr, shift left, "F(2)"] \arrow[rr, shift right, "F(3)"'] & & F(1),
\end{tikzcd}
\tag{98}
$$

where $E = \{\pi_0 \in F(0) \mid \pi_0 \circ F(2) = \pi_0 \circ F(3)\}$ and $\pi_1 = F(2) \circ \pi_0 = F(3) \circ \pi_0$ and $g = F(2) \circ f = F(3) \circ f$.

4. The limit of $F : \mathbf{0} \to \mathcal{C}$, where $\mathbf{0}$ is an empty category, is $X \xrightarrow{!} \mathrm{Lim}F$. Hence, the limit of F is a *terminal object* $\mathbf{1}$. In particular, for $\mathcal{C} = \mathbf{Set}$ we have $X \xrightarrow{!} \mathrm{Lim}F = \{*\}$.

diagram: cone: limit:

$$
C
\qquad
\begin{tikzcd}
X \arrow[d, dotted, "!"'] \\
\mathrm{Lim}F.
\end{tikzcd}
\tag{99}
$$

A ***cocone*** and a ***colimit*** are defined by dualisation, that is, by reversing the arrows in (92) and (95). In the same say, one can and show that coequaliser, coproduct, pushout and initial object are examples of colimits. Dually, when every diagram $D : \mathcal{J} \to \mathcal{C}$, where $\mathcal{J}$ is a (finite) category, has a colimit, it is said that the category $\mathcal{C}$ ***has (finite) colimits*** or is (***finitely***) ***cocomplete***. A category is called ***right exact*** iff it is finitely cocomplete.

Definition 4.12 *Let $\mathcal{J}$ be a directed poset (considered as a small category with arrows $i \to j$ defined by $i \leq j$) and $F' : \mathcal{J} \to \mathcal{C}$ be a contravariant functor. The colimit of F' is called a **directed limit** or **inductive limit**, and is denoted $\varinjlim_{x_i} F'$.*

Definition 4.13 *A functor $F : \mathcal{C} \to \mathcal{D}$ **preserves (all) limits** and is called **left exact** iff it sends all limiting cones (limits) in $\mathcal{C}$ into limiting cones (limits) in $\mathcal{D}$. Dually, a functor $F : \mathcal{C} \to \mathcal{D}$ **preserves (all) colimits** and is called **right exact** iff it sends all colimiting cones (colimits) in $\mathcal{C}$ into colimiting cones (colimits) in $\mathcal{D}$.*

The following propositions will be important when considering toposes later on. For proofs see [104] or [2].

Proposition 4.14 *A category $\mathcal{C}$ is (finitely) complete if it has a terminal object, equalisers and (finite) products, or if it has a terminal object and (finite) pullbacks. Dually, a category $\mathcal{C}$ is (finitely) cocomplete if it has an initial object, coequalisers and (finite) coproducts, or if it has an initial object and (finite) pushouts.*

Proposition 4.15 *If category $\mathcal{D}$ is complete and category $\mathcal{C}$ is small, then the functor category $\mathcal{D}^{\mathcal{C}}$ is complete. Dually, if category $\mathcal{D}$ is cocomplete and category $\mathcal{C}$ is small, then the functor category $\mathcal{D}^{\mathcal{C}}$ is cocomplete.*

Examples

1. **Set** is complete and cocomplete.

2. Any small category $\mathcal{C}$ is complete iff it is cocomplete.
3. The representable functors $\mathrm{Hom}_{\mathcal{C}}(C,-) : \mathcal{C} \to \mathbf{Set}$ and $\mathrm{Hom}_{\mathcal{C}}(-,C) : \mathcal{C}^{op} \to \mathbf{Set}$ preserve limits and colimits.
4. **Top** and **Grp** are complete categories.
5. The forgetful functor $U : \mathbf{Grp} \to \mathbf{Set}$ preserves limits.
6. The free group functor $F : \mathbf{Set} \to \mathbf{Grp}$ preserves colimits.
7. The forgetful functor $U : \mathbf{Top} \to \mathbf{Set}$ preserves limits and colimits.

5 Adjunctions

The notion of an adjoint functor is one of the most important notions in category theory. If some mathematical construction is formulated in terms of a functor between two categories, then the functor which is (left or right) adjoint of this functor provides an important related structure that allows to understand better the original construction. By the same reasons, it is considered as mathematically elegant to define the mathematical notions in terms of adjoint functors, if it is possible. In some sense, while categories, functors and natural transformations are basic notions of category theory, limits and adjoints are its basic tools.

Apart from other reasons, we introduce adjoints in order to define the notions of exponential objects and cartesian closed categories, which are necessary for the definition of a topos. However, these two notions can be also defined without using the adjunction explicitly. In the Section 5.2 we introduce these terms in both ways, hence reader might skip Section 5.1 at the first reading, if he or she will find it too hard.

5.1 Adjoints and (co)units

Definition 5.1 *If $\mathcal{C}$ and $\mathcal{D}$ are categories and $L : \mathcal{C} \to \mathcal{D}$ and $R : \mathcal{D} \to \mathcal{C}$ are functors, then L is called **left adjoint** to R and R is called **right adjoint** to L, which is denoted by $L \dashv R$, iff there exists a bijection*

$$\sigma_{AB} : \mathrm{Hom}_{\mathcal{D}}(L(A), B) \cong \mathrm{Hom}_{\mathcal{C}}(A, R(B)), \tag{100}$$

which is natural in the sense that σ_{AB} is a component of a natural transformation σ of hom-bifunctors

$$\begin{tikzcd}
\mathcal{C}^{op} \times \mathcal{D} \arrow[rr, bend left, "\mathrm{Hom}_{\mathcal{D}}(L(-),-)"{name=U}] \arrow[rr, bend right, "\mathrm{Hom}_{\mathcal{C}}(-,R(-))"'{name=D}] & & \mathbf{Set} \arrow[from=U, to=D, Rightarrow, "\sigma"]
\end{tikzcd} \tag{101}$$

This means that for any $A'B' \in \mathrm{Ob}(\mathcal{C})$ and for any arrows $a : A' \to A$ in $\mathcal{C}$ and $b : B \to B'$ and $f : F(A) \to B$ in $\mathcal{D}$ the diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{D}}(L(A),B) \arrow[rrr, "\mathrm{Hom}_{\mathcal{D}}(L(a),b)"] \arrow[dddd, "\sigma_{AB}"'] & & & \mathrm{Hom}_{\mathcal{D}}(L(A'),B') \arrow[dddd, "\sigma_{A'B'}"] \\
& f \arrow[r, mapsto] \arrow[dd, mapsto] & b \circ f \circ L(a) \arrow[d, mapsto] & \\
& & \sigma_{A'B'}(b \circ f \circ L(a)) \arrow[d, equal] & \\
& \sigma_{AB}(f) \arrow[r, mapsto] & R(b) \circ \sigma_{AB}(f) \circ a & \\
\mathrm{Hom}_{\mathcal{C}}(A,R(B)) \arrow[rrr, "\mathrm{Hom}_{\mathcal{C}}(a,R(b))"'] & & & \mathrm{Hom}_{\mathcal{C}}(A',R(B'))
\end{tikzcd} \tag{102}$$

commutes. This is summarised in the equation

$$R(b) \circ \sigma_{AB}(f) \circ a = \sigma_{A'B'}(b \circ f \circ L(a)), \tag{103}$$

which is often schematically denoted by ***transposition diagram***, given by

$$\begin{array}{ccccccc} A' & \xrightarrow{a} & A & \xrightarrow{\sigma_{AB}(f)} & R(B) & \xrightarrow{R(b)} & R(B') \\ \hline L(A') & \xrightarrow{L(a)} & L(A) & \xrightarrow{f} & B & \xrightarrow{b} & B' \end{array} \tag{104}$$

or simply by

$$\frac{A \to R(B)}{L(A) \to B.} \tag{105}$$

If $L \dashv R$ is an adjoint pair of functors, then there exists a natural bijection between the set of elements of $R(B)$ defined on stage A and the set of elements of B defined on stage $L(A)$. For an arrow (element) $f : A \to R(B)$ in $\mathcal{C}$ we call the ***left transpose*** or ***left adjoint transpose*** or ***left adjunct*** the arrow $f^\sharp : L(A) \to B$ in $\mathcal{D}$ which corresponds to f under the natural isomorphism: $\sigma(A, B)(f^\sharp) = f$. For an arrow $g : C \to L(D)$ in $\mathcal{D}$ we denote its adjoint transposed arrow in $\mathcal{C}$ as $g^\flat : R(C) \to D$, and call it the ***right transpose*** or ***right adjoint transpose*** or ***right adjunct***.

Definition 5.2 *Let $L : \mathcal{C} \to \mathcal{D}$ and $R : \mathcal{D} \to \mathcal{C}$ be a pair of adjoint functors such that $L \dashv R$ is given by the natural transformation σ with components $\sigma_{AB} : \mathrm{Hom}_{\mathcal{D}}(L(A), B) \to \mathrm{Hom}_{\mathcal{C}}(A, R(B))$. Then*

$$\eta_A := \sigma_{AL(A)}(\mathrm{id}_{L(A)}) \in \mathrm{Hom}_{\mathcal{C}}(A, RL(A)) \tag{106}$$

is called a ***unit*** *over A, while*

$$\varepsilon_B := \sigma^{-1}_{R(B)B}(\mathrm{id}_{R(B)} \in \mathrm{Hom}_{\mathcal{D}}(LR(B), B) \tag{107}$$

is called a ***counit*** *over B.*

The bijection (100) implies that

$$\forall g \in \mathrm{Hom}_{\mathcal{C}}(A, R(B)) \;\; \exists f \in \mathrm{Hom}_{\mathcal{D}}(L(A), B) \quad g = \sigma_{AB}(f). \tag{108}$$

the naturality (101) of this bijection allows to select a unique f that satisfies the additional property, specified using the unit over A

Proposition 5.3 *An arrow η_A is universal, that is,*

$$\forall g \in \mathrm{Hom}_{\mathcal{C}}(A, R(B)) \;\; \exists! f \in \mathrm{Hom}_{\mathcal{D}}(L(A), B) \tag{109}$$

such that the diagram

$$\begin{tikzcd} A \arrow[r, "\eta_A"] \arrow[dr, "g"'] & R(L(A)) \arrow[d, dotted, "R(f)"] & L(A) \arrow[d, dotted, "f"] \\ & R(B) & B \end{tikzcd} \tag{110}$$

commutes, and f is given by $\sigma^{-1}_{AB}(g) = f$. Moreover, and arrow ε_B is universal, that is

$$\forall f \in \mathrm{Hom}_{\mathcal{D}}(L(A), B) \;\; \exists! g \in \mathrm{Hom}_{\mathcal{D}}(A, R(B)) \tag{111}$$

such that the diagram

$$\begin{tikzcd} R(B) & L(R(B)) \arrow[r, "\varepsilon_B"] & B \\ A \arrow[u, dotted, "g"] & L(A) \arrow[u, dotted, "L(g)"] \arrow[ur, "f"'] & \end{tikzcd} \tag{112}$$

commutes, and g is given by $\sigma_{AB}(f) = g$.

Proof. This result follows from naturality of bijection σ_{AB}. Let us substitute $a : A' \to A$, $f : L(A) \to B$, and $b : B \to B'$ in (104) by $\mathrm{id}_A : A \to A$, $\mathrm{id}_{L(A)} : L(A) \to L(A)$, and $f : L(A) \to B$. Then

$$\frac{A \xrightarrow{\mathrm{id}_A} A \xrightarrow{\sigma_{AL(A)}(\mathrm{id}_{L(A)})} R(L(A)) \xrightarrow{R(f)} R(B)}{L(A) \xrightarrow{L(\mathrm{id}_A)} L(A) \xrightarrow{\mathrm{id}_{L(A)}} L(A) \xrightarrow{f} B,} \tag{113}$$

hence

$$R(f) \circ \sigma_{AL(A)}(\mathrm{id}_{L(A)}) = \sigma_{AB}(f \circ \mathrm{id}_{L(A)} \circ L(\mathrm{id}_A)) = \sigma_{AB}(f \circ \mathrm{id}_{L(A)} \circ \mathrm{id}_{L(A)}) = \sigma_{AB}(f). \tag{114}$$

Using the notation

$$\sigma_{AB}(f) =: g, \tag{115}$$
$$\sigma_{AL(A)}(\mathrm{id}_{L(A)}) =: \eta_A, \tag{116}$$

we obtain

$$R(f) \circ \eta_A = g. \tag{117}$$

The proof of universality of counit follows by duality. □

Proposition 5.4 *The arrows η_A and ε_B are components of the natural transformations*

$$\begin{aligned} \eta &: \mathrm{id}_{\mathcal{C}} \to R \circ L, \\ \varepsilon &: L \circ R \to \mathrm{id}_{\mathcal{D}}, \end{aligned} \tag{118}$$

respectively. Moreover, the diagrams

$$\begin{tikzcd} & L \circ R \circ L \arrow[dr, "\varepsilon_L"] & \\ L \arrow[ur, "L(\eta)"] \arrow[rr, "\mathrm{id}_L"'] & & L \end{tikzcd} \tag{119}$$

and

$$\begin{tikzcd} & R \circ L \circ R \arrow[dr, "R(\varepsilon)"] & \\ R \arrow[ur, "\eta_R"] \arrow[rr, "\mathrm{id}_R"'] & & R \end{tikzcd} \tag{120}$$

commute.

Proof. We have

$$\begin{gathered} R(L(k)) \circ \eta_A = \sigma_{AL(A')}(L(k)) = \sigma_{AL(A')}(\mathrm{id}_{L(A')} \circ \mathrm{id}_{L(A')} \circ L(k)) = \\ R(\mathrm{id}_{L(A')}) \circ \sigma_{A'L(A)}(\mathrm{id}_{L(A')}) \circ k = \eta_{A'} \circ k, \end{gathered} \tag{121}$$

where the first equality follows from (110) and (115), the third equality follows from naturality of σ, while the last equality follows from the definition of η_A. In consequence, the diagram

$$\begin{tikzcd} A \arrow[r, "\eta_A"] \arrow[d, "k"'] & R(L(A)) \arrow[d, "R(L(k))"] \\ A' \arrow[r, "\eta_{A'}"'] & R(L(A')) \end{tikzcd} \tag{122}$$

is commutative, hence $\eta : \mathrm{id}_{\mathcal{C}} \to R \circ L$ is a natural transformation. The proof for $\varepsilon : L \circ R \to \mathrm{id}_{\mathcal{D}}$ follows from dualisation. The commutativity of diagram (119) follows from the equation

$$\varepsilon_{L(C)} \circ L(\eta_C) = \mathrm{id}_{L(C)}, \tag{123}$$

which in turn follows from transpositions

$$\frac{L(C) \xrightarrow{L(\eta_C)} L(R(L(C))) \xrightarrow{\varepsilon_{L(C)}} L(C)}{C \xrightarrow{\eta_C} R(L(C)) \xrightarrow{\mathrm{id}_{R(L(C))}} R(L(C))} \tag{124}$$

and

$$\frac{C \xrightarrow{\eta_C} R(L(C))}{L(C) \xrightarrow{\mathrm{id}_{L(C)}} L(C).} \tag{125}$$

The commutativity of the second diagram follows from duality. □

Definition 5.5 *The natural transformation $\eta : \mathrm{id}_{\mathcal{C}} \to R \circ L$ is called a* ***unit of adjunction*** *$L \dashv R$, while the natural transformation $\varepsilon : L \circ R \to \mathrm{id}_{\mathcal{D}}$ is called a* ***counit of adjunction*** *$L \dashv R$.*

The notions of unit and counit of adjunction allow an alternative definition of left and right adjoint functors.

Definition 5.6 *If $\mathcal{C}$ and $\mathcal{D}$ are categories and $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ are functors, then F is called* ***left adjoint*** *to G and G is called* ***right adjoint*** *to F, which is denoted as $F \dashv G$, iff there exist the natural transformations $\eta : \mathrm{id}_{\mathcal{C}} \to G \circ F$ and $\varepsilon : F \circ G \to \mathrm{id}_{\mathcal{D}}$, satisfying*

$$\begin{aligned} \varepsilon_D \circ F(G(f)) \circ F(\eta_C) = f, \\ G(\varepsilon_D) \circ G(F(g)) \circ \eta_C = g, \end{aligned} \tag{126}$$

for $f : F(C) \to D$ and $g : C \to G(D)$, and called the ***unit*** *and* ***counit*** *of the adjunction $L \dashv R$.*

Proposition 5.7 *Definitions 5.1 and 5.6 are equivalent.*

Proof. The derivation of Definition 5.6 from Definition 5.1 was already constructed. Following in the opposite direction, we define the natural transformations σ and τ by their components

$$\begin{aligned} \sigma_{AB}(f) := G(f) \circ \eta_A, \\ \tau_{AB}(g) := \varepsilon_B \circ F(g). \end{aligned} \tag{127}$$

Because η_A and ε_B satisfy (126), the commuting diagrams (110) and (112) hold, hence σ_{AB} and τ_{AB} are bijections, and $\sigma_{AB}^{-1} = \tau_{AB}$. □

Hence, by Definition 5.2, the natural transformation σ is uniquely determined by its values on identities in $\mathcal{C}$ and $\mathcal{D}$.

Proposition 5.8 *A left or right adjoint, if it exists, is unique up to natural isomorphism.*

Proof. Let F and F' be two functors $\mathcal{C} \to \mathcal{D}$ that are left adjoint to $G : \mathcal{D} \to \mathcal{C}$ with the corresponding units of adjunction given by η and η'. From universality of η_A it follows that there exists a unique, up to isomorphism, isomorphism $\tau_C : F(C) \to F'(C)$ such that the diagram

$$\begin{tikzcd} & C \arrow[dl, "\eta'_C"'] \arrow[dr, "\eta_C"] & \\ G(F(C)) \arrow[rr, "G(\tau_C)"'] & & G(F'(C)) \end{tikzcd} \tag{128}$$

commutes. It remains to show naturality of this isomorphim. This is equivalent ot showing that the diagram

$$\begin{tikzcd} F(C) \arrow[r, "\tau_C"] \arrow[d, "F(f)"'] & F'(C) \arrow[d, "F'(f)"] \\ F(C') \arrow[r, "\tau_{C'}"'] & F'(C') \end{tikzcd} \tag{129}$$

commutes. In order to prove this, consider the transpose of this diagram, together with the arrow $f : C \to C'$ and associated commuting trianges,

$$
\begin{tikzcd}
 & C \arrow[dl, "\eta_C"'] \arrow[dr, "\eta'_C"] \arrow[ddd, bend left=60, "f"] & \\
G(F(C)) \arrow[rr, "G(\tau_C)"'] \arrow[d, "G(F(f))"'] & & G(F'(C)) \arrow[d, "G(F'(f))"] \\
G(F(C')) \arrow[rr, "G(\tau_{C'})"] & & G(F'(C')) \\
 & C'. \arrow[ul, "\eta_{C'}"] \arrow[ur, "\eta'_{C'}"'] &
\end{tikzcd}
\tag{130}
$$

From the definition and properties of the unit of adjunction, we obtain

$$
\begin{aligned}
G(\tau_{C'} \circ F(f)) \circ \eta_C = G(\tau_{C'}) \circ G(F(f)) \circ \eta_C = G(\tau_{C'}) \circ \eta_{C'} \circ f = \eta'_{C'} \circ f, \\
G(F'(f) \circ \tau_C) \circ \eta_C = G(F'(f)) \circ G(\tau_C) \circ \eta_C = G(F'(f)) \circ \eta_{C'} = \eta'_{C'} \circ f.
\end{aligned}
\tag{131}
$$

Hence, the internal square of the diagram (130) commutes, so does it the transposed diagram (129). The proof for right adjoint follows by duality. □

Proposition 5.9 *Let $G : \mathcal{C} \to \mathcal{D}$ be a functor which has a left adjoint H. If some functor $F : \mathcal{J} \to \mathcal{C}$ has a limit in $\mathcal{C}$ given by a vertex* $\mathrm{Lim}F$ *and a family of projections π, then the functor $G \circ F : \mathcal{J} \to \mathcal{D}$ has a limit in $\mathcal{D}$ given by the vertex* $\mathrm{Lim}GF$ *and the family of projections $G\pi$.*

Proof. Consider the diagrams

$$
\begin{tikzcd}
 & HX \arrow[d, dotted, "h"] \arrow[ddl, bend right, "\sigma^\flat_J"'] \arrow[ddr, bend left, "\sigma^\flat_K"] & \\
 & \mathrm{Lim}F \arrow[dl, "\pi_J"'] \arrow[dr, "\pi_K"] & \\
F(J) \arrow[rr, "F(j)"'] & & F(K)
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
 & X \arrow[d, dotted, "h^\sharp"] \arrow[ddl, bend right, "\sigma_J"'] \arrow[ddr, bend left, "\sigma_K"] & \\
 & G(\mathrm{Lim}F) \arrow[dl, "G(\pi_J)"'] \arrow[dr, "G(\pi_K)"] & \\
G(F(J)) \arrow[rr, "G(F(j))"'] & & G(F(K)).
\end{tikzcd}
\tag{132}
$$

Recall that the category $\mathbf{Cone}(F)$ consists of objects given by cones (C, π) over functor $F : \mathcal{J} \to \mathcal{C}$ and arrows $f : (C, \pi) \to (C', \pi')$ such that $\pi_J = \pi'_K \circ j$, where $j : J \to K$ is an arrow in $\mathcal{J}$. By applying the functor G to the limit $(\mathrm{Lim}F, \pi)$ in $\mathcal{C}$, we get a cone $(G(\mathrm{Lim}F), G(\pi))$ in $\mathcal{D}$. $H \dashv G$, so there is an isomorphism between the cones $\sigma : X \to G \circ F$ and $\sigma^b : H(X) \to H \circ G \circ F \cong F$ from Definition 4.1 and the definition of a right adjoint. However, $(\mathrm{Lim}F, \pi)$ is a universal cone (a terminal obect in the category $\mathbf{Cone}(F)$), so there exists a unique arrow $h : H(X) \to \mathrm{Lim}F$ such that $\pi \circ h = \sigma^\flat$. Thus, there exists a unique arrow $h^\sharp : G(H(X)) = X \to G(\mathrm{Lim}F)$ such that $(\pi \circ h)^\sharp = (\sigma^\flat)^\sharp = \sigma$ from left adjunction and uniqueness of h. Hence $G(\pi) : G(\mathrm{Lim}F) \to F$ is a limiting cone. In other words, $G(\mathrm{Lim}F) = \mathrm{Lim}G \circ F$. □

Corollary 5.10 *Right adjoint functors preserve limits (in particular: products, pullbacks, equalisers and the terminal object). By duality, left adjoint functors preserve colimits (in particular: coproducts, pullbacks, coequalisers and the initial object).*

Definition 5.11 *The categories $\mathcal{C}$ and $\mathcal{D}$ are called* ***equivalent*** *iff there exists a pair of adjoint functors $L \dashv R$, $L : \mathcal{C} \to \mathcal{D}$ and $R : \mathcal{D} \to \mathcal{C}$, such that the unit $\eta : \mathrm{id}_{\mathcal{C}} \to L \circ R$ and the counit $\varepsilon : R \circ L \to \mathrm{id}_{\mathcal{D}}$ are natural isomorphisms $\mathrm{id}_{\mathcal{C}} \cong L \circ R$ and $R \circ L \cong \mathrm{id}_{\mathcal{D}}$.*

Definition 5.12 *The subcategory $\mathcal{C}$ of $\mathcal{D}$ is called* ***reflective*** *in $\mathcal{D}$ iff the inclusion functor $F : \mathcal{C} \to \mathcal{D}$ has a left adjoint $G : \mathcal{D} \to \mathcal{C}$. The functor G is called a* ***reflector****. Dually, $\mathcal{C}$ is called* ***coreflective*** *in $\mathcal{D}$ iff the inclusion functor $F : \mathcal{C} \to \mathcal{D}$ has a right adjoint, which is called a* ***coreflector****.*

