Postquantum Brègman relative entropies

Ryszard Paweł Kostecki
International Center for Theory of Quantum Technologies, University of Gdańsk
ul. Jana Bażyńskiego 1a, 80–309 Gdańsk, Poland
kostecki@fuw.edu.pl
https://www.fuw.edu.pl/~kostecki

1 March 2024

Abstract

We develop a new approach to construction of Brègman relative entropies over nonreflexive Banach spaces, based on nonlinear mappings into reflexive Banach spaces. We apply it to derive few families of Brègman relative entropies over several radially compact base normed spaces in spectral duality. In particular, we prove generalised pythagorean theorem and norm-to-norm continuity of the corresponding entropic projections for a family induced on preduals of any W*-algebras and of semifinite JBW-algebras using Mazur maps into corresponding noncommutative and nonassociative L_p spaces. We also prove generalised pythagorean theorem for a family induced using Kaczmarz maps into Orlicz spaces over semifinite W*-algebras, and for a family over generalised spin factors. Additionally, we establish Lipschitz–Hölder continuity of the nonassociative Mazur map on positive parts of unit balls, characterise several geometric properties of the Morse–Transue–Nakano–Luxemburg norm on noncommutative Orlicz spaces, and introduce a new family of L_p spaces over order unit spaces.

1 Introduction

We present some basic elements of the theory of generalised Brègman relative entropies over nonreflexive Banach spaces. Using nonlinear embeddings of Banach spaces together with the Euler–Legendre functions, this approach unifies two former approaches to Brègman relative entropy: one based on reflexive Banach spaces, another based on differential geometry. This construction allows to extend Brègman relative entropies, and related geometric and operator structures, to arbitrary-dimensional state spaces of probability, quantum, and postquantum theory. We give several examples, not considered previously in the literature.

If $\varnothing \neq K \subseteq Z$, $x \in Z$, and $\arg\inf_{y\in K}\{D(y,x)\}$ (resp., $\arg\inf_{y\in K}\{D(x,y)\}$) is a singleton set, then we will denote the element of this set by $\overleftarrow{\mathfrak{P}}^D_K(x)$ (resp., $\overrightarrow{\mathfrak{P}}^D_K(x)$), while the map $x \mapsto \overleftarrow{\mathfrak{P}}^D_K(x)$ [124, p. 32] [88, Ch. 3.2] (resp., $x \mapsto \overrightarrow{\mathfrak{P}}^D_K(x)$ [33, Eqn. (16)]) will be called a ***left*** (resp., ***right***) D***-projection*** of x onto K.

For a convex closed $C \subseteq M \subseteq \mathbb{R}^n$, D_Ψ given by (2) exhibits [23, Lemm. 1],

$$D_\Psi(x, \overleftarrow{\mathfrak{P}}^{D_\Psi}_C(y)) + D_\Psi(\overleftarrow{\mathfrak{P}}^{D_\Psi}_C(y), y) \geq D_\Psi(x,y)\ \forall (x,y) \in C \times M \tag{1}$$

(and analogously for $\overrightarrow{\mathfrak{P}}^{D_\Psi}_C$ [98, Prop. 4.11]; cf. also [33, Thm. 1]), with $\geq$ replaced by $=$ for affine closed C. This property is a nonlinear generalisation of a pythagorean theorem, and is interpreted as an additive decomposition of an (information about) "data" into "signal" and "noise". It is a fundamental feature of D_Ψ, characterising $\overleftarrow{\mathfrak{P}}^{D_\Psi}_C$ [16, Cor. 3.35] and $\overrightarrow{\mathfrak{P}}^{D_\Psi}_C$ [98, Prop. 4.11].

We introduce a generalisation, $D_{\ell,\Psi}$, of a family of Brègman informations D_Ψ on reflexive Banach spaces $(X, \|\cdot\|_X)$, applicable to a wide range of nonreflexive Banach spaces $(Y, \|\cdot\|_Y)$. (E.g., to postquantum state spaces, given by bases $V^+_1 \subseteq V^+$ of positive cones V^+ of radially compact base normed spaces in spectral duality, $(V, \|\cdot\|_V) = (Y, \|\cdot\|_Y)$.) The main idea is to pull back the properties exhibited by D_Ψ with Euler–Legendre Ψ acting on $(X, \|\cdot\|_X)$ into the properties exhibited by $D_{\ell,\Psi}(\cdot,\cdot) := D_\Psi(\ell(\cdot), \ell(\cdot))$, where $\ell : Z \to X$ and $Z \subseteq Y$.

1.1 Brègman vs Brunk–Ewing –Utz

Given a strictly convex, differentiable function $\Psi : \mathbb{R}^n \to \mathbb{R}$ (or $\Psi : M \to \mathbb{R}$ with convex $M \subseteq \mathbb{R}^n$), there are two approaches to construction of a functional encoding the first order Taylor expansion of Ψ (together with its further use in optimisation problems): one going back to Brègman's [23, p. 1021]

$$D_\Psi(x, y) := \Psi(x) - \Psi(y) - \sum\nolimits_{i=1}^{n} (x_i - y_i)(\mathrm{grad}\Psi(y_i)), \tag{2}$$

for $x, y \in \mathbb{R}^n$ (or $x, y \in M$), another going back to the Brunk–Ewing–Utz [26, Eqn. (4.4)]

$$D^\mu_\Psi(x, y) := \int_{\mathcal{X}\subseteq\mathbb{R}^m} \mu(\chi) D_\Psi(x(\chi), y(\chi)), \tag{3}$$

for $x, y : \mathcal{X} \to \mathbb{R}$, $n = 1$, and a measure μ on the Borel subsets of $\mathbb{R}^m$. The former approach has been generalised and widely developed for $\mathbb{R}^n$ replaced by a reflexive Banach space $(X, \|\cdot\|_X)$ (see Section 2.1). On the other hand, the latter approach was generalised and further developed for $(\mathcal{X}, \mu)$ given by any countably finite nonzero measure space (see [41] and references therein).

The passage from probabilistic to quantum theoretic setting corresponds to replacing $(L_1(\mathcal{X}, \mu), \|\cdot\|_1)$ by the Banach predual $\mathcal{N}_\star$ of a W*-algebra $\mathcal{N}$ (all of these spaces are nonreflexive). The noncommutative analogue $D^{\mathrm{tr}_\mathcal{H}}_\Psi$ of D^μ_Ψ was introduced in [135, §2.2] for finite dimensional real Hilbert spaces, and in [115, pp. 127–129][1] for type I W*-algebras (see also [69, §V] for type I_n JBW-algebras). However, due to nonreflexivity of $\mathcal{N}_\star$, this definition shares the same optimisation-theoretic problems as D^μ_Ψ, is incapable of utilising the vast body of reflexive Banach space theoretic results obtained for D_Ψ, and it is also unclear how to extend the definition of $D^{\mathrm{tr}_\mathcal{H}}_\Psi$ to arbitrary W*-algebras.

In Section 3 we present a new approach to extension of D_Ψ to nonreflexive Banach spaces $(Y, \|\cdot\|_Y)$, by means of nonlinear embedding $\ell : Z \to X$, where $Z \subseteq Y$ and $(X, \|\cdot\|_X)$ is a reflexive Banach space. The main idea is to pull back the properties exhibited by D_Ψ on $(X, \|\cdot\|_X)$ into the corresponding properties exhibited by

$$D_{\ell,\Psi}(\cdot, \cdot) := D_\Psi(\ell(\cdot), \ell(\cdot)) \tag{4}$$

on $(Y, \|\cdot\|_Y)$. In order to express topological behaviour of $D_{\ell,\Psi}$ in terms of $(Y, \|\cdot\|_Y)$, without relativisation to $(X, \|\cdot\|_X)$, ℓ has to additionally preserve the corresponding continuity properties. Hence, the best behaved sector of the theory of generalised Brègman information $D_{\ell,\Psi}$ consists of a fusion of nonlinear convex analysis on reflexive Banach spaces with a nonlinear homeomorphy of Banach spaces. On the other hand, the relativisation of convexity is unavoidable, and as a result we generically deal with *ℓ-convex* sets in $(Y, \|\cdot\|_Y)$ (i.e. the sets which are mapped by ℓ into convex sets in $(X, \|\cdot\|_X)$).

In Section 4 we apply this approach to derive a new family of Brègman relative entropies over preduals of any W*-algebras and of semifinite JBW-algebras, implementing ℓ by the generalisations of a Mazur map [100, p. 83] into L_p spaces over these algebras. The nonassociative Mazur maps have not been considered before. We prove their Lipschitz–Hölder continuity on positive parts of unit balls.

This paper can be seen as a concrete functional analytic implementation (and clarification) of an idea presented in [83, Eqns. (24), (31)] (inspired by [76, §6–§8]), and as a prequel to upcoming series of papers on $D_{\ell,\Psi}$ and related geometric structures. As for integration on W*-algebras (resp., JBW-algebras), we refer to [42, 58, 132] (resp., [73, 1, 13, 74]) as standard references. Cf. [84] for a review of the W*-algebraic case.

[1] More precisely, $D^{\mathrm{tr}_\mathcal{H}}_\Psi(x, y) := \mathrm{tr}_\mathcal{H}(D_\Psi(x, y))$ for a convex and Gateaux differentiable $\Psi : W \to \mathfrak{B}(\mathcal{H})$, where W is a convex subset of a Banach space, e.g. $W = (\mathfrak{B}(\mathcal{H}))^+_\star$. The evaluation of $D^{\mathrm{tr}_\mathcal{H}}_\Psi(x, y)$ is thus defined by spectral calculus applied to Ψ.

2 Two approaches to Brègman functionals

Definition 2.1. *(cf.* [23, p. 1019] [54, p. 794] [40, p. 161]*) For any set Z, $D : Z \times Z \to [0,\infty]$ will be called an* ***information*** *on Z (and $-D$ will be called a* ***relative entropy***[2] *on Z) iff $D(x,y) = 0 \iff x = y \; \forall x,y \in Z$.*

2.1 Brègman functionals on reflexive Banach spaces

In what follows, $(X, \|\cdot\|_X)$ will denote a Banach space over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ [14, §1], $B(X,\|\cdot\|_X) := \{x \in X \mid \|x\|_X \leq 1\}$, $S(X,\|\cdot\|_X) := \{x \in X \mid \|x\|_X = 1\}$. $(X^\star, \|\cdot\|_{X^\star})$ will denote a Banach space of continuous linear functions $X \to \mathbb{K}$, equipped with a norm $\|y\|_{X^\star} := \sup\{|y(x)| \mid x \in B(X,\|\cdot\|_X)\}$ $\forall y \in X^\star$ [70, p. 62], and will be called a ***Banach dual*** of $(X, \|\cdot\|_X)$ (with respect to a bilinear duality $[\![x,y]\!]_{X\times X^\star} := y(x) \in \mathbb{K} \; \forall x \in X \; \forall y \in X^\star$). If $(Y, \|\cdot\|_Y)$ and $(X, \|\cdot\|_X)$ are such that $(Y^\star, \|\cdot\|_{Y^\star}) = (X, \|\cdot\|_X)$, then $Y =: X_\star$ is called a ***predual*** of X. $(X, \|\cdot\|_X)$ is called ***reflexive*** [68, pp. 219–220] iff $X \ni x \mapsto [\![x, \cdot\,]\!]_{X\times X^\star} \in X^{\star\star}$ is an isometric isomorphism. Given $(X, \|\cdot\|_X)$, for any $Y \subseteq X$, $\mathrm{int}(Y)$ (resp., $\overline{Y}^{\|\cdot\|_X}$) will denote a topological interior (resp., closure) of Y with respect to the topology of $\|\cdot\|_X$.

