New approaches to resource theories:
relative entropic transmitters & epistemic adjointness

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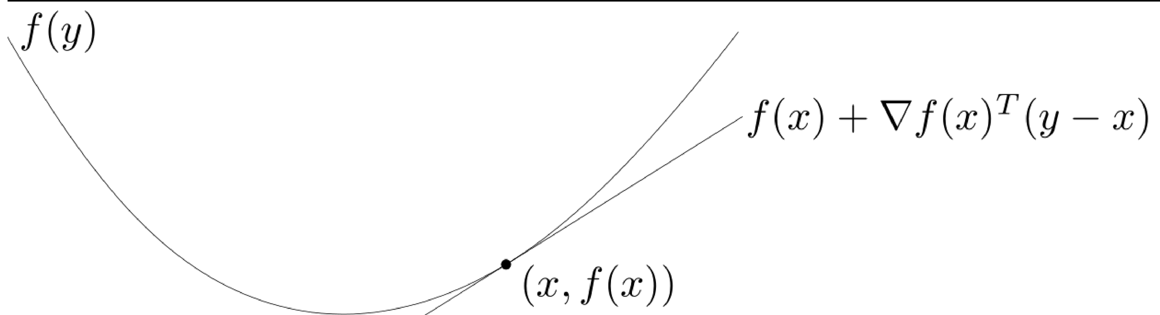
I present two new approaches to resource theories: first provides an implementation of Mielnik's idea of nonlinear transmitters, with ℓ-pullbacks of left and right Brègman strongly quasi-nonexpansive operations replacing the CPTP maps, and concrete examples constructed using duality mapping together with the Mazur (resp., Kaczmarz) map on L_p (resp., Orlicz) spaces over JBW (resp., W*) algebras; second provides a categorical framework for resource theory of epistemic knowledge, with facts (resp., actions) available to a given user encoded in terms of comonad (resp., monad). The first approach provides some special cases of the latter framework.

D_Ψ, $\overleftarrow{\mathfrak{P}}^{D_\Psi}$, $\overrightarrow{\mathfrak{P}}^{D_\Psi}$, LSQ($\Psi$), RSQ($\Psi$) [0]



Let X be a reflexive Banach space (hence, $X^{\star\star}\cong X$), let $\Psi:X\to]-\infty,\infty]$ be Legendre (:= essentially strictly convex and essentially Gateaux differentiable). Then:

1) Gateaux derivative of Ψ, $\mathfrak{D}^{\mathrm{G}}\Psi:\mathrm{int}(\mathrm{efd}(\Psi))\to\mathrm{int}(\mathrm{efd}(\Psi^{\mathbf{F}}))$, is a bijection, where: int := topological interior, $\varnothing\neq\mathrm{efd}(\Psi):=\{x\in X\mid\Psi(x)\neq\infty\}$, $\Psi^{\mathbf{F}}(y):=\sup_{x\in X}\{[\![x,y]\!]-\Psi(x)\}$, $[\![x,y]\!]:=y(x)\;\forall x,y\in X\times X^\star$.
2) ***Brègman information***, defined as
$$D_\Psi(y,x):=\Psi(y)-\Psi(x)-[\![y-x,\mathfrak{D}^{\mathrm{G}}\Psi(x)]\!]$$
for $x\in\mathrm{int}(\mathrm{efd}(\Psi))$ and ∞ otherwise, satisfies:
(i) $D_\Psi(x,y)\geq0$, with $=0$ iff $x=y$;
(ii) if $x\in\mathrm{int}(\mathrm{efd}(\Psi))$ and $\varnothing\neq Q\subseteq\mathrm{int}(\mathrm{efd}(\Psi))$ is convex closed, then $\arg\inf_{z\in Q}\{D_\Psi(z,x)\}$ is a singleton, $\{\overleftarrow{\mathfrak{P}}^{D_\Psi}_Q(x)\}$, called ***left D_Ψ-projection of x onto Q***;
(iii) ***left generalised pythagorean theorem***:
$$D_\Psi(y,x)\geq D_\Psi(y,\overleftarrow{\mathfrak{P}}^{D_\Psi}_Q(x))+D_\Psi(\overleftarrow{\mathfrak{P}}^{D_\Psi}_Q(x),x)\;\forall y\in Q,$$
with $=$ iff Q is affine, characterises $\overleftarrow{\mathfrak{P}}^{D_\Psi}_Q$;
(iv) if $\varnothing\neq K\subseteq\mathrm{int}(\mathrm{efd}(\Psi))$ is such that $\mathfrak{D}^{\mathrm{G}}\Psi(K)$ is convex closed, then $\arg\inf_{y\in K}\{D_\Psi(x,y)\}$ is a singleton, and its element, the ***right D_Ψ-projection of x onto*** K, satisfies $\overrightarrow{\mathfrak{P}}^{D_\Psi}_K(x)=\mathfrak{D}^{\mathrm{G}}\Psi^{\mathbf{F}}\circ\overleftarrow{\mathfrak{P}}^{D_{\Psi^{\mathbf{F}}}}_{\mathfrak{D}^{\mathrm{G}}\Psi(K)}\circ\mathfrak{D}^{\mathrm{G}}\Psi(x)$;
(v) ***right generalised pythagorean theorem***:
$$D_\Psi(x,y)\geq D_\Psi(x,\overrightarrow{\mathfrak{P}}^{D_\Psi}_K(x))+D_\Psi(\overrightarrow{\mathfrak{P}}^{D_\Psi}_K(x),y)\;\forall y\in K,$$
with $=$ iff $\mathfrak{D}^{\mathrm{G}}\Psi(K)$ is affine, characterises $\overrightarrow{\mathfrak{P}}^{D_\Psi}_Q$.
3) Given $\varnothing\neq M\subseteq\mathrm{int}(\mathrm{efd}(\Psi))$ and a function $T:M\to\mathrm{int}(\mathrm{efd}(\Psi))$, $\mathrm{Fix}(T):=\{x\in M\mid T(x)=x\}\neq\varnothing$, while $\widehat{\mathrm{Fix}}(T)$ is defined as a set of such $x\in M$ that there exists a sequence $\{x_n\}_{n\in\mathbb{N}}\subseteq M$ weakly convergent to x with $\lim_{n\to\infty}\|x_n-Tx_n\|_X=0$. In general, $\mathrm{Fix}(T)\subseteq\widehat{\mathrm{Fix}}(T)$.
4) We will call $T:M\to\mathrm{int}(\mathrm{efd}(\Psi))$: **(i)** ***LSQ***($\Psi$) iff $D_\Psi(x,T(y))\leq D_\Psi(x,y)\;\forall(x,y)\in\widehat{\mathrm{Fix}}(T)\times M$ and ($p\in\widehat{\mathrm{Fix}}(T)$, $\{y_n\}_{n\in\mathbb{N}}$ is bounded, $\lim_{n\to\infty}(D_\Psi(p,y_n)-D_\Psi(p,Ty_n))=0$) $\Rightarrow$ $\lim_{n\to\infty}D_\Psi(Ty_n,y_n)=0$; **(ii)** ***RSQ***($\Psi$) iff $D_\Psi(T(x),y)\leq D_\Psi(x,y)\;\forall(x,y)\in M\times\widehat{\mathrm{Fix}}(T)$ and ($p\in\widehat{\mathrm{Fix}}(T)$, $\lim_{n\to\infty}(D_\Psi(y_n,p)-D_\Psi(T(y_n),p))=0$, $\{y_n\}_{n\in\mathbb{N}}$ is bounded) $\Rightarrow$ $\lim_{n\to\infty}(y_n,T(y_n))=0$.
5) Under some conditions on Ψ (we will call such Ψ to be ***LSQ-adapted***) one has: **(i)** if $\varnothing\neq K\subseteq\mathrm{int}(\mathrm{efd}(\Psi))$ and $\{T_1,\ldots,T_n\}$ are LSQ(Ψ) functions $K\to K$ such that $\widehat{F}:=\bigcap_{i=1}^n\widehat{\mathrm{Fix}}(T_i)\neq\varnothing$ and $T:=T_n\circ\cdots\circ T_1$, then $\widehat{\mathrm{Fix}}(T)\subseteq\widehat{F}$, and if $\widehat{\mathrm{Fix}}(T)\neq\varnothing$ then T is LSQ(Ψ); **(ii)** $\overleftarrow{\mathfrak{P}}^{D_\Psi}_Q\in\mathrm{LSQ}(\Psi)$ with $\widehat{\mathrm{Fix}}(\overleftarrow{\mathfrak{P}}^{D_\Psi}_Q)=\mathrm{Fix}(\overleftarrow{\mathfrak{P}}^{D_\Psi}_Q)=Q$.
6) **(i)** Under some conditions on Ψ (we will call such Ψ ***RSQ-compositional***): if $\varnothing\neq K\subseteq X$, $\{T_1,\ldots,T_n\}$ are RSQ(Ψ) functions $K\to K$ such that $\widehat{F}:=\bigcap_{i=1}^n\widehat{\mathrm{Fix}}(T_i)\neq\varnothing$ and $T:=T_n\circ\cdots\circ T_1$, then $\widehat{\mathrm{Fix}}(T)\subseteq\widehat{F}$, and if $\widehat{\mathrm{Fix}}(T)\neq\varnothing$ then T is RSQ(Ψ); **(ii)** under some additional conditions (making Ψ to be ***RSQ-adapted***) we get $\overrightarrow{\mathfrak{P}}^{D_\Psi}_{\mathfrak{D}^{\mathrm{G}}\Psi^{\mathbf{F}}(M)}\in\mathrm{RSQ}(\Psi)$ with $\widehat{\mathrm{Fix}}(\overrightarrow{\mathfrak{P}}^{D_\Psi}_{\mathfrak{D}^{\mathrm{G}}\Psi^{\mathbf{F}}(M)})=\mathrm{Fix}(\overrightarrow{\mathfrak{P}}^{D_\Psi}_{\mathfrak{D}^{\mathrm{G}}\Psi^{\mathbf{F}}(M)})=\mathfrak{D}^{\mathrm{G}}\Psi^{\mathbf{F}}(M)$.