Examples

1. The functor $\mathcal{C} \to \mathbf{1}$ has a right adjoint iff $\mathcal{C}$ has terminal object, and has left adjoint iff $\mathcal{C}$ has initial object. The components of the unit of the right adjunction are given by the morphisms $\mathrm{Ob}(\mathcal{C}) \ni C \mapsto \mathbf{1} \in \mathbf{1}$, while the components of the counit of the left adjunction are given by the morphism $\mathbf{1} \ni 0 \mapsto C \in \mathrm{Ob}(\mathcal{C})$.
2. If a category $\mathcal{C}$ has binary products, then the ***cartesian product bifunctor*** $- \times - : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is right adjoint to diagonal functor $\Delta : \mathcal{C} \to \mathcal{C} \times \mathcal{C}$. Because right adjoint is unique if it exists, then the category $\mathcal{C}$ is said to ***have binary products*** iff the functor Δ has a right adjoint. The unit of the right adjoint of Δ is given by the ***diagonal arrow*** $\delta_A : A \to (A, A)$, while its counit is given by the pair of projections $\pi_A : A \times B \to A$ and $\pi_B : A \times B \to B$. The left adjoint of Δ, if it exists, is a ***coproduct bifunctor*** $\sqcup : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ assigning to each pair (A, B) of objects its coproduct $\sqcup(A, B) = A+B$. The unit of the left adjunction of Δ is given by the pair of injections $i : A \to A+B$ and $j : B \to A + B$, while its counit is given by the arrow $A + A \ni (i(x), j(x)) \mapsto x \in A$.
3. The constant functor $\Delta : \mathcal{C} \to \mathcal{C}^{\mathcal{D}}$ has right adjoint iff $\mathcal{C}$ has limits indexed by $\mathcal{D}$ and has left adjoint iff $\mathcal{C}$ has colimits indexed by $\mathcal{D}$. The unit of left adjunction and the counit of right adjunction are given bythe universal cone.
4. The forgetful functor $U : \mathbf{Grp} \to \mathbf{Set}$ has a left adjoint which is a free group functor $F : X \to F(X)$ that assigns to X a free group $F(X)$ which generators are the elements of X. The unit of an adjunction is an insertion of these generators from **Set** to **Set** given by $X \to U(F(X))$.
5. The forgetful functor $U : \mathbf{Top} \to \mathbf{Set}$ jas a left adjoint which assigns to every set X the *discrete topology* on X, defined as such topology that all subsets of X are open.
6. The inclusion functor $U : \mathbf{CompHaus} \to \mathbf{Top}$ has a left adjoint which is the Čech–Stone compactification functor $\beta : \mathbf{Top} \to \mathbf{CompHaus}$. The category **CompHaus** is a subcategory of the category **CplRegHaus** of all completely regular Hausdorff spaces. As a subcategory, it is full and reflexive. The reflector $\beta : \mathbf{CplRegHaus} \to \mathbf{CompHaus}$ sends each $X \in \mathrm{Ob}(\mathbf{CplRegHaus})$ to its Čech–Stone compactification. By the Uryhson lemma, an universal arrow $X \to U(\beta(X))$ of this adjunction is an injection. In the general case of inclusion $U : \mathbf{CompHaus} \to \mathbf{Top}$ the universal arrow of the Čech–Stone compactification left adjoint functor is no longer an injection. The forgetful functor $U : \mathbf{CompHaus} \to \mathbf{Set}$ has a left adjoint whic assigns to each set X the Čech–Stone compactification of the discrete topology on X.
7. The functor $\mathrm{Hom}_{\mathbf{Vect}_K^{fin}}(V, -) : \mathbf{Vect}_K^{fin} \to \mathbf{Vect}_K^{fin}$, $V \in \mathrm{Ob}(\mathbf{Vect}_K^{fin})$, has left adjoint given by the tensor multiplication $(-) \otimes V : \mathbf{Vect}_K^{fin} \to \mathbf{Vect}_K^{fin}$. More generally, an adjoint pair $(- \otimes_K V) \dashv \mathrm{Hom}_{K\text{-}\mathbf{Mod}}(V, -)$ of functors K-**Mod** $\to$ K-**Mod** is determined by a counit $\mathrm{Hom}_{K\text{-}\mathbf{Mod}}(V, W) \otimes_K V \to V$, given by evaluation.
8. $\mathbf{Vect}_{\mathbb{R}}^{fin}$ and $(\mathbf{Vect}_{\mathbb{R}}^{fin})^{op}$ are equivalent. **Rel** and $\mathbf{Rel}^{op}$ are equivalent.
9. (Stone duality.) Let **BoolSp** denote the category of boolean spaces, defined as zero-dimensional compact Hausdorff spaces, and continuous functions between them. Then the Stone theorem asserts that to each boolean space on can associate the boolean algebra of its closed-and-open subsets. From this it follows that the categories $\mathbf{Bool}^{op}$ and **BoolSp** are equivalent.
10. (Pontryagin duality.) Let **CompHausAb** denote the category of compact Hausdorff abelian groups and their continuous homomorphisms. Then, by Pontryagin duality theorem, to each $G \in \mathrm{Ob}(\mathbf{CompHausAb})$ there can be associated an abelian group of characters $\mathrm{Hom}_{\mathbf{CompHausAb}}(G, \mathbb{R}/\mathbb{Z})$. In consequence, $\mathbf{CompHausAb}^{op}$ is equivalent to **Ab**.
11. Let $(P, \leq)$ be a preorder, and define an equivalence relation $\sim$ on P by $x \sim y \iff (x \leq y$ and $y \leq x)$. The function $f_\sim : X \ni x \mapsto [x] \in X/\sim$ induces the morphism $(X, \leq) \mapsto (X/\sim, (f_\sim \times f_\sim) \circ \leq)$ which defines a reflector $G : \mathbf{Preord} \to \mathbf{Poset}$ of the inclusion $\mathbf{Poset} \to \mathbf{Preord}$.
12. The forgetful functor $U : \mathbf{Poset} \to \mathbf{Set}$ has left adjoint but has no right adjoint.
13. Let G be the group and let $G^\bullet$ be its commutator subgroup[5]. The maps $G \mapsto G/G^\bullet$ defines a reflector $L : \mathbf{Grp} \to \mathbf{Ab}$ of the inclusion $\mathbf{Ab} \to \mathbf{Grp}$.
14. For a topological space $(X, \mathcal{O})$ define a new topology on set X given by $\mathcal{O}' := \{A \subseteq X \mid X \setminus A$ is sequentially closed in $(X, \mathcal{O})\}$. Then the map $(X, \mathcal{O}') \to (X, O)$ defines a coreflector $\mathbf{Top} \to \mathbf{Seq}$ of the inclusion $\mathbf{Seq} \to \mathbf{Top}$. Hence, **Seq** is full coreflexive subcategory of **Top**.

[5] [*A commutator subgroup is defined as...*].

15. Let $(P, \leq)$ and $(Q, \leq')$ be two preorders, regarded as categories, and let $F : P \to Q$ and $G : Q \to P$ be two order-preserving functions, regarded as functors. Then F is left adjoint to G iff

$$\forall p \in P\ \forall q \in Q \quad p \leq' F(p) \iff p \leq G(q). \tag{133}$$

This condition imposes a bijection between hom-sets in P and in Q, but it also imposes a naturality of this bijection, because for preorders each hom-set has at most one element. The unit of this adjunction is given by $p \leq G(F(p))\ \forall p \in P$, while the counit is given by $q \leq' F(G(q))\ \forall q \in Q$. The pair of adjoint order-preserving functors between preorders is called a ***Galois connection***.

16. Let $X, Y, A \in \mathrm{Ob}(\mathbf{Set})$ and let $f : X \to Y$ in $\mathbf{Set}$. Then X^A denotes the set of all functions $g : A \to X$ and $X^A \in \mathrm{Ob}(\mathbf{Set})$. Consider the functors $A \times (-) : \mathbf{Set} \to \mathbf{Set}$ and $(-)^A : \mathbf{Set} \to \mathbf{Set}$, defined by

$$\begin{gathered}(A \times (-))(X) = A \times X,\\ (A \times (-))(f) = (\mathrm{id}_A \times f) : A \times X \ni (a, x) \mapsto (a, f(x)) \in A \times Y,\\ (-)^A(X) = X^A,\\ (-)^A(f) = f^A : X^A \ni g \mapsto f \circ g \in Y^A.\end{gathered} \tag{134}$$

These functors induce two hom-bifunctors and a natural transformation,

$$\begin{tikzcd} \mathbf{Set}^{op} \times \mathbf{Set} \arrow[r, bend left, "\mathrm{Hom}_{\mathbf{Set}}(A\times(-),-)"] \arrow[r, bend right, "\mathrm{Hom}_{\mathbf{Set}}(-,(-)^A)"'] \arrow[r, phantom, "\Downarrow\sigma"] & \mathbf{Set} \end{tikzcd} \tag{135}$$

such that

$$\sigma_{XY} : \mathrm{Hom}_{\mathbf{Set}}(A \times X, Y) \to \mathrm{Hom}_{\mathbf{Set}}(X, Y^A) \tag{136}$$

is a bijection given by

$$\sigma_{XY}(h)(x)(a) = h(a, x) \tag{137}$$

for $h : A \times Y \to Y$, with naturality property given by

$$l^A \circ \sigma_{XY}(h) \circ k = \sigma_{X'Y'}(l \circ h \circ (\mathrm{id}_A \times k), \tag{138}$$

what corresponds to the transposition

$$\begin{array}{ccccccc} A \times X' & \xrightarrow{\mathrm{id}_A \times k} & A \times X & \xrightarrow{h} & Y & \xrightarrow{l} & Y' \\ \hline X' & \xrightarrow{k} & X & \xrightarrow{\sigma_{XY}(h)} & Y^A & \xrightarrow{l^A} & Y'^A \end{array} \tag{139}$$

The unit and counit of this adjunction,

$$\begin{gathered}\eta : \mathrm{id}_{\mathbf{Set}} \to (A \times (-))^A,\\ \varepsilon : A \times (-)^A \to \mathrm{id}_{\mathbf{Set}},\end{gathered} \tag{140}$$

are given by

$$\begin{gathered}\eta_X(\cdot)(a) : X \ni x \mapsto (a, x) \in A \times X,\\ \varepsilon_Y : A \times Y^A \ni (a, f) \mapsto f(a) \in Y.\end{gathered} \tag{141}$$

It is also worth to observe that $A \times (-)$ has left adjoint iff $A \cong \{*\}$, while $(-)^A$ has right adjoint iff $A \cong \{*\}$.

5.2 Exponentials and cartesian closed categories

Note that in the category $\mathbf{Set}$ one can consider the object C^A, defined as the set of all functions from A to C:

$$C^A := \{f \mid f : A \to C\}, \tag{142}$$

and define the arrow

$$\mathrm{ev} : A \times C^A \to A, \tag{143}$$

such that

$$\mathrm{ev}(\langle x, f\rangle) = f(x). \tag{144}$$

Such an arrow is called an ***evaluation arrow***. This property can be generallised in strictly categorical terms.

Definition 5.13 *An object $A \in \mathrm{Ob}(\mathcal{C})$ is called **exponentiable** iff for every $C \in \mathrm{Ob}(\mathcal{C})$ there exists an object C^A, called **exponential**, and a morphism $\mathrm{ev} : C^A \times A \to C$, called **evaluation**, such that for any $f : B \times A \to C$ there exists a unique $f^\flat : B \to C^A$, called **exponential transpose**, for which the diagram*

$$\begin{tikzcd} A \times C^A \arrow[r, "\mathrm{ev}"] & C \\ A \times B. \arrow[u, dotted, "! \, \mathrm{id}_A \times f^\flat"] \arrow[ur, "f"'] & \end{tikzcd} \tag{145}$$

*commutes. We say that category $\mathcal{C}$ **has exponentials** iff every object has its exponential.*

Definition 5.14 *A category $\mathcal{C}$ is called **cartesian closed** iff all objects of $\mathcal{C}$ are exponentiable (equivalently, $\mathcal{C}$ has exponentials) and has finite products.*

In the last example of the preceding subsection we have shown that exponentiation in **Set** can be characterised as the left adjoint to the cartesian product functor. This observation allows to provide an alternative definition of exponentiation in a category $\mathcal{C}$, which is based on adjoint functors. If $\mathcal{C}$ is a category with binary products, we can define for every A the ***cartesian product functor*** $A \times (-) : \mathcal{C} \to \mathcal{C}$, with the following action:

$$(A \times (-))(B) = A \times B, \tag{146}$$
$$(A \times (-))(f) = \mathrm{id}_A \times f. \tag{147}$$

If there exists a functor R right adjoint to $A \times (-)$, then there is a natural isomorphism of bifunctors $\mathrm{Hom}(A \times (-), -) \cong \mathrm{Hom}(-, R(-))$, so for any arrow $f : A \times B \to C$ there is a unique arrow $f^\flat : B \to R(C)$. Such an adjoint functor R is denoted as $(-)^A$, write C^A for $R(C)$, and call it the ***exponential*** or the ***exponential object***. We say that category $\mathcal{C}$ ***has exponentials*** if for any $A \in \mathrm{Ob}(\mathcal{C})$ the functor $A \times (-) : \mathcal{C} \to \mathcal{C}$ has a right adjoint $(-)^A : \mathcal{C} \to \mathcal{C}$. Hence, ***cartesian closed categories*** are defined as those which have finite products together with the functor $(-)^A$ which is right adjoint to the cartesian product functor $A \times -$.

It is easy to see that the evaluation arrow ev is a counit ε_A of the adjunction $A \times (-) \dashv (-)^A$. The bijective correspondence (natural isomorphism) between $\mathrm{Hom}(A \times (-), -)$ and $\mathrm{Hom}(-, (-)^A)$ can be expressed as $\mathrm{Hom}(A \times B, C) \cong \mathrm{Hom}(B, C^A)$ and denoted as a transposition

$$\frac{B \to C^A}{A \times B \to C.} \tag{148}$$

The diagrams of transposition of arrows in cartesian closed categories as called λ-***conversion diagrams***.

Examples

1. **Set** is a cartesian closed category.
2. **Set**^{op} *is not* a cartesian closed category. This can be shown by checking that the functor $X + (-) :$ **Set** $\to$ **Set** does not preserve colimits.
3. **Ab**, **Vect**$_K$, **Vect**$_K^{fin}$ are not cartesian closed categories.
4. A poset $(P, \leq)$, viewed as category, is cartesian closed iff it has finite products, $(p \times q) = (p \wedge q)$, for every $p, q \in P$. Then the set $\{r \in P \mid r \wedge p \leq q\}$ has always a largest element, and the exponentiation q^p may be defined by (148) as
$$r \wedge p \leq q \iff r \leq q^p. \tag{149}$$
The exponential q^p in poset is denoted as $p \Rightarrow q$.

5. **Poset** is cartesian closed. The object $(P, \leq)^{(Q,\leq')}$ is the set of all order preserving functions from P to Q with order defined by $f \leq g \iff f(p) = g(p)\ \forall p \in P$. The map $(\hat{f}(r))(p)$ is given by $f(p, r)$, while ev is an ordinary evaluation map.
6. **Top** is not cartesian closed. This can be shown by observing that $(\mathbb{Q} \times -) : \mathbf{Top} \to \mathbf{Top}$ is not left adjoint, because it does not preserve quotients of topological spaces (hence, it does not preserve coequalisers). In consequence, the space of all continuous maps between topological spaces is not a topological space.
7. **Seq** is cartesian closed (with respect to products of sequential spaces and not with respect to the usual product of topological spaces).
8. One of the most significant examples of a cartesian closed category is a functor category $\mathbf{Set}^{\mathcal{C}^{op}}$. We will explore its cartesian closed properties later, considering it as a major example of a topos.

The following properties of cartesian closed categories make clear why they are called 'cartesian closed'.

Proposition 5.15 *For any cartesian closed category $\mathcal{C}$, and any objects X, Y, Z of $\mathcal{C}$, we have*

1. $\mathbf{0} \times X \cong \mathbf{0}$ *if* $\mathbf{0}$ *exists in* $\mathcal{C}$,
2. $\mathbf{1} \times X \cong X$,
3. $X^{\mathbf{0}} \cong \mathbf{1}$ *if* $\mathbf{0}$ *exists in* $\mathcal{C}$,
4. $X^{\mathbf{1}} \cong X$,
5. $\mathbf{1}^X \cong \mathbf{1}$,
6. $X \times Y \cong Y \times X$,
7. $(X \times Y) \times Z \cong X \times (Y \times Z)$,
8. $Y^X \times Z^X \cong (Y \times Z)^X$,
9. $Z^{X \times Y} \cong (Z^X)^Y$,
10. $X + Y \cong Y + X$,
11. $(X + Y) + Z \cong X + (Y + Z)$,
12. $Z^X \times Z^Y \cong Z^{X+Y}$ *if* $\mathcal{C}$ *has binary coproducts,*
13. $(X \times Y) + (X \times Z) \cong X \times (Y + Z)$ *if* $\mathcal{C}$ *has binary coproducts.*

Proof.

1. For any $A \in \mathrm{Ob}(\mathcal{C})$ the set $\mathrm{Hom}_{\mathcal{C}}(\mathbf{0}, A^X)$ has only one element, because $\mathbf{0}$ is initial object. By definition of exponential, $\mathrm{Hom}_{\mathcal{C}}(\mathbf{0}, A^X) \cong \mathrm{Hom}_{\mathcal{C}}(\mathbf{0} \times X, A)$. Hence, for any $A \in \mathrm{Ob}(\mathcal{C})$ there exists a unique arrow $\mathbf{0} \times X \to A$. In consequence, $\mathbf{0} \times X$ is an initial object. Because initial object is unique up to isomorphism, we obtain $\mathbf{0} \times X \cong \mathbf{0}$.
2. By equipping the product diagram of $X \times \mathbf{1}$ with an arbitrary arrow : $X \times \mathbf{1} \to X$, we obtain a commutative diagram

$$\begin{tikzcd}
& E \arrow[dl, "f"'] \arrow[d, dotted, "!\langle f,g\rangle"] \arrow[dr, "g"] & \\
X & X \times \mathbf{1} \arrow[l, "p_X"'] \arrow[r, dotted, "p_1"] \arrow[d, "h"] & \mathbf{1} \\
& X. \arrow[ul, equal] \arrow[ur, dotted, "!"'] &
\end{tikzcd} \tag{150}$$

   Since the maps p_1 and ! are unique, h has to be an isomorphism.

3. Let us consider and arrow $Y \to X^{\mathbf{0}}$. From the diagram (148) there is a λ-conversion $\frac{Y \to X^{\mathbf{0}}}{Y \times \mathbf{0} \to X}$. The morphism $Y \times \mathbf{0} \to X$ can be considered in the context of the product diagram of $Y \times \mathbf{0}$, leading to a commutative diagram

$$\begin{tikzcd}
& E \arrow[dl, "f"'] \arrow[d, dotted, "!\langle f,g\rangle"] \arrow[dr, "g"] & \\
Y & Y \times \mathbf{0} \arrow[l, "p_Y"'] \arrow[r, "p_{\mathbf{0}}"] \arrow[d] & \mathbf{0} \arrow[dl, dotted] \\
& X &
\end{tikzcd} \tag{151}$$

Hence, for *any* Y there is a bijection $\mathrm{Hom}_{\mathcal{C}}(Y \times \mathbf{0}, X) \cong \mathrm{Hom}_{\mathcal{C}}(\mathbf{0}, X)$, since the arrow $p_{\mathbf{0}}$ is unique. So, $\frac{Y \to X^{\mathbf{0}}}{\mathbf{0} \to X}$ for *any* Y, thus there must be an isomorphism $X^{\mathbf{0}} \cong \mathbf{1}$ (by the definition of an initial object).

4. Also from (148) it follows that $\begin{array}{c} (Y, X^{\mathbf{1}}) \\ \hline (\mathbf{1} \times Y, X) \\ \hline (Y, X). \end{array}$

5. Right adjoint preserves terminal object.

6. By definition of a product.

7. By definition of a product.

8. Right adjoint preserves products.

9. $\begin{array}{c} A \to (Z^X)^Y \\ \hline A \times Y \to Z^X \\ \hline A \times Y \times X \to Z \\ \hline A \times (Y \times X) \to Z \\ \hline A \to Z^{(Y \times X)}. \end{array}$

10. By definition of a coproduct.

11. By definition of a coproduct.

12. Left adjoint preserves coproducts.

13. Left adjoint preserves coproducts.

□

Remark 5.16 *If $X = Y^A$, then its global elements, that is, the maps $\mathbf{1} \to X$, satisfy*

$$\begin{array}{c} \mathbf{1} \to Y^A \\ \hline A \times \mathbf{1} \to Y \\ \hline A \to Y. \end{array} \tag{152}$$

This means that global elements (points) of Y^A are naturally isomorphic to the maps $A \to Y$. Hence, the exponential Y^A can be thought of as the object of all maps from A to Y, like in the case of **Set**, which we considered earlier in (142).

6 Subobject classifier

Now we turn back to the idea of a subobject. We had defined a subobject as a class of equivalent monic arrows (see (31)). We will now refine the characterisation of subobjects by introducing the notion of *subobject classifier*.

6.1 Subobjects in Set

Let us first recall some properties of the pullback. From

$$\{x \mid x \in A,\ f(x) = 1\} = f^{-1}(\{1\}) \tag{153}$$

it follows that the diagram

$$\begin{tikzcd} f^{-1}(\{1\}) \arrow[r, dotted, "!"] \arrow[d, hook] & \{1\} \arrow[d, hook] \\ A \arrow[r, "f"'] & \{0,1\} \end{tikzcd} \tag{154}$$

is a pullback, and the function $f|_{f^{-1}(\{1\})}$ is a unique arrow $! : f^{-1}(\{1\}) \to \{1\}$ for which this diagram is a pullback. If one would replace $\{1\}$ in (154) by $\{0\} = \mathbf{1}$ in **Set** and replace the inclusions $\{1\} \hookrightarrow \{0,1\} =: 2$ by the function $\top : \{0\} \ni 0 \mapsto 1 \in 2$, then the diagram

$$\begin{tikzcd} f^{-1}(\{1\}) \arrow[r, dotted, "!"] \arrow[d, hook] & \mathbf{1} \arrow[d, hook, "\top"] \\ A \arrow[r, "f"'] & 2 \end{tikzcd} \tag{155}$$

is again a pullback.

Definition 6.1 *A* ***characteristic function*** *is defined as an arrow $\chi_B : A \to 2$ in* **Set** *such that, for any given subset $B \subseteq A$,*

$$\chi_B(x) = \begin{cases} 0 & : x \notin B \\ 1 & : x \in B. \end{cases} \tag{156}$$

Hence, for every $B \subseteq A$, we have a commutative diagram

$$\begin{tikzcd} B \arrow[r, dotted, "!"] \arrow[d, hook] & \mathbf{1} \arrow[d, hook, "\top"] \\ A \arrow[r, "\chi_B"'] & 2 \end{tikzcd} \tag{157}$$

This makes one wonder if this diagram is a pullback. The set of all functions $A \to 2$ in **Set** is given by the exponential $2^A \in \mathrm{Ob}(\mathbf{Set})$. On the other hand, the ***power set*** $P(A)$ of some set A is defined as the set of all subsets of A,

$$P(A) := \{B \mid B \subseteq A\}. \tag{158}$$

Proposition 6.2 *The characteristic function sets up an isomorphism $P(A) \cong 2^A$, given by $P(A) \ni B \mapsto \chi_B \in 2^A$.*

Proof. For any $B \subseteq A$ and $C \subseteq A$ there exists two unique functions $B \to C$ and $C \to B$ such that $\chi_B = \chi_C$, because $C \hookrightarrow A$ and $B \hookrightarrow A$ are mono equalisers for χ_B and χ_C. These functions are injective because their compositions are injective, and they are surjective **[***expain why***]**. From this follows that $B \cong C$. Hence, the morphism $P(A) \ni B \mapsto \chi_B \in 2^A$ is injective. It is also surjective, because

$$\forall f \in 2^A\ \ \exists B \subseteq A\ \ f = \chi_B \tag{159}$$

given by

$$B = \{x \mid x \in A,\ f(x) = 1\}. \tag{160}$$

□

From this property it follows that the commutative diagram (157) is a pullback. Moreover,

Proposition 6.3 *The characteristic function χ_B is a unique function $A \to 2$ in* **Set** *such that the diagram*

$$\begin{tikzcd} B \arrow[r, dotted, "!"] \arrow[d, hook] & \mathbf{1} \arrow[d, hook, "\top"] \\ A \arrow[r] & 2 \end{tikzcd} \tag{161}$$

is a pullback for any $B \subseteq A$.