A Banach space $(X, \|\cdot\|_X)$ is said to satisfy the ***Radon–Riesz property*** [129, Thm. 5] iff, for any $\{x_n \in X \mid n \in \mathbb{N}\}$, convergence of x_n to $x \in X$ in a weak topology together with $\lim_{n\to\infty} \|x_n\|_X = \|x\|_X$ implies $\lim_{n\to\infty} \|x_n - x\|_X = 0$. A Banach space $(X, \|\cdot\|_X)$ is called: ***strictly convex*** iff [61, p. 39]

$$\forall x,y \in X \; (\|x+y\|_X = \|x\|_X + \|y\|_X, \; x \neq 0 \neq y) \;\Rightarrow\; \exists \lambda > 0 \; y = \lambda x; \tag{5}$$

uniformly convex iff [39, Def. 1]

$$\forall \epsilon_1 > 0 \; \exists \epsilon_2 > 0 \; \forall x,y \in S(X,\|\cdot\|_X) \; \|x-y\|_X \geq \epsilon_1 \;\Rightarrow\; \frac{1}{2}\|x+y\|_X \leq 1 - \epsilon_2; \tag{6}$$

locally uniformly convex [93, Def. 0.2] iff

$$\forall \epsilon_1 > 0 \;\; \forall x \in S(X,\|\cdot\|_X) \;\; \exists \epsilon_2 > 0 \;\; \forall y \in S(X,\|\cdot\|_X) \;\; \|x-y\|_X \geq \epsilon_1 \;\Rightarrow\; \frac{1}{2}\|x+y\|_X \leq 1 - \epsilon_2; \tag{7}$$

uniformly Fréchet differentiable iff [46, p. 375]

$$\forall \epsilon_1 > 0 \; \exists \epsilon_2 > 0 \; \forall x,y \in S(X,\|\cdot\|_X) \; \|x-y\|_X \leq \epsilon_1 \;\Rightarrow\; 1 - \frac{1}{2}\|x+y\|_X \leq \epsilon_2 \|x-y\|_X. \tag{8}$$

Given Banach spaces $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$, $Z \subseteq X$, $W \subseteq Y$, $t \in]0,\infty[$, a function $f : Z \to W$ is said to be t-***Lipschitz–Hölder continuous*** on Z iff

$$\exists c > 0 \; \forall x,y \in Z \; \|f(x) - f(y)\|_Y \leq c\|x-y\|_X^t. \tag{9}$$

For convenience of notation, assume that $\mathbb{K} = \mathbb{R}$ (all results and formulas below are applicable for $\mathbb{K} = \mathbb{C}$ under replacement of $[\![\cdot,\cdot]\!]_{X\times X^\star}$ by $\mathrm{re}\,[\![\cdot,\cdot]\!]_{X\times X^\star}$). We assume $\inf \varnothing := \infty$.

Given a Banach space $(X, \|\cdot\|_X)$, $\Psi : X \to]-\infty,\infty]$ will be called: ***proper*** iff $\mathrm{efd}(\Psi) := \{x \in X \mid \Psi(x) \neq \infty\} \neq \varnothing$; ***convex*** (resp., ***strictly convex***) iff

$$x \neq y \;\Rightarrow\; \Psi(\lambda x + (1-\lambda)y) \leq \; (\text{resp., } <\,) \; \lambda\Psi(x) + (1-\lambda)\Psi(y) \; \forall x,y \in \mathrm{efd}(\Psi) \; \forall \lambda \in]0,1[\tag{10}$$

(this is equivalent to the definition based on the same inequality, with quantifiers changed to $\forall x,y \in X \; \forall \lambda \in [0,1]$, with the conventions $\infty + \infty \equiv \infty$, $0 \cdot \infty \equiv \infty$, $0 \cdot (-\infty) = 0$, and without assuming $x \neq y$). The set of all proper, convex, lower semicontinuous functions $\Psi : X \to]-\infty,\infty]$ will be

[2] Cf. «information is the negative of the quantity (...) defined as entropy» [139, p. 76].

denoted by $\Gamma(X,\|\cdot\|_X)$. If $\Psi : X \to]-\infty,\infty]$ is proper, then the ***right Gateaux derivative*** of Ψ at $x \in \mathrm{efd}(\Psi)$ in the direction $h \in X$ reads [10, p. 53]

$$\mathrm{efd}(\Psi)\times X \ni (x,h) \mapsto \mathfrak{D}^{\mathrm{G}}_{+}\Psi(x,h) := \lim_{t\to^{+}0}\left(\Psi(x+th)-\Psi(x)\right)/t \in]-\infty,\infty], \tag{11}$$

and it exists $\forall h \in X$. $\Psi \in \Gamma(X,\|\cdot\|_X)$ is called ***Gateaux differentiable*** at $x \in \mathrm{int}(\mathrm{efd}(\Psi))$ [62, p. 311] iff $\mathfrak{D}^{\mathrm{G}}_{+}\Psi(x,y) = -\mathfrak{D}^{\mathrm{G}}_{+}\Psi(x,-y)$ $\forall y \in X$. In such case $\mathfrak{D}^{\mathrm{G}}_{+}\Psi(x,\cdot)$ is linear, so it defines a bounded linear operator $\mathfrak{D}^{\mathrm{G}}_{+}\Psi(x,y) =: \left[\!\left[y,\mathfrak{D}^{\mathrm{G}}\Psi(x)\right]\!\right]_{X\times X^\star}$ $\forall y \in X$. A set of all $\Psi \in \Gamma(X,\|\cdot\|_X)$ which are Gateaux differentiable on $\mathrm{int}(\mathrm{efd}(\Psi)) \neq \varnothing$ will be denoted $\Gamma^{\mathrm{G}}(X,\|\cdot\|_X)$. A Banach space $(X,\|\cdot\|_X)$ is called: ***Gateaux differentiable*** [101, p. 78] iff $\|\cdot\|_X$ is Gateaux differentiable at every $x \in X\setminus\{0\}$; ***Fréchet differentiable*** [102, p. 129] iff for any fixed $x \in X\setminus\{0\}$ $\mathfrak{D}^{\mathrm{G}}\|h\|_X(x)$ exist in uniform convergence $\forall h \in S(X,\|\cdot\|_X)$. In the latter case $\mathfrak{D}^{\mathrm{G}}\|\cdot\|_X$ will be denoted by $\mathfrak{D}^{\mathrm{F}}\|\cdot\|_X$.

For a proper $\Psi : X \to]-\infty,\infty]$,

$$X^\star \ni y \mapsto \Psi^{\mathbf{F}}(y) := \sup_{x\in X}\{[\![x,y]\!]_{X\times X^\star} - \Psi(x)\} \in]-\infty,\infty], \tag{12}$$

called a ***Fenchel dual*** of Ψ [59, p. 75] [104, p. 8], satisfies $\Psi^{\mathbf{F}} \in \Gamma(X^\star,\|\cdot\|_{X^\star})$ [24, Thm. 3.6]. If $(X,\|\cdot\|_X)$ is reflexive and $\Psi \in \Gamma^{\mathrm{G}}(X,\|\cdot\|_X)$, then Ψ will be called ***Euler–Legendre***[3] [15, Def. 5.2.(iii), Thm. 5.4, Thm. 5.6] [119, §2.1] iff $\Psi^{\mathbf{F}} \in \Gamma^{\mathrm{G}}(X^\star,\|\cdot\|_{X^\star})$ and

$$\begin{cases} \mathrm{efd}(\mathfrak{D}^{\mathrm{G}}\Psi) := \{x \in \mathrm{efd}(\Psi) \mid \exists\, \mathfrak{D}^{\mathrm{G}}\Psi(x)\} = \mathrm{int}(\mathrm{efd}(\Psi)), \\ \mathrm{efd}(\mathfrak{D}^{\mathrm{G}}\Psi^{\mathbf{F}}) = \mathrm{int}(\mathrm{efd}(\Psi^{\mathbf{F}})). \end{cases} \tag{13}$$

For $X = \mathbb{R}^n$, the above definition of Euler–Legendre functions goes back to Rockafellar, who showed [122, Thm. C-K] [123, Thm. 1] that if $\varnothing \neq U \subseteq \mathbb{R}^n$ is open and convex, while $\Psi : U \to]-\infty,\infty]$ is strictly convex, differentiable on U, and

$$\lim_{t\to^{+}0}\tfrac{\mathrm{d}}{\mathrm{d}t}\Psi(tx+(1-t)y) = -\infty \quad \forall (x,y) \in U \times (\overline{U}^{\|\cdot\|_{\mathbb{R}^n}} \setminus U), \tag{14}$$

then $\mathrm{grad}\,\Psi$ is a bijection on U, $\mathrm{grad}(\Psi^{\mathbf{F}}) = (\mathrm{grad}\,\Psi)^{-1}$ on $(\mathrm{grad}\,\Psi)(U)$, and $\Psi^{\mathbf{F}}$ on $(\mathrm{grad}\,\Psi)(U)$ satisfies the same conditions as Ψ on U.

$\Psi \in \Gamma(X,\|\cdot\|_X)$ is called: ***totally convex*** at $x \in \mathrm{efd}(\Psi)$ iff [28, 2.2] [29, p. 62]

$$\nu_\Psi(x,t) := \inf\{D^{+}_{\Psi}(y,x) \mid y \in \mathrm{efd}(\Psi),\ \|y-x\|_X = t\} > 0 \quad \forall t \in]0,\infty[, \tag{15}$$

where [29, Eqn. (2)]

$$D^{+}_{\Psi} : X\times X \ni (x,y) \mapsto \begin{cases} \Psi(x)-\Psi(y)-\mathfrak{D}^{\mathrm{G}}_{+}\Psi(y;x-y) & : y \in \mathrm{efd}(\Psi) \\ \infty & : \text{otherwise} \end{cases} \in [0,\infty]. \tag{16}$$

Definition 2.2. [3, Eqn. (1)] *For any $\Psi \in \Gamma^{\mathrm{G}}(X,\|\cdot\|_X)$ and any $x, y \in X$, the* ***Brègman function*** *on $(X,\|\cdot\|_X)$ is defined as*

$$D_\Psi : X\times X \ni (x,y) \mapsto \begin{cases} \Psi(x)-\Psi(y)-\left[\!\left[x-y,\mathfrak{D}^{\mathrm{G}}\Psi(y)\right]\!\right]_{X\times X^\star} & : y \in \mathrm{int}(\mathrm{efd}(\Psi)) \\ \infty & : \textit{otherwise} \end{cases} \in [0,\infty]. \tag{17}$$

Proposition 2.3. [30, Prop. 1.1.9] *If $\Psi \in \Gamma^{\mathrm{G}}(X,\|\cdot\|_X)$, then D_Ψ is an information on X iff Ψ is strictly convex on* $\mathrm{int}(\mathrm{efd}(\Psi))$.

[3] These functions are usually called "Legendre" (although they were introduced namelessly for $X = \mathbb{R}^n$ in [122, Thm. C-K]). Yet, the transformation $\mathrm{d}(z(x,y)-px-qy) = -x\mathrm{d}p - y\mathrm{d}q$, with $p = \frac{\partial z(x,y)}{\partial x}$ and $q = \frac{\partial z(x,y)}{\partial y}$, was introduced first by Euler [56, Part I, Probl. 11], and only 17 years later by Legendre [92, p. 347].

Definition 2.4. *Let $\Psi \in \Gamma^{\mathrm{G}}(X,\|\cdot\|_X)$, $y \in \mathrm{int}(\mathrm{efd}(\Psi))$, and $K \subseteq X$ with $\varnothing \neq K \cap \mathrm{int}(\mathrm{efd}(\Psi))$. If the set* $\arg\inf_{x\in K}\{D_\Psi(x,y)\}$ *(resp.,* $\arg\inf_{x\in K\subseteq\mathrm{int}(\mathrm{efd}(\Psi))}\{D_\Psi(y,x)\}$*) is a singleton, then its element will be denoted* $\overleftarrow{\mathfrak{P}}^{D_\Psi}_K(y)$ *(resp.,* $\overrightarrow{\mathfrak{P}}^{D_\Psi}_K(y)$*), and called a* ***left*** *(resp.,* ***right****)* D_Ψ***-projection*** *of y onto K* [23, p. 1019][4] *(resp.,* [18, Def. 3.1, Lemm. 3.5][5]*), while K will be called a* ***left*** *(resp.,* ***right****)* D_Ψ***-Chebyshëv*** *set* [16, Def. 3.28] *(resp.,* [17, Def. 1.7]*).*

Proposition 2.5. [16, Cor. 3.35] *If $(X,\|\cdot\|_X)$ is reflexive, Ψ is Euler–Legendre, $\varnothing \neq K \subseteq X$ is convex and closed, and $K \cap \mathrm{int}(\mathrm{efd}(\Psi)) \neq \varnothing$, then K is left D_Ψ-Chebyshëv, and, for any $w \in K$ and any $x \in \mathrm{int}(\mathrm{efd}(\Psi))$, w is the unique solution of*

$$D_\Psi(y,z) + D_\Psi(z,x) \leq D_\Psi(y,x) \;\; \forall y \in K \tag{18}$$

(with respect to z) iff $w = \overleftarrow{\mathfrak{P}}^{D_\Psi}_K(x)$. Furthermore, in 'then' case of (18)*, if K is affine, then $\leq$ in* (18) *turns into $=$.*

Proposition 2.6. [98, Prop. 4.11] *If $(X,\|\cdot\|_X)$ is reflexive, $\Psi \in \Gamma^{\mathrm{G}}(X,\|\cdot\|_X)$ and $\mathrm{efd}(\Psi) = X$, $\Psi^{\mathbf{F}} \in \Gamma^{\mathrm{G}}(X^\star,\|\cdot\|_{X^\star})$ is totally convex, $\varnothing \neq K \subseteq \mathrm{int}(\mathrm{efd}(\Psi))$, and $\mathfrak{D}^{\mathrm{G}}\Psi(K)$ is convex and closed, then K is right D_Ψ-Chebyshëv, and, for any $w \in K$ and $x \in \mathrm{int}(\mathrm{efd}(\Psi))$, w is the unique solution of*

$$D_\Psi(x,z) + D_\Psi(z,y) \leq D_\Psi(x,y) \;\; \forall y \in K \tag{19}$$

(with respect to z) iff $w = \overrightarrow{\mathfrak{P}}^{D_\Psi}_K(x)$. Furthermore, in 'then' case of (19)*, if K is affine, then $\leq$ in* (19) *turns into $=$.*