[0] Key works:

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2) Reich S., 1996, *A weak convergence theorem for the alternating method with Bregman distances*, in: Kartsatos A.G. (ed.), *Theory and applications of nonlinear operators of accretive and monotone type*, Dekker, New York, pp.313–318;
3) Bauschke H.H., Borwein J.M., Combettes P.L., 2001, *Essential smoothness, essential strict convexity, and Legendre functions in Banach spaces*, Commun. Contemp. Math. **3**, 615–647;
4) Sabach S., 2012, *Iterative methods for solving optimization problems*, Ph.D. thesis, Technion, Haifa.

$D_{\ell,\Psi}$ and (ℓ,Ψ)-transmitters [6, 2]

- Given $Z\subseteq\mathrm{int}(\mathrm{efd}(\Psi))$, a set U, and a bijection $\ell:U\to Z$, we define the ***Brègman ℓ-information*** on U as $D_{\ell,\Psi}(\phi,\psi):=D_\Psi(\ell(\phi),\ell(\psi))\;\forall\phi,\psi\in U$.
- For $C\subseteq U$, if $\ell(C)$ is convex (resp., affine; closed), then C will be called ***ℓ-convex*** (resp., ***ℓ-affine***; ***ℓ-closed***).
- For any ℓ-closed ℓ-convex (resp., $(\mathfrak{D}^{\mathrm{G}}\Psi\circ\ell)$-closed $(\mathfrak{D}^{\mathrm{G}}\Psi\circ\ell)$-convex) set C and any $\psi\in U$, a ***left*** (resp., ***right***) ***$D_{\ell,\Psi}$-projection*** is $\overleftarrow{\mathfrak{P}}^{D_{\ell,\Psi}}_C(\psi):=\overleftarrow{\mathfrak{P}}^{D_\Psi}_{\ell(C)}(\ell(\psi))$ (resp., $\overrightarrow{\mathfrak{P}}^{D_{\ell,\Psi}}_C(\psi):=\overrightarrow{\mathfrak{P}}^{D_\Psi}_{\ell(C)}(\ell(\psi))$).
- For $\varnothing\neq W\subseteq U$ and $T:\ell(W)\to Z$, $T^\ell:\ell^{-1}\circ T\circ\ell:W\to U$ will be called an ***ℓ-operation*** (or an ***ℓ-transmitter***).
- This gives classes LSQ(ℓ,Ψ) and RSQ(ℓ,Ψ) of ℓ-operations. Few other classes are also available. We will denote $\widehat{\mathrm{Fix}}(T^\ell):=\ell^{-1}(\widehat{\mathrm{Fix}}(T))$.