Proof. In order to show this, one has to use the properties of subsets in **Set**. Consider the diagram

$$\begin{tikzcd} f^{-1}(\{1\}) = \{x \mid x \in A,\ f(x) = 1\} \arrow[dr, dotted, "k"] \arrow[drr, dotted, "!"] \arrow[ddr, "j"'] & & \\ & B \arrow[r, dotted, "!"] \arrow[d, hook, "i"'] & \mathbf{1} \arrow[d, hook, "\top"] \\ & A \arrow[r, "f"'] & 2. \end{tikzcd} \tag{162}$$

If for some $f : A \to 2$ the internal square of this diagram is a pullback the for any $f(x) = 1$ for any $x \in B$, hence $x \in f^{-1}(\{1\})$, so $B \subseteq f^{-1}(\{1\})$. On the other hand, because the external diagram is a pullback, there exists a unique k such that $i \circ k = j$. Because i and j are inclusions, k has also to be an inclusion, so $f^{-1}(\{1\}) \subseteq B$. In consequence, $f^{-1}(\{1\}) = B$. Observing that $f = \chi_{f^{-1}(\{1\})}$, one obtains $f = \chi_B$. $\square$

In this sense, the set $2 \in \mathrm{Ob}(\mathbf{Set})$ together with an arrow $\top : \mathbf{1} \to 2$ in **Set** plays a crucial role on the passage between the subsets and the characteristic functions. In this sense, the set 2 equipped with the map $\top$ allows to *classify subobjects*. The pullback diagram (157) describes the correspondence between the subobjects of a set A and the characteristic morphisms in a purely categorical way. This allows to generalise this characterisation from **Set** to any category $\mathcal{C}$ with a terminal object.

[*Show* $\mathrm{Sub}(A) \cong P(A)$.**]**

6.2 Subobjects in $\mathcal{C}$

Definition 6.4 *A* ***subobject classifier*** *or a* ***generalised truth-value object***[6] *is an object Ω in $\mathcal{C}$, together with an arrow $\top : \mathbf{1} \to \Omega$, called the* ***true arrow***, *such that for each monic arrow $m : B \rightarrowtail A$ there is a unique arrow $\chi_B = \chi_m = \chi(m) : A \to \Omega$, called the* ***characteristic arrow*** *of m (or of B), such that the diagram*

$$\begin{tikzcd} B \arrow[r, dotted, "!"] \arrow[d, tail, "m"'] & \mathbf{1} \arrow[d, tail, "\top"] \\ A \arrow[r, "\chi(m)"'] & \Omega \end{tikzcd} \tag{163}$$

is a pullback. The arrow $(\top \circ !) : B \xrightarrow{!} \mathbf{1} \rightarrowtail \Omega$ is often denoted as $\top_B : B \to \Omega$.

Proposition 6.5 *χ_m does not depend on the choice of the arrow from the class of all equivalent monic arrows describing the same subobject. That is for any two monic arrows f and g with the same codomain*

$$f \sim g \iff \chi_f = \chi_g. \tag{164}$$

[6] We will explain the reason for this name later.

Proof. Consider the diagram

$$\begin{tikzcd}
C \arrow[dr, "k"] \arrow[ddr, tail, "g"'] \arrow[drr, dotted, "!"] & & \\
& B \arrow[r, dotted, "!"] \arrow[d, tail, "f"] & \mathbf{1} \arrow[d, hook, "\top"] \\
& A \arrow[r, "\chi_f"'] & \Omega
\end{tikzcd} \tag{165}$$

and assume that the inner square in (165) is a pullback. If f and g are two equivalent monic arrows, then there is an isomorphism $k : B \cong C$ such that the left traingle in (165) commutes. From this it follows that the outer square in (165) is also a pullback diagram. Hence, from Definition 6.4 applied to g, we obtain $\chi_f = \chi_g$. In order to prove implication in the opposite direction, let us assume that $\chi_f = \chi_g$ and the outer square is a pullback. Then, (by universal character of a pullback) there exists an arrow k such that $f \circ k = g$. On the other hand, by interchanging f and g, we obtain $g \circ k = f$. This implies that $f \sim g$. □

Proposition 6.6 *The subobject classifier is unique up to isomorphism.*

Proof. Let $\top : \mathbf{1} \to \Omega$ and $\top' : \mathbf{1} \to \Omega'$ be two different subobject classifiers. Then there is a commutative diagram

$$\begin{tikzcd}
\mathbf{1} \arrow[r, dotted, "!"] \arrow[d, tail, "\top"'] & \mathbf{1} \arrow[r, dotted, "!"] \arrow[d, tail, "\top'"'] & \mathbf{1} \arrow[d, tail, "\top"'] \\
\Omega \arrow[r, "(\chi_\top)'"'] & \Omega' \arrow[r, "\chi_{\top'}"'] & \Omega,
\end{tikzcd} \tag{166}$$

where the left and right squares are pullbacks. The external square given by composition !∘! and $\chi_{\top'} \circ (\chi_\top)'$ is also a pullback, because every external square of a composition of two pullback squares is a pullback. Hence, by the uniqueness of characteristic arrow, $\chi_{\top'} \circ (\chi_\top)'$ is an identity arrow of Ω, which is invertible, so $\chi_{\top'} \circ (\chi_\top)' = \mathrm{id}_\Omega$ and $(\chi_\top)' \circ (\chi'_\top) = \mathrm{id}_{\Omega'}$. □

Proposition 6.7 *In any category $\mathcal{C}$ with a subobject classifier Ω,*

$$\boxed{\mathrm{Sub}(A) \cong \mathrm{Hom}_{\mathcal{C}}(A, \Omega)} \tag{167}$$

Proof. A map

$$\mathrm{Sub}(A) \ni [f] \mapsto \chi_f \in \mathrm{Hom}_{\mathcal{C}}(A, \Omega) \tag{168}$$

is a surjection. Moreover, for any $h : A \to \Omega$ and a pullback diagram

$$\begin{tikzcd}
B \arrow[r, dotted, "!"] \arrow[d, "f"'] & \mathbf{1} \arrow[d, tail, "\top"] \\
A \arrow[r, "h"'] & \Omega
\end{tikzcd} \tag{169}$$

an arrow f is mono, becayse $\top : \mathbf{1} \rightarrowtail \Omega$ is mono and a pullback of mono arrow is mono (see Proposition 2.13). In consequence, $\chi_f = h$. Hence, the map (168) is also an injection. □

Definition 6.8 *The* ***power object*** *$P(A)$ of an object A in a cartesian closed category $\mathcal{C}$ with subobject classifier Ω is defined as the object Ω^A. If Ω^A exists for any A in $\mathcal{C}$, we say that $\mathcal{C}$* ***has power objects****.*

It is clear that if a cartesian closed category has a subobject classifier, then it has power objects. One can also define the contravariant ***power object functor*** $P : \mathcal{C}^{op} \to \mathcal{C}$, given by $P : A \mapsto \Omega^A$ for $A \in \mathrm{Ob}(\mathcal{C})$, and such that $P(f) : \Omega^B \mapsto \Omega^A$ for $f : A \to B$ in $\mathcal{C}$ is given by $P(f)(B) = \{x \in A | f(x) \in B\}$. When power object is defined for cartesian closed categories, we have

$$\frac{A \times B \to \Omega}{B \to \Omega^A}, \tag{170}$$

thus for every category with power objects

$$\mathrm{Hom}(A \times B, \Omega) \cong \mathrm{Hom}(B, \Omega^A). \tag{171}$$

This equation, together with (167) written in the form $\mathrm{Sub}(A \times B) \cong \mathrm{Hom}(A \times B, \Omega)$, gives the isomorphism

$$\mathrm{Sub}(A \times B) \cong \mathrm{Hom}(B, P(A)). \tag{172}$$

7 Toposes

7.1 The Lawvere–Tierney elementary topos $\mathcal{E}$

Definition 7.1 *A **topos**[7] or **elementary topos** is a category satisfying one of these equivalent conditions:*

1. *it is a complete category with exponentials and subobject classifier,*
2. *it is a complete category with subobject classifier and its power object,*
3. *it is a cartesian closed category with equalisers and subobject classifier.*

The definition of an *elementary topos*, given originally by Lawvere and Tierney in 1969, also assumed *cocompleteness*. It was later shown by Mikkelsen [117] (see also [129], [57] or [107] for more contemporary proof) that the completeness of a category with subobject classifier implies its cocompleteness. A topos has then not only all finite limits, but also all finite colimits, as well as a subobject classifier, exponentials and power objects.

This means that topos is such category which has, in particular,

(i) terminal object,

(ii) equalisers,

(iii) pullbacks,

(iv) all other limits,

(v) exponential objects,

(vi) subobject classifier.

[7]The name *topos*, introduced by Grothendieck, caused a certain controversy regarding its plural form. Some authors use exclusively either the form *toposes* or the form *topoi*. The modern English naming convention is intended to be suggestive of the fact that Grothendieck toposes generalise topological spaces. The ancient Greek naming convention is intended to reflect the fact that a topos is a $\tau o \pi o \varsigma$, that is, a *place* of geometry, and at the same time as a *place* of logic. Because all these meanings are valuable, we use both plural forms interchangeably.

It is worth to note that another ingenious idea of Grothendieck, named by him a *scheme*, also reveals a deep relationship with ancient Greek. In Plato's dialogue *Meno*, the notion of $\sigma\chi\eta\mu\alpha$ appears as representing the idea of a general shape of a geometrical figure. Due to such a correlation it is hard to believe that the names 'topos' and 'scheme' were proposed by Grothendieck in a non-conscious way. It is interesting that, following these ideas, one can ask what the relationship is between geometric shape ($\sigma\chi\eta\mu\alpha$) and geometric place ($\tau o \pi o \varsigma$)? The answer given by the Grothendieck school is that any category of schemes embeds into a suitable topos [30] (see also [121], [67]).

Broadly speaking, the terminal object lets us consider global elements (global sections) of objects; the subobject classifier Ω lets us consider subobjects and — as we will see — generalised truth-values of assertions; exponentials make it possible to consider objects comprised of all arrows from one object to another. Together with cartesian closedness and existence of power objects, these properties allow to deal with arrows in objects in topos in a way that is very similar to the usual properties of the category **Set**. So, "anything you really needed to do in the category of sets can be done in any topos" [6].

Examples

1. **Set** is a topos.
2. $\mathbf{Set}^{\mathcal{C}^{op}}$ is a topos for any category $\mathcal{C}$ (for a proof, see Section 7.3).
3. $\mathbf{Sh}_J(\mathcal{C})$, a category of sheaves over arbitrary category $\mathcal{C}$ equipped with a Grothendieck topology J, is a topos (for a definition and a proof, see Section 10.1).

One can consider the morphisms between toposes and construct a category of toposes and their morphisms.

Definition 7.2 *If $\mathcal{E}_1$ and $\mathcal{E}_2$ are toposes, then a* ***geometric morphism*** $g : \mathcal{E}_1 \to \mathcal{E}_2$ *is defined as a pair of adjoint functors $g^* \dashv g_*$ between $\mathcal{E}_1$ and $\mathcal{E}_2$, such that g^* preserves finite limits (which in turn implies that g_* preserves colimits). The category of toposes and their geometric morphisms is denoted* **Topoi**.

One can define also the natural numbers object in topos. The natural numbers $\mathbb{N}$ are defined by Peano axioms.

Definition 7.3 (Peano)

1. *0 is a natural number,*
2. *every natural number n has its successor, denoted $s(n)$,*
3. *0 is not a successor of any natural number,*
4. *if n and m are different natural numbers, then they has different successor.*

The third and fourth of these axioms uses the binary equality ($=$) relation of natural numbers, defined as follows:

1. $n = n$ $\forall n \in \mathbb{N}$ (*reflexivity*),
2. if $n = m$ then $m = n$ $\forall n, m \in \mathbb{N}$ (*symmetry*),
3. if $n = m$ and $m = k$ then $m = k$ $\forall n, m, k \in \mathbb{N}$ (*transitivity*),
4. if $n \in \mathbb{N}$ and $n = m$ then $m \in \mathbb{N}$.

Usually these axioms are interpreted in a language of sets and functions (what means that natural numbers are interpreted as forming a set, while s is interpreted as function on this set). The topos theoretic definition of natural numbers does not require any set-theoretic notions.

Definition 7.4 (Lawvere) *A* ***natural numbers object*** *in topos $\mathcal{E}$ is such object $\mathbb{N}$ together with arrows $\mathbf{1} \xrightarrow{0} \mathbb{N} \xrightarrow{s} \mathbb{N}$, that for every diagram $\mathbf{1} \xrightarrow{x} X \xrightarrow{u} X$ in $\mathcal{E}$ there exists a unique morphism $\mathbb{N} \xrightarrow{f} X$ such that the diagram*

$$\begin{tikzcd} \mathbf{1} \arrow[r, "0"] \arrow[dr, "x"'] & \mathbb{N} \arrow[r, "s"] \arrow[d, dotted, "f"] & \mathbb{N} \arrow[d, dotted, "f"] \\ & X \arrow[r, "u"'] & X \end{tikzcd} \tag{173}$$

commutes.

This axiom gives an ability of definte the ***arithmetical operations*** on $\mathbb{N}$ in $\mathcal{E}$,

$$+ : \mathbb{N} \times \mathbb{N} \to \mathbb{N}, \tag{174}$$
$$\cdot : \mathbb{N} \times \mathbb{N} \to \mathbb{N}, \tag{175}$$
$$\exp : \mathbb{N} \times \mathbb{N} \to \mathbb{N}, \tag{176}$$

through their exponential transposed arrows $+^\flat : \mathbb{N} \to \mathbb{N}^\mathbb{N}$, $\cdot^\flat : \mathbb{N} \to \mathbb{N}^\mathbb{N}$, and $\exp^\flat : \mathbb{N} \to \mathbb{N}^\mathbb{N}$, by asserting that the appropriate diagrams should commute. These diagrams express categorically the usual meaning of operations $+$, $\cdot$ and exp. For example, the commuting diagram defining the addition of natural numbers is given by

$$\begin{tikzcd}
\mathbf{1} \arrow[r, "0"] \arrow[dr, "\mathbf{1}^\flat_\mathbb{N}"'] & \mathbb{N} \arrow[r, "s"] \arrow[d, dotted, "+^\flat"] & \mathbb{N} \arrow[d, dotted, "+^\flat"] \\
& \mathbb{N}^\mathbb{N} \arrow[r, "s^\mathbb{N}"'] & \mathbb{N}^\mathbb{N},
\end{tikzcd} \tag{177}$$

what corresponds precisely to equational conditions of the Peano definition of addition in $\mathbb{N}$, given by

$$0 + a = a, \tag{178}$$
$$(s \circ a) + b = s \circ (a + b). \tag{179}$$

For the prupose of later sections, we will need also the following two properties.

Proposition 7.5 *An arrow in a topos is iso iff it is mono and epic.*

Proof. For any arrow in any category if it is mono and epic, then it is iso. In the opposite direction, let us first prove that any mono arrow $f : A \to B$ in a topos $\mathcal{E}$ is an equaliser of χ_f and $\top_A$. Consider the pullback diagram

$$\begin{tikzcd}
C \arrow[dr, dotted] \arrow[drr, dotted, "!_C"] \arrow[ddr, "g"'] & & \\
& B \arrow[r, dotted, "!_B"] \arrow[d, tail, "f"] & \mathbf{1} \arrow[d, tail, "\top"] \\
& A \arrow[ur, dotted, "!_A"] \arrow[r, "\chi_f"'] & \Omega.
\end{tikzcd} \tag{180}$$

From $!_A =!_B \circ f$ and commutativity of the pullback square it follows that $\chi_f \circ f = \top_A \circ f \equiv \top \circ !_A \circ f$. Moreover, from $\chi_f \circ g = \top_A \circ g$ and the commuting of the external pullback square it follows that

$$\begin{tikzcd}
B \arrow[r, tail, "f"] & A \arrow[r, shift left, "\chi_f"] \arrow[r, shift right, "\top_A"'] & \Omega \\
C \arrow[u, dotted] \arrow[ur, "g"'] & &
\end{tikzcd} \tag{181}$$

□

Proposition 7.6 (Kelly–Tierney) *Any arrow in a topos can be uniquely (up to isomorphism) decomposed into an epic arrow followed by a monic arrow.*

Proof. See e.g. ... □

7.2 Sieves

Consider the small category **2**:

$$\begin{tikzcd}
0 \arrow[loop above, "\mathrm{id}_0"] \arrow[r, "2"] & 1 \arrow[loop above, "\mathrm{id}_1"]
\end{tikzcd} \tag{182}$$

object	sieves on it
0	$\{\mathrm{id}_0\} = \uparrow 0$
0	$\emptyset$
1	$\{\{\mathrm{id}_1\}, \{2\}\} = \uparrow 1$
1	$\{2\}$
1	$\emptyset$

Table 1: All sieves on objects of the category **2**.

There are two generalised elements of 1: $\mathrm{id}_1 \in_1 1$ and $2 \in_0 1$, and one generalised element of 0: $\mathrm{id}_0 \in_0 0$. Instead of 'global' set-theoretic question which elements of 1 belong also to 0, in category theory we may ask *when* (at which stage) elements of 1 belong to 0. The second question is far more general, because it admits answers, which do not secessarily belong to the set $\{0, 1\} \cong \{\mathrm{true}, \mathrm{false}\}$. The answers can have partial truth values: an element which does not belong now (at given stage), may belong tomorrow (at further stage). Notice that $2 \in_0 1$ and $\mathrm{id}_1 \in_1 1$ implies that $\mathrm{id}_1 \circ 2 \in_0 1$, so, by the abuse of notation, $\mathrm{id}_1 \in_0 1$. In a more complicated category we could classify the arrows (elements) by considering sets of arrows which have the the same domain. However, as will be shown in the following pages, it is even more useful to ask about sets of arrows with the same codomain, because a *subobject* is defined as a class of equivalent monic arrows with the same codomain.

Definition 7.7 *A* ***sieve*** *on B in $\mathcal{C}^{op}$ is a collection S of arrows such that:*

1. $f \in S \Rightarrow \mathrm{cod}(f) = B$,
2. $(f \in S \wedge \mathrm{cod}(g) = \mathrm{dom}(f)) \Rightarrow f \circ g \in S$.

A ***maximal*** *(or* ***principal****)* ***sieve*** *on B, called* $\max_B$ *or $\uparrow B$, is a set of all arrows such that* $\mathrm{cod}(f) = B$.

Example A sieve in **Poset** is an upper set. If P is a poset, and $p, r \in P$, then a maximal sieve on p is given by $\uparrow p = \{r \in P \mid p \leq q\}$.

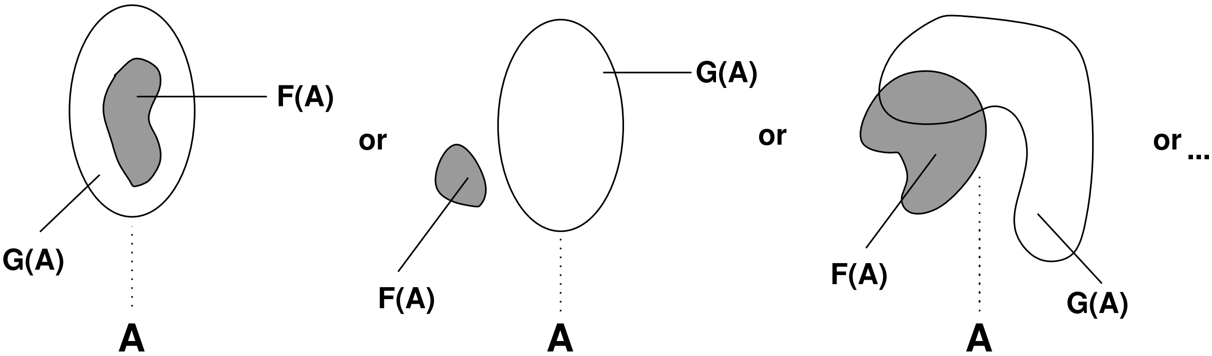
Note that different sieves are generated by different objects which we choose to 'pick up' the first arrow. It is easy to see that the maximal sieve along all sieves of A is always a sieve which contains an identity morphism of A.

Consider now again the diagram (182). There are two sieves on the object 0: the empty sieve $\emptyset$ and the maximal sieve $\uparrow 0 = \{\mathrm{id}_0\}$, generated by the identity id_0. There are *three* sieves on the object 1: the empty sieve $\emptyset$, the maximal sieve $\uparrow 1 = \{\{\mathrm{id}_1\}, \{2\}\}$ generated by the identity $\{\mathrm{id}_1\}$, and the sieve $\{2\}$ generated by the map $2 : 0 \to 1$. This means that taking into account the morphisms between objects of a category leads to appearance of non-maximal and non-empty sieves. In other words, an additional sieve $\{2\}$ on the object 1 in the category **2** has appeared because an arrow $2 : 0 \to 1$ *cannot be considered as a global element.* If this arrow could be considered as a global element, it would mean that there exists an arrow $1 \to 0$. But then 2 would 'fall into' the maximal sieve. This leads to quite an interesting observation, that the question about whether some element x belongs to an object X has two possible answers ('*yes*' and '*no*'), only if *all elements* of X are global, that is, $x \in_{\mathbf{1}} X$. In such case, there are only two sieves on every object: a maximal one ('true') and an empty one ('false'). However, in general, category theory gives us the ability to deal also with essentially non-global elements (arrows), and, consequently, to ask *when* (*at which stage*) an element or subobject belongs to an object.

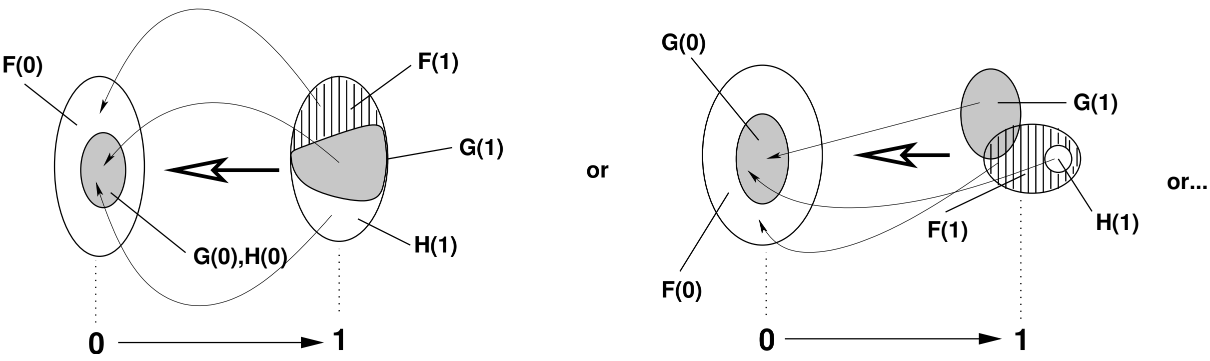
These considerations show that there is an intimate connection between sieves and truth values. In fact, we will see later that sieves consitute algebraic (and topological) models for logic. However, in order to understand the relationship between sieves and logic, we need first to understand independently the role of sieves in topoi and the role of algebra and topology in logic and its models.