Proposition 2.7. [15, Lemm. 6.2] *Let $\Psi = \Psi_{1,\beta} := \beta\|\cdot\|_X^{1/\beta}$, $\beta \in]0,1[$, for a reflexive $(X,\|\cdot\|_X)$. Then $\Psi_{1,\beta}$ is Euler–Legendre iff $(X,\|\cdot\|_X)$ is Gateaux differentiable and strictly convex. Furthermore, in such case $\Psi_{1,\beta}$ is also strictly convex on $\mathrm{int}(\mathrm{efd}(\Psi_{1,\beta})) = X$.*

Proposition 2.8. [2, §7](+[38, I.4.7.(f)]) *If $(X,\|\cdot\|_X)$ is Gateaux differentiable, and $\Psi = \Psi_{1,\beta} := \beta\|\cdot\|_X^{1/\beta}$, then $\mathfrak{D}^{\mathrm{G}}\Psi_{1,\beta}(x) = \|x\|_X^{1/\beta-2}j(x)$, and*

$$D_{\Psi_{1,\beta}}(x,y) = \beta\|x\|_X^{1/\beta} + (1-\beta)\|y\|_X^{1/\beta} - \|y\|_X^{1/\beta-2}\,[\![x,j(y)]\!]_{X\times X^\star} \in \mathbb{R}^+ \;\forall x,y \in X, \tag{20}$$

where $j(x)$ is defined as [80, p. 35] *$z \in X^\star$ such that $[\![x,z]\!]_{X\times X^\star} = \|x\|_X\|z\|_{X^\star}$ and $\|z\|_{X^\star} = \|x\|_X$.*

Proposition 2.9. [120, Cor. 4.4.(ii)] *If $(X,\|\cdot\|_X)$ is reflexive, strictly convex, Fréchet differentiable, and has a Radon–Riesz property, $\varnothing \neq K \subseteq X$ is convex and closed, and $\Psi = \Psi_{\beta,\beta} := \|\cdot\|_X^{1/\beta}$, $\beta \in]0,1[$, then $\overleftarrow{\mathfrak{P}}^{D_{\Psi_{\beta,\beta}}}_K$ is norm-to-norm continuous on $\mathrm{int}(\mathrm{efd}(\Psi_{\beta,\beta})) = X$.*

Proposition 2.10. [31, Prop. 2.4] *If $(X,\|\cdot\|_X)$ is locally uniformly convex and $\beta \in]0,1[$, then $\Psi = \Psi_{\beta,\beta} := \|\cdot\|_X^{1/\beta}$ is totally convex.*

Remark 2.11. If $(X,\|\cdot\|_X)$ is a Banach space over $\mathbb{C}$, then Propositions 2.3, 2.5, 2.6, 2.7, 2.8, 2.9, 2.10 hold under replacing $[\![\cdot,\cdot]\!]_{X\times X^\star}$ in Definition 2.2 by $\mathrm{re}\,[\![\cdot,\cdot]\!]_{X\times X^\star}$. In Section 3 we will keep working with $(X,\|\cdot\|_X)$ over $\mathbb{R}$, while in Section 4 we will make use also of the case of $(X,\|\cdot\|_X)$ over $\mathbb{C}$.

[4] First special case of left D_Ψ-projection for nonsymmetric D_Ψ, with D_Ψ given by the Kullback–Leibler information, was introduced in [124, p. 32] [88, Ch. 3.2].

[5] First special case of right D_Ψ-projection for nonsymmetric D_Ψ, with D_Ψ given by the Kullback–Leibler information, was introduced in [33, Eqn. (16)] [34, Def. 22.2]. See also [5, §3.6].

2.2 Brègman functionals on dually flat manifolds

Let M be a C^3-manifold with a tangent bundle $\mathbf{T}M$, a C^3 riemannian metric tensor $\mathbf{g}$ on $\mathbf{T}M$, and a pair $(\nabla,\widetilde{\nabla})$ of C^3 affine connections on $\mathbf{T}M$ (with an arbitrary torsion). Let $\mathbf{t}_c^{\nabla}$ denote a ∇-parallel transport in $\mathbf{T}M$ along a curve c in M. Then the ***Norden–Sen geometry*** is defined as a quadruple $(M,\mathbf{g},\nabla,\widetilde{\nabla})$ satisfying any of the equivalent conditions [110, pp. 205–206, §2, §4] [126, p. 46]:[6]

$$\mathbf{g}(\mathbf{t}_c^{\nabla}(\cdot),\mathbf{t}_c^{\widetilde{\nabla}}(\cdot))=\mathbf{g}, \tag{21}$$

$$\mathbf{g}(\nabla_u v,w)+\mathbf{g}(v,\widetilde{\nabla}_u w)=u(\mathbf{g}(v,w))\ \forall u,v,w\in\mathbf{T}M. \tag{22}$$

If Z is a finite dimensional C^3-manifold and an information $D\in\mathrm{C}^3(Z\times Z;\mathbb{R}^+)$ has a positive definite hessian matrix, then a third order Taylor expansion of D on Z induces [54, pp. 795–796] [55, p. 357] a riemannian metric $\mathbf{g}^D$ on $\mathbf{T}Z$ and a pair $(\nabla^D,\widetilde{\nabla}^D)$ of torsion-free affine connections on $\mathbf{T}Z$, satisfying the characteristic property (22) of the Norden–Sen geometry. This way the global geometric properties of D can be analysed in local terms of its torsion-free Norden–Sen differential geometry.[7]

The ***dually flat*** (a.k.a. ***hessian***) geometry [127, Prop. (p. 213)] is characterised among all torsion-free Norden–Sen geometries by the flatness of ∇ and $\widetilde{\nabla}$. This is equivalent with existence of two coordinate systems, $\{\theta_i\mid i\in\{1,\ldots,n\}\}:M\to\mathbb{R}^n$ and $\{\eta_i\mid i\in\{1,\ldots,n\}\}:M\to\mathbb{R}^n$, such that, $\forall\rho\in M$,

$$\begin{cases}\eta_i(\rho)=\dfrac{\partial\Psi(\theta(\rho))}{\partial\theta^i},\quad\theta_i(\rho)=\dfrac{\partial\Psi^{\mathbf{F}}(\eta(\rho))}{\partial\eta^i} & (23)\\ \Psi^{\mathbf{F}}(y)=\sup\limits_{x\in\mathbb{R}^n}\left\{\displaystyle\sum_{i=1}^n x_iy_i-\Psi(x)\right\}\ \forall x\in\mathbb{R}^n, & (24)\end{cases}$$

and, for $D_{\theta,\Psi}(\rho,\sigma):=D_\Psi(\theta(\rho),\theta(\sigma))$ with D_Ψ defined by (2),

$$\begin{cases}\Gamma^{\nabla^{D_{\theta,\Psi}}}_{ijk}(\theta(\rho))=0,\quad\Gamma^{\widetilde{\nabla}^{D_{\eta,\Psi}}}_{ijk}(\eta(\rho))=0 & (25)\\ \mathbf{g}^{D_{\theta,\Psi}}_{ij}(\theta(\rho))=\dfrac{\partial^2\Psi(\theta(\rho))}{\partial\theta^i\partial\theta^j}, & (26)\end{cases}$$

where $\Gamma^{\nabla}(u,v,w):=\mathbf{g}(\nabla_uv,w)\ \forall u,v,w\in\mathbf{T}M$, while the subscript $_i$ denotes evaluation at the i-th component of a basis in $\mathbf{T}M$ given by coordinate system differentials (i.e., setting $u=\frac{\partial}{\partial\theta^i}$, etc., in (25)). (By (24), this implies $\mathbf{g}^{D_{\eta,\Psi}}_{ij}(\eta(\rho))=\frac{\partial^2\Psi^{\mathbf{F}}(\eta(\rho))}{\partial\eta^i\partial\eta^j}$.) When reconsidered in this setting, the left (resp., right) generalised pythagorean theorem (i.e., (18) (resp., (19)) for affine K) is equivalent with: a projection of $y\in M$ onto $\nabla^{D_{\theta,\Psi}}$-(resp., $\nabla^{D_{\theta,\Psi}}$-)autoparallel submanifold C along $\widetilde{\nabla}^{D_{\eta,\Psi}}$-(resp., $\nabla^{D_{\theta,\Psi}}$-)geodesics is $\mathbf{g}^{D_{\theta,\Psi}}$-orthogonal ($=\mathbf{g}^{D_{\eta,\Psi}}$-orthogonal) to C [6, Cor. 3.5].

Equation (24) is a special case of (12). Furthermore, (23) require only C^1-differentiability. The approach presented in Section 3 is rooted in an observation that the correct generalisation of (23) requires two components: Euler–Legendre Ψ on a reflexive Banach space $(X,\|\cdot\|_X)$, and nonlinear embeddings into $(X,\|\cdot\|_X)$ and $(X^\star,\|\cdot\|_{X^\star})$, replacing, respectively, θ and η.

3 Extension to nonreflexive Banach spaces

Definition 3.1. *Let $(Y,\|\cdot\|_Y)$ be a Banach space, let $(X,\|\cdot\|_X)$ be a reflexive Banach space, let $\Psi\in\Gamma^{\mathrm{G}}(X,\|\cdot\|_X)$ be strictly convex on* $\mathrm{int}(\mathrm{efd}(\Psi))$*, let $\varnothing\neq Z\subseteq Y$, and let $\ell:Z\to\ell(Z)\subseteq X$ be a*

[6] In comparison, given $(M,\mathbf{g})$, the Levi-Civita affine connection $\nabla^{\mathbf{g}}$ is characterised among all torsion-free affine connections on $\mathbf{T}M$ by $\mathbf{g}(\mathbf{t}_c^{\nabla^{\mathbf{g}}}(\cdot),\mathbf{t}_c^{\nabla^{\mathbf{g}}}(\cdot))=\mathbf{g}$. Each torsion-free Norden–Sen geometry determines $\nabla^{\mathbf{g}}$ by $\nabla^{\mathbf{g}}=\frac{1}{2}(\nabla+\widetilde{\nabla})$ [110, p. 211].

[7] Following [91, §4], the torsion-free Norden–Sen geometries are sometimes called "statistical manifolds". Apart from not crediting the original authors of this structure, such terminology is misleading, since these geometries are independent of any notion of statistics.

bijection such that $\ell(Z)\cap\mathrm{int}(\mathrm{efd}(\Psi))\neq\varnothing$*. Then:*

(i) *an* (ℓ,Ψ)***-information*** *(or a **generalised Brègman information**) on* Z *is defined by*

$$D_{\ell,\Psi}(\phi,\psi):=D_{\Psi}(\ell(\phi),\ell(\psi))\ \ \forall(\phi,\psi)\in Z\times\ell^{-1}(\ell(Z)\cap\mathrm{int}(\mathrm{efd}(\Psi)));\tag{27}$$

(ii) *a triple* (Z,ℓ,Ψ) *will be called a **generalised pythagorean geometry;***

(iii) *if* $\varnothing\neq C\subseteq Y$ *and* $\ell(C)$ *is convex (resp., closed; affine), then* C *will be called* ℓ***-convex*** *(resp.,* ℓ***-closed;*** ℓ***-affine****).*

Proposition 3.2. *Let* $(Y,\|\cdot\|_Y)$ *be a Banach space, let* $(X,\|\cdot\|_X)$ *be a reflexive Banach space,* $\varnothing\neq Z\subseteq Y$*,* $\ell:Z\to X$*,* $\Psi\in\Gamma^{\mathrm{G}}(X,\|\cdot\|_X)$*, let* (Z,ℓ,Ψ) *be a generalised pythagorean geometry, and* $\psi\in\ell^{-1}(\ell(Z)\cap\mathrm{int}(\mathrm{efd}(\Psi)))$*. Then*

(i) $D_{\ell,\Psi}$ *is an information on* Z*.*

If Ψ *is Euler–Legendre and* $\varnothing\neq C\subseteq Z$ *is* ℓ*-convex and* ℓ*-closed, then:*

(ii) there exists the unique solution of $\arg\inf_{\phi\in C}\{D_{\ell,\Psi}(\phi,\psi)\}$ *(denoted by* $\overleftarrow{\mathfrak{P}}^{D_{\ell,\Psi}}_C(\psi)$*), i.e.* C *is left* $D_{\ell,\Psi}$*-Chebyshëv;*

(iii) $\omega=\overleftarrow{\mathfrak{P}}^{D_{\ell,\Psi}}_C(\psi)$ *iff* ω *is the unique solution of*

$$D_{\ell,\Psi}(\phi,\zeta)+D_{\ell,\Psi}(\zeta,\psi)\leq D_{\ell,\Psi}(\phi,\psi)\ \ \forall\phi\in C;\tag{28}$$