Resource theories for (ℓ,Ψ)-transmitters [1]

Given a set U (of states), a ***resource theory of states*** is a triple (P,S,R), where P is a submonoid of a monoid of endomorphisms of U, $\varnothing\neq S\subseteq U$ satisfies $P(S)\subseteq S$, and $R:=\{r:U\to\mathbb{R}^+\mid(r\circ p)(\phi)\leq r(\phi)\;\forall\phi\in U\;\forall p\in P\}$. The elements of P (resp., S; R) are called ***free operations*** (resp., ***free states***; ***resource monotones***). We introduce:

$1^{\mathrm{L}}_{\mathrm{R}}$) $(\mathcal{T},\widehat{\mathrm{Fix}}(\mathcal{T}),\bigcup_{\phi\in\widehat{\mathrm{Fix}}(\mathcal{T})}\{D_{\ell,\Psi}(\phi,\cdot)\})$ (resp., $(\mathcal{T},\widehat{\mathrm{Fix}}(\mathcal{T}),\bigcup_{\phi\in\widehat{\mathrm{Fix}}(\mathcal{T})}\{D_{\ell,\Psi}(\cdot,\phi)\})$): if $\varnothing\neq K\subseteq U$, $\mathcal{T}\subseteq\mathrm{LSQ}(\ell,\Psi)$ (resp., $\mathcal{T}\subseteq\mathrm{RSQ}(\ell,\Psi)$) is a monoid such that $T^\ell:K\to K$ $\forall T^\ell\in\mathcal{T}$, $\bigcap_{i=1}^n\widehat{\mathrm{Fix}}(T_i)\neq\varnothing$ and $\widehat{\mathrm{Fix}}(T_1\circ\cdots\circ T_n)\neq\varnothing$ $\forall\{T_1^\ell,\ldots,T_n^\ell\}\subseteq\mathcal{T}$, then $D_{\ell,\Psi}(\phi,\cdot)$ (resp., $D_{\ell,\Psi}(\cdot,\phi)$) is a resource monotone for any $\phi\in\widehat{\mathrm{Fix}}(\mathcal{T})$; this holds if Ψ is LSQ-adapted (resp., RSQ-compositional);

$2^{\mathrm{L}}_{\mathrm{R}}$) $(\mathcal{T},K,\bigcup_{\phi\in K}\{D_{\ell,\Psi}(\phi,\cdot)\})$ (resp., $(\mathcal{T},K,\bigcup_{\phi\in K}\{D_{\ell,\Psi}(\cdot,\phi)\})$): for any ℓ-closed ℓ-convex (resp., $(\mathfrak{D}^{\mathrm{G}}\Psi\circ\ell)$-closed $(\mathfrak{D}^{\mathrm{G}}\circ\ell)$-convex) set $\varnothing\neq K\subseteq U$, if $\mathcal{T}$ is given by the set of all $\overleftarrow{\mathfrak{P}}^{D_{\ell,\Psi}}_Q$ (resp., $\overrightarrow{\mathfrak{P}}^{D_{\ell,\Psi}}_Q$) with ℓ-closed ℓ-convex (resp., $(\mathfrak{D}^{\mathrm{G}}\Psi\circ\ell)$-closed $(\mathfrak{D}^{\mathrm{G}}\Psi\circ\ell)$-convex) Q such that $K\subseteq Q$, with the composition $\overleftarrow{\mathfrak{P}}^{D_{\ell,\Psi}}_{Q_1}\diamond\overleftarrow{\mathfrak{P}}^{D_{\ell,\Psi}}_{Q_2}:=\overleftarrow{\mathfrak{P}}^{D_{\ell,\Psi}}_{Q_1\cap Q_2}$ (resp., $\overrightarrow{\mathfrak{P}}^{D_{\ell,\Psi}}_{Q_1}\diamond\overrightarrow{\mathfrak{P}}^{D_{\ell,\Psi}}_{Q_2}:=\overrightarrow{\mathfrak{P}}^{D_{\ell,\Psi}}_{Q_1\cap Q_2}$).