7.3 The topos of presheaves $\mathbf{Set}^{\mathcal{C}^{op}}$

Recall Definition 4.9 of a cone, where the functor category $\mathcal{C}^{\mathcal{J}}$ was regarded as a category of diagrams in $\mathcal{C}$ *indexed* by the elements of a small category $\mathcal{J}$, and consider $\mathbf{Set}^{\mathcal{C}^{op}}$ as the category of diagrams in $\mathbf{Set}$ indexed by the elements of $\mathcal{C}^{op}$ (when $\mathcal{C}^{op} = \mathbf{1}$, this functor category is just $\mathbf{Set}^{\mathbf{1}^{op}} \cong \mathbf{Set}^{\mathbf{1}} \cong \mathbf{Set}$.) For a given $A \in \mathrm{Ob}(\mathcal{C})$, the different possible relationships between functors would result in different variations of subsets, indexed by the object A. For example,



The presheaves on the category $\mathcal{C}$ are given by sets varying over each object (recall the picture at the end of Section 4.1), but there are also arrows between objects of $\mathcal{C}$, which are translated into morphisms between varying sets.



Proposition 7.8 $\mathbf{Set}^{\mathcal{C}^{op}}$ *is a topos.*

Proof. We are going now to show that $\mathbf{Set}^{\mathcal{C}^{op}}$ is a topos. This means that we have to prove that $\mathbf{Set}^{\mathcal{C}^{op}}$ has finite limits, exponentials and subobject classifier.

- **Finite limits**
  The terminal object of $\mathbf{Set}^{\mathcal{C}^{op}}$ is a constant functor $\mathbf{1} : \mathcal{C}^{op} \to \mathbf{Set}$, assigning the terminal object $\mathbf{1}$ in $\mathbf{Set}$ (the set $\{0\}$) to each object in $\mathcal{C}^{op}$ and an identity arrow $\mathrm{id}_{\mathbf{1}}$ for every arrow in $\mathcal{C}^{op}$. The product in $\mathbf{Set}^{\mathcal{C}^{op}}$ of two functors $F, G \in \mathrm{Ob}(\mathbf{Set}^{\mathcal{C}^{op}})$ for every $f : B \to A$ in $\mathcal{C}^{op}$ is given pointwise by the image of functors of $\mathbf{Set}^{\mathcal{C}^{op}}$ in $\mathbf{Set}$:

$$(F \times G)(A) = F(A) \times G(A), \tag{183}$$
$$(F \times G)(f) = F(f) \times G(f). \tag{184}$$

  In the same way, by pointwise construction in $\mathbf{Set}$, one forms finite products and other limits, including equalisers. From Proposition 4.14 it follows that $\mathbf{Set}^{\mathcal{C}^{op}}$ is finitely complete, that is, it has finite limits.

- **Subobject classifier**
In order to identify the subobject classifier Ω in $\mathbf{Set}^{\mathcal{C}^{op}}$, we should first identify the subobjects, which by definition are given by nonequivalent classes of monic arrows with the same codomain. Note that the action of a monic arrow $i : F \rightarrowtail G$ on any A in $\mathcal{C}^{op}$ results in the ordinary set-inclusion $i_A : F(A) \hookrightarrow G(A)$ in **Set**. Such an F is a subobject of G, namely a *subfunctor* and its image $F(A)$ is a subset of $G(A)$.

Definition 7.9 *A functor $F : \mathcal{C}^{op} \to$ **Set** is a **subfunctor** of $G : \mathcal{C}^{op} \to$ **Set** and we denote it $F \subseteq G$, if for any $f : B \to A$ in $\mathcal{C}^{op}$ there is a commutative diagram*

$$\begin{tikzcd} F(A) \arrow[r, "F(f)"] \arrow[d, hook] & F(B) \arrow[d, hook] \\ G(A) \arrow[r, "G(f)"'] & G(B). \end{tikzcd} \tag{185}$$

Consequently, we seek such functor Ω which may be regarded as a subfunctor classifier. From (167) we have

$$\mathrm{Sub}(A) \cong \mathrm{Hom}(A, \Omega). \tag{186}$$

The object A in $\mathbf{Set}^{\mathcal{C}^{op}}$ is not only an object, but also a contravariant functor. The Yoneda lemma, together with equation (**??**), implies that $\Omega : \mathcal{C}^{op} \to \mathbf{Set}$ should satisfy

$$\mathrm{Sub}(\mathrm{Hom}(-, B)) \cong \mathrm{Hom}(\mathrm{Hom}(-, B), \Omega) \equiv \mathrm{Nat}(\mathrm{Hom}(-, B), \Omega) \cong \Omega(B) \tag{187}$$

for every $B \in \mathrm{Ob}(\mathcal{C})$. Hence, if we where to identify the $\mathrm{Sub}(\mathrm{Hom}(-, B))$, we would automatically identify the subobject classifier $\Omega(B)$. As noted earlier, the candidate for an object which may classify the subobjects (thus, the 'sets' of all nonequivalent classes of monic arrows with the same codomain) is a sieve, or rather the set of different sieves, because the sieve is defined as a set of arrows with the same codomain (with the composition property). The following proposition states it more precisely.

Proposition 7.10 *There is a natural bijection between the subfunctor $F \subseteq \mathrm{Hom}(-, B)$ and a sieve on B.*

Proof. Let us take $\mathrm{Hom}(C, B)$ and $F \subseteq \mathrm{Hom}(-, B)$. If $f \in F(C)$, then $f \in \mathrm{Hom}(C, B)$ and $\mathrm{cod}(f) = B$. For any $g : D \to C$ we have $f \circ g \in \mathrm{Hom}(D, B)$, and the commutative diagram

$$\begin{tikzcd} C & & & D \arrow[lll, "g"'] \\ & & & \\ F(C) \ni f \arrow[rrr, mapsto, "F(g)"] \arrow[d, hook] \arrow[d, mapsto] & & & F(g)(f) \in F(D) \arrow[d, mapsto] \arrow[d, hook] \\ \mathrm{Hom}(C, B) \ni f \arrow[rrr, mapsto, "\ldots \circ g"'] & & & f \circ g \in \mathrm{Hom}(D, B), \end{tikzcd} \tag{188}$$

$$\Downarrow \mathrm{Hom}(-,B)$$

so $f \in$ "sieve on B generated by F" and $f \circ g \in$ "sieve on B generated by F". □

This means that there is a natural isomorphism between sieves over a given stage (that is, the sets S of arrows with codomain B such that $f \in S \Rightarrow f \circ g \in S$) and the subfunctors of the Yoneda functor of the stage (that is, the equivalence classes of arrows in $\mathbf{Set}^{\mathcal{C}^{op}}$ with the codomain $\mathrm{Hom}(-, B)$). Hence, for every subfunctor there exists a sieve which corresponds to it, and there is a bijection between $\mathrm{Sub}(\mathrm{Hom}(-, B))$ and the set of all sieves on B.

$$\boxed{\text{set of all sieves on } B \cong \mathrm{Sub}(\mathrm{Hom}(-, B)) \cong \mathrm{Nat}(\mathrm{Hom}(-, B), \Omega) \cong \Omega(B)} \tag{189}$$

There is also analogous bijective correspondence between Ω in $\mathbf{Set}^{\mathcal{C}}$ and the set of all sieves on $\mathcal{C}^{op}$, so we could start from consideration of sieves on $\mathcal{C}^{op}$ (that is, cosieves on $\mathcal{C}$). This means that the topos of presheaves is not something really different then the topos of varying sets. The practical

difference between them is the need of thinking in terms of codomain instead of domain (the former is in many cases more natural). Recall now that for the full identification of a subobject classifier we should identify not only Ω, but also the truth arrow $\top : \mathbf{1} \to \Omega$.
The truth arrow $\top : \mathbf{1} \to \Omega$ in $\mathbf{Set}^{\mathcal{C}^{op}}$ is defined by its action on any object $A \in \mathcal{C}$ as $\top_A :=\uparrow A$, so it is given by a maximal sieve on every object of $\mathcal{C}^{op}$. In other words, this is the set of all arrows which have the object A as codomain. We should check that the pair $(\Omega, \top)$ defined in this way really is a subobject classifier. For this purpose, let us consider the monic map of two functors $f : F \rightarrowtail G$ in $\mathbf{Set}^{\mathcal{C}^{op}}$. We need to define the characteristic arrow χ_f in such a way that for every monic map $f : F \rightarrowtail G$ the map χ_f is unique, and satisfies the pullback diagram (163). Consider then a pullback diagram in $\mathbf{Set}$:

$$\begin{tikzcd} F(A) \arrow[r] \arrow[d, hook, "f(A)"'] & \mathbf{1}_A \cong \{0\} \arrow[d, hook, "\top(A)"] \\ G(A) \arrow[r, "\chi_f(A)"'] & \Omega(A). \end{tikzcd} \tag{190}$$

This diagram says that $\chi_f(A)$ assigns to every element x of a subset $F(A) \subset G(A)$ a unique sieve in the set of sieves $\Omega(A)$. The definition of a sieve implies that the diagram

$$\begin{tikzcd} F(A) \arrow[r, "F(f)"] \arrow[d, hook] & F(B) \arrow[d, hook] \\ G(A) \arrow[r, "G(f)"] & G(B) \end{tikzcd} \tag{191}$$

should commute. One can then define the natural transformation $\chi_f : G \to \Omega$ as follows:

$$\forall A \in \mathcal{C}^{op} \;\; \forall x \in F(A) \qquad \chi_{f(A)}(x) = (\chi_f)(A)(x) := \{f : B \to A \mid G(f)(x) \in G(B)\}. \tag{192}$$

Such a χ_f makes the diagram

$$\begin{tikzcd} F \arrow[r, dotted, "!"] \arrow[d, tail] & \mathbf{1} \arrow[d, tail, "\top"] \\ G \arrow[r, "\chi_f"'] & \Omega \end{tikzcd} \tag{193}$$

into a pullback square for every $A \in \mathcal{C}$, so $(\Omega, \top)$ is a subobject classifier in $\mathbf{Set}^{\mathcal{C}^{op}}$.

- **Exponentials**
  By the exponential functorial version of diagram (145) and equation (142), one can define an exponential G^F in $\mathbf{Set}^{\mathcal{C}^{op}}$ as

$$\begin{tikzcd} F \times G^F \arrow[r, "\mathrm{ev}"] & G \\ F \times \mathrm{Hom}(-, A), \arrow[u, dotted, "! \, \mathrm{id}_F \times \tau^\flat"] \arrow[ur, "\tau"'] & \end{tikzcd} \tag{194}$$

$$G^F(A) := \{\tau \mid \tau \in \mathrm{Nat}(F \times \mathrm{Hom}(-, A), G)\} = \mathrm{Nat}(F \times \mathrm{Hom}(-, A), G), \tag{195}$$

  such that for any $f : B \to A$

$$G^F(f) : \mathrm{Nat}(F \times \mathrm{Hom}(-, A), G) \to \mathrm{Nat}(F \times \mathrm{Hom}(-, B), G) \tag{196}$$

  and for any τ

$$G^F(f)(\tau) = \tau \circ (\mathrm{id}_F \times [\mathrm{Hom}(-, A) \mapsto \mathrm{Hom}(-, B)]) = \tau \circ (\mathrm{id}_F \times Y_f), \tag{197}$$

  where Y is the Yoneda embedding functor.

This ends the proof that $\mathbf{Set}^{\mathcal{C}^{op}}$ is a topos. $\square$

7.4 An example: sieves and subobjects in the topos $\mathbf{Set}^{\mathbf{2}^{op}}$

Consider once again the category

$$\mathbf{2} \equiv \begin{tikzcd} 0 \arrow[loop above, "\mathrm{id}_0"] \arrow[r, "2"] & 1. \arrow[loop above, "\mathrm{id}_1"] \end{tikzcd} \tag{198}$$

Using this category, we will analyse in detail the correspondence between sieves on $\mathcal{C}$ and related subfunctors of $\mathbf{Set}^{\mathcal{C}^{op}}$. Let us start from the identification of the elements of hom-functors,

functor	elements $\in$ hom-set
$\mathrm{Hom}(-,0)$	$[0 \mapsto \{\mathrm{id}_0\}] \in \mathrm{Hom}(0,0)$
$\mathrm{Hom}(-,0)$	$[1 \mapsto \{\emptyset\}] \in \mathrm{Hom}(1,0)$
$\mathrm{Hom}(-,1)$	$[0 \mapsto \{2\}] \in \mathrm{Hom}(0,1)$
$\mathrm{Hom}(-,1)$	$[1 \mapsto \{\mathrm{id}_1\}] \in \mathrm{Hom}(1,1)$

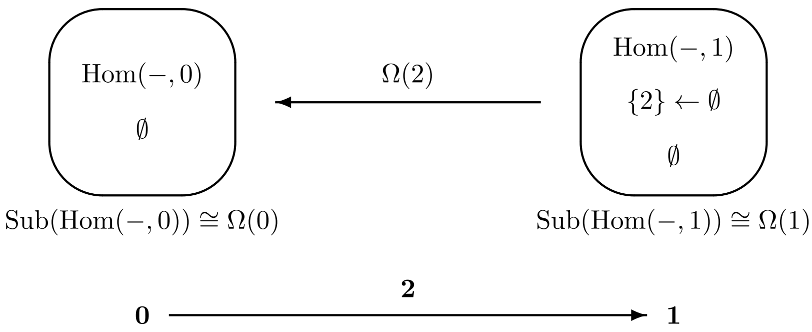
and of the subfunctors (subobjects) of $\mathrm{Hom}(-,0)$ and $\mathrm{Hom}(-,1)$.

functor	subfunctors
$\mathrm{Hom}(-,0)$	$\mathrm{Hom}(-,0)$
$\mathrm{Hom}(-,0)$	$\ulcorner\emptyset\urcorner$
$\mathrm{Hom}(-,1)$	$\mathrm{Hom}(-,1)$
$\mathrm{Hom}(-,1)$	$\ulcorner\emptyset\urcorner$
$\mathrm{Hom}(-,1)$	'$\{2\} \leftarrow \emptyset$'

$\ulcorner\emptyset\urcorner$ is a constant functor of an empty set. The important nontrivial subfunctor is '$\{2\} \leftarrow \emptyset$' is defined as:

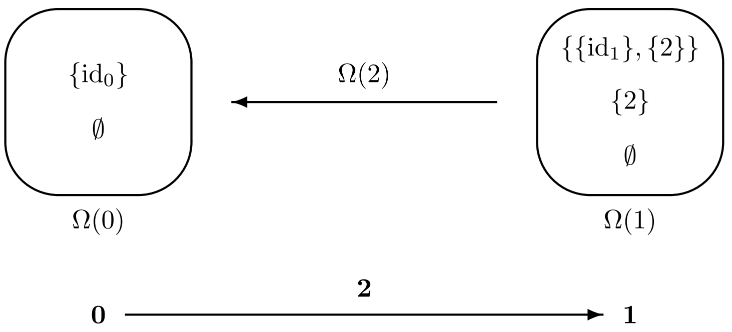
$$\begin{aligned} (\{2\} \leftarrow \emptyset)(0) &= \{2\}, \\ (\{2\} \leftarrow \emptyset)(1) &= \emptyset, \\ (\{2\} \leftarrow \emptyset)(2) &= \emptyset \mapsto \{2\}. \end{aligned} \tag{199}$$

It is a well-defined subfunctor of $\mathrm{Hom}(-,1)$, because the value of '$\{2\} \leftarrow \emptyset$' on 0 belongs to $\mathrm{Hom}(0,1)$, the value on 1 belongs to $\mathrm{Hom}(1,1)$, and the value on 2 is a morphism between them. The picture of subfunctors on this category looks then as follows:



(200)

On the other hand, $\Omega(C)$ is the set of all sieves on C, that is, the set of all sets $S_C = \{f \mid \mathrm{cod}(f) = C,\ f \circ g \in S_C\}$. Is it really the same object as the one presented in the subfunctor picture above? Let us check it by using the table of sieves over objects of $\mathbf{2}$ (see Table 1 in the preceding section), and drawing a picture of sieves on (over) this category.



(201)

This picture is very similar to the picture (200). Indeed, these are *the same* pictures, due to (189). Hence, one can identify particular subfunctors with the corresponding sieves:

subfunctors	sieves	on stage
$\mathrm{Hom}(-,0)$	$\{\mathrm{id}_0\} = \uparrow 0$	0
$\emptyset$	$\emptyset$	0
$\mathrm{Hom}(-,1)$	$\{\{2\},\{\mathrm{id}_1\}\} = \uparrow 1$	1
$\{2\} \leftarrow \emptyset$	$\{2\}$	1
$\emptyset$	$\emptyset$	1

One can also ask, which subsets of $\Omega(0)$ correspond to which subsets of $\Omega(1)$, or – in other words – how does the map $\Omega(2) : \Omega(1) \to \Omega(0)$ act? We will examine it by calculating the values of $\Omega(2)$ on the individual subfunctors (sieves) of $\Omega(1)$,

$$\begin{aligned}
\Omega(2)(\mathrm{Hom}(-,1))(0) &= \{f : 0 \to 0,\ \ 2 \circ f \in \mathrm{Hom}(0,1)\} = \{\mathrm{id}_0\},\\
\Omega(2)(\mathrm{Hom}(-,1))(1) &= \{f : 1 \to 0,\ \ 2 \circ f \in \mathrm{Hom}(1,1)\} = \emptyset,\\
\Omega(2)(\{2\} \leftarrow \emptyset)(0) &= \{f : 0 \to 0,\ \ 2 \circ f \in (\{2\} \leftarrow \emptyset)(0)\} = \{\mathrm{id}_0\},\\
\Omega(2)(\{2\} \leftarrow \emptyset)(1) &= \{f : 1 \to 0,\ \ 2 \circ f \in (\{2\} \leftarrow \emptyset)(1)\} = \emptyset,\\
\Omega(2)(\ulcorner\emptyset\urcorner)(0) &= \{f : \mathrm{cod}(f) = 0,\ \ 2 \circ f \in \ulcorner\emptyset\urcorner(0)\} = \emptyset,\\
\Omega(2)(\ulcorner\emptyset\urcorner)(1) &= \{f : \mathrm{cod}(f) = 0,\ \ 2 \circ f \in \ulcorner\emptyset\urcorner(1)\} = \emptyset,
\end{aligned} \tag{202}$$

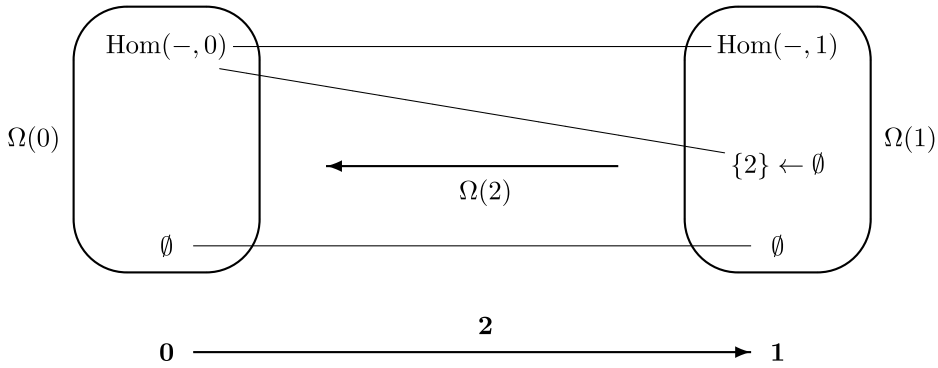
and comparing it with the values of $\Omega(0)$:

$$\begin{aligned}
\mathrm{Hom}(0,0) &= \{\mathrm{id}_0\},\\
\mathrm{Hom}(1,0) &= \emptyset,\\
\ulcorner\emptyset\urcorner(0) &= \emptyset,\\
\ulcorner\emptyset\urcorner(1) &= \emptyset.
\end{aligned} \tag{203}$$

This leads us to the following conclusion:

$$\begin{aligned}
\Omega(2)(\ulcorner\emptyset\urcorner) &= \ulcorner\emptyset\urcorner,\\
\Omega(2)(\{2\} \leftarrow \emptyset) &= \mathrm{Hom}(-,0),\\
\Omega(2)(\mathrm{Hom}(-,1)) &= \mathrm{Hom}(-,0),
\end{aligned} \tag{204}$$

This can, in turn, be can be drawn as:



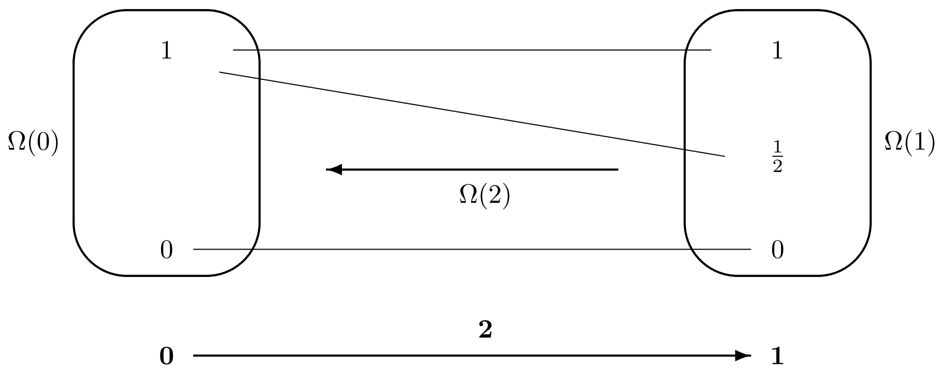
(205)

Recall that the subobject classifier Ω was defined by the pullback diagram

$$\begin{tikzcd} B \arrow[r, dotted, "!"] \arrow[d, tail] & \mathbf{1} \arrow[d, tail, "\top"] \\ A \arrow[r, "\chi_B"'] & \Omega, \end{tikzcd} \tag{206}$$

where $\top : \mathbf{1} \to \Omega$ was a monic truth arrow. One may wish to consider Ω as generalised truth-value object ('set'), composed of generalised truth values indicating if the elements of A belong to B. These truth values may be different than simply 0 or 1, because the elements of A can belong to B at different stages. Hence, there are more possibilities then 'belongs' or 'does not belong'. Such elements of A which belong to B on all stages should be regarded as those whose characteristic arrows have truth value equal to 1 ('absolutely true'), while elements of A which do not belong to B at any stage naturally should have truth value of a characteristic arrow equal to 0 ('absolutely false'). But there are also elements of A which belong to B at some stage, thus the truth-value of assertion about these elements is partial and depends on the particular stage.