(iv) if C *is* ℓ*-affine, then* $\leq$ *in 'then' case of (iii) turns to* $=$*;*

(v) if ℓ *is norm-to-norm continuous and* $\overleftarrow{\mathfrak{P}}^{D_\Psi}_K$ *is norm-to-norm continuous for any convex closed* $\varnothing\neq K\subseteq\mathrm{int}(\mathrm{efd}(\Psi))$*, then* $\overleftarrow{\mathfrak{P}}^{D_{\ell,\Psi}}_C$ *is norm-to-norm continuous for any* ℓ*-convex* ℓ*-closed* $\varnothing\neq C\subseteq\ell^{-1}(\ell(Z)\cap\mathrm{int}(\mathrm{efd}(\Psi)))$*.*

If $\Psi^{\mathbf{F}}\in\Gamma^{\mathrm{G}}(X^\star,\|\cdot\|_{X^\star})$ *is totally convex,* $\mathrm{efd}(\Psi)=X$*,* $\varnothing\neq C\subseteq Z\cap\ell^{-1}(\mathrm{int}(\mathrm{efd}(\Psi)))$*, and* $\mathfrak{D}^{\mathrm{G}}\Psi(C)$ *is* ℓ*-convex and* ℓ*-closed, then:*

(vi) there exists the unique solution of $\arg\inf_{\phi\in C}\{D_{\ell,\Psi}(\psi,\phi)\}$ *(denoted by* $\overrightarrow{\mathfrak{P}}^{D_{\ell,\Psi}}_C(\psi)$*), i.e.* C *is right* $D_{\ell,\Psi}$*-Chebyshëv;*

(vii) $\omega=\overrightarrow{\mathfrak{P}}^{D_{\ell,\Psi}}_C(\psi)$ *iff* ω *is the unique solution of*

$$D_{\ell,\Psi}(\phi,\zeta)+D_{\ell,\Psi}(\zeta,\psi)\leq D_{\ell,\Psi}(\phi,\psi)\ \ \forall\psi\in C;\tag{29}$$

(viii) if $\mathfrak{D}^{\mathrm{G}}\Psi(C)$ *is* ℓ*-affine, then* $\leq$ *in 'then' case of (viii) turns to* $=$*.*

Proof. (i) and (ii)–(iv) (resp., (vi)–(viii)) follow from Propositions 2.3 and 2.5 (resp., 2.6), combined with bijectivity of ℓ, while (v) follows from bijectivity of ℓ and compositionality of norm-to-norm continuous maps. □

Remark 3.3. For $X=\mathbb{R}^n$, $D_{\ell,\Psi}$ recovers the setting of Brègman information $D_{\theta,\Psi}$ on an n-dimensional C^1-manifold (hence, in particular, C^∞-manifold) M, with the map $\ell:M\to\mathbb{R}^n$ (resp., $\mathfrak{D}^{\mathrm{G}}\Psi\circ\ell:M\to\mathbb{R}^n$) given by the coordinate system $\{\theta_i\}$ (resp., $\{\eta_i\}$). More specifically, a domain M of a dually flat geometry is assumed to be a (suitably differentiable) manifold, covered by two global maps $\{\theta_i\}$ and $\{\eta_i\}$, *without* assuming $M\subseteq\mathbb{R}^n$, cf. [6, 128]. This is not addressed by (2), and is addressed (up to a weaker assumption on the order of differentiability) by (27).

This way the framework of generalised Brègman information $D_{\ell,\Psi}$ unifies reflexive Banach space theoretic and finite dimensional smooth information geometric approaches to Brègman information.

If ℓ is a norm-to-norm continuous homeomorphism, then the ℓ-closed sets in Z are closed in terms of topology of $\|\cdot\|_Y$. This fragment of a theory provides a fusion of nonlinear convex analysis with nonlinear homeomorphic theory of Banach spaces. In particular, if ℓ is Lipschitz–Hölder continuous, then it allows to pull back the conditions on Lipschitz–Hölder continuity of $\overleftarrow{\mathfrak{P}}^{D_\Psi}_K$ and $\overrightarrow{\mathfrak{P}}^{D_\Psi}_K$ into results on Lipschitz–Hölder continuity of $\overleftarrow{\mathfrak{P}}^{D_{\ell,\Psi}}_C$ and $\overrightarrow{\mathfrak{P}}^{D_{\ell,\Psi}}_C$. Generalised pythagorean geometry (Z,ℓ,Ψ) is a more general object than $D_{\ell,\Psi}$, and (as we will show in another paper) allows to suitably generalise also the affine connections (25).

In this context, our approach arises partially from an observation that the ℓ_γ (resp., ℓ_Υ) embeddings, cf. Definition 4.1 (resp., 4.5) below, used in [107, Eqn. (2.7)] (resp., [63, §7.2]), are finite dimensional Mazur (resp., Kaczmarz) maps [100, p. 83] (resp., [78, p. 148]) on $(L_1(\mathcal{X},\mu))^+$. Drawing from rethinking of an important example in [76, §6–§8] (see Remark 4.4), an abstract framework aiming at this unification was proposed in [83, Eqns. (24), (31)]. Definition 3.1 and Proposition 3.2 provide concrete functional analytic implementation of this framework, based on the use of Euler–Legendre Ψ and totally convex $\Psi^{\mathbf{F}}$.

Remark 3.4. The proofs of Propositions 2.7, 2.9, and 2.10 hold, without any additional alteration, under replacing Ψ in each of these Propositions by $\Psi_{\alpha,\beta} := \frac{\beta}{\alpha}\|\cdot\|_X^{1/\beta}$, with $\beta \in]0,1[$ and $\alpha \in]0,\infty[$. ($\Psi_{\alpha,\beta}$ has appeared earlier in [75, p. 616].) In such case $\mathfrak{D}^{\mathrm{G}}\Psi_{\alpha,\beta}(x) = \frac{1}{\alpha}\|x\|_X^{1/\beta-2}j(x)$, and

$$D_{\Psi_{\alpha,\beta}}(x,y) = \frac{1}{\alpha}\left(\beta\|x\|_X^{1/\beta} + (1-\beta)\|y\|_X^{1/\beta} - \|y\|_X^{1/\beta-2}\,[\![x,j(y)]\!]_{X\times X^\star}\right) \in \mathbb{R}^+ \;\forall x,y\in X. \tag{30}$$

Proposition 3.5. *If $(X,\|\cdot\|_X)$ is a strictly convex, Gateaux differentiable, reflexive Banach space, $(Y,\|\cdot\|_Y)$ is a Banach space, $\varnothing \neq Z \subseteq Y$, $\Psi = \Psi_{\alpha,\beta} := \frac{\beta}{\alpha}\|\cdot\|_X^{1/\beta}$, $\beta \in]0,1[$, $\alpha \in]0,\infty[$, $\ell : Z \to \ell(Z) \subseteq X$ is a bijection, $\varnothing \neq C \subseteq Z$ is ℓ-convex and ℓ-closed, then:*

(i) $\mathrm{int}(\mathrm{efd}(\Psi_{\alpha,\beta})) = X$;

(ii) $D_{\ell,\Psi_{\alpha,\beta}}$ *is an information on* Z*;*

(iii) $\forall\psi\in Z\;\exists!\;\overleftarrow{\mathfrak{P}}^{D_{\ell,\Psi_{\alpha,\beta}}}_C(\psi)$;

(iv) $\forall(\phi,\psi)\in C\times Z$

$$D_{\ell,\Psi_{\alpha,\beta}}(\phi,\overleftarrow{\mathfrak{P}}^{D_{\ell,\Psi_{\alpha,\beta}}}_C(\psi)) + D_{\ell,\Psi_{\alpha,\beta}}(\overleftarrow{\mathfrak{P}}^{D_{\ell,\Psi_{\alpha,\beta}}}_C(\psi),\psi) \leq D_{\ell,\Psi_{\alpha,\beta}}(\phi,\psi); \tag{31}$$

(v) if C is ℓ-affine, then $\leq$ in 'then' case of (31) turns into $=$;

(vi) if $(X,\|\cdot\|_X)$ is Fréchet differentiable, and has a Radon–Riesz property, and ℓ is norm-to-norm continuous, then $\overleftarrow{\mathfrak{P}}^{D_{\ell,\Psi_{\alpha,\beta}}}_C$ is norm-to-norm continuous on Z.

If, furthermore, $(X^\star,\|\cdot\|_{X^\star})$ is locally uniformly convex, $\varnothing \neq \widetilde{C} \subseteq Z$, and $\mathfrak{D}^{\mathrm{G}}\Psi_{\alpha,\beta}(\widetilde{C})$ is ℓ-convex and ℓ-closed, then:

(vii) $\forall\psi\in Z\;\exists!\;\overrightarrow{\mathfrak{P}}^{D_{\ell,\Psi_{\alpha,\beta}}}_{\widetilde{C}}(\psi)$;

(viii) $\forall(\phi,\psi)\in Z\times\widetilde{C}$

$$D_{\ell,\Psi_{\alpha,\beta}}(\phi,\overleftarrow{\mathfrak{P}}^{D_{\ell,\Psi_{\alpha,\beta}}}_{\widetilde{C}}(\psi)) + D_{\ell,\Psi_{\alpha,\beta}}(\overleftarrow{\mathfrak{P}}^{D_{\ell,\Psi_{\alpha,\beta}}}_{\widetilde{C}}(\psi),\psi) \leq D_{\ell,\Psi_{\alpha,\beta}}(\phi,\psi); \tag{32}$$

(ix) if $\mathfrak{D}^{\mathrm{G}}\Psi_{\alpha,\beta}(\widetilde{C})$ is ℓ-affine, then $\leq$ in 'then' case of (32) turns into $=$.

Proof. (i) follows from the finiteness of the values of $\Psi_{\alpha,\beta}$; (ii)–(ix) follows from Propositions 2.7, 2.9, and 2.10, combined with Remark 3.4 and Proposition 3.2. ∎

Remark 3.6. The results in Propositions 4.2, 4.7, 4.15, and 4.18 do not depend explicitly on the particular form of $\Psi = \Psi_{\alpha,\beta}$, but only on the fact that its further properties (including the properties of $\overleftarrow{\mathfrak{P}}^{D_\Psi}$ and $\overrightarrow{\mathfrak{P}}^{D_\Psi}$) are determined, via Propositions 2.7–2.10, by the norm geometric properties of an underlying reflexive Banach space. Hence, it is natural to ask about more general class of functions on reflexive Banach spaces $(X, \|\cdot\|_X)$, which would allow for a suitable control by means of the differentiability and convexity properties of norm geometry of $(X, \|\cdot\|_X)$. This can be achieved by consideration of a class of functions [11, p. 200]

$$\Psi_\varphi(x) := \int_0^{\|x\|_X} \mathrm{d}t\, \varphi(t), \tag{33}$$

where $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is positive, strictly increasing, continuous, $\varphi(0) = 0$, and $\lim_{t\to\infty} \varphi(t) = \infty$ [25, p. 348]. (In particular, $\Psi_{\alpha,\beta} = \Psi_{\varphi_{\alpha,\beta}}$ with $\varphi_{\alpha,\beta}(t) = \frac{1}{\alpha} t^{1/\beta - 1}$.) However, since this requires us to develop suitable generalisations of the convex analytic results contained in Propositions 2.7–2.10, these resuls will be provided in another paper.

4 Application to nonreflexive base normed spaces

If $(Y, \|\cdot\|_Y)$ is partially ordered by $\geq$, then $Y^+ := \{x \in Y \mid x \geq 0\}$. All examples below feature $(Y, \|\cdot\|_Y)$ given by some kind of a radially compact base normed space $(V, \|\cdot\|_V)$. Such spaces provide the setting for the (linear) convex operational generalisation of quantum theory (a.k.a. "generalised probability theory" or "postquantum theory"), with state space given by $V_1^+ := \{\phi \in V^+ \mid \|x\|_V = 1\}$. For the sake of generality, we deal with $Z \subseteq V$ whenever possible. However, the restriction of these results to $Z \subseteq V^+$ or $Z \subseteq V_1^+$ is straightforward.

4.1 L_p spaces and Mazur maps

Definition 4.1. [118, p. 58] *For any W^*-algebra $\mathcal{N}$, and $\gamma_1, \gamma_2 \in]0, \infty[$, a **noncommutative Mazur map** is defined as*

$$\ell_{\gamma_1,\gamma_2} : L_{1/\gamma_1}(\mathcal{N}) \ni x = u_x|x| \mapsto u_x|x|^{\gamma_2/\gamma_1} \in L_{1/\gamma_2}(\mathcal{N}), \tag{34}$$

where $x = u_x|x|$ is the unique polar decomposition of x, while the functional analytic meaning of the symbol '$|x|^{\gamma_2/\gamma_1}$' is given in [58, p. 196]. Also, $\ell_{\gamma_2} := \ell_{1,\gamma_2}$.