Each $y\in\mathrm{int}(\mathrm{efd}(\Psi^{\mathbf{F}}))$ defines an ***observable*** on U, given by $y\circ\ell:U\to\mathbb{R}$. The (linear) ***witnesses*** of S are defined as the elements of $\{y\in\mathrm{int}(\mathrm{efd}(\Psi^{\mathbf{F}}))^+\mid[\![x,y]\!]\geq0\;\forall x\in S\}$.

Categories, functors, (co)monads [1]

1. Allowing $\varnothing$ as object and empty arrows as morphisms, we obtain a category $\mathtt{lCvx}(\ell,\Psi)$ (resp., $\mathtt{LSQ}_{\mathtt{cvx}}(\ell,\Psi)$) of ℓ-convex ℓ-closed sets and left $D_{\ell,\Psi}$-projections (resp., LSQ(ℓ,Ψ) transmitters), composed by $\diamond$ (resp., $\circ$).
2. For any LSQ-adapted Ψ, there is an embedding functor $\iota^{\mathrm{L}}_{\ell,\Psi}:\mathtt{lCvx}(\ell,\Psi)\to\mathtt{LSQ}_{\mathtt{cvx}}(\ell,\Psi)$, and a functor $\mathrm{Fix}^{\mathrm{L}}_{\ell,\Psi}:\mathtt{LSQ}_{\mathtt{cvx}}(\ell,\Psi)\to\mathtt{lCvx}(\ell,\Psi)$, defined by identity on objects and by $T\mapsto\overleftarrow{\mathfrak{P}}^{D_{\ell,\Psi}}_{\mathrm{Fix}(T)}$ on arrows.
3. For $\mathrm{int}(\mathrm{efd}(\Psi))=X=\mathrm{ran}(\ell)$, let $\mathtt{Pow}(X)$ be the category of subsets of X and functions between them. A map $\mathrm{co}^{\mathrm{L}}_{\ell,\Psi}(\cdot)$, assigning to each $Y\in\mathrm{Ob}(\mathtt{Pow}(X))$ the norm closure of a convex hull of Y, and to each $f\in\mathrm{Arr}(\mathtt{Pow}(X))$ the map $\overleftarrow{\mathfrak{P}}^{D_\Psi}_Q$, $Q=\overline{\mathrm{co}}^{\mathrm{L}}_\Psi(\mathrm{ran}(f))$, determines a functor $\mathrm{co}^{\mathrm{L}}_{\ell,\Psi}:\mathtt{Pow}(\ell^{-1}(X))\to\mathtt{lCvx}(\ell,\Psi)$.
4. The adjunctions, $\iota^{\mathrm{L}}_{\ell,\Psi}\dashv\mathrm{Fix}^{\mathrm{L}}_{\ell,\Psi}$ and $\mathrm{co}^{\mathrm{L}}_{\ell,\Psi}\dashv\mathrm{Frg}_{\mathtt{Set}}$, where $\mathrm{Frg}_{\mathtt{Set}}$ is a functor forgetting about convexity and topology, equip $\mathtt{lCvx}(\ell,\Psi)$ with a comonad $\mathrm{co}^{\mathrm{L}}_{\ell,\Psi}\circ\mathrm{Frg}_{\mathtt{Set}}$ and a monad $\mathrm{Fix}^{\mathrm{L}}_{\ell,\Psi}\circ\iota^{\mathrm{L}}_{\ell,\Psi}$.
5. Under some additional conditions, $\mathfrak{D}^{\mathrm{G}}\Psi$ sets functorial equivalence between the above categories and their right versions, inducing the corresponding (comonad, monad) pair on $\mathtt{rCvx}(\ell,\Psi)$.