The presheaf Ω is a functor in the topos of presheaves $\mathbf{Set}^{\mathcal{C}^{op}}$, so it is *varying* on a category $\mathcal{C}^{op}$, with truth values also varying from object to object. The correspondence between sieves and subfunctors of Ω leads to assignment of generalised (truth) values to every sieve (and corresponding subfunctor of Ω). By the definition of a truth arrow $\top : \mathbf{1} \to \Omega$ in $\mathbf{Set}^{\mathcal{C}^{op}}$, the truth value 1 at every stage is assigned to a maximal sieve at that stage. The value 0 is given naturally to the empty sieve. Other sieves can be given some other truth values. In the topos $\mathbf{Set}^{\mathbf{2}^{op}}$ on a stage 1 there is one additional "unexpected" sieve (subfunctor of Ω), namely $\{2\}$ ('$\{2\} \leftarrow \emptyset$'). We can regard it as the truth-value '1/2' in $\Omega(1)$, and redraw the picture (205), replacing the subfunctors (sieves) by their corresponding truth-values:



(207)

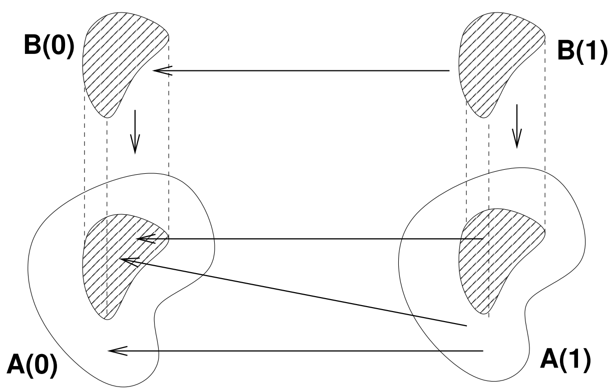
In **Set** we have $\Omega = \{0, 1\}$, which means that every element of a set, $x \in A$, can be regarded only as belonging to a subset $B \subseteq A$ *or not*. When one considers sets (presheaves) varying on a category, each element of the set may be regarded as belonging to a subset in *some* degree, *at some stage*. This is a consequence of the fact that elements are identified with the arrows, arrows belong to sieves, and the set of all sieves is (naturally isomorphic to) a subobject classifier, therefore different sieves in Ω correspond to different 'degrees of belonging'. Such 'partiality' of the logical value of the assertion '$x \in B$' is represented, in the above example, by the value '1/2'. This means that $x \in A(1) \wedge x \notin B(1)$ on the stage 1, but $x \in A(0) \wedge x \in B(0)$ on the stage 0, which is represented by an arrow $\frac{1}{2} \rightarrow 1$. In other words, the subset inclusion diagram (185)

$$\begin{tikzcd} F(A) \arrow[r, "F(f)"] \arrow[d, hook] & F(B) \arrow[d, hook] \\ G(A) \arrow[r, "G(f)"'] & G(B) \end{tikzcd} \tag{208}$$

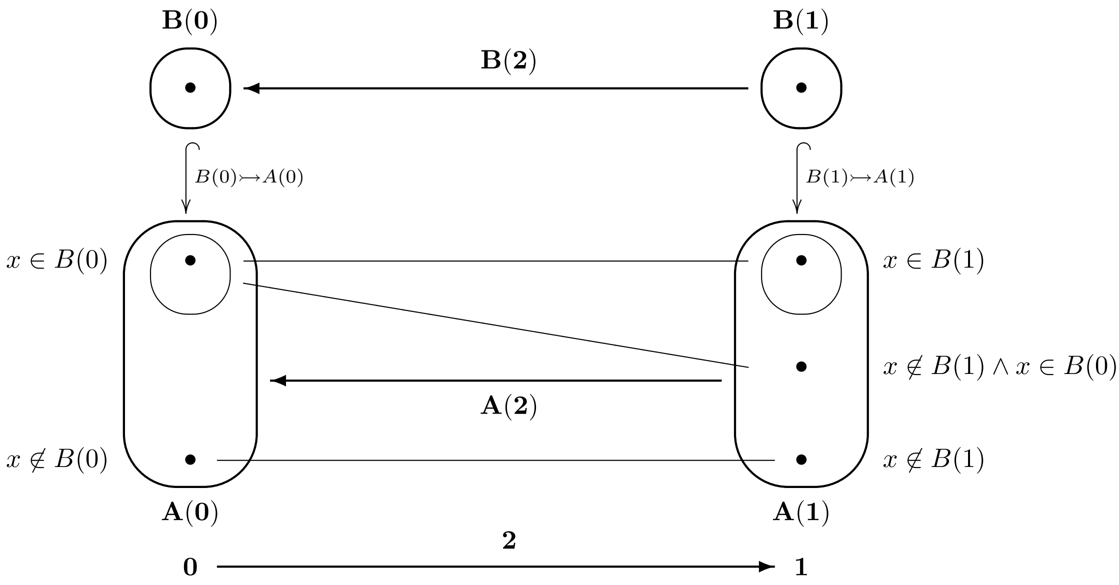
becomes in this case a diagram

$$\begin{tikzcd} B(0) \arrow[d, hook] & B(1) \arrow[l] \arrow[d, hook] \\ A(0) & A(1), \arrow[l] \end{tikzcd} \tag{209}$$

where A and B are set-valued contravariant functors, while 0 and 1 are elements of $\mathbf{2} = \{0 \rightarrow 1\}$. We may draw the diagram (209) in a set-theoretic form, which is easier to understand:



what is the same as



Coming back to consideration of the functor category $\mathbf{Set}^{\mathcal{C}^{op}}$ as category of diagrams in **Set** indexed by the elements of $\mathcal{C}^{op}$, note that one can not speak about the Ω-pullback diagram in $\mathbf{Set}^{\mathcal{C}^{op}}$,

$$\begin{tikzcd} B \arrow[r, dotted, "!"] \arrow[d, tail] & \mathbf{1} \arrow[d, tail, "\top"] \\ A \arrow[r, "\chi_B"'] & \Omega, \end{tikzcd} \tag{210}$$

but rather about the family of diagrams in **Set** indexed by $\mathcal{C}^{op}$:

$$\begin{tikzcd} B(0) \arrow[r, dotted, "!"] \arrow[d, tail] & \mathbf{1}(0) \arrow[d, tail, "\top(0)"] \\ A(0) \arrow[r, "\chi_{B(0)}"'] & \Omega(0) \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} B(1) \arrow[r, dotted, "!"] \arrow[d, tail] & \mathbf{1}(1) \arrow[d, tail, "\top(1)"] \\ A(1) \arrow[r, "\chi_{B(1)}"'] & \Omega(1) \end{tikzcd} \quad \text{and...} \tag{211}$$

where 0 and 1 and ... are the objects of $\mathcal{C}^{op}$. These diagrams are not independent from each other, but related in the same way as the objects in $\mathcal{C}^{op}$. For example, for a topos of presheaves over

$$\mathcal{C} = \mathbf{2} = \{0 \xrightarrow{2} 1\}, \tag{212}$$

the image of the action of the subobject classifier diagram is given by $\Omega(0) \xleftarrow{\Omega(2)} \Omega(1)$, as it was presented in the pictures (200), (205) and (207). (The inversion of the arrows comes from contravariance of the functor Ω.) The complete diagram of the subobject classifier in $\mathbf{Set}^{\mathcal{C}^{op}}$ is then 'three-dimensional'. For example, the full diagram of classification of subobjects in $\mathbf{Set}^{\mathbf{2}^{op}}$ is following:

$$\begin{tikzcd}
 & & 1 \arrow[loop above, "\mathrm{id}_1"] & \\
 & 0 \arrow[loop above, "\mathrm{id}_0"] \arrow[ur, "2"] & & \\
 & & {} \arrow[d, Rightarrow, "\mathbf{Set}^{(-)^{op}}"] & \\
 & B(1) \arrow[dl, "B(2)"'] \arrow[rr, dotted, "!(1)"] \arrow[dd, hook] & {} & \mathbf{1}(1) = \{\mathrm{id}_1\} \cong \{0\} \arrow[dl, "\mathbf{1}(2)"] \arrow[dd, hook, "\top(1)"] \\
B(0) \arrow[rr, dotted, "!(0)"] \arrow[dd, hook] & & \mathbf{1}(0) = \{\mathrm{id}_0\} \cong \{0\} \arrow[dd, hook, "\top(0)"'] & \\
 & A(1) \arrow[rr, "\chi_A(1)"'] \arrow[dl, "A(2)"] & & \Omega(1) = \{\{\{\mathrm{id}_1\}, \{2\}\}, \{2\}, \emptyset\} \cong \{0, \frac{1}{2}, 1\} \arrow[dl, "\Omega(2)"] \\
A(0) \arrow[rr, "\chi_A(0)"'] & & \Omega(0) = \{\{\mathrm{id}_0\}, \emptyset\} \cong \{0, 1\}. &
\end{tikzcd} \tag{213}$$

7.5 Presheaves over posets

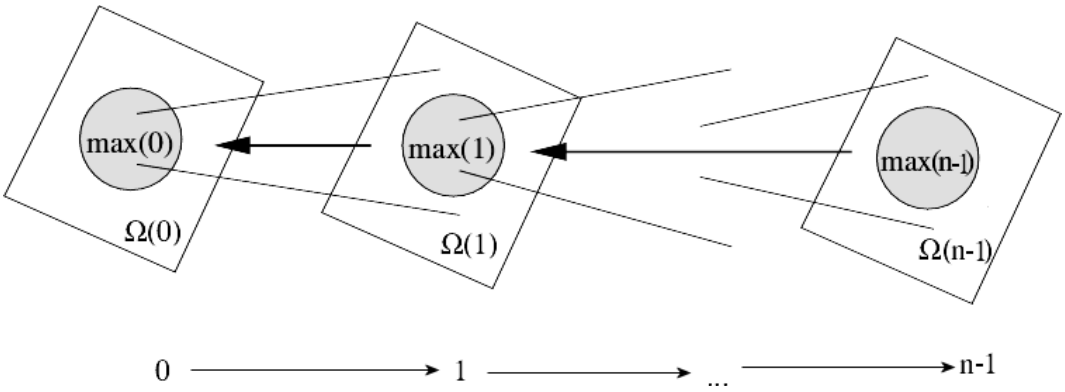
So far we have investigated the internal structure of Ω in $\mathbf{Set}^{\mathbf{2}^{op}}$, but obviously there are many other different presheaf categories. Quite similar is the structure of a topos $\mathbf{Set}^{\mathbf{n}^{op}}$, with $\mathbf{n}$ defined as

$$\begin{tikzcd}
0 \arrow[r, "f_{0,1}"] \arrow[loop below, "\mathrm{id}_0"] \arrow[rr, bend left, "f_{0,2}"] \arrow[rrr, bend left=50, "f_{0,\dots}"] \arrow[rrrr, bend left=60, "f_{0,n-1}"] & 1 \arrow[r, "f_{1,2}"] \arrow[loop below, "\mathrm{id}_1"] \arrow[rr, bend left, "f_{1,\dots}"] \arrow[rrr, bend right=60, "f_{1,n-1}"'] & 2 \arrow[r, "f_{2,\dots}"] \arrow[loop below, "\mathrm{id}_2"] \arrow[rr, bend right, "f_{2,n-1}"'] & \dots \arrow[r, "f_{\dots,n-1}"] & n-1. \arrow[loop below, "\mathrm{id}_{n-1}"]
\end{tikzcd} \tag{214}$$

The sieves over $\mathbf{n}^{op}$ are the following (where $f_{0,2} = f_{1,2} \circ f_{0,1}$):

stage	sieves	'truth-values'
0	$\{\mathrm{id}_0\}$	1
$-\shortparallel-$	$\emptyset$	0
1	$\{\{\mathrm{id}_1\},\{f_{0,1}\}\}$	1
$-\shortparallel-$	$\{f_{0,1}\}$	1/2
$-\shortparallel-$	$\emptyset$	0
2	$\{\{\mathrm{id}_2\},\{f_{1,2}\},\{f_{0,2}\}\}$	1
$-\shortparallel-$	$\{\{f_{1,2}\},\{f_{0,2}\}\}$	1/2
$-\shortparallel-$	$\{f_{0,2}\}$	1/4
$-\shortparallel-$	$\emptyset$	0
...	...	...
n	$\{\{\mathrm{id}_n\},\{f_{n-1,n}\},\ldots,\{f_{0,n}\}\}$	1
$-\shortparallel-$	$\{\{f_{n-1,n}\},\ldots,\{f_{0,n}\}\}$	1/2
$-\shortparallel-$	...	...
$-\shortparallel-$	$\{f_{0,n}\}$	$1/2^n$
$-\shortparallel-$	$\emptyset$	0

and can be represented in a form similar to (7.4) as



This picture symbolically shows that the number of different truth values reduces, when one moves backward to earlier stages. Conversely, one can say that with the 'flow of time', the notion of truth obtains more and more different flavours (or meanings, or values). The logical value 1, defined by the truth arrow $\top : \mathbf{1} \to \Omega$, is always provided by a maximal sieve, which is chosen by the identity arrow of a given object.

In the above topos one can more directly recognise what the partiality of truth-values means: for example a sieve $\{f_{0,2}\}$ which belongs to Ω at stage 2 corresponds to the element of the set A which will 'fall into' the subset $B \subseteq A$ only at 0, and not earlier. Hence its truth value is partial, namely 1/4. It might be handy to think here about a chain of events in the Minkowski spacetime, where some events have been felt earlier in the light cone and so will stay forever inside, some events will fall in the 'near' time (e.g. 1/2), some will fall even later, and some in the very far 'future', i.e., $1/2^n$ for some big n. Of course, $\mathcal{C}$ does not have to be 'linear'. The simplest example of such category is

$$0 \underset{g}{\overset{f}{\rightrightarrows}} 1, \tag{215}$$

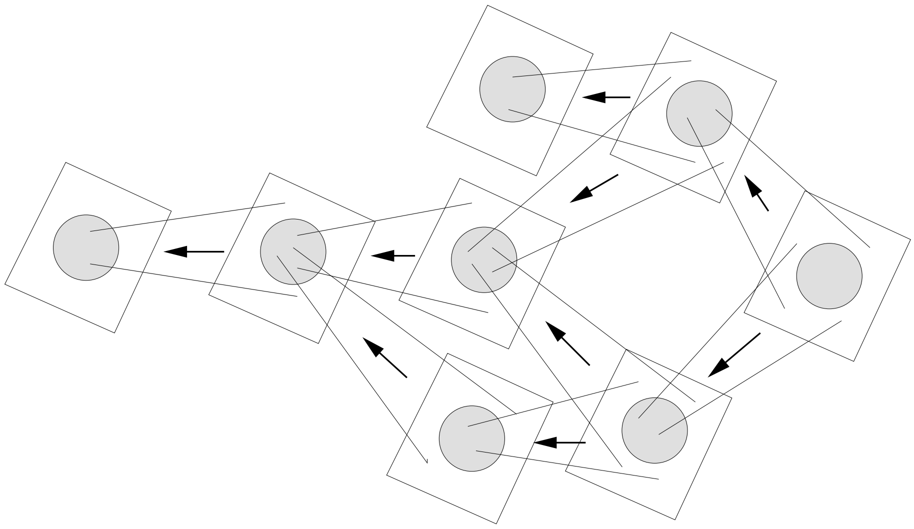
with five different sieves on the stage 1:

$$\mathrm{Hom}(-,1) \cong \{\{\mathrm{id}_1, f, g\},\ \emptyset,\ \{f\},\ \{g\},\ \{f,g\}\}\,. \tag{216}$$

We can assign with these sieves the logical values 1, 0, a, b, and ab, respectively. A more complicated structure of $\mathcal{C}$ is for example

$$\begin{tikzcd}
& & E \arrow[r] & A \arrow[dr] & \\
H \arrow[r] & G \arrow[r] \arrow[dr] & D \arrow[ur] \arrow[dr] & & C, \\
& & F \arrow[r] & B \arrow[ur] &
\end{tikzcd} \tag{217}$$

but the way to construct the sieves is the same. They form the following diagram:



The above category is a good example of a situation when $\mathcal{C}$ is a poset. In such case the sieves varying over $\mathcal{C}$ are 'falling into each other'. In other words, the sieves over poset are *downward closed.*

Definition 7.11 *Let $(P, \leq)$ be a partially ordered set. $Q \subseteq P$ is* ***downward closed*** *on $p \in P$, if*

1. $\forall q \in Q \quad q \leq p$,

2. $\forall q \in Q \ \forall p' \in P \quad (p' \leq q) \Rightarrow (p' \in Q)$.

8 Logic and language

The discussion has so far uncovered the fact that there are a variety of different categories of presheaves with corresponding sieves and generalised truth-values. This raises an important question: how to manage all these different truth-values on a fixed stage and how to manage their variaton over the stages of base category? This question is important since, if we want to make any statements and handle them (do the *predicate calculus* of these statements), we have to fix the logic and the corresponding algebra of propositions. Two-valued classical propositional logic (and, more generally, boolean algebra valued classical propositional logic) corresponds to boolean algebra, but neither two-valued classical propositional logic nor boolean algebra seem to be appropriate for the multivalued object Ω of generalised truth values. Very fortunately, it turns out that Ω has the natural structure of a Heyting algebra, which implements the intuitionistic logic and directly generalises boolean algebra. We will now explore some notions from formal logic and model theory which will help to clarify these terms and statements.

8.1 Language and interpretation

In this section we will discuss the formalisation of a language of a mathematical theory. The main objects of investigation of every mathematical theory are mathematical ***structures***. However, a mathematical theory also consists of certain *metamathematical* contents, which determine the possible ways in which the mathematical structures can be analysed. One can say that a mathematical theory consists of text, given by its mathematical structures, and context, provided by its metamathematical language. The original development of formalisation of a language was advocated by the need of distinction between the contents of a mathematical theory and the metamathematical description of its usage, brought into the light of attention by paradoxes in the foundations of set theory. However, such a distinction can be also be made consistently in an informal way. The main virtue of the formalisation of mathematical theories is that this procedure gives an ability to express a given mathematical theory in an abstract form. This enables the construction of different *intepretations* (*models*) of the same (formal, abstract) mathematical theory in other mathematical theories. For example, theories initially formulated in terms of sets can be formalised and interpreted in terms of other structures (in particular, toposes). In order to provide a formalisation of a theory, one has to specify not only its set of axioms, but also its language. The language of a theory is understood as a collection of rules by which the sentences and formulae of a theory can be built, as well as a collection of rules of deductive inference of statements and formulae of a theory as logical consequences of given sentences and formulae.

Definition 8.1 *A mathematical **structure** (an object of interest of a mathematical theory) is a universe consisting of:*

- *a set of **individuals** (eg. elements of a group),*
- *a set of **properties** of (or relations between) individuals (eg. equality of between elements of a group),*
- *a set of **functions** or operations on individuals (eg. composition in a group).*

The mathematical theory of a given kind of structure contains sentences and propositions (propositional functions) built from the above constituents. A set of signs of a formal language $\mathcal{L}$, corresponding to a given structure, consists of:

- a set of individual constants (denoting particular individuals of a structure),
- propositional variables (varying over particular individuals of a structure),
- n-argument predicative symbols (denoting properties of a structure),
- n-argument function symbols (denoting functions of a structure).
- auxiliary signs (like comma, and parentheses) with the rules of their usage.

Individual constants are identified with 0-argument (constant) function symbols. Expressions in $\mathcal{L}$ built from variables, function and predicate symbols are called the ***formulas***, while expressions in $\mathcal{L}$ built from variables and function symbols are called ***terms***. The ***sentences*** are formulae with propositional variables replaced by individual constants. A simple formula is

Socrates is a man,

where 'Socrates' is an individual constant, 'a man' is a propositional variable (it can vary over some designated individuals of a corresponding mathematical structure), 'is' is a predicative symbol, and there is no function symbol. Another quite famous sentence is

$$2 + 2 = 4, \tag{218}$$

where '2' and '4' both are constants, '+' is a function symbol, and '=' is a relation symbol. Note that only our ***interpretation*** says that '+' means addition. There is nothing to stop us interpreting '2' as *a snake*, '4' as *a tree*, '+' as *is speaking to*, and '=' as *while laying on.* This *Gedankenexperiment* is by itself not very deep, but it leads to quite deep corollary: *the same formal language $\mathcal{L}$ may have different **interpretations*** (in different structures). So, for a given very simple language $\mathcal{L}_{\text{very simple}}$ consisting of

1. variables: 2, 4,
2. function symbol: $+$,
3. relation symbol: $=$,

one can consider different interpretations of it. The interpretation in terms of a structure $\mathfrak{U}_{\mathbb{N}}$, where $\mathbb{N}$ are natural numbers, can be given by:

1. '2' and '4' *mean* the natural numbers 2 and 4, respectively,
2. '+' *means* the addition in the natural numbers,
3. '=' *means* the equality relation in the natural numbers.

On the other hand, the alternative interpretation of $\mathcal{L}_{\text{very simple}}$, in terms of $\mathfrak{U}_{\mathcal{E}}$, where $\mathcal{E}$ is a cartesian closed category, can be given by:

1. '2' and '4' *mean* the objects A and B of $\mathcal{E}$,
2. '+' *means* the cartesian product of objects,
3. '=' *means* isomorphism of objects.

The interpretation in terms of structure $\langle$snake, tree, is speaking to, while laying on$\rangle$ was given above. Consider now two formulae:

$$2+2=4, \tag{219}$$
$$x+x=y. \tag{220}$$

The first formula contains the variables which are individual constants, i.e., which are *intended* to denote the designated individuals. On the other hand, the variables in the second formula are intended to ***range over*** individuals of some structure. In this sense these two formulae are of a different *kind.* The second formula differs from the first one because it expresses certain ***conditions***. This means that when ranging over different individuals of the structure, this second formula generates different sentences, each one with its own possible truth value. When such a formula is endowed with a meaning, given by the interpretation of a language in some mathematical structure, it is called a ***proposition***. Note that this means that different structures can provide different meanings of the same sentences.

We can extend this picture and introduce in $\mathcal{L}$ the ***connective symbols***, which will be used to connect formulae together, in order to form new formulae. If $\mathcal{L}_{\text{very simple}}$ were equipped with one more individual constant '1', then one could write these sentences:

$$1+1=2 \tag{221}$$
$$2+2=4 \tag{222}$$
$$1+2=1 \tag{223}$$

In such case, introduction of the 2-ary (w-argument) connective symbol '$\wedge$' to $\mathcal{L}_{\text{very simple}}$ allows to use $\mathcal{L}_{\text{very simple}}$ in order to form formulae like

$$1+1=2 \quad \wedge \quad 2+2=4, \tag{224}$$
$$1+1=2 \quad \wedge \quad 1+2=1. \tag{225}$$

Note that the sentences (221) and (222) under the standard interpretation in the structure $\mathfrak{U}_{\mathbb{N}} := \langle \mathbb{N}, +_{\mathbb{N}}, \cdot_{\mathbb{N}} \rangle$ are *considered* to posses the *truth value* 'truth', while the sentence (223) is *considered* to posses the truth value 'false'. The notion of 'being *meaningful*' is closely related to the concept of truth values (or truth value evaluation). A proposition is said to be ***meaningful*** if it is possible to assign truth values to it. The assignment of truth values (***degrees of meaning***) depends on the structure $\mathfrak{U}$ which is chosen for an interpretation of an (abstract) formal language. In other words, by saying that the meaning of the given sentence may vary under the interpretation, we say that the meaning of it, assigned by an interpretation in a particular structure, belongs to some specified object of this structure. For example, the meaning of the formula interpreted in the the classical universe of sets (the category **Set**) is specified by the elements of the set {true, false}. We *intend* that connective symbols should produce meaningful sentences from meaningful sentences, so the formulae (224) and (225), which in fact are sentences by definition, should possess their own truth values under any interpretation. This means that if any two sentences α and β of $\mathcal{L}$ have concrete truth values under some interpretation $\mathfrak{U}$, then their logical connection

$$\alpha \wedge \beta \tag{226}$$

must also have a concrete truth value under $\mathfrak{U}$. But what is this value? It must be assigned by the interpretation! Hence, *the interpretation handles the assignment of truth values to logically connected sentences.* It may be presented in so-called ***truth tables*** or ***logical tables***. For example

if α is interpreted to be	then $\neg\alpha$ is interpreted to be
true	false
false	true

if α is interpreted to be	and β is interpreted to be	then $\alpha \wedge \beta$ is interpreted to be
true	true	true
false	true	false
true	false	false
false	false	false

It follows that connectives can considered to be as a special type of relation symbols (acting on formulae of $\mathcal{L}$), with their meaning varying on the interpretation.

Let us now considering a simple propositional language $\mathcal{L}_{SP}$ which *consists* of:

1. collection of symbols: x, y, ..., called *variables*,
2. symbols: $\vee$, $\wedge$, $\neg$, $\Rightarrow$, called *connectives*,
3. symbols: (,), called *brackets*,

and *has* the following *rules* of creating of the sentences (denoted as α, β, ...):

1. every variable is a sentence,
2. if α is a sentence, then $\neg\alpha$ is a sentence,
3. if α and β are sentences, then $(\alpha \wedge \beta)$, $(\alpha \vee \beta)$ and $(\alpha \Rightarrow \beta)$ are sentences.

This language is still quite simple. In order to describe a language which is required to develop rich mathematical theory, we need more rich structure.