Proposition 4.2. *Let $\mathcal{N}$ be a W^*-algebra, $\gamma, \beta \in]0, 1[$, $\lambda, \alpha \in]0, \infty[$, $\varnothing \neq C \subseteq \mathcal{N}_\star$, and let $\Psi \in \Gamma^{\mathrm{G}}(L_{1/\gamma}(\mathcal{N}), \|\cdot\|_{1/\gamma})$ be strictly convex on $\mathrm{efd}(\Psi) = L_{1/\gamma}(\mathcal{N})$. Then:*

- *(i) $D_{\lambda\ell_\gamma,\Psi}$ is an information on $\mathcal{N}_\star$;*
- *(ii) if Ψ is Euler–Legendre and C is $\lambda\ell_\gamma$-convex and closed, then C is left $D_{\lambda\ell_\gamma,\Psi}$-Chebyshëv, while $\overleftarrow{\mathfrak{P}}_C^{D_{\lambda\ell_\gamma,\Psi}}$ satisfies*

$$D_{\lambda\ell_\gamma,\Psi}(\phi, \overleftarrow{\mathfrak{P}}_C^{D_{\lambda\ell_\gamma,\Psi}}(\psi)) + D_{\lambda\ell_\gamma,\Psi}(\overleftarrow{\mathfrak{P}}_C^{D_{\lambda\ell_\gamma,\Psi}}(\psi), \psi) \leq D_{\lambda\ell_\gamma,\Psi}(\phi, \psi)\ \forall(\phi, \psi) \in C \times \mathcal{N}_\star, \tag{35}$$

  with $\leq$ replaced by $=$ if C is $\lambda\ell_\gamma$-affine;
- *(iii) if $\Psi^{\mathbf{F}} \in \Gamma^{\mathrm{G}}(L_{1/(1-\gamma)}(\mathcal{N}), \|\cdot\|_{1/(1-\gamma)})$ is totally convex, $\mathfrak{D}^{\mathrm{G}}\Psi(C)$ is $\lambda\ell_\gamma$-convex and closed, then C is right $D_{\lambda\ell_\gamma,\Psi}$-Chebyshëv, while $\overrightarrow{\mathfrak{P}}_C^{D_{\lambda\ell_\gamma,\Psi}}$ satisfies*

$$D_{\lambda\ell_\gamma,\Psi}(\phi, \overleftarrow{\mathfrak{P}}_C^{D_{\lambda\ell_\gamma,\Psi}}(\psi)) + D_{\lambda\ell_\gamma,\Psi}(\overleftarrow{\mathfrak{P}}_C^{D_{\lambda\ell_\gamma,\Psi}}(\psi), \psi) \leq D_{\lambda\ell_\gamma,\Psi}(\phi, \psi)\ \forall(\phi, \psi) \in \mathcal{N}_\star \times C, \tag{36}$$

  with $\leq$ replaced by $=$ if $\mathfrak{D}^{\mathrm{G}}\Psi(C)$ is $\lambda\ell_\gamma$-affine;

(iv) if $\Psi = \Psi_{\alpha,\beta} = \frac{\beta}{\alpha}\|\cdot\|_{1/\gamma}^{1/\beta}$, *then:*

a) (i)–(iii) hold for $D_{\lambda\ell_\gamma,\Psi} = D_{\lambda\ell_\gamma,\Psi_{\alpha,\beta}}$*;*

b) $\overleftarrow{\mathfrak{P}}_C^{D_{\lambda\ell_\gamma,\Psi_{\alpha,\beta}}}$ *is norm-to-norm continuous on* $\mathcal{N}_\star$*;*

c) $\forall(\phi,\psi)\in\mathcal{N}_\star\times\mathcal{N}_\star$

$$D_{\lambda\ell_\gamma,\Psi_{\alpha,\beta}}(\phi,\psi) = \frac{\lambda^{1/\beta}}{\alpha}\left(\beta\|\phi\|_1^{\gamma/\beta} + (1-\beta)\|\psi\|_1^{\gamma/\beta} - \|\psi\|_1^{\gamma(\frac{1}{\beta}-\frac{1}{\gamma})}\mathrm{re}\int u_\phi|\phi|^\gamma u_\psi|\psi|^{1-\gamma}\right)\in\mathbb{R}^+. \quad (37)$$

where the symbol $\int$ *is understood in the sense of* [58, Eqn. (3.12')]*;*

Proof. (i)–(iii) ℓ_γ is a norm-to-norm homeomorphism from $(\mathcal{N}_\star,\|\cdot\|_1)\cong(L_1(\mathcal{N}),\|\cdot\|_1)$ to $(L_{1/\gamma}(\mathcal{N}),\|\cdot\|_{1/\gamma})$ for any $\gamma\in]0,1[$ [118, Lemm. 3.2]. The rest follows from Proposition 3.2.

(iv) Since $\mathrm{int}(\mathrm{efd}(\Psi_{\alpha,\beta})) = L_{1/\gamma}(\mathcal{N})$, we have $(\lambda\ell_\gamma)^{-1}(\mathrm{int}(\mathrm{efd}(\Psi_{\alpha,\beta}))) = \mathcal{N}_\star$. Equation (37) follows by a direct calculation from the formula (30), using [82, Lemm. 3.1],

$$L_{1/\gamma}(\mathcal{N})\ni x\mapsto j(x) = \|x\|_{1/\gamma}^{2-1/\gamma}u_x|x|^{1/\gamma-1}\in L_{1/(1-\gamma)}(\mathcal{N}) \quad (38)$$

(the latter following from [133, Prop. 24] [71, p. 162]; cf. also [76, Eqn. (11)]), with $x = u_x|x|$. For any $\gamma\in]0,1[$, $(L_{1/\gamma}(\mathcal{N}),\|\cdot\|_{1/\gamma})$ is uniformly convex [99, Lemm. 8.1, 8.2] [57, Thm. 5.3]. Together with the Banach duality [81, Thm. 3.4.3] [71, Thm. 10.(2)] [133, Thm. 32.(2)]

$$(L_{1/\gamma}(\mathcal{N}),\|\cdot\|_{1/\gamma})^\star\cong(L_{1/(1-\gamma)}(\mathcal{N}),\|\cdot\|_{1/(1-\gamma)})\ \forall\gamma\in]0,1[, \quad (39)$$

this implies uniform Fréchet differentiability of $(L_{1/\gamma}(\mathcal{N}),\|\cdot\|_{1/\gamma})$ for $\gamma\in]0,1[$. Uniform convexity of a Banach space entails its Radon–Riesz property, local uniform convexity, reflexivity, and strict convexity, while uniform Fréchet differentiability entails (Fréchet differentiability and thus) Gateaux differentiability. Hence, $\Psi_{\alpha,\beta}$ is Euler–Legendre on any $(L_{1/\gamma}(\mathcal{N}),\|\cdot\|_{1/\gamma})$, and $\Psi^{\mathbf{F}}_{\alpha,\beta}$ is totally convex on any $(L_{1/\gamma}(\mathcal{N}),\|\cdot\|_{1/\gamma})$, by means of Propositions 2.7, 2.10, and Remark 3.4. The rest follows from Proposition 3.5.(ii)–(ix) and Remark 2.11.

□

Corollary 4.3. *(i)* $D_{\lambda\ell_\gamma,\Psi_{\alpha,\beta}} = D_{\ell_\gamma,\Psi_{\alpha\lambda^{-1/\beta},\beta}}$.

(ii) For $\lambda=1$, $\beta=\gamma$, $\alpha=\gamma(1-\gamma)$, *we obtain* $\Psi_{\gamma(1-\gamma),\gamma}(x) = \frac{1}{1-\gamma}\|x\|_{1/\gamma}^{1/\gamma}$ $\forall x\in L_{1/\gamma}(\mathcal{N})$, *and* $\forall\phi,\psi\in\mathcal{N}_\star$

$$D_{\ell_\gamma,\Psi_{\gamma(1-\gamma),\gamma}}(\phi,\psi) = D_{\frac{1}{\gamma}\ell_\gamma,\Psi_{\gamma^{1-1/\gamma}(1-\gamma),\gamma}}(\phi,\psi) \quad (40)$$

$$= \frac{\|\phi\|_1}{1-\gamma} + \frac{\|\psi\|_1}{\gamma} + \frac{\mathrm{re}\int u_\phi|\phi|^\gamma u_\psi|\psi|^{1-\gamma}}{\gamma(1-\gamma)} =: D_\gamma(\phi,\psi). \quad (41)$$

Proof. Follows from (37) by a direct calculation. □

Remark 4.4. Identification of D_γ as $D_{\ell_\gamma,\Psi_{\gamma(1-\gamma),\gamma}}$, provided in Corollary 4.3.(ii), is new. Up to reformulation in weight-independent terms, provided in [83, Eqn. (41)], the formula (41) was obtained in [76, §8] (cf. also [111, Eqn. (42)]) as $D_\Psi(\frac{1}{\gamma}\ell_\gamma(\phi),\frac{1}{\gamma}\ell_\gamma(\psi))$ with Ψ equal to $\Psi_{\gamma^{1-1/\gamma}(1-\gamma),\gamma}$ (but not identified there as an example of $\Psi_{\alpha,\beta}$, although the corresponding D_Ψ was explicitly identified as a Brègman functional). Hence, Proposition 4.2.(iii) provides a generalisation of [76, Prop. 8.1.(i)–(ii), Prop. 8.2.(ii)] to all pairs $(\ell_\gamma,\Psi_{\alpha,\beta})$ with any (α,β,γ), not necessarily $(\gamma(1-\gamma),\gamma,\gamma)$.

Definition 4.5. *Let A be a semifinite JBW-algebra with a Jordan product $\bullet$ and a unit $\mathbb{I}$, let τ be a faithful normal semifinite trace on A, $\gamma_1,\gamma_2\in]0,\infty[$. Then we define a* ***nonassociative Mazur map*** *as*

$$\ell_{\gamma_1,\gamma_2}:L_{1/\gamma_1}(A,\tau)\ni x=s_x\bullet|x|\mapsto s_x\bullet|x|^{\gamma_2/\gamma_1}\in L_{1/\gamma_2}(A,\tau), \tag{42}$$

where $x=s_x\bullet|x|$ is a polar decomposition with $s_x\in A$ such that $s_x^2=\mathbb{I}$. Also, $\ell_{\gamma_2}:=\ell_{1,\gamma_2}$.

Proposition 4.6. *Let A be a semifinite JBW-algebra with a unit $\mathbb{I}$, τ a faithful normal semifinite trace on A, $\gamma_1,\gamma_2\in]0,1]$. Then ℓ_{γ_1,γ_2} is $\min\{\frac{\gamma_2}{\gamma_1},1\}$-Lipschitz–Hölder continuous on $(B(L_{1/\gamma_1}(A,\tau),\|\cdot\|_{1/\gamma_1}))^+:=\{x\geq 0\mid x\in B(L_{1/\gamma_1}(A,\tau),\|\cdot\|_{1/\gamma_1})\}$. In particular, ℓ_γ is γ-Lipschitz–Hölder continuous on $(B(A_\star,\|\cdot\|_1))^+$.*

Proof. Any faithful normal semifinite trace $\bar{\tau}$ on a semifinite JBW-algebra J can be extended to a faithful normal semifinite trace $\tilde{\tau}$ on an enveloping von Neumann algebra $\tilde{J}$ of J [12, Thm. 2]. The type of $\tilde{J}$ is the same as the type of J [12, Thm. 8]. Let now J be a JBW-subalgebra of A, generated by $\mathbb{I}$ and $x,y\in A$. Given any formula of inequality involving τ and $\{x,y,\mathbb{I}\}$, this formula holds if it is true under replacing J and $\bar{\tau}=\tau$ by $\tilde{J}$ and $\tilde{\tau}$, respectively [13, Rem. (p. 94)]. The inequality formula (9) of t-Lipschitz–Hölder continuity of ℓ_{γ_1,γ_2} on $(B(L_{1/\gamma_1}(A,\tau),\|\cdot\|_{1/\gamma_1}))^+$ is

$$\exists c>0\;\forall x,y\in L_{1/\gamma_1}(A,\tau)\;\left(\tau\left(\left|x^{\gamma_2/\gamma_1}-y^{\gamma_2/\gamma_1}\right|^{1/\gamma_2}\right)\right)^{\gamma_2}\leq c(\tau(|x-y|^{1/\gamma_1}))^{t\gamma_1}. \tag{43}$$

Since $x,y\in A$, the result follows from the fact that ℓ_{γ_1,γ_2} is $\min\{\frac{\gamma_2}{\gamma_1},1\}$-Lipschitz–Hölder continuous on $B(L_{1/\gamma_1}(\mathcal{N}),\|\cdot\|_{1/\gamma_1})$ for any W*-algebra $\mathcal{N}$ [121, Thm. (p. 37)]. □