Examples of $D_{\ell,\Psi}$ [2, 1]

1) For a Hilbert space $\mathcal{H}$, $\dim\mathcal{H}=:n<\infty$, the Umegaki relative entropy $D_1(\rho,\phi):=\mathrm{tr}_{\mathcal{H}}(h_\rho(\log h_\rho-\log h_\phi))$ equals $D_{\ell=\mathrm{id},\Psi=\Phi\circ\Lambda}(\rho,\phi)$, where $\psi=\mathrm{tr}_{\mathcal{H}}(h_\psi\cdot)\in\mathfrak{B}(\mathcal{H})_\star$, Λ is a nonincreasing rearrangement of eigenvalues, while $\Phi(x):=\sum_{i=1}^n(x_i\log(x_i)-x_i)$ for $x\geq0$ and ∞ otherwise [Bauschke–Borwein'97]. So, Lüders' and quantum Jeffrey's rules [4], as well as a partial trace [5], being derived from special cases of $\overleftarrow{\mathfrak{P}}^{D_1}$, belong to $\overleftarrow{\mathfrak{P}}^{D_{\ell,\Psi}}$.

If $(X,\|\cdot\|_X)$ is strictly convex, uniformly Fréchet differentiable, and satisfies the Radon–Riesz property, then $\Psi_\varphi:=\int_0^{\|\cdot\|_X}\mathrm{d}t\varphi(t):X\to\mathbb{R}^+$ is Legendre, LSQ-adapted, and RSQ-compositional for any strictly increasing, continuous $\varphi:\mathbb{R}^+\to\mathbb{R}^+$, with $\varphi(0)=0$, and $\lim_{t\to\infty}(t)=\infty$. This gives us:

2) for any semi-finite JBW-algebra A with a faithful normal semi-finite trace τ, the nonassociative $(L_{1/\gamma}(A,\tau),\|\cdot\|_{1/\gamma})$ spaces [Abdullaev'84, Iochum'84] are uniformly convex and uniformly Fréchet differentiable for any $\gamma\in]0,1[$ [Ayupov'86, Iochum'86]. Introducing the nonassociative Mazur map $\ell_\gamma:A_\star\ni\phi=|h_\phi|\circ s_\phi\mapsto|h_\phi|^\gamma\circ s_\phi\in L_{1/\gamma}(A,\tau)$ for $\phi=\tau(h_\phi\circ\cdot)$, $|h_\phi|\in A^+$, $s_\phi^2=\mathbb{I}$, we obtain the class $D_{\ell_\gamma,\Psi_\varphi}:A_\star\times A_\star\to[0,\infty]$. Due to isometric isomorphism of $L_{1/\gamma}(A,\tau)$ for different τ [Ayupov–Abdullaev'89], $D_{\ell_\gamma,\Psi_\varphi}$ do not depend on τ, only on (A,γ,φ). For $\varphi(t)=t^{1/\beta-1}/\beta$, $\beta\in]0,1[$, we get $D_{\gamma,\beta}(\omega,\phi)=(\tau(h_\omega))^{\gamma/\beta}+\frac{1-\beta}{\beta}(\tau(h_\phi))^{\gamma/\beta}-\frac{1}{\beta}(\tau(h_\phi))^{\gamma/\beta-1}\tau(h_\omega^\gamma\circ h_\phi^{1-\gamma})\;\forall\omega,\phi\in A_\star^+$.
3) A function $\Upsilon:\mathbb{R}\to\mathbb{R}^+$ is called ***Orlicz*** iff it is convex, $\Upsilon(0)=0$, $\Upsilon\not\equiv0$, and $\Upsilon(-u)=\Upsilon(u)$. Any Υ and a semi-finite W*-algebra $\mathcal{N}$ with a faithful normal semi-finite trace τ determine a noncommutative Orlicz space $(L_\Upsilon(\mathcal{N},\tau),\|\cdot\|_\Upsilon)$ [Kunze'90]. In [2] we characterise (in terms of conditions on Υ and a type of $\mathcal{N}$) strictly convex, uniformly Fréchet differentiable $L_\Upsilon(\mathcal{N},\tau)$, satisfying the Radon–Riesz property. By replacing $[\![\cdot,\cdot]\!]$ with $\mathrm{re}\,[\![\cdot,\cdot]\!]$, and introducing the noncommutative Kaczmarz map, $\ell_\Upsilon:\mathcal{N}_\star\ni\phi\mapsto u_\phi\Upsilon^{-1}(|h_\phi|)\in L_\Upsilon(\mathcal{N},\tau)$, where $\phi=\tau(u_\phi|h_\phi|\cdot)$ is a unique polar decomposition, we get $D_{\ell_\Upsilon,\Psi_\varphi}:\mathcal{N}_\star\times\mathcal{N}_\star\to[0,\infty]$ (due to isometric isomorphism of $L_\Upsilon(\mathcal{N},\tau)$ for different τ [Ayupov–Chilin–Abdullaev'12], it does not depend on τ).