Definition 8.2 *The **first order formal propositional language** $\mathcal{L}_{FOFP}$ consists of an **alphabet** A, given by the disjoint sets of*

- *symbols of individual variables (eg. $x, y, z, \ldots$),*

- *n-argument function symbols (eg. $=, \leq, \ldots$),*
- *n-argument predicate symbols (eg. $\wedge, \vee, \Rightarrow, \neg, \ldots$),*
- *1-argument quantifier symbols (eg. $\forall, \exists, \ldots$),*
- *auxiliary signs (eg. $(,), \ldots$),*

a set T of all terms such that

- *all individual variables and 0-argument functions belong to T,*
- *all terms built from function symbols substituted with elements of T belong to T,*

and a set F of all formulae such that

- *all predicate symbols with arguments substituted by elements of T belong to F,*
- *all connectives with arguments substituted by elements of F belong to F,*
- *all quantifiers with arguments substituted by individual variables and acting on formulae f of F belong to F.*

In order to deduce mathematical formulae as logical consequences of others, one has to introduce the ***rules of inference***, that is, the operations which associate some formula of $\mathcal{L}$ with a given finite sequence of formulae. The associated formula is called a ***logical consequence*** or just a ***consequence*** of an initial sequence of formulae, called ***premises***. The map $C : X \to C(X)$ which to every set X of formulae associates a set of *all* its logical consequences is called a ***consequence operator***. A pair $(\mathcal{L}, C)$ is called a (formal) ***deductive system***. If a given set A of formulae is *assumed* to be true (to be satisfied) in *every* admissible interpretation, then it is called a set of ***axioms***. A triple $(\mathcal{L}, C, A)$ is called a (formal) ***theory***. The formulae of a set $C(A)$ are called the ***theorems*** of a theory $(\mathcal{L}, C, A)$.

Now we would like to interpret the language $\mathcal{L}$ in some structure $\mathfrak{U}$, that is, to specify the ***semantics*** of $\mathcal{L}$ in $\mathfrak{U}$. It means also that we would like to assign truth values to sentences of $\mathcal{L}$.[8] This leads to specification of the object (or 'set') of available truth-values, and specification of the morphism (or 'function'), called the ***truth valuation*** on $\mathcal{L}$, from the sentences of language $\mathcal{L}$ to this object. A ***truth value*** is just an element of an ***object of truth values***. By denoting the object of all sentences as A and the object of truth values as $\widetilde{\Omega}$,[9] one can regard the truth valuation of sentences as the map

$$\sigma : A \to \widetilde{\Omega}. \tag{227}$$

If a truth valuation σ of a sentence α gives truth value 'true' (denoted as $\top$), then we say that σ ***satisfies*** α and denote it as

$$\sigma \vDash \alpha. \tag{228}$$

Hence,

$$\sigma(\alpha) = \top \quad \text{iff} \quad \sigma \vDash \alpha. \tag{229}$$

One should note that some sentences of a given formal language $\mathcal{L}$ may have the same truth value 'truth' independently of the choice of different truth valuations $\sigma_1, \sigma_2, \ldots$ Every such sentence α is called a ***tautology*** and is denoted as $\vDash \alpha$. One can consider some tautologies of a given formal language $\mathcal{L}$ as *axioms* and use them to create an ***axiomatic system*** by specifying these axioms together with the rules of inference, as well as create *theory* by specifying these axioms together with the consequence operator.

[8] From the semantic point of view, it is completely irrelevant *what* the individuals of $\mathfrak{U}$ *are*. In semantics we are interested only in the *meaning* of sentences of $\mathcal{L}$ given by the *interpretation* $\mathfrak{U}$, and not in their ontology. Hence, the assignment of truth values answers only on the questions of *being meaningful* and not the questions of *being*.

[9] In this section we will exceptionally consider $\widetilde{\Omega}$ *not* as subobject classifier in a topos, but just as *some* object of truth values. The interpretation of a language and logic in a topos, that is, the construction showing that the subobject classifier Ω of a topos *is* a model of $\widetilde{\Omega}$, will be discussed in the next section.

In order to define the meaning of the sentences $\neg\alpha$, $(\alpha \wedge \beta)$, $(\alpha \vee \beta)$ and $(\alpha \Rightarrow \beta)$, it suffices to observe, that any logical table is a map from the object of truth vales to itself. Indeed,

$$\neg : \widetilde{\Omega} \to \widetilde{\Omega}, \tag{230}$$
$$\wedge : \widetilde{\Omega} \times \widetilde{\Omega} \to \widetilde{\Omega}. \tag{231}$$

Hence, we may just define the maps

$$\neg_{\mathfrak{U}} : \widetilde{\Omega} \to \widetilde{\Omega}, \tag{232}$$
$$\wedge_{\mathfrak{U}} : \widetilde{\Omega} \times \widetilde{\Omega} \to \widetilde{\Omega}, \tag{233}$$
$$\vee_{\mathfrak{U}} : \widetilde{\Omega} \times \widetilde{\Omega} \to \widetilde{\Omega}, \tag{234}$$
$$\Rightarrow_{\mathfrak{U}} : \widetilde{\Omega} \times \widetilde{\Omega} \to \widetilde{\Omega}, \tag{235}$$

by

$$\sigma(\neg\alpha) =: \neg_{\mathfrak{U}}(\sigma(\alpha)), \tag{236}$$
$$\sigma(\alpha \wedge \beta) =: \sigma(\alpha) \wedge_{\mathfrak{U}} \sigma(\beta), \tag{237}$$
$$\sigma(\alpha \vee \beta) =: \sigma(\alpha) \vee_{\mathfrak{U}} \sigma(\beta), \tag{238}$$
$$\sigma(\alpha \Rightarrow \beta) =: \sigma(\alpha) \Rightarrow_{\mathfrak{U}} \sigma(\beta). \tag{239}$$

Hence, for a fixed interpretation $(\mathfrak{U}, \sigma)$, we may omit the signs $\mathfrak{U}$ and σ. This leads us back to formal algebraic structures. The problem of the interpretation σ of the theory $(\mathcal{L}, C, A)$ constructed in a formal language $\mathcal{L}$ can be then considered as a problem of representation of one algebraic structure, related with the given propositional language $\mathcal{L}$, in another algebraic structure $\mathfrak{U}$. But such perspective requires us to understand better the way the formal languages and algebraic structures are related. We will start from consideration of sentences of the form

$$((\alpha \wedge \beta) \vee \gamma) \Rightarrow \delta \wedge (\neg\gamma), \tag{240}$$

as elements of algebras of signs equipped in operations $\Rightarrow$, $\vee$, $\wedge$ and $\neg$. Such algebras are called ***algebras of logic***.

8.2 Algebras of logic

Definition 8.3 *An algebra $L = \langle L, \wedge, \vee, 0, 1\rangle$ is a **lattice** if for every $a, b \in L$*

1. *$\wedge, \vee$ are both commutative and associative,*
2. *$a \wedge 1 = a, \quad a \vee 0 = a$,*
3. *$a \wedge a = a, \quad a \vee a = a$,*
4. *$(a \wedge b) \vee b = b, \quad (a \vee b) \wedge b = b$.*

This is an equational definition, which means that every algebra L satisfying these equations is a lattice. A lattice on a poset can be considered also as a category. For $x, y \in L$ we have $x \leq y \iff x \to y$. The product is identified with the infinum (or the greatest lower bound) $x \wedge y$, while the coproduct with the supremum (or the least upper bound) $x \vee y$. Elements $0, 1 \in L$ such that $0 \leq x \leq 1$ for every $x \in L$ are initial and terminal objects, respectively. This yields us to the alternative, strictly categorical definition of a lattice.

Definition 8.4 *A **lattice** $\mathbf{L}$, is a poset which has finite products and coproducts.*

Note that this definition requires not only the existence of products and coproducts. It requires all finite products and coproducts, what includes also 0-ary product, given by $0/1$ and 0-ary coproduct, given by $1/0$. From this definition it follows that the category **Latt** consisting of all lattices and all homomorphisms of lattices is a subcategory of **Poset**. One may check that it is not full subcategory.

Definition 8.5 *A **distributive lattice** is a lattice satisfying the identity*

$$\forall a, b, c \qquad a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c). \tag{241}$$

Definition 8.6 *A **complement** of an element $a \in L$ is such x that*

1. $a \wedge x = 0$,
2. $a \vee x = 1$.

Proposition 8.7 *A complement in a distributive lattice is unique (if it exists).*

Proof. Let x and x' be two different complements to a. Then $x' = 1 \wedge x' = (a \vee x) \wedge x' = (a \wedge x') \vee (x \wedge x') = (a \wedge x) \vee (x \wedge x') = x \wedge (a \vee x') = x$. $\square$

Definition 8.8 *A **boolean algebra** is a distributive lattice in which every element a has a complement, denoted as $\neg a$.*

This means that in a boolean algebra we have

1. $\forall a \;\; a \wedge \neg a = 0$,
2. $\forall a \;\; a \vee \neg a = 1$.

Example The set of all subsets of a given set X, partially ordered by an inclusion, is a boolean algebra:

$$\begin{aligned} A \wedge B &:= A \cap B, \\ A \vee B &:= A \cup B, \\ 1 &:= X, \\ 0 &:= \emptyset, \\ \neg A &:= X \setminus A. \end{aligned} \tag{242}$$

The ***implication*** operation in boolean algebra can be defined algebraically as

$$a \Rightarrow b := \neg a \vee b, \tag{243}$$

which corresponds to the logical truth table

α	β	$\neg\alpha$	$\neg\alpha \vee \beta$	$\alpha \Rightarrow \beta$
0	0	1	1	1
0	1	1	1	1
1	0	0	0	0
1	1	0	1	1

However, there is also the possibility to first define an algebra containing an implication, and later force it to have the property that every a specifies uniquely a complement $\neg a$.

Definition 8.9 *A **Heyting algebra** is a lattice which is cartesian closed. The exponential b^a, existing for every two elements a and b, is denoted as $a \Rightarrow b$.*

The definition (149) of an exponential $c^b \iff b \Rightarrow c$ in poset implies the following:

$$a \leq b \Rightarrow c \iff a \wedge b \leq c. \tag{244}$$

This implies that the equational definition of Heyting algebra is the following:

Proposition 8.10 *The algebra $H = \langle H, \wedge, \vee, 0, 1, \Rightarrow\rangle$ is a Heyting algebra if for every $a, b, c, \in H$,*

1. $\wedge, \vee$ *are both commutative and associative,*
2. $a \wedge 1 = a, \quad a \vee 0 = a,$
3. $a \wedge a = a, \quad a \vee a = a,$
4. $(a \wedge b) \vee b = b, \quad (a \vee b) \wedge b = b,$
5. $(a \Rightarrow a) = 1,$
6. $a \wedge (a \Rightarrow b) = a \wedge b, \quad b \wedge (a \Rightarrow b) = b,$
7. $a \Rightarrow (b \wedge c) = (a \Rightarrow b) \wedge (a \Rightarrow c).$

Proposition 8.11 *Every boolean algebra is a Heyting algebra.*

Proof. Equational proof of properties involving $\wedge, \vee, 0$ and 1 is obvious. We have to prove that $\Rightarrow$ defined for the boolean algebra is the same as $\Rightarrow$ given in the definition of the Heyting algebra. It is sufficient to show that (244) holds for (243), i.e., that $a \leq \neg b \vee c \iff a \wedge b \leq c$. We will prove this implication in both directions. In one direction: $a \wedge b \leq (\neg b \vee c) \wedge b \leq c \wedge b \leq c$. In opposite direction: $a = 1 \wedge a = (\neg b \vee b) \wedge a = (\neg b \wedge a) \vee (b \wedge a) \leq \neg b \vee c$. $\square$

This means that the implication defined for boolean algebra is an implication defined for Heyting algebra. We can define the negation in every Heyting algebra as

$$\neg a := (a \Rightarrow 0). \tag{245}$$

This leads to an interesting observation, that for Heyting algebras implication is more fundamental then negation.

Proposition 8.12 *A complement of an element a of a Heyting algebra, if exists, is $\neg a$.*

Proof. If x is a complement of a, then $a \wedge x = 0$, so $x \wedge a \leq 0 \iff x \leq a \Rightarrow 0 \iff x \leq \neg a$. From $a \vee 1$ we get $\neg a = \neg a \wedge 1 = \neg a \wedge (a \vee x) = \neg a \wedge x \leq x$, so finally $x = \neg a$. $\square$

A Heyting algebra is a distributive lattice. This follows immediately from its definition as a lattice which is cartesian closed. For any cartesian closed category we have $(a + b) \times c = a \times c + b \times c$, hence $(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c)$. So, the difference between Heyting and boolean algebras consists in the fact that in the latter each element a has a unique complement $\neg a$. This yields to the famous proposition.

Proposition 8.13 *A Heyting algebra is a boolean algebra iff any of the following statements are satisfied:*

$$\forall a \quad \neg\neg a = a, \tag{246}$$
$$\forall a \quad a \vee \neg a = 1. \tag{247}$$

Hence, in every Heyting-not-boolean algebra there is an a such that $\neg\neg a \neq a$, and the law of excluded middle does not hold. This means that generally in a Heyting algebra (246) and (247) do not hold. An element a of a Heyting algebra such that $a = \neg\neg a$ is called the ***regular element***.

Due to their properties, Heyting and boolean algebras correspond to different axiomatic systems of logic, that is, to different set of sentences considered to be tautological and equipped with the rules of inference. In what follows we will present two axiomatic systems, classical and intuitionistic, whose axioms and rules produce exactly the spectrum of the sentences available in the corresponding algebra.

8.3 Classical and intuitionistic logic

> *Indeed, classical logic can by all means be seen as a standard wherever there is a need for precise reasoning, especially in mathematics and computer science. The principles of the classical logic are extremely useful as a tool to describe and classify the common-sense reasoning patterns occurring both in everyday life and mathematics. It is however important to understand the following. First of all, no system of rules can capture all of the rich and complex world of human thoughts, and thus every logic can merely be used as a limited-purpose tool rather then as an ultimate oracle, responding to all possible questions. In addition, the principles of classical logic, although easily acceptable by our intuition, are not the only possible reasoning principles.* [153]

Definition 8.14 *An axiomatic system of **classical logic** (**CL**) consists of twelve axioms:*

1. $\vdash \alpha \Rightarrow (\alpha \wedge \alpha)$,
2. $\vdash (\alpha \wedge \beta) \Rightarrow (\beta \wedge \alpha)$,
3. $\vdash (\alpha \Rightarrow \beta) \Rightarrow ((\alpha \wedge \gamma) \Rightarrow (\beta \Rightarrow \gamma))$,
4. $\vdash ((\alpha \Rightarrow \beta) \wedge (\beta \Rightarrow \gamma)) \Rightarrow (\alpha \Rightarrow \gamma)$,
5. $\vdash \beta \Rightarrow (\alpha \Rightarrow \beta)$,
6. $\vdash (\alpha \wedge (\alpha \Rightarrow \beta)) \Rightarrow \beta$,
7. $\vdash \alpha \Rightarrow (\alpha \vee \beta)$,
8. $\vdash (\alpha \vee \beta) \Rightarrow (\beta \vee \alpha)$,
9. $\vdash ((\alpha \Rightarrow \gamma) \wedge (\beta \Rightarrow \gamma)) \Rightarrow ((\alpha \wedge \beta) \Rightarrow \gamma)$,
10. $\vdash \neg\alpha \Rightarrow (\alpha \Rightarrow \beta)$,
11. $\vdash ((\alpha \Rightarrow \beta) \wedge (\alpha \Rightarrow \neg\beta)) \Rightarrow \neg\alpha$,
12. $\vdash \alpha \vee \neg\alpha$,

*and one rule of propositional calculus called **modus ponens**:*

$$\text{if} \quad \alpha = (\alpha \Rightarrow \beta) = \top \quad \text{then} \quad \beta = \top. \tag{248}$$

It can be shown that this axiomatic system is equivalent with boolean algebra. In Heyting algebras we have $\nvdash \alpha \vee \neg\alpha$, so the last axiom of CL cannot hold. Generally, ***intuitionistic logic*** (***IL***) is defined as an axiomatic system which corresponds to Heyting algebra and for which the axiom of excluded middle does not hold. However, there can be different axiomatisations of intuitionistic logic. An example of axiomatic system of IL is the above axiomatic system of CL, but with the 12th axiom *removed*. In both intuitionistic and classical cases there is the same rule of propositional calculus i.e., *modus ponens*.[10]

The intuitionistic logic is not only the axiomatic system corresponding to a different (weaker, hence more rich) algebra of propositions, but it is also a formalisation of a language of a whole approach to (and philosophy of) mathematics, called *intuitionism* or *constructivism*. In brief, intuitionism denies any preëxistence of the mathematical structures, saying that mathematicians just *construct* them, and do not pull them out of the (trancendent or mythical, but—as a rule—undefined) 'platonic reality'. In other words, *there is no hidden ideal truth, there are only explicit constructions*. Hence, everything which can be explicitly stated, has to be constructed. Consequently, *to prove* something means *to construct* it. The immediate consequence of this approach is a change of the meaning associated to logical symbols $\wedge, \vee, \neg, \Rightarrow$: they cannot be understood as expressions of the relations between some

[10] There are also other rules, designated to handle the quantifiers $\forall$ and $\exists$. However we do not want to give neither the classical or intuitionistic description of quantifiers, nor their categorical definition (although it is an elegant construction, provided in terms of adjoint functors).

preëxisting objects. They have to be understood as expressions of the relations between *constructions* which may be performed. The so-called ***Brouwer–Heyting–Kolmogorov interpretation*** (BHK) expresses intuitionistic approach to logic by giving the following meaning to logical connectives:

1. a proof (construction) of $(\alpha \wedge \beta)$ is a construction consisting of a construction/proof of α *and* the construction/proof of β,
2. a proof (construction) of $(\alpha \vee \beta)$ is a construction consisting of a construction/proof of α *or* the construction/proof of β,
3. a proof (construction) of $(\alpha \Rightarrow \beta)$ is a construction transforming every construction/proof of α *into* the construction/proof of β,
4. a proof (construction) of $(\neg\alpha)$ is a construction that shows that there is *no* construction/proof of α.

Given the above intepretation of connectives, the negation $\neg$ in BHK interpretation is not an *involution*, like it was in CL. In particular, $\neg\neg\alpha$ means here *the construction that shows that (there is no construction that shows that (there is no construction of α)).* This is not the same as the statement *there is a construction of α*. So, under the BHK interpretation of logical connectives, one has $\nvdash \neg\neg\alpha = \alpha$, and so $\nvdash \alpha \vee \neg\alpha$ too. IL is then understood as an axiomatisation corresponding to BHK interpretation. The algebra of propositions of IL is a Heyting algebra. This way we obtain the following correspondence:

$$\frac{\text{classical logic}}{\text{intuitionistic logic}} = \frac{\text{boolean algebras}}{\text{Heyting algebras}} \tag{249}$$

Returning from somewhat philosophical considerations to the issues of logic and semantics, recall that the logical object of truth values $\widetilde{\Omega}$ is identified in classical logic with the set $2 = \{0, 1\} = \{\text{truth}, \text{false}\}$. We have shown that $\widetilde{\Omega} = 2$ together with the maps $\neg$, $\wedge$, $\vee$, $\Rightarrow$ has a natural boolean algebra structure. Since a boolean algebra is a special case of a Heyting algebra, it follows that $\widetilde{\Omega} = 2$ also has the structure of a Heyting algebra.

Examples

1. (*Tarski–Stone representation theorem*) The algebra $\mathrm{Opens}(X)$ of all open subsets of a topological space X is a Heyting algebra, i.e., if A, B, C are open subsets of X, then

$$\begin{aligned} A \wedge B &:= A \cap B, \\ A \vee B &:= A \cup B, \\ 1 &:= X, \\ 0 &:= \emptyset, \\ \neg A &:= \mathrm{Int}(X \setminus A), \\ A \Rightarrow B &:= \mathrm{Int}((X \setminus A) \cup B), \end{aligned} \tag{250}$$

   where $\mathrm{Int}(C)$ means the maximal open subset of C. From this definition we see that $\neg\neg A$, the open interior around the open interior of A, may not be the same as A, and $A \vee \neg A$ may not be equal to 1. For example, if $X = \mathbb{R}$, then $A = \{x \in \mathbb{R} \mid x > 0\}$ and $\neg A = \{x \in \mathbb{R} \mid x < 0\}$, hence $A \vee \neg A \neq 1$. Thus, Heyting algebra may be called an *algebra of topology.*

2. The partially ordered set $\mathrm{Sub}(X)$ of subobjects of an object $X \in \mathrm{Ob}(\mathbf{Set}^{\mathcal{C}^{op}})$ is also a Heyting algebra, i.e., if F, G are subfunctors of X and A, B are objects of $\mathcal{C}^{op}$, then

$$\begin{aligned} (F \wedge G)(A) &:= F(A) \cap G(A), \\ (F \vee G)(A) &:= F(A) \cup G(A), \\ 1 &:= X, \\ 0 &:= \ulcorner \emptyset \urcorner, \\ (\neg F)(A) &:= \{a \in X(A) \mid \forall f : B \to A \quad F(f)(a) \notin F(B)\}, \\ (F \Rightarrow G)(A) &:= \{a \in X(A) \mid \forall f : B \to A \quad F(f)(a) \in F(B) \Rightarrow F(f)(a) \in G(B)\}. \end{aligned} \tag{251}$$

   From this definition we see that $\neg F \vee F$, specified by $\neg F(A) \cup F(A)$ may not be the same as X.

The second example above is a simple illustration of a general statement about every topos.

Proposition 8.15 *For any object A in some topos $\mathcal{E}$, the poset* $(\mathrm{Sub}(A), \subseteq)$ *of subobjects of A in $\mathcal{E}$ has the structure of a Heyting algebra.* ***[****, and this structure is natural in the sense that, for a morphism $f : A \to B$ in $\mathcal{E}$, the induced map* $f^{-1} : \mathrm{Sub}(B) \to \mathrm{Sub}(A)$ *is a homomorphism of Heyting algebras.****]***

Proof. See e.g. [107]. □

Recall from (167) that in every topos there exists an isomorphism $\mathrm{Sub}(A) \cong \mathrm{Hom}(A, \Omega)$, where Ω is the subobject classifier. Thus, the set of arrows from any object A to the subobject classifier Ω has the structure of Heyting algebra. Because Hom and Sub are sets, hence the statement of this isomorphism belongs to language of set theory. In other words, the above isomorphism is an ***external statement***, which is a statement of the certain property of the topos described in set-theoretical language. However, every external statement corresponds naturally to some ***internal statement***, which describes facts only in the internal language of topos, refering to its objects and arrows. The corresponding internal statement in this case says that Ω^A is a Heyting algebra. In particular, also $\Omega \cong \Omega^{\mathbf{1}}$ is a Heyting algebra. In the next section we will show that the subobject classifier Ω in any topos may be successfully used to *interpret* the *logical* object of truth values $\widetilde{\Omega}$.

8.4 Semantics for logic

[*To do!***]**

8.5 Semantics for language

[*To do!***]**

9 Logic and language in a topos

[*Say about categorical logic of Heyting algebras and its topos models.***]**

According to distinction between syntactic (formal) and semantic (interpreted) aspects of a language, when dealing with linguistic properties of topoi, one has to consider

- Mitchell–Bénabou language of a topos [119], [10], [11],
- Kripke–Joyal semantics of a topos [65], [**?**].

Mitchell–Bénabou language is an internal language of a topos. In other words, it is a formal language which results from formalisation of naïve language used to speak about the internal objects and arrows of a topos. On the other hand, Kripke–Joyal semantics is an interpretation of a formal language in a topos.

9.1 Categorical logic of Heyting algebra

We will first analyse how to express logical operations on boolean and Heyting algebra in categorical language.