Proposition 4.7. *Let A be a semifinite JBW-algebra with a Jordan product $\bullet$ and a faithful normal semifinite trace τ, $\varnothing\neq C\subseteq A_\star$, $\gamma,\beta\in]0,1[$, $\lambda,\alpha\in]0,\infty[$. Let $\Psi\in\Gamma^{\mathrm{G}}(L_{1/\gamma}(A,\tau),\|\cdot\|_{1/\gamma})$ be strictly convex on $\mathrm{efd}(\Psi)=L_{1/\gamma}(A,\tau)$. Then:*

- *(i) $D_{\lambda\ell_\gamma,\Psi}$ is an information on $A_\star$;*
- *(ii) if Ψ is Euler–Legendre and C is $\lambda\ell_\gamma$-convex and $\lambda\ell_\gamma$-closed, then C is left $D_{\lambda\ell_\gamma,\Psi}$-Chebyshëv, and $\overleftarrow{\mathfrak{P}}^{D_{\lambda\ell_\gamma,\Psi}}_C$ satisfies (35) under replacement of $\mathcal{N}_\star$ by $A_\star$, and with $\leq$ replaced by $=$ if C is $\lambda\ell_\gamma$-affine;*
- *(iii) if $\Psi^{\mathbf{F}}\in\Gamma^{\mathrm{G}}(L_{1/(1-\gamma)}(\mathcal{N}),\|\cdot\|_{1/(1-\gamma)})$ is totally convex, and $\mathfrak{D}^{\mathrm{G}}\Psi(C)$ is $\lambda\ell_\gamma$-convex and $\lambda\ell_\gamma$-closed, then C is right $D_{\lambda\ell_\gamma,\Psi}$-Chebyshëv, and $\overrightarrow{\mathfrak{P}}^{D_{\lambda\ell_\gamma,\Psi}}_C$ satisfies (36) under replacement of $\mathcal{N}_\star$ by $A_\star$, and with $\leq$ replaced by $=$ if $\mathfrak{D}^{\mathrm{G}}\Psi(C)$ is $\lambda\ell_\gamma$-affine;*
- *(iv) in particular, if $\Psi=\Psi_{\alpha,\beta}=\frac{\beta}{\alpha}\|\cdot\|^{1/\beta}_{1/\gamma}$, then (i)–(iii) hold for $D_{\lambda\ell_\gamma,\Psi}=D_{\lambda\ell_\gamma,\Psi_{\alpha,\beta}}=D_{\ell_\gamma,\Psi_{\alpha\lambda^{-1/\beta},\beta}}$, where $\forall\omega,\phi\in A_\star$ $D_{\lambda\ell_\gamma,\Psi_{\alpha,\beta}}(\omega,\phi)=$*

$$\frac{\lambda^{1/\beta}}{\alpha}\left(\beta(\tau(\omega))^{\gamma/\beta}+(1-\beta)(\tau(\phi))^{\gamma/\beta}-(\tau(\phi))^{\gamma/\beta-1}\tau((s_\omega\bullet|\omega|^\gamma)\bullet(s_\phi\bullet|\phi|^{1-\gamma}))\right)\in\mathbb{R}^+; \tag{44}$$

- *(v) if $\varnothing\neq K\subseteq L_{1/\gamma}(A,\tau)$ is convex and closed, then $\overleftarrow{\mathfrak{P}}^{D_{\Psi_{\alpha,\beta}}}_K$ is norm-to-norm continuous on $L_{1/\gamma}(A,\tau)$;*
- *(vi) if $C\subseteq(B(A_\star,\|\cdot\|_1))^+$ is $\lambda\ell_\gamma$-convex and closed, then $\overleftarrow{\mathfrak{P}}^{D_{\lambda\ell_\gamma,\Psi_{\alpha,\beta}}}_C$ is norm-to-norm continuous on $(B(A_\star,\|\cdot\|_1))^+$.*

Proof. The Banach space duality [1, Thm. 2.1.10] [73, Thm. V.3.2]

$$(L_{1/\gamma}(A,\tau),\|\cdot\|_{1/\gamma})^\star\cong(L_{1/(1-\gamma)}(A,\tau),\|\cdot\|_{1/(1-\gamma)})\;\forall\gamma\in]0,1[, \tag{45}$$

together with the uniform convexity of $(L_{1/\gamma}(A,\tau),\|\cdot\|_{1/\gamma})$ $\forall\gamma\in]0,1[$ [13, Thm. 2.5] [74, Cor.12, Cor. 13], imply uniform Fréchet differentiability of $(L_{1/\gamma}(A,\tau),\|\cdot\|_{1/\gamma})$ $\forall\gamma\in]0,1[$. Given a polar decomposition $x=s_x\bullet|x|$ with $s_x\in A$ such that $s_x^2=\mathbb{I}$, the formula $\|x\|_{1/\gamma}^{1-1/\gamma}s_x\bullet|x|^{1/\gamma-1}$ [1, p. 51] [73, Lemm. V.3.3.2^o] (cf. [13, p. 101] and [74, p. 420]) equals to $\mathfrak{D}^{\mathrm{F}}\|x\|_{1/\gamma}$ by [74, Lemm. 14]. Hence, using $j(x)=\frac{1}{2}\mathfrak{D}^{\mathrm{F}}(\|x\|_X^2)=\|x\|_X\mathfrak{D}^{\mathrm{F}}\|x\|_X$, which is valid for any Fréchet differentiable $(X,\|\cdot\|_X)$, we obtain

$$j(x)=\|x\|_{1/\gamma}^{2-1/\gamma}s_x\bullet|x|^{1/\gamma-1}. \tag{46}$$

Furthermore, γ-Lipschitz–Hölder continuity of ℓ_γ, proved in Proposition 4.6, implies (uniform continuity, hence also) norm-to-norm continuity of ℓ_γ on $(B(A_\star,\|\cdot\|_1))^+$. The rest of the proof follows from Propositions 3.2 and 3.5.(ii)–(ix) in the same way as in the Proposition 4.2 and Corollary 4.3.(i). □

Remark 4.8. Any JBW-algebra (hence, also a self-adjoint part of any W*-algebra) is a special case of an archimedean order unit space $(A,\|\cdot\|_A)$ with a distinguished order unit e, which is Banach dual to the radially compact base normed space $(V,\|\cdot\|_V)\cong(A_\star,\|\cdot\|_{A_\star})$. Hence, it is natural to ask whether the above results can be extended to radially compact base normed spaces. If these spaces satisfy an additional spectral duality condition [4, Def. (p. 55)], then they admit spectral theory and functional calculus [4, §7–§8]. The notion of a finite trace τ_{AS} on such $(A,\|\cdot\|_A)$ has been introduced in [4, Def. (p. 107)], and was extended beyond finite case in [134, Def. 2.2]. Construction of a corresponding norm $\|\cdot\|_{1/\gamma}:=(\tau_{\mathrm{AS}}(|\cdot|^{1/\gamma}))^\gamma$ on $A_{1/\gamma,\tau_{\mathrm{AS}}}:=\{x\in A\mid x\geq0,(\tau_{\mathrm{AS}}(x))^\gamma<\infty\}$, implying the construction of Banach spaces $(L_{1/\gamma}(A,\tau_{\mathrm{AS}}):=\overline{A_{1/\gamma,\tau_{\mathrm{AS}}}}^{\|\cdot\|_{1/\gamma}},\|\cdot\|_{1/\gamma})$ with $\gamma\in]0,1[$, was provided in [134, Cor. 3.12]. However, $(L_1(A,\tau_{\mathrm{AS}}),\|\cdot\|_1)\cong(A_\star,\|\cdot\|_{A_\star})$ iff A is a JBW-algebra with $e=\mathbb{I}$ [20, Thm. 6]. An alternative notion of a trace on A, τ_{B}, has been proposed in [19, Def. 1], together with a corresponding norm $\|\cdot\|_1$ on (A,e) [19, Thm. 1], and with a proof that $(L_1(A,\tau_{\mathrm{B}}),\|\cdot\|_1)\cong(A_\star,\|\cdot\|_{A_\star})$ for any order unit A in spectral duality [19, Thm. 2]. Hence, in order to use the Mazur map $\ell_\gamma:V^+\ni x\mapsto x^\gamma\in(L_{1/\gamma}(A,\tau_{\mathrm{B}}),\|\cdot\|_{1/\gamma})^+$, $\gamma\in]0,1[$, to establish a generalisation of our results for $(V,\|\cdot\|_V)$ in spectral duality, the following statements have to be proved: (i) $\|\cdot\|_{1/\gamma}$ determined by a faithful τ_{B} is a norm on $A_{1/\gamma,\tau_{\mathrm{B}}}$; (ii) $(L_{1/\gamma}(A,\tau_{\mathrm{B}}),\|\cdot\|_{1/\gamma})$ are reflexive, Gateaux differentiable, and strictly convex (cf. Proposition 2.7); (iii) they are also Fréchet differentiable and have the Radon–Riesz property (cf. Proposition 2.9); (iv) they are also locally uniformly convex (cf. Proposition 2.10); (v) ℓ_γ is norm-to-norm continuous (cf. Proposition 3.2.(v)). Below we make a first step in this direction, proving (i), which allows us to establish the suitable definitions of $(L_{1/\gamma}(A,\tau_{\mathrm{B}}),\|\cdot\|_{1/\gamma})$ and of the corresponding Mazur map.

Proposition 4.9. *Let $(A,\|\cdot\|_A)$ be an archimedean order unit space, which is in spectral duality with a radially compact base normed space $(V,\|\cdot\|_V)\cong(A_\star,\|\cdot\|_{A_\star})$. Let $\tau:A^+\to\mathbb{R}^+$ be a finite (resp., finite and faithful) Berdikulov trace, as defined by* [19, Def. 1]. *If $\gamma\in]0,\infty]$, then the function $x\mapsto\|x\|_{1/\gamma}:=(\tau(|x|^{1/\gamma}))^\gamma$ is a seminorm (resp., norm) on $A_{1/\gamma}:=\{x\in A\mid\|x\|_{1/\gamma}<\infty\}$.*

Proof. Follows from [134, Cor. 3.12], combined with the fact that a finite Berdikulov trace is a finite Alfsen–Shultz trace [19, Lemm. 1]. □

Definition 4.10. *Let $(A,\|\cdot\|_A)$ be an archimedean order unit space, which is in spectral duality with a radially compact base normed space $(V,\|\cdot\|_V)\cong(A_\star,\|\cdot\|_{A_\star})$. Let τ be a finite faithful Berdikulov trace. Let $\gamma,\gamma_1,\gamma_2\in]0,1]$. Then:*

*(i) the $L_{1/\gamma}(A,\tau)$ **space** is defined as a completion of $A_{1/\gamma}$ in the norm $\|\cdot\|_{1/\gamma}$. Furthermore, $(L_\infty(A,\tau),\|\cdot\|_\infty):=(A,\|\cdot\|_A)$;*

*(ii) (a positive part of) the **postquantum Mazur map** is defined as*

$$(L_{1/\gamma_1}(A,\tau))^+\ni\phi\mapsto\phi^{\gamma_2/\gamma_1}\in(L_{1/\gamma_2}(A,\tau))^+. \tag{47}$$

4.2 Orlicz spaces and Kaczmarz maps

Remark 4.11. In what follows, we will say that a W*-algebra is of type $\mathrm{I}_\infty^{\mathrm{s.f.}}$ iff it is a separable factor of type I_∞.