Epistemic adjointness [3, 1]

We define an ***epistemic inference theory*** as a triple $(\mathtt{IndInf},E,J)$, where $\mathtt{IndInf}$ is a category of information state spaces as objects and information processings (inductive inferences) as morphisms, J is a monad encoding operations on $\mathtt{IndInf}$, while E is a comonad, encoding the range of possible adjunctions $I\dashv P$ (with a category $\mathtt{ExpDes}$ of experimental designs (e.g., spaces of configuration parameters) and experimental procedures, $I:\mathtt{ExpDes}\to\mathtt{IndInf}$ encoding the method of model construction from data, and $P:\mathtt{IndInf}\to\mathtt{ExpDes}$ encoding the criteria of (ideal) experimental verification).

- Identifying J with ***agent***, E with ***coagent***, and (E,J) with a ***subject/user***, the ***multi-(co)agent*** epistemic inference theory is given by $(\mathtt{IndInf},\{E_i\mid i\in\mathcal{I}\},\{J_j\mid j\in\mathcal{J}\})$, and becomes ***multi-user*** under pairing $\mathcal{I}=\mathcal{J}$.
- An ***intersubjective commensurability*** of two subjects/users is given by a groupoid between them, understood internally in the 2-category $\mathtt{Comonads}\times\mathtt{Monads}$ over $\mathtt{IndInf}$, with the (co)lax×(co)lax morphisms as 1-cells, and the choice of lax vs colax (as well as weak vs strong) dependent on the purposes.
- Ex.1: $(\mathtt{lCvx}(\ell,\Psi),\mathrm{co}^{\mathrm{L}}_{\ell,\Psi}\circ\mathrm{Frg}_{\mathtt{Set}},\mathrm{Fix}^{\mathrm{L}}_{\ell,\Psi}\circ\iota^{\mathrm{L}}_{\ell,\Psi})$ is an epistemic information theory with a single user.
- Ex.2: $(\mathtt{Pow}(\mathcal{N}_\star),\mathrm{id}_{\mathtt{Pow}(\mathcal{N}_\star)},\{\mathrm{Frg}_{\mathbf{Set}}\circ\mathrm{co}^{\mathrm{L}}_{\ell_\Upsilon,\Psi_\varphi}(\cdot)^{\ell}\})$, with Υ and φ varying as in Ex.3) above, is a multi-agent epistemic inference theory. Each agent corresponds to a family of resource theories of states of type 2^{L}), parametrised by ℓ_Υ-closed ℓ_Υ-convex sets of free states.
- Ex.3: If $\mathtt{IndInf}$ has a terminal object $\mathbf{1}$, then any agent (J,μ^J,ν^J) determines a monoid $(M_J:=\mathrm{Nat}(\mathrm{id}_{\mathtt{IndInf}},J),\mu^J(\cdot\circ\cdot),\eta^J)$ of ***free operations***, with ***resource spaces*** (resp., ***free resources***) as elements of $\mathrm{Ob}(\mathtt{IndInf})$ (resp., $\{\sigma_1(\mathbf{1})\in\mathrm{Ob}(\mathtt{IndInf})\mid\sigma\in M_J\}$), and functors $r:\mathtt{IndInf}\to[0,\infty]$ with $r\circ\sigma_A(A)\leq\sigma_A(A)$ $\forall\sigma\in M_J\;\forall A\in\mathrm{Ob}(\mathtt{IndInf})$ as ***resource monotones***.

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