Let us begin with the categorical interpretation of the negation, i.e., the map $\neg : 2 \to 2$. It can be considered as a characteristic function of the set $\mathbf{1} = \{0\} \subset \{0, 1\}$, because

$$\neg(1) \mapsto 0 \in \{0, 1\}, \tag{252}$$
$$\neg(0) \mapsto 1 \in \{0, 1\}. \tag{253}$$

Thus, one can define negation $\neg$ as the characteristic map of the *boolean false arrow* $\bot \colon \{0\} \hookrightarrow 2$, given by the pullback

$$\begin{tikzcd} \mathbf{1} \arrow[r, dotted, "!"] \arrow[d, tail, "\bot"'] & \mathbf{1} \arrow[d, "\top"] \\ 2 \arrow[r, "\neg"'] & 2. \end{tikzcd} \tag{254}$$

Conjunction $\wedge$ has the properties

$$\begin{aligned} \wedge(0,0) &= 0, \\ \wedge(0,1) &= 0, \\ \wedge(1,0) &= 0, \\ \wedge(1,1) &= 1, \end{aligned} \tag{255}$$

so it can be represented as the characteristic arrow of the set $\{(1,1)\} \subset 2 \times 2$, or, equivalently, of the map

$$\langle \top, \top \rangle : \mathbf{1} \rightarrowtail 2 \times 2 \tag{256}$$

given by

$$\langle \top, \top \rangle : \{0\} \mapsto \{(1,1)\}. \tag{257}$$

Hence, categorical definition of logical conjunction states that it is a characteristic map in the pullback diagram

$$\begin{tikzcd} \mathbf{1} \arrow[r, dotted, "!"] \arrow[d, tail, "\langle \top,\top \rangle"'] & \mathbf{1} \arrow[d, "\top"] \\ 2 \times 2 \arrow[r, "\wedge"'] & 2. \end{tikzcd} \tag{258}$$

Implication is defined similarly, as the characteristic arrow of the set

$$\{\leq\} := \{(0,0), (0,1), (1,1)\} \tag{259}$$

given by the pullback diagram

$$\begin{tikzcd} \{\leq\} \arrow[r, dotted, "!"] \arrow[d, hook] & \mathbf{1} \arrow[d, "\top"] \\ 2 \times 2 \arrow[r, "\Rightarrow"'] & 2. \end{tikzcd} \tag{260}$$

In order to characterise the object $\{\leq\}$ in categorical terms, one needs to prove the following proposition:

Proposition 9.1 *The set $\{\leq\}$ is an equaliser*

$$\begin{tikzcd} \{\leq\} \arrow[r, hook] & 2 \times 2 \arrow[r, shift left, "\wedge"] \arrow[r, shift right, "\pi_1"'] & 2, \\ X \arrow[u, dotted, "!"] \arrow[ur] & & \end{tikzcd} \tag{261}$$

where π_1 is a projection of the first component of the product.

Proof.

$$\begin{aligned} 2 \times 2 \ni (0,1) &\overset{\wedge}{\longmapsto} 0 \wedge 1 = 0 = \pi_1(0,1), \\ 2 \times 2 \ni (0,0) &\overset{\wedge}{\longmapsto} 0 \wedge 0 = 0 = \pi_1(0,0), \\ 2 \times 2 \ni (1,1) &\overset{\wedge}{\longmapsto} 1 \wedge 1 = 1 = \pi_1(1,1). \end{aligned} \tag{262}$$

On the other hand, $1 \wedge 0 = 0 \neq 1 = \pi_1(1,0)$, so any subset of 2×2 which contains pair $(1,0)$ is not an equaliser. $\square$

Definition 9.2 *The partial order $\leq$ on a lattice object L is an equaliser*

$$\leq \xrightarrow{e} L \times L \underset{\pi_1}{\overset{\wedge}{\rightrightarrows}} L, \tag{263}$$

where π_1 is the canonical projection of the first component of the product.

The operation $\vee$ is characterised in a slightly more sophisticated way. In order to identify $\vee$ with the characteristic function of the set $U = \{(0,1),(1,0),(1,1)\}$, we first of all need to express U in a categorical way. To do so, we will first construct the union $\{(0,1),(1,1)\} \cup \{(1,0),(1,1)\}$. The first component of this union can be identified with the map $\langle \top, \mathrm{id}\rangle : 2 \to 2\times 2$, while latter with the map $\langle \mathrm{id}, \top\rangle : 2 \to 2\times 2$. Thus, one can draw a diagram:

$$\begin{tikzcd} & 2 \times 2 & \\ 2 \arrow[ur, "\langle\top,\mathrm{id}\rangle"] \arrow[r] & 2+2 \arrow[u, dotted, "f"] & 2, \arrow[ul, "\langle\mathrm{id},\top\rangle"'] \arrow[l] \end{tikzcd} \tag{264}$$

where f is the unique arrow $\langle\langle \top, \mathrm{id}\rangle, \langle \mathrm{id}, \top\rangle\rangle$ such that this diagram commutes. Using the epi-mono-factorisation (see proposition 7.6) we get the unique purely categorical description of the ***image*** object U:

$$\begin{tikzcd} 2+2 \arrow[rr, dotted, "\langle\langle\top,\mathrm{id}\rangle,\langle\mathrm{id},\top\rangle\rangle"] \arrow[dr, two heads] & & 2 \times 2 \\ & U. \arrow[ur, hook] & \end{tikzcd} \tag{265}$$

This enables us to define $\vee$ by the pullback diagram

$$\begin{tikzcd} U \arrow[r, dotted, "!"] \arrow[d, hook] & \mathbf{1} \arrow[d, "\top"] \\ 2 \times 2 \arrow[r, "\vee"'] & 2. \end{tikzcd} \tag{266}$$

Up to now we have defined what a Heyting algebra is from the point of view of the external 'Sub-Hom' language. Now we would like to work 'inside' a topos, i.e., we would like to have a description of Heyting algebra in terms of arrows and objects.

9.2 Categorical logic in a topos

Definition 9.3 *A* ***lattice object*** *or an* ***internal lattice*** *in a category with finite limits is an object L together with arrows*

$$\begin{aligned} \wedge &: L \times L \to L, \\ \vee &: L \times L \to L, \\ \top &: \mathbf{1} \to L, \\ \bot &: \mathbf{1} \to L, \end{aligned} \tag{267}$$

such that algebra $L = \langle L, \wedge, \vee, \top, \bot\rangle$ is a lattice. A lattice object L is called a ***Heyting algebra object*** *or an* ***internal Heyting algebra*** *in the category with finite limits if it is cartesian closed. The element a^b of an exponential L^L is denoted as $b \Rightarrow a$ and defines the map $\Rightarrow: L \times L \to L$.*

The external statement 8.15 can be stated in the internal terms of topos in the form of an internal statement.

Proposition 9.4 *For any object A in a topos, the object Ω^A is an internal Heyting algebra.*

We will prove slightly less general statement.

Proposition 9.5 *The subobject classifier Ω in a topos is an internal Heyting algebra.*

Proof. Using the analogy between the object 2 in **Set** and the subobject classifier Ω in a topos, the above categorical definitions of connectives can be generalised to give the definition of connectives in topos, by replacing 2 by Ω. Conjunction is defined as the characteristic arrow of a pullback diagram

$$\begin{tikzcd} \mathbf{1} \arrow[r, dotted, "!"] \arrow[d, tail, "\langle\top,\top\rangle"'] & \mathbf{1} \arrow[d, "\top"] \\ \Omega\times\Omega \arrow[r, "\wedge"'] & \Omega. \end{tikzcd} \tag{268}$$

Disjunction is defined by the pullback diagram

$$\begin{tikzcd} U \arrow[r, dotted, "!"] \arrow[d, tail] & \mathbf{1} \arrow[d, "\top"] \\ \Omega\times\Omega \arrow[r, "\vee"'] & \Omega, \end{tikzcd} \tag{269}$$

where U is the image defined by the epi-mono-factorisation given by the diagram:

$$\begin{tikzcd} \Omega+\Omega \arrow[rr, dotted, "\langle\langle\top,\mathrm{id}\rangle,\langle\mathrm{id},\top\rangle\rangle"] \arrow[dr, two heads] & & \Omega\times\Omega \\ & U. \arrow[ur, tail] & \end{tikzcd} \tag{270}$$

Implication is defined by the pullback diagram

$$\begin{tikzcd} \leq \arrow[r, dotted, "!"] \arrow[d, tail] & \mathbf{1} \arrow[d, "\top"] \\ \Omega\times\Omega \arrow[r, "\Rightarrow"'] & \Omega, \end{tikzcd} \tag{271}$$

where $\leq$ is the partial order object defined by the equaliser

$$\leq \overset{e}{\rightarrowtail} \Omega\times\Omega \underset{\pi_1}{\overset{\wedge}{\rightrightarrows}} \Omega, \tag{272}$$

with π_1 denoting a canonical projection on the first component of the product. The negation is defined by the pullback diagram

$$\begin{tikzcd} \mathbf{1} \arrow[r, dotted, "!"] \arrow[d, tail, "\bot"'] & \mathbf{1} \arrow[d, tail, "\top"] \\ \Omega \arrow[r, "\neg"'] & \Omega, \end{tikzcd} \tag{273}$$

where the *false arrow* $\bot\colon \mathbf{1} \rightarrowtail \Omega$ is defined by the pullback diagram

$$\begin{tikzcd} \mathbf{0} \arrow[r, "!"] \arrow[d] & \mathbf{1} \arrow[d, tail, "\top"] \\ \mathbf{1} \arrow[r, tail, "\bot"'] & \Omega. \end{tikzcd} \tag{274}$$

By cartesian closedness of a topos, the map $\Omega\times\Omega \Rightarrow \Omega$ uniquely defines an exponential Ω^Ω. So, *if* $\langle\Omega,\wedge,\vee,\top,\bot\rangle$ is a lattice, then it is an internal Heyting algebra. In order to show that it is a lattice, it is sufficient to show that it is a poset, with the arrows of this poset given by the equivalence classes of monic arrows of subfunctors (subsieves) of Ω. We leave this task for the reader. Alternatively, one can also check that all equational definitions of a lattice, when stated in the form of diagrams, are satisfied. □

In the above proof we have introduced a general notion of a false arrow.

Definition 9.6 *A **false arrow** in any topos is a unique arrow $\perp\colon \mathbf{1} \rightarrowtail \Omega$ which is a characteristic map of an initial object.*

In other words, a false arrow is given by the condition that there is a pullback diagram

$$\begin{tikzcd} \mathbf{0} \arrow[r, "!"] \arrow[d] & \mathbf{1} \arrow[d, "\top"] \\ \mathbf{1} \arrow[r, "\perp"'] & \Omega. \end{tikzcd} \tag{275}$$

Note that this is a straightforward generalisation of the boolean false arrow $\perp\colon \{0\} \hookrightarrow 2$. As a byproduct of this definition, we get the notion of a *boolean topos*.

Definition 9.7 *An **internal boolean algebra** or **boolean object** is an internal Heyting algebra L such that $\neg\neg = \mathrm{id}_L$. A **boolean topos** is topos whose subobject classifier Ω is a boolean object.*

Examples

1. **Set** is a boolean topos.
2. $\mathbf{Set}^{\mathbf{2}^{op}}$ is not a boolean topos.

9.3 Language of a topos

This way we have established topos-theoretic interpretation of symbols of logical connectives $\wedge, \vee, \Rightarrow$ and $\neg$. In other words, we have specified their semantics in topos. The interpretation of a sentence α of a formal language $\mathcal{L}$ in the semantics of a topos $\mathcal{E}$ is denoted as

$$\mathcal{E} \models \alpha. \tag{276}$$

But the knowledge of topos semantics for logical connectives is not sufficient for most of our purposes. In order to deal with sentences such as

$$(x \in A) \wedge \neg(x \in B), \tag{277}$$

we need to know how to incorporate set-theoretic construction into the statements of the formal language, and we also need to know its topos semantics. The crucial problem is to formalise and provide semantics for the sign '$\in$'. Recall that in the framework of topos theory we have a possibility to consider not only the global assertions (on the state $\mathbf{1}$ of terminal object), such as

$$(x \in_{\mathbf{1}} A) \wedge \neg(x \in_{\mathbf{1}} B), \tag{278}$$

but also *local* ones, like, for example

$$(x \in_C A) \wedge \neg(x \in_C B). \tag{279}$$

Certainly, there is an essential difference between terms $x \in_{\mathbf{1}} A$ and $x \in_C A$. This difference can be appropriately managed using the perspective of ***type theory***. The (syntactic aspect of the) idea of type theory, as formulated by Russell, is to consider different collections of letters to denote variables of different types (or sorts). For example, $f, g, \ldots$ should be used to denote variables of function type, while $n, m, \ldots$ should be used to denote variables of natural type. In the context of topos theory, different types are specified by particular objects in topoi. Because we do not insist on the validity of a law of excluded middle, we will deal with *intuitionistic type theory*.

[*Continue about the Mitchell–Bénabou language*]

9.4 Semantics of a topos

[*Say something more about the Kripke–Joyal semantics*]

If a given topos is a Grothendieck topos, Kripke–Joyal semantics reduces to so-called sheaf semantics. The main properties of this semantics are the following:

- individual constant (element) b of type X is interpreted as a morphism $b : \mathbf{1} \to X$,
- n-argument property P of type $X_1 \times \ldots \times X_n$ is interpreted as a subobject P of $X_1 \times \ldots \times X_n$,
- n-argument function f from $X_1 \times \ldots \times X_n$ to Y is interpreted as a morphism $f : X_1 \times \ldots \times X_n \to Y$.

Moreover,

- a term $t(x_1, \ldots, x_n)$ with free variables among $x_1, \ldots, x_n$ ranging over $X_1, \ldots, X_n$ is interpreted as a morphism from object $X_1 \times \ldots \times X_n$ to an appropriate codomain.

In the case of a topos of presheaves $\mathbf{Set}^{\mathcal{C}^{op}}$ this semantics takes more explicit form. The term $t(x_1, \ldots, x_n)$ is specified on a *stage* $C \in \mathcal{C}$ as

$$t_C(a_1, \ldots, a_n) := t(x_1, \ldots, x_n)[a_1 \in X_1(C), \ldots, a_n \in X_n(C)], \tag{280}$$

while the formula $\phi(x_1, \ldots, x_n)$ is specified on a stage $C \in \mathcal{C}$ as

$$C \Vdash \phi(a_1, \ldots, a_n) \iff \vdash \phi(x_1, \ldots, x_n)[a_1 \in X_1(C), \ldots, a_n \in X_n(C)]. \tag{281}$$

In general, the following facts hold [78]:

- For every topos $\mathcal{T}$ there is an associated type theory $\mathcal{L}(\mathcal{T})$ which is the ***internal language*** of $\mathcal{T}$.
- For every type theory $\mathcal{L}$ there is an associated topos $\mathcal{T}(\mathcal{L})$ ***generated*** by $\mathcal{L}$.

Hence, any given type theory $\mathcal{L}$ can be expressed in some topos $\mathcal{T}(\mathcal{L})$, and for a given topos $\mathcal{T}$ we may express the internal operations on its arrows and objects in the associated type theory $\mathcal{L}(\mathcal{T})$, namely in the internal language of $\mathcal{T}$. In the previous section we introduced, more or less implicitly, the elements of both approaches, by identifying arrows with *elements*, and expressing the propositional intuitionistic logic in the internal language of a topos. Generally, we can take some type theory and model (express) it in some topos, or alternatively take some topos and translate its internal structure to some corresponding type theory, which is its internal language.[11] This correspondence between $\mathcal{T}$ and $\mathcal{L}$ implies that given any topos $\mathcal{T}$ there is a language $\mathcal{L}(\mathcal{T})$ such that if we were to take some sentence $\varphi(x)$, where x is a variable, then there exists a unique arrow $f : A \to \Omega$ in $\mathcal{T}$ such that $\varphi(x) \equiv f(x)$. For example, if $a : C \to A$ is some generalised element, then $\varphi(a) \equiv f(a)$. We say that $\varphi(a)$ ***holds at stage*** C or – equivalently – that C ***forces*** $\varphi(a)$, and denote it as

$$C \Vdash \varphi(a), \tag{282}$$

if

$$f(a) = \top \circ !(C) = \top_C. \tag{283}$$

This can be written also as

$$\vdash_A \varphi(x) \iff A \Vdash \varphi(a) \text{ and } A = \mathrm{dom}(a), \tag{284}$$

where $\vdash_A \varphi(x)$ is a local assertion of formula $\varphi(x)$ which holds at the stage A. Due to the correspondence between type theory and the internal language of a topos, we can consider the generalisation of set theory

[11] Strictly speaking, the interpretation of a intuitionistic type theory in topos is not *exactly* the same as the internal intuitionistic language of a topos, but there are strict correspondences (adjoint functors) between them. See [78] for a comprehensive treatment of this theme.

called ***local set theory*** (build in a formal language called a ***local language***), in which set-theoretic operations can be performed only on such sets and elements, which are of the same *type* [9]. This corresponds to speaking about validity of sentences *at the same stage*.

Indeed, we have seen that in the $\mathbf{Set}^{\mathcal{C}^{op}}$ there can be situations such that given $B \subseteq A$

$$\mathbf{Set}^{\mathcal{C}^{op}} \models (x \in_C A \wedge \neg x \in_C B) \iff C \Vdash (x \in A \wedge x \notin B), \tag{285}$$

$$\mathbf{Set}^{\mathcal{C}^{op}} \models (x \in_D A \wedge x \in_D B) \iff D \Vdash (x \in A \wedge x \in B). \tag{286}$$

It follows that for a given $A \in \mathrm{Ob}(\mathcal{E})$ we may fix some stage B and speak about all arrows $B \xrightarrow{x} A$ and subobjects $\mathrm{Sub}(A)$ just as if they were elements of sets and subsets of a set-theory. There is however one restriction: we are working still within the intuitionistic logic (fixing the stage does not change the fact that $(\mathrm{Sub}(A), \subseteq)$ has the structure of a Heyting algebra). It means that all reasoning in our 'naïve set theory on a fixed stage' has to be performed constructively, without using the double negation or excluded middle laws. Thus, on any fixed stage we can *speak* about elements and subobjects of any object A in some topos $\mathcal{E}$ using the naïve intuitionistic set theory (hence, using its natural Heyting algebra of logic).

For more complete discussion of the lingustic and logical aspects of topoi see [78], [111], [9], [47], and [59].

10 Geometry in a topos

The geometrical aspects of a topos, contrary to categorical or even logical ones, are hard to access without some initial knowledge. Providing self-contained discussion of these issues would lead to an uncontrolled growth of this introduction. Hence, we will just mention here the basic line along which the geometric pespective of a topos is emerging.

10.1 The Grothendieck topos $\mathbf{Sh}(\mathcal{C})$ of sheaves over category $\mathcal{C}$

There might be many different algebraic structures varying over topological space, like rings, modules, groups, etc. The notion of presheaf and sheaf are intended to describe the collections of algebraic structures which are varying over base space in well-behaved way. The adjective 'well-behaved' means that a certain kind of consistency condition must hold between the structure of the base space and the varying structure.

Definition 10.1 *A* ***presheaf of structures*** *over a topological space X is a contravariant functor F from the category* $\mathbf{Open}(X)$ *of open subsets of topological spaces with arrows given by inclusion to the category* $\mathbf{Struct}$ *of (some) algebraic structures and their structure-preserving morphisms.*

Definition 10.2 *A presheaf over topological space X is called a* ***sheaf*** *if for each open cover $U = \bigcup_{i \in I} U_i$ of $U \in \mathrm{Ob}(\mathbf{Open}(X))$ and each collection of elements $\{f_i \in F(U_i)\}$ such that*

$$\forall i, j \in I \quad f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j} \tag{287}$$

there exists a unique element $f \in F(U)$ such that $f|_{U_i} = f_i$ for all $i \in I$.

The notion of a sheaf and presheaf can be linked with each other through the notion of a stalk, which can be considered as a collection of germs of a presheaf.

Definition 10.3 *Let $x \in X$. The stalk F_x of a presheaf F at x is the direct limit of the structure $F(U)$ over all open neighbourhoods U of x:*

$$F_x := \varinjlim_{x \in U} F(U) = \left\{ \left(\bigcup_U F(U) \right) / \sim \mid x \in U,\ U_i \ni f_i \sim g_j \in U_j \iff \left(\exists \widetilde{U} \subset U_i \cap U_j \mid x \in \widetilde{U}, f_i|_{\widetilde{U}} = g_j|_{\widetilde{U}} \right) \right\}. \tag{288}$$

Using stalks, one can show that for a given presheaf F there exists a unique corresponding sheaf $\widetilde{F}$, defined by a morphism of presheaves $s : F \to \widetilde{F}$ such that $s : F_x \to \widetilde{F}_x$ is an isomorphism for every $x \in X$.

Now, the topos $\mathbf{Set}^{\mathcal{C}^{op}}$ is a category of presheaves of sets over any arbitrary *category*. Hence, it is a generalisation of the notion of a category of presheaves over topological spaces. The ***Grothendieck topos*** is a category of presheaves of sets over any arbitrary category, equipped with a ***Grothendieck topology***. The latter notion is a very general notion of topology, which allows us to 'topologise' any category. A category $\mathcal{C}$ equipped with a Grothendieck topology J is called a ***site*** and denoted $(\mathcal{C}, J)$. Hence, a Grothendieck topos is a category of contravariant functors from a given site $(\mathcal{C}, J)$ to the category **Set**. It can be shown that every Grothendieck topos is an elementary topos. For extensive discussion of Grothendieck toposes and Grothendieck topology read [5] and [107]. Some very interesting applications are presented in [121].

[*Define Grothendieck topologies and Grothendieck toposes, introduce Giraud axioms, discuss sheafification and relationships with the Lawvere–Tierney topology.***]**

10.2 From infinitesimals to microlinear spaces

[*Say some general things about SDG/SIA and its relationship to post-Cauchy-Weierstraß analysis.***]**

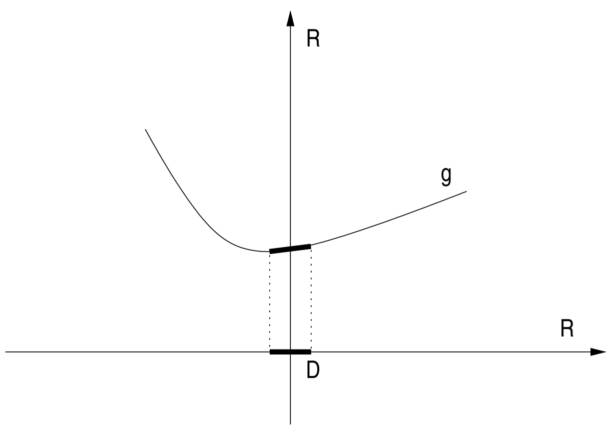
The infinitesimal analysis (and differential geometry) can be developed as an algebraic system based on an assumption that the structure of the real line may be modeled by such commutative unital ring R, that there exists the object of nilpotent infinitesimals

$$D := \{x \in R | x^2 = 0\} \subset R, \tag{289}$$

and $D \neq \{0\}$. This implies that R cannot be considered to be equal to the set-theoretic field $\mathbb{R}$. Condition $D \neq \{0\}$ forces the existence of some *infinitesimal* $x \in R$ that is not equal to zero, but is 'such small' that $x^2 = 0$. The *Kock–Lawvere axiom*

$$\forall g : D \to R \ \exists! b : D \to R \ \forall d \in D \quad g(d) = g(0) + d \cdot b \tag{290}$$

imposed on the structure of this ring says that every function on R is differentiable. On the plane $R \times R$ this axiom means that the graph of g *coincides* on D with a straight line with the slope $g'(0) := b$ going through $(0, g(0))$,



where the line tangent to g at 0 is tangent on an infinitesimal part of domain $D \subset R$ (and *not* in the sense of a limit in a point!). This implies that every function is infinitesimally linear. However, the Kock–Lawvere axiom is not compatible with the law of excluded middle. Let's define a function $g : D \to R$ such that

$$\begin{cases} g(d) = 1 & \text{iff} \quad d \neq 0, \\ g(d) = 0 & \text{iff} \quad d = 0. \end{cases} \tag{291}$$

The Kock–Lawvere axiom implies that $D \neq \{0\}$, because otherwise b would be not unique. So (using the law of excluded middle) we may assume that there exists such $d_0 \in D$ that $d_0 \neq 0$. From the Kock–Lawvere axiom we have immediately $1 = g(d_0) = 0 + d_0 \cdot b$. After squaring both sides we receive $1 = 0$.

This result means that we cannot make anything meaningful based on Kock–Lawvere axiom using the classical logic in which the law of excluded middle holds. However, we are not restricted to such logic, and we can use the weaker logic which does not use this law. After the substraction of this law from the set of axioms of classical logic, we obtain the set of axioms of intuitionistic logic. The usage of the latter is very similar to the usage of classical logic. The only difference is the need of performing all proofs in the constructive way, without assuming the existence of objects which cannot be explicitly constructed, hence without the axiom of choice. Of course, the post-Weierstrassian set-theoretic line $\mathbb{R}$ is not a good model of R. However, we will show later that there are good models of R which reexpress all constructive contents of post-Weierstrasian analysis.