Definition 4.12. *Let $\mathcal{N}$ be a semifinite W*-algebra, let τ be a faithful normal semifinite trace on $\mathcal{N}$. Let $\mathscr{M}(\mathcal{N},\tau)$ denote the space of all τ-measurable operators affiliated with $\mathcal{N}$* [108, §2] [140, p. 91]. *Then:*

*(i) if $\Upsilon:\mathbb{R}\to\mathbb{R}^+$ is even, convex, and $\Upsilon(u)=0 \iff u=0$, then it will be called an **Orlicz** **function** (cf.* [112, p. 208]*);*

*(ii) a **Young–Birnbaum–Orlicz dual** of an Orlicz function Υ is defined as* [141, p. 226] [22, Eqn. (5)] *(cf.* [97, Eqn. (1)]*)*

$$\mathbb{R}\ni y\mapsto \Upsilon^{\mathbf{Y}}(y):=\sup\{x|y|-\Upsilon(x)\mid x\geq 0\}\in[0,\infty]; \tag{48}$$

we will also denote, for any Orlicz function Υ:

$$\Upsilon'_+ := \text{a right derivative of } \Upsilon, \tag{49}$$

$$\varpi_\Upsilon(\lambda) := \sup\{t>0\mid \Upsilon^{\mathbf{Y}}(\Upsilon'_+(t))\leq\lambda\}, \tag{50}$$

$$\Upsilon\in\mathrm{N} :\iff \lim_{u\to^+0}\frac{\Upsilon(u)}{u}=0 \text{ and } \lim_{u\to\infty}\frac{\Upsilon(u)}{u}=\infty \text{ [22, Def. I.§1.5]}, \tag{51}$$

$$\Upsilon\in\triangle_2^0 :\iff \lim_{u\to^+0}\frac{\Upsilon(2u)}{\Upsilon(u)}<\infty \text{ [22, Eqn. }(\triangle_2)], \tag{52}$$

$$\Upsilon\in\triangle_2^\infty :\iff \limsup_{u\to\infty}\frac{\Upsilon(2u)}{\Upsilon(u)}<\infty \text{ [22, p. 36]}, \tag{53}$$

$$\Upsilon\in\triangle_2 :\iff \sup_{u>0}\frac{\Upsilon(2u)}{\Upsilon(u)}<\infty \text{ [27, p. 494]} \iff \Upsilon\in\triangle_2^0\cap\triangle_2^\infty, \tag{54}$$

$$\Upsilon\in\mathrm{SC}(I) :\iff \Upsilon \text{ is strictly convex on an interval } I\subseteq\mathbb{R}, \tag{55}$$

$$\Upsilon\in\mathrm{C}^1(I) :\iff \Upsilon \text{ is continuously differentiable on an interval } I\subseteq\mathbb{R}. \tag{56}$$

(iii) [117, §2] [106, p. 6] *(=*[105, Def. 2.3.19, p. 111]*)* [131, p. 91] [89, p. 126] *(cf. also* [90, Prop. 2.2]*) a **noncommutative Orlicz space** is defined as*

$$L_\Upsilon(\mathcal{N},\tau):=\{x\in\mathscr{M}(\mathcal{N},\tau)\mid \exists\lambda>0\;\tau(\Upsilon(\lambda x))<\infty\}; \tag{57}$$

*a **noncommutative Morse–Transue–Nakano–Luxemburg norm** on $L_\Upsilon(\mathcal{N},\tau)$ is defined as*

$$\|x\|_\Upsilon:=\inf\{\lambda\geq 0\mid \tau(\Upsilon(x/\lambda))\leq 1\}; \tag{58}$$

*a **noncommutative Orlicz norm** on $L_\Upsilon(\mathcal{N},\tau)$ is defined as*

$$\|x\|_\Upsilon^{\mathrm{O}}:=\sup\{\tau(|xy|)\mid y\in\mathscr{M}(\mathcal{N},\tau),\;\tau(\Upsilon^{\mathbf{Y}}(|y|)\leq 1\}; \tag{59}$$

*(iv) if Υ_1 and Υ_2 are Orlicz functions, then we define a **noncommutative Kaczmarz map** as*

$$\ell_{\Upsilon_1,\Upsilon_2}:L_{\Upsilon_1}(\mathcal{N},\tau)\ni x=u_x|x|\mapsto u_x(\Upsilon_2^{-1}\circ\Upsilon_1)(|x|)\in L_{\Upsilon_2}(\mathcal{N},\tau), \tag{60}$$

where $x=u_x|x|$ is the unique polar decomposition of x;

(v) if $\mathcal{N}$ is either of type $I_\infty^{s.f.}$, or type II_1, or type II_∞, then

$$\widetilde{\mathrm{type}}(\mathcal{N}):=\begin{cases} I_\infty^{s.f.} & : \mathcal{N} \text{ is noncommutative of type } I_\infty^{s.f.}, \text{ or } \mathcal{N}=L_\infty(\mathcal{X},\mu) \text{ with purely atomic and infinite } (\mathcal{X},\mu)\\ II_1 & : \mathcal{N} \text{ is noncommutative of type } II_1, \text{ or } \mathcal{N}=L_\infty(\mathcal{X},\mu) \text{ with nonatomic and finite } (\mathcal{X},\mu)\\ II_\infty & : \mathcal{N} \text{ is noncommutative of type } II_\infty, \text{ or } \mathcal{N}=L_\infty(\mathcal{X},\mu) \text{ with nonatomic and infinite } (\mathcal{X},\mu).\end{cases}$$

Proposition 4.13. *Let $\mathcal{N}$ be a W*-algebra either of type $I_\infty^{s.f.}$, or type II_1, or type II_∞, let τ be a faithful normal semifinite trace on $\mathcal{N}$. Let Υ be an Orlicz function such that $\Upsilon^{\mathbf{Y}}$ is an Orlicz function, and $\Upsilon \in \triangle_2^0$ (resp., $\triangle_2^\infty$; $\triangle_2$) if $\mathcal{N}$ is of type $I_\infty^{s.f.}$ (resp., II_1; II_∞). Then*

$$(L_\Upsilon(\mathcal{N},\tau), \|\cdot\|_\Upsilon)^\star \cong (L_{\Upsilon^{\mathbf{Y}}}(\mathcal{N},\tau), \|\cdot\|^{\mathrm{O}}_{\Upsilon^{\mathbf{Y}}}). \tag{61}$$

Proof. The proof is based on mutual relationship of commutative [125, p. 1292] [86, II.§4.1] and noncommutative [114, 3^o] [103, p. 10] rearrangement invariant spaces, denoted $(E(\mathcal{X},\mu), \|\cdot\|_{E(\mathcal{X},\mu)})$ and $(E(\mathcal{N},\tau), \|\cdot\|_{E(\mathcal{N},\tau)})$, respectively, with $\widetilde{\mathrm{type}}(\mathcal{N}) = \widetilde{\mathrm{type}}(L_\infty(\mathcal{X},\mu))$, $E(\mathcal{N},\tau) := \{x \in \mathscr{M}(\mathcal{N},\tau) \mid x^\tau \in E(\mathcal{X},\mu)\}$, and $\|\cdot\|_{E(\mathcal{N},\tau)} := \|(\cdot)^\tau\|_{E(\mathcal{X},\mu)}$, where x^τ denotes a rearrangement of $x \in E(\mathcal{N},\tau)$ [65, §4] [113, p. 79] [140, Def. 2.2]. If $(E(\mathcal{X},\mu), \|\cdot\|_{E(\mathcal{X},\mu)})$ is order continuous, then it is strongly symmetric [86, Cor. (p. 142)]. If $(E(\mathcal{X},\mu), \|\cdot\|_{E(\mathcal{X},\mu)})$ is strongly symmetric, then $(E(\mathcal{N},\tau), \|\cdot\|_{E(\mathcal{N},\tau)})$ is strongly symmetric [49, Thm. 51] (=[50, Thm. 6.1.2, Prop. 6.8.13.(i)]). If $(E(\mathcal{N},\tau), \|\cdot\|_{E(\mathcal{N},\tau)})$ is strongly symmetric and $\mathcal{N}$ is of type $\mathrm{I}_\infty^{\mathrm{s.f.}}$, II_1, or II_∞, then its noncommutative Köthe dual $((E(\mathcal{N},\tau))^\times, \|\cdot\|^\times_{E(\mathcal{N},\tau)})$ [49, p. 227] satisfies $((E(\mathcal{N},\tau))^\times, \|\cdot\|^\times_{E(\mathcal{N},\tau)}) = ((E(\mathcal{N},\tau))^\star, \|\cdot\|^\star_{E(\mathcal{N},\tau)})$ [49, Thm. 27]. For any Orlicz Υ, if $\Upsilon \in \triangle_2^0$ (resp., $\triangle_2^\infty$; $\triangle_2$) for $\widetilde{\mathrm{type}}(L_\infty(\mathcal{X},\mu)) = \mathrm{I}_\infty^{\mathrm{s.f.}}$ (resp., II_1; II_∞), then $(L_\Upsilon(\mathcal{X},\mu), \|\cdot\|_\Upsilon)$ is order continuous [96, Thm. 2.3.6] and $(L_\Upsilon(\mathcal{X},\mu), \|\cdot\|_\Upsilon)^\star \cong (L_{\Upsilon^{\mathbf{Y}}}(\mathcal{X},\mu), \|\cdot\|^{\mathrm{O}}_{\Upsilon^{\mathbf{Y}}})$ [53, Cor. 2.2.10, Thm. 2.2.11, Thm. 2.1.17]. Hence, $(L_\Upsilon(\mathcal{N},\tau), \|\cdot\|_\Upsilon)^\star \cong (L_{\Upsilon^{\mathbf{Y}}}(\mathcal{N},\tau), \|\cdot\|^\times_\Upsilon = \|(\cdot)^\tau\|^{\mathrm{O}}_{\Upsilon^{\mathbf{Y}}})$. From $\|x\|_\Upsilon \leq 1 \iff \tau(\Upsilon(|x|)) \leq 1\ \forall x \in L_\Upsilon(\mathcal{N},\tau)$ [64, Lemm. 5.40] and a definition of $\|\cdot\|^\times_{E(\mathcal{N},\tau)}$ [49, p. 227] it follows that $\|\cdot\|^{\mathrm{O}}_{\Upsilon^{\mathbf{Y}}} = \|\cdot\|^\times_\Upsilon$. □

Proposition 4.14. *Let $\mathcal{N}$ be a W*-algebra either of type $I_\infty^{s.f.}$, or type II_1, or type II_∞, let τ be a faithful normal semifinite trace on $\mathcal{N}$, let $(\mathcal{X},\mu)$ be a countably finite measure space such that $\widetilde{\mathrm{type}}(L_\infty(\mathcal{X},\mu)) = \widetilde{\mathrm{type}}(\mathcal{N})$. Let $\Upsilon : \mathbb{R} \to \mathbb{R}^+$ be an Orlicz function. Then:*

(i) equivalent are:

1) $(L_\Upsilon(\mathcal{N},\tau), \|\cdot\|_\Upsilon)$ *is strictly convex;*

2) $(L_\Upsilon(\mathcal{X},\mu), \|\cdot\|_\Upsilon)$ *is strictly convex;*

3) $$\begin{cases} \Upsilon \in \triangle_2^0 \cap \mathrm{SC}([0,\Upsilon^{-1}(\frac{1}{2})]) & : \widetilde{\mathrm{type}}(\mathcal{N}) = I_\infty^{s.f.} \\ \Upsilon \in \triangle_2^\infty \cap \mathrm{SC}(\mathbb{R}) & : \widetilde{\mathrm{type}}(\mathcal{N}) = II_1 \\ \Upsilon \in \triangle_2 \cap \mathrm{SC}(\mathbb{R}) & : \widetilde{\mathrm{type}}(\mathcal{N}) = II_\infty; \end{cases}$$

(ii) if $\Upsilon^{\mathbf{Y}} \in \triangle_2^\infty$ (resp., $\triangle_2$) for $\widetilde{\mathrm{type}}(\mathcal{N}) = II_1$ (resp., II_∞), then equivalent are:

1) $(L_\Upsilon(\mathcal{N},\tau), \|\cdot\|_\Upsilon)$ *is Gateaux differentiable;*

2) $(L_\Upsilon(\mathcal{X},\mu), \|\cdot\|_\Upsilon)$ *is Gateaux differentiable;*

3) $$\begin{cases} \Upsilon \in \triangle_2^0 \cap \mathrm{C}^1([0,\Upsilon^{-1}(1)[) & : \widetilde{\mathrm{type}}(\mathcal{N}) = I_\infty^{s.f.} \\ \Upsilon \in \triangle_2^\infty \cap \mathrm{C}^1(\mathbb{R}) & : \widetilde{\mathrm{type}}(\mathcal{N}) = II_1 \\ \Upsilon \in \triangle_2 \cap \mathrm{C}^1(\mathbb{R}) & : \widetilde{\mathrm{type}}(\mathcal{N}) = II_\infty; \end{cases}$$

(iii) equivalent are:

1) $(L_\Upsilon(\mathcal{N},\tau), \|\cdot\|_\Upsilon)$ *has Radon–Riesz property;*

2) $(L_\Upsilon(\mathcal{X},\mu), \|\cdot\|_\Upsilon)$ *has Radon–Riesz property;*

3) $$\begin{cases} \Upsilon \in \triangle_2^0 & : \widetilde{\mathrm{type}}(\mathcal{N}) = I_\infty^{s.f.} \\ \Upsilon \in \triangle_2^\infty \cap \mathrm{SC}(\mathbb{R}) & : \widetilde{\mathrm{type}}(\mathcal{N}) = II_1 \\ \Upsilon \in \triangle_2 \cap \mathrm{SC}(\mathbb{R}) & : \widetilde{\mathrm{type}}(\mathcal{N}) = II_\infty; \end{cases}$$

(iv) if $\Upsilon \in \mathrm{N}$ for $\widetilde{\mathrm{type}}(\mathcal{N}) = II$, then equivalent are:

1) $(L_\Upsilon(\mathcal{N},\tau), \|\cdot\|^O_\Upsilon)$ *is locally uniformly convex;*

2) $(L_\Upsilon(\mathcal{X},\mu),\|\cdot\|^{O}_{\Upsilon})$ is locally uniformly convex;

3) $\begin{cases} \Upsilon\in\triangle_2^0\cap\mathrm{SC}([0,\varpi(1)]),\ \Upsilon^{\mathbf{Y}}\in\triangle_2^0,\ \exists u>0\ \Upsilon^{\mathbf{Y}}(\Upsilon'_+(u))\geq\frac{1}{2} & :\ \widetilde{\mathrm{type}}(\mathcal{N})=I_\infty^{s.f.}\\ \Upsilon\in\triangle_2^\infty\cap\mathrm{SC}(\mathbb{R}),\ \Upsilon^{\mathbf{Y}}\in\triangle_2^\infty & :\ \widetilde{\mathrm{type}}(\mathcal{N})=II_1\\ \Upsilon\in\triangle_2\cap\mathrm{SC}(\mathbb{R}),\ \Upsilon^{\mathbf{Y}}\in\triangle_2 & :\ \widetilde{\mathrm{type}}(\mathcal{N})=II_\infty;\end{cases}$

(v) if $\widetilde{\mathrm{type}}(\mathcal{N})=II$ then equivalent are:

1) $(L_\Upsilon(\mathcal{N},\tau),\|\cdot\|_{\Upsilon})$ is reflexive;

2) $(L_\Upsilon(\mathcal{X},\mu),\|\cdot\|_{\Upsilon})$ is reflexive;

3) $\begin{cases}\Upsilon,\Upsilon^{\mathbf{Y}}\in\triangle_2^\infty & :\ \widetilde{\mathrm{type}}(\mathcal{N})=II_1\\ \Upsilon,\Upsilon^{\mathbf{Y}}\in\triangle_2 & :\ \widetilde{\mathrm{type}}(\mathcal{N})=II_\infty;\end{cases}$

(vi) if $\widetilde{\mathrm{type}}(\mathcal{N})=I_\infty^{s.f.}$ then: $\Upsilon,\Upsilon^{\mathbf{Y}}\in\triangle_2^0 \iff$ (v).2) $\Rightarrow$ (v).1).