We call a property P of an object A *decidable* if $\vdash (P(x) \vee \neg P(x))$ for every $x \in A$ (the $\vdash$ sign reads as 'is satisfied'). The Kock–Lawvere axiom implies that equality in R is not decidable:

$$\exists x, y \in R \quad \nvdash (x = y) \vee (x \neq y) \tag{292}$$

(we will see later that R is not decidable not only for the property of equality of elements, but also for other properties, such as the ordering). Clearly, this non-decidability is introduced by the subobject D of the infinitesimal elements. The non-decidability of infinitesimal objects can be interpreted as their 'non-observability' (in the Boolean frames)[12]. Infinitesimals appear then with a perfect agreement with Leibniz' idea of "auxiliary variables" [?]. But this leads to an important conclusion: we cannot think about the real line R as consisting of equally 'observable' points laying infinitesimally close to each other (also because so far we have not defined any ordering structure). Only *some* elements of space are measurable and decidable, and only these elements can be "pointed" while one considers some kind of movement (this is very important issue for dissolving Zeno paradoxes). This means also that the differentiability and smoothness of functions and curves on R actually relies on 'unobservable' elements of R.

To enhance differentiability to any order of Taylor series, we have to consider also an object

$$D_n := \{x \in R | x^{n+1} = 0\} \subset R. \tag{293}$$

I would be good if sums and multiplications of infinitesimals would be infinitesimal too. However, D is not an ideal of R (e.g. $(d_1 + d_2)^2 = 2d_1 d_2 \neq 0$). Hence, in order to hold infinitesimality, we have to consider more wide class of infinitesimal objects, such that appropriate polynomials of infinitesimal elements are forced to cease. These are so-called *spaces of formal infinitesimals* (called also *nilpotent objects*, *infinitesimal affine schemes*, or just *infinitesimal spaces*), defined as spectra of Weil algebras,

$$D(W) := Spec_R(W) = \{(d_1, \ldots, d_n) \in R^n | p_1(d_1, \ldots, d_n) = \ldots = p_m(d_1, \ldots, d_n) = 0\}, \tag{294}$$

where W is a Weil R-algebra with a finite presentation in terms of an R-algebra with n generators divided by the ideal generated by the polynomials $p_1, \ldots, p_m$:

$$R[X_1, \ldots, X_n]/(p_1(X_1, \ldots, X_n), \ldots, p_m(X_1, \ldots, X_n)).$$

For example, one can consider

$$D := Spec_R(R[X]/(X^2)) = \{d \in R | d^2 = 0\},$$

$$Spec_R(R[X, Y]/(X^2 + Y^2 - 1)) = \{(x, y) \in R^2 | x^2 + y^2 - 1 = 0\}.$$

The structure $R[X]/(X^2)$ appears very naturally if one defines the multiplication in $R \times R$ by the rule $(a_1, b_1)(a_2, b_2) := (a_1 a_2, a_1 b_2 + a_2 b_1)$. In such case the Kock-Lawvere axiom (290) can be stated as requirement that the map $\alpha : R[X]/(X^2) \to R^D$ such that $\alpha : (a, b) \mapsto [d \mapsto a + db]$ should be an

[12] We will show below that the non-observability of infinitesimals is strenghtened by the generalized Kock–Lawvere axiom and the way of construction of the microlinear (differentiable) spaces.

isomorphism. Noticing that $D = Spec_R(R[X]/(X^2))$, one can use the formal infinitesimals for the generalization of the Kock–Lawvere axiom into form: *For any Weil algebra W the R-algebra homomorphism $\alpha : W \to R^{D(W)}$ is an isomorphism.* This generalization allows to solve the problem that the result of addition or multiplication of infinitesimals is not an infinitesimal, equivalent with the fact that (formal) infinitesimals do not form an ideal of R. Consider the addition $d_1 + d_2$ and the multiplication $d_1 \cdot d_2$. They can be formulated in categorical terms as commutative diagrams

$$\begin{tikzcd} D \times D \arrow[r, shift left, "id"] \arrow[r, shift right, "r"'] & D \times D \arrow[r, "+"] & D_2 \end{tikzcd} \tag{295}$$

and

$$\begin{tikzcd} D \times D \arrow[r, shift left, "id"] \arrow[r, shift right, "r"'] & D \times D \arrow[r, "\cdot"] & D, \end{tikzcd} \tag{296}$$

where $r(d_1, d_2) = (d_2, d_1)$. We can say, that functions 'perceive' the multiplication and addition of infinitesimal as surjective, if the diagrams

$$\begin{tikzcd} R^{D\times D} & R^{D\times D} \arrow[l, shift right, "R^r"'] \arrow[l, shift left, "R^{id}"] & R^{D_2} \arrow[l, "R^+"] \end{tikzcd} \tag{297}$$

and

$$\begin{tikzcd} R^{D\times D} & R^{D\times D} \arrow[l, shift right, "R^r"'] \arrow[l, shift left, "R^{id}"] & R^{D} \arrow[l, "R^\cdot"] \end{tikzcd} \tag{298}$$

are commuting equalizer diagrams. However, this is true thanks to the generalized Kock-Lawvere axiom and the fact that the diagrams of Weil algebras which generate (295) and (296) are commuting equalizer limit diagrams. So, despite (295) and (296) *are not* coequalizer diagrams, they are 'perceived' to be such by all functions on R. This is a basic example of an operation on infinitesimals which is 'thought to be surjective' by functions on R. Another operations are given by another *limit* diagrams of Weil algebras. In general, the generalized Kock-Lawevere axiom forces that functions which work on infinitesimals 'perceive' their multiplication, addition and other operations as surjective, because it establishes that for the limit diagrams of Weil algebras

$$\begin{tikzcd} & \text{Lim } W \arrow[dl, "\mu_i"'] \arrow[dr, "\mu_j"] & \\ W_i \arrow[rr, "f"'] & & W_j, \end{tikzcd} \tag{299}$$

the diagram

$$\begin{tikzcd} & R^{D(\text{Lim } W)} \arrow[dl, "R^{D(\mu_i)}"'] \arrow[dr, "R^{D(\mu_j)}"] & \\ R^{D(W_i)} \arrow[rr, "R^{D(f)}"'] & & R^{D(W_j)}, \end{tikzcd} \tag{300}$$

is also a limit, despite that the diagram

$$\begin{tikzcd} & D(\text{Lim } W) & \\ D(W_i) \arrow[ur, "D(\mu_i)"] & & D(W_j). \arrow[ul, "D(\mu_j)"'] \arrow[ll, "D(f)"'] \end{tikzcd} \tag{301}$$

is not a colimit. Here we symbolically write $f : W_i \to W_j$ to denote any diagram which can be a base for some limit cone of Weil algebras $\{W_i\}_{i\in I}$, with Lim W given by projective limit $\varprojlim_{i\in I} W_i$. This way all algebra of infinitesimal elements works 'invisible' (and 'unobservable'!) from the point of view of ordinary functions, which are differentiable thanks to these infinitesimals. In such case R is called to 'perceive (301) as colimit', and (301) is called a quasi-colimit. The procedure of proving that concrete quasi-colimit diagrams of formal infinitesimal objects are perceived as colimits is a basic type of proof in infinitesimal geometry, and it plays here the same role as the procedure of proving that the higher-order terms in standard differential geometry are becoming negligible. However, in the case of infinitesimal geometry all operations are performed purely geometrically and categorically, without any consideration of the infinitesimal limit (in an analytic sense) or a system of local coordinates.

We can consider now the formal infinitesimals $D(W)$ as an image of the covariant functor $D = Spec_R$ from the category $\mathcal{W}$ of Weil finitely generated R-algebras to the cartesian closed category $\mathcal{E}$ of intuitionistic sets, which contains models of such objects like R, $D(W)$, etc. Moreover, we have the functor $R^{(-)}$: $\mathcal{E} \to \mathcal{E}$. The generalized Kock–Lawvere axiom says that the covariant composition of functors

$$\mathcal{W}^{op} \xrightarrow{Spec_R} \mathcal{E} \xrightarrow{R^{(-)}} \mathcal{E}$$

sends limit diagrams in $\mathcal{W}$ to limit diagrams in $\mathcal{E}$. This axiom allows to define the notion of infinitesimally differentiable (microlinear) manifold, without use of topology or coordinates. Let $\mathcal{D}$ be a finite inverse diagram (cocone) of infinitesimal spaces which is an image of a functor $D = Spec_R$ applied to some finite diagram in the category of Weil algebras, and is send by $R^{(-)}$ to a limit diagram in a cartesian closed category $\mathcal{E}$. An object M of $\mathcal{E}$ is called the *microlinear space* if the functor $M^{(-)}$ sends every $\mathcal{D}$ in $\mathcal{E}$ into a limit diagram. In such case M is said to *perceive* $\mathcal{D}$ as a colimit diagram, and $\mathcal{D}$ is called a *quasi-colimit*. Hence, if M is a microlinear space and X is any object, then M^X is microlinear. Any finite limit of microlinear objects is microlinear. R and its finite limits as well as its exponentials are microlinear. And any infinitesimal space is microlinear too.

The notion of microlinear space encodes the full *differential* content of the post-Weierstrassian notion of 'differential manifold'. It clearly needs no topology, neither the local covering and coordinates, in order to construct and handle the differential geometric objects. Later we will see how one can add topological structure to microlinear spaces (what will result in the definition of a *formal manifold*). This separation of topological and arithmetical constructions from algebraic, geometric and differential ones is the key and striking achievement of infinitesimal analysis.

10.3 Well-adapted topos models of differential geometry and smooth analysis

In general, we may try to interpret the above system (called *synthetic differential geometry* (SDG) or *smooth infinitesimal analysis* (SIA)) in some topos, particularly in some topos of sheaves over a given site, thanks to the inner Heyting algebra structure of the subobject classifier of an elementary topos. This means that instead of the system of classical differential calculus and geometry based on the concept of limits and interpretation of this system in set theory, we can use a system of synthetic differential calculus and geometry based on the concept of infinitesimals and interpretation of this system in topos theory.

Every interpretation of axioms of SDG/SIA in a particular category is called a *model*. By the obvious reasons, we are at most interested in such models of SDG/SIA which allow to establish the link between the 'classical' analytic post-Weierstassian differential calculus and geometry with the synthetic one. The structure of SDG/SIA implies that we have to work in complete cartesian closed category, but the fact, that we have to interpret the intuitionistic logic of statements somehow 'naturally' inside this category, leads us to the assumption that we will work in a topos, which is complete and cocomplete cartesian closed category with the subobject classifier.

One of the simplest models of SIA/SDG is the topos $\mathbf{Set}^{\mathbb{R}\text{-}\mathbf{Alg}}$ of set-valued functors from the category $\mathbb{R}$-**Alg** of (finitely presented) $\mathbb{R}$-algebras to the category **Set** of sets[13]. Each such functor is a forgetful functor, which associates to an $\mathbb{R}$-algebra the set of its elements, and to every homomorphism f of $\mathbb{R}$-algebras the same f as function on sets. We induce commutative unital ring structure on functors $R \in \mathbf{Set}^{\mathbb{R}\text{-}\mathbf{Alg}}$ in the following natural way: for every $A \in \mathbb{R}\text{-}\mathbf{Alg}$ we consider a ring $R(A)$, together with operations of addition $+_A : R(A) \times R(A) \to R(A)$ and multiplication $\cdot_A : R(A) \times R(A) \to R(A)$, which are natural in the sense, that they are preserved by the homomorphisms in $\mathbb{R}$-**Alg**, thus also by the corresponding functors $\mathbb{R}\text{-}\mathbf{Alg} \to \mathbf{Set}$. The functor R is a model of a synthetic real line R, while an

[13]We consider $\mathbb{R}$-algebras based on $\mathbb{R} \in \mathbf{Set}$, but we could consider also $\mathbb{R}_C$- or $\mathbb{R}_D$-algebras build from Cauchy or Dedekind reals of some topos. We will assume also that all algebras and rings considered in this section are finitely presented.

object $D \subset R$ has the following interpretation in $\mathbf{Set}^{\mathbf{R\text{-}Alg}}$:

$$\mathbf{R\text{-}Alg} \ni A \overset{D}{\longmapsto} D(A) = \{a \in A | a^2 = 0\}. \tag{302}$$

The functorial construction of models may look quite esotherical at first sight, but in fact it strictly expresses the difference and the link between our *concepts* and their *models*. One should note that our concepts are formulated in abstract and 'background-free' way: as some relations between objects and elements. For example, the *concept* (an algebraic *locus*) of a sphere S^2 is

$$S^2 = \{(x, y, z) | x^2 + y^2 + z^2 = 1\}. \tag{303}$$

We may now take different backgrounds to express S^2, for example, by saying that elements of S^2 should belong to some commutative $\mathbb{R}$-algebra (to some object in the category $\mathbb{R}$-**Alg**). To 'see' somehow 'naturally' how such sphere S^2, expressed in terms of $\mathbb{R}$-**Alg**, 'looks like' we use the set-theoretical 'eyes' or 'screen'. This leads us to demand that S^2 should give as an output the *set* of triples of elements of $A \in \mathbf{R\text{-}Alg}$ which satisfy the 'conditions' given in the definition of S^2. So, the *interpretation* (*model*) of the concept (locus) S^2 is a set-valued *functor* $\mathbf{Set}^{\mathbb{R}\text{-}\mathbf{Alg}}$:

$$\mathbb{R}\text{-}\mathbf{Alg} \ni A \overset{S^2}{\longmapsto} S^2(A) = \{(x, y, z) \in A^3 | x^2 + y^2 + z^2 = 1\} \in \mathbf{Set}, \tag{304}$$

which means that S^2 is modelled by the functor which takes these elements from the ring A which fit the pattern $x^2+y^2+z^2 = 1$, and produces a set which contains them. Recall that the global elements of $R(A)$ are the arrows $\mathbf{1} \to R(A)$. The $\mathbb{R}$-algebra corresponding to $\mathbf{1} \cong \{*\}$ is the $\mathbb{R}$-algebra with one generator $\mathbb{R}[X]$, while the arrow corresponding to $\mathbf{1} \xrightarrow{\ulcorner x \urcorner} R(A)$ is an $\mathbb{R}$-algebra homomorphism $\mathbb{R}[X] \xrightarrow{\phi_x} A$. This means that

$$R \cong Hom_{\mathbb{R}\text{-}\mathbf{Alg}}(\mathbb{R}[X], -), \tag{305}$$

is a representable functor:

$$R(A) \cong Hom_{\mathbb{R}\text{-}\mathbf{Alg}}(\mathbb{R}[X], A). \tag{306}$$

By the Yoneda Lemma

$$\mathrm{Hom}(R, R) \cong \mathrm{Nat}(\mathrm{Hom}(\mathbb{R}[X], -), \mathrm{Hom}(\mathbb{R}[X], -)) \cong \mathrm{Hom}(\mathbb{R}[X], \mathbb{R}[X]), \tag{307}$$

so the maps $f : R \to R$ on the ring R (from the synthetic point of view) are the maps of polynomials with coefficients in $\mathbb{R}$ (from the interpretational point of view). It can be shown (see [62] for details), that $\mathbf{Set}^{\mathbb{R}\text{-}\mathbf{Alg}}$ satisfies the generalized Kock-Lawvere axiom (and weak version of integration axiom), but it does not satisfy other axioms. Thus there is a need to consider different models (universes of interpretation).

Note that $\mathbb{R}$-**Alg** of $\mathbb{R}$-algebras is defined as a category of arrows $f_A : \mathbb{R} \to A$, where commutative rings A are such that $xy = yx$ for every $x \in f_A(\mathbb{R})$ and for every $y \in \mathbb{R}$. Hence, we may consider the category $\mathbb{R}$-**Alg** as the category of rings A equipped with the additional structure given by the maps $A^n \xrightarrow{f_A(p)} A^m$ preserving the structure of polynomials $p = (p_1, \ldots, p_n) : \mathbb{R}^n \to \mathbb{R}^m$ in such way that identities, projections and compositions are preserved: $f_A(\mathrm{id}) = \mathrm{id}$, $f_A(\pi) = \pi$ and $f_A(p \circ q) = f_A(p) \circ f_A(q)$. This means that construction of $\mathbb{R}$-algebras and C^∞-algebras is similar.

There should be many algebraic theories $\mathcal{A}$ intermediate between only polynomials as operations and all C^∞ functions as operations, pehaps satisfying some suitable closure conditions, in particular the $\mathcal{A}$ generated by cos, sin, exp, e^{-1/x^2}. [?]

It can be shown, using the Hadamard lemma, that C^∞-ring divided by an ideal is C^∞-ring.

Another important example of C^∞-ring is a ring of *germs* of smooth functions.

Definition 10.4 *A **germ** at $x \in \mathbb{R}^n$ is an equivalence class of such $\mathbb{R}$-valued functions which coincide on some open neighbourhood U of x, and is denoted as $f|_x$ for some $f : U \to \mathbb{R}$. We denote a ring of germs at x as $C^\infty_x(\mathbb{R}^n)$. If I is an ideal, then $I|_x$ is the object of germs at x of elements of I.*

Of course, $C^\infty_x(\mathbb{R}^n)$ is a C^∞-ring and $I|_x$ is an ideal of $C^\infty_x(\mathbb{R}^n)$. The ***object of zeros*** $Z(I)$ of an ideal I is defined as

$$Z(I) = \{x \in \mathbb{R}^n | \forall f \in I \quad f(x) = 0\}. \tag{308}$$

We may introduce the notion of ***germ-determined ideal*** as such I that

$$\forall f \in C^\infty(\mathbb{R}^n)\ \forall x \in Z(I) \quad f|_x \in I|_x \Rightarrow f \in I. \tag{309}$$

The *dual* to the full subcategory of (finitely generated) C^∞-rings whose objects are of form $C^\infty(\mathbb{R}^n)/I$ such that I is germ-determined ideal is denoted by $\mathbf{G}$ (we take the *dual* category, because we want to make a topos of presheaves $\mathbf{Set}^{\mathbf{G}^{op}}$, where sets will be varying on the (finitely generated) C^∞-rings and not on their duals). Recall that for $\mathbb{R}$-algebras we used the functor

$$\mathbb{R}\text{-}\mathbf{Alg} \ni A \longmapsto R(A) \in \mathbf{Set}, \tag{310}$$

as the model (interpretation) of the naive-SDG ring R in the topos $\mathbf{Set}^{\mathbb{R}\text{-}\mathbf{Alg}}$. In the same way we may define the intepretation of the ring R in the topos $\mathbf{Set}^{\mathbf{G}^{op}}$:

$$\mathbf{G}^{op} \ni A \longmapsto R(A) \in \mathbf{Set}. \tag{311}$$

The topos $\mathbf{Set}^{\mathbf{G}^{op}}$ of presheaves over the category of germ-determined C^∞-rings equipped with the Grothendieck topology is called the ***Dubuc topos***, and is denoted by $\mathcal{G}$.[14] This topos is not only very good well-adapted model of SDG, but it also has a good representation of classical paracompact C^∞-manifolds.

For some purposes we can also use the larger topos $\mathbf{Set}^{\mathbf{L}^{op}} := \mathbf{Set}^{C^\infty}$. It does not have the interpretation for an axiom R2 of local ring and an axiom N3 of Archimedean ring, but it is a good toy-model, easier to concern than $\mathcal{G}$ is. Note that the equation (305):

$$R(A) \cong \mathbb{R}\text{-}\mathbf{Alg}(\mathbb{R}[X], A) \tag{312}$$

has an analogue in case of the intepretation of SDG in topos $\mathbf{Set}^{\mathbf{L}^{op}}$:

$$R(\ell A) \cong \mathbf{Set}^{\mathbf{L}^{op}}(\ell A, \ell C^\infty(\mathbb{R})), \tag{313}$$

where $\ell C^\infty(\mathbb{R})$ is the C^∞-ring (the symbol ℓ denotes here the fact, that we are working within the category which is dual to that of C^∞ rings). Thus, a real line of an axiomatic SDG becomes now

$$R \cong \mathrm{Hom}_{\mathbf{Set}^{\mathbf{L}^{op}}}(-, \ell C^\infty(\mathbb{R})), \tag{314}$$

or, using the formal logical sign which denotes interpretation in the model,

$$\mathbf{Set}^{\mathbf{L}^{op}} \models R \cong \mathrm{Hom}_{\mathbf{Set}^{\mathbf{L}^{op}}}(-, \ell C^\infty(\mathbb{R})). \tag{315}$$

This means that the element of ring R, the real number of naive intuitionistic set theory, is some morphism $\ell A \to \ell C^\infty(\mathbb{R})$. We say that we have a ***real at stage*** ℓA. Thus, our *concept* of the real line R of Synthetic Differential Geometry can be *modelled* (*interpreted*) by the different rings (stages) of smooth functions on the *classical* space $\mathbb{R}^n$ (which can be, however, defined categorically, as an n-ary product of an object $\mathbb{R}_D$ of Dedekind reals in the Boolean topos $\mathbf{Set}$). For example, at the stage $\ell A = C^\infty(\mathbb{R}^n)/I$, where I is some ideal of the ring $C^\infty(\mathbb{R}^n)$, a *real* (*real variable*, *real number*) is an equivalence class $f(x)$ mod I, where $f \in C^\infty(\mathbb{R}^n)$. An interpretation of the most important (naive) objects of SDG is following ([121]):

smooth real line	$R = Y(\ell C^\infty(\mathbb{R})) = s(\mathbb{R})$
point	$\mathbf{1} = Y(\ell(C^\infty(\mathbb{R})/(x))) = s(\{*\}) = \{x \in R \| x = 0\}$
first-order infinitesimals	$D = Y(\ell(C^\infty(\mathbb{R})/(x^2))) = \{x \in R \| x^2 = 0\}$
k$^{\text{th}}$-order infinitesimals	$D_k = Y(\ell(C^\infty(\mathbb{R})/(x^{k+1}))) = \{x \in R \| x^{k+1} = 0\}$
infinitesimals	$\mathbb{\Delta} = Y(\ell C_0^\infty(\mathbb{R})) = \{x \in R \| \forall n \in N \quad -\frac{1}{n+1} < x < \frac{1}{n+1}\}$

[14] More precisely, the topos $\mathcal{G}$ is a subcategory of $\mathbf{Set}^{\mathbf{G}^{op}}$ obtained by sheafification, and we have an inclusion $\mathcal{G} \hookrightarrow \mathbf{Set}^{\mathbf{G}^{op}}$. The left adjoint $a : \mathbf{Set}^{\mathbf{G}^{op}} \to \mathcal{G}$ is called the ***sheafification functor***. In other words, $\mathcal{G}$ is the topos of *sheaves* on $\mathbf{G}$.

The symbol Y denotes the Yoneda functor $\mathrm{Hom}(-, \ell A) =: Y(\ell A)$, while s denotes the functor $s : \mathbf{Man}^\infty \to \mathbf{Set}^{\mathbf{L}^{op}}$, introduced in the proposition **??** (the symbol Y is often ommited, so one writes $\ell C^\infty(\mathbb{R})/I$ instead of $Y(\ell C^\infty(\mathbb{R})/I)$).

It seems that the 'heaven of total smoothness' of SDG should be somehow paid for. And indeed, it is. The simplification of a structure of geometrical theory raises the complication of its interpretation: we have to construct special toposes for intepreting SDG, going beyond set theory and the topos **Set**. However, such complication may unexpectedly become a solution of many of our problems. Particularly, the well-adapted model $\mathcal{G}$ of SDG is a topos of functors from (sheafified germ-determined duals of) C^∞-rings to **Set**, which means that we express differential geometry not in terms of points on manifold, but through such smooth functions on it, which have the same *germ*, what means that they coincide on some neigbourhood. In the Dubuc topos $\mathcal{G}$ we have the interpretation (identification):

$$\boxed{\text{the real line } R \cong \text{a functor } \ R : C^\infty \supset \mathbf{G}^{op} \longrightarrow \mathbf{Set}}$$

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