Proof. In what follows, we restrict the mutual characterisations of norm geometric properties of corresponding commutative and noncommutative rearrangement invariant Banach spaces (denoted, respectively, $(E(\mathcal{X},\mu),\|\cdot\|_{E(\mathcal{X},\mu)})$ and $(E(\mathcal{N},\tau),\|\cdot\|_{E(\mathcal{N},\tau)})$) to the case of noncommutative Orlicz spaces, and combine them with the corresponding characterisations of norm geometric properties of commutative Orlicz spaces by means of the properties of Orlicz function.

(i) 3) $\iff$ 2) is proved in: [79, Rem. 2] for $\widetilde{\mathrm{type}}(\mathcal{N})=\mathrm{I}_\infty^{\mathrm{s.f.}}$; [136, Cor. 5] for $\widetilde{\mathrm{type}}(\mathcal{N})=\mathrm{II}_1$; [60, Thm. 1.7] for $\widetilde{\mathrm{type}}(\mathcal{N})=\mathrm{II}_\infty$. 1) $\iff$ 2) follows as a special case of: [9, Cor. 2.5.(i)] for $(E(\mathcal{N},\tau),\|\cdot\|_{E(\mathcal{N},\tau)})$ with $\widetilde{\mathrm{type}}(\mathcal{N})=\mathrm{I}_\infty^{\mathrm{s.f.}}$; [36, Thm. 1.1] [44, Cor. 5.6] for $(E(\mathcal{N},\tau),\|\cdot\|_{E(\mathcal{N},\tau)})$ with $\widetilde{\mathrm{type}}(\mathcal{N})\in\{\mathrm{II}_1,\mathrm{II}_\infty\}$.

(ii) 3) $\iff$ 2) is proved in: [67, Thm. 13] for $\widetilde{\mathrm{type}}(\mathcal{N})=\mathrm{I}_\infty^{\mathrm{s.f.}}$; [67, Thm. 11] for $\widetilde{\mathrm{type}}(\mathcal{N})\in\{\mathrm{II}_1,\mathrm{II}_\infty\}$. 1) $\iff$ 2) follows as a special case of: [9, Cor. 2.5.(ii)] for $(E(\mathcal{N},\tau),\|\cdot\|_{E(\mathcal{N},\tau)})$ with $\widetilde{\mathrm{type}}(\mathcal{N})=\mathrm{I}_\infty^{\mathrm{s.f.}}$; [45, Cor. 2.13] for $(E(\mathcal{N},\tau),\|\cdot\|_{E(\mathcal{N},\tau)})$ with $\widetilde{\mathrm{type}}(\mathcal{N})\in\{\mathrm{II}_1,\mathrm{II}_\infty\}$. Both [9, Cor. 2.5.(ii)] and [45, Cor. 2.13] require order continuity of $(E(\mathcal{X},\mu),\|\cdot\|_{E(\mathcal{X},\mu)})$. For $(L_\Upsilon(\mathcal{X},\mu),\|\cdot\|_\Upsilon)$ this is imposed by $\Upsilon\in\triangle_2^0$ (resp., $\triangle_2^\infty$; $\triangle_2$) for $\widetilde{\mathrm{type}}(\mathcal{N})=\mathrm{I}_\infty^{\mathrm{s.f.}}$(resp., II_1; II_∞). [45, Cor. 2.13] requires also $\lim_{t\to\infty}x^\mu(t)=0\ \forall x\in(E(\mathcal{X},\mu))^\times$, which is satisfied if $((E(\mathcal{X},\mu))^\times,\|\cdot\|_{(E(\mathcal{X},\mu))^\times})$ is order continuous (cf., e.g., [52, p. 730]).

(iii) 3) $\iff$ 2) is proved in: [72, Thm. 2.8] for $\widetilde{\mathrm{type}}(\mathcal{N})=\mathrm{I}_\infty^{\mathrm{s.f.}}$; [138, Thm. (p. 341)] for $\widetilde{\mathrm{type}}(\mathcal{N})\in\{\mathrm{II}_1,\mathrm{II}_\infty\}$. 1) $\iff$ 2) follows as a special case of: [8, Thm. I] for $(E(\mathcal{N},\tau),\|\cdot\|_{E(\mathcal{N},\tau)})$ with $\widetilde{\mathrm{type}}(\mathcal{N})=\mathrm{I}_\infty^{\mathrm{s.f.}}$; [35, Thm. 2.7] (cf. [44, Thm. 16.4]) for $(E(\mathcal{N},\tau),\|\cdot\|_{E(\mathcal{N},\tau)})$ with $\widetilde{\mathrm{type}}(\mathcal{N})\in\{\mathrm{II}_1,\mathrm{II}_\infty\}$.

(iv) 3) $\iff$ 2) is proved in: [43, Cor. 2.18] for $\widetilde{\mathrm{type}}(\mathcal{N})=\mathrm{I}_\infty^{\mathrm{s.f.}}$; [32, Thm. 1] for $\widetilde{\mathrm{type}}(\mathcal{N})=\mathrm{II}_1$; [109, Thm. 3.5] for $\widetilde{\mathrm{type}}(\mathcal{N})=\mathrm{II}_\infty$. 1) $\iff$ 2) follows as a special case of: [37, Thm. 2.2] (cf. [87, Rem. 3]) for $(E(\mathcal{N},\tau),\|\cdot\|_{E(\mathcal{N},\tau)})$ with $\widetilde{\mathrm{type}}(\mathcal{N})=\mathrm{I}_\infty^{\mathrm{s.f.}}$; [37, Cor. 2.1]+[87, Thm. 2] (cf. [37, Thm. 2.1]) with $\widetilde{\mathrm{type}}(\mathcal{N})\in\{\mathrm{II}_1,\mathrm{II}_\infty\}$.

(v)–(vi) Equivalence of (v).2) with the corresponding conditions on Υ and $\Upsilon^{\mathbf{Y}}$ is proved in [96, Thm. 5] for $\widetilde{\mathrm{type}}(\mathcal{N})\in\{\mathrm{I}_\infty^{\mathrm{s.f.}},\mathrm{II}_1,\mathrm{II}_\infty\}$. If $\widetilde{\mathrm{type}}(\mathcal{N})=\mathrm{II}_1$ (resp., II_∞), then (v).1) $\iff$ (v).2) follows as a special case of [130, Thm. 1.3.6] (resp., [51, Thm. 4.8]). Otherwise, (v).1) $\Rightarrow$ (v).2) follows as a special case of: [9, p. 153] (cf. also [50, Prop. 6.8.15]) for $(E(\mathcal{N},\tau),\|\cdot\|_{E(\mathcal{N},\tau)})$ with $\widetilde{\mathrm{type}}(\mathcal{N})=\mathrm{I}_\infty^{\mathrm{s.f.}}$; [49, Thm. 54.(v)] (cf. [52, Cor. 5.16]) for $(E(\mathcal{N},\tau),\|\cdot\|_{E(\mathcal{N},\tau)})$ with $\widetilde{\mathrm{type}}(\mathcal{N})=\mathrm{II}_\infty$. [51, Thm. 4.8] requires $(E(\mathcal{X},\mu),\|\cdot\|_{E(\mathcal{X},\mu)})$ to be strongly symmetric. This is implied by the order continuity of $(E(\mathcal{X},\mu),\|\cdot\|_{E(\mathcal{X},\mu)})$ [86, Cor. (p. 142)]. The latter is implied for $(L_\Upsilon(\mathcal{X},\mu),\|\cdot\|_\Upsilon)$ for $\widetilde{\mathrm{type}}(\mathcal{N})=\mathrm{II}_\infty$ by $\Upsilon\in\triangle_2$ due to [96, Thm. 2.3.6].

□

Proposition 4.15. *Let Υ be an Orlicz function, let $\mathcal{N}$ be a W^*-algebra, either of type $I_\infty^{s.f.}$, or of type II_1, or of type II_∞, and let $\Upsilon, \Upsilon^{\mathbf{Y}} \in \triangle_2^0$ (resp., $\triangle_2^\infty$; $\triangle_2$) if $\mathcal{N}$ is of type $I_\infty^{s.f.}$ (resp., II_1; II_∞). Let τ be a faithful normal semifinite trace on $\mathcal{N}$. Let $\beta \in]0,1[$, $\alpha \in]0,\infty[$. Let $\Psi \in \Gamma^{\mathrm{G}}(L_\Upsilon(\mathcal{N},\tau), \|\cdot\|_\Upsilon)$ be strictly convex on $\mathrm{efd}(\Psi) = L_\Upsilon(\mathcal{N},\tau)$. Then:*

(i) $D_{\ell_\Upsilon,\Psi}$ is an information on $\mathcal{N}_\star$;

(ii) if Ψ is Euler–Legendre, and $\varnothing \neq C \subseteq \mathcal{N}_\star$ is ℓ_Υ-convex and ℓ_Υ-closed, then C is left $D_{\ell_\Upsilon,\Psi}$-Chebyshëv, while $\overleftarrow{\mathfrak{P}}_C^{D_{\ell_\Upsilon,\Psi}}$ satisfies

$$D_{\ell_\Upsilon,\Psi}(\phi, \overleftarrow{\mathfrak{P}}_C^{D_{\ell_\Upsilon,\Psi}}(\psi)) + D_{\ell_\Upsilon,\Psi}(\overleftarrow{\mathfrak{P}}_C^{D_{\ell_\Upsilon,\Psi}}(\psi), \psi) \leq D_{\ell_\Upsilon,\Psi}(\phi,\psi) \; \forall(\phi,\psi) \in C \times \mathcal{N}_\star, \tag{62}$$

with $\leq$ replaced by $=$ if C is ℓ_Υ-affine;

(iii) if $\Upsilon^{\mathbf{Y}}$ is an Orlicz function, $\Psi^{\mathbf{F}} \in \Gamma^{\mathrm{G}}(L_{\Upsilon^{\mathbf{Y}}}(\mathcal{N},\tau), \|\cdot\|^{\mathrm{O}}_{\Upsilon^{\mathbf{Y}}})$ is totally convex, $\varnothing \neq C \subseteq \mathcal{N}_\star$, and $\mathfrak{D}^{\mathrm{G}}\Psi(C)$ is ℓ_Υ-convex and ℓ_Υ-closed, then C is right $D_{\ell_\Upsilon,\Psi}$-Chebyshëv, while $\overrightarrow{\mathfrak{P}}_C^{D_{\ell_\Upsilon,\Psi}}$ satisfies

$$D_{\ell_\Upsilon,\Psi}(\phi, \overrightarrow{\mathfrak{P}}_C^{D_{\ell_\Upsilon,\Psi}}(\phi)) + D_{\ell_\Upsilon,\Psi}(\overrightarrow{\mathfrak{P}}_C^{D_{\ell_\Upsilon,\Psi}}(\phi), \psi) \leq D_{\ell_\Upsilon,\Psi}(\phi,\psi) \; \forall(\phi,\psi) \in \mathcal{N}_\star \times C, \tag{63}$$

with $\leq$ replaced by $=$ if $\mathfrak{D}^{\mathrm{G}}\Psi(C)$ is ℓ_Υ-affine;

(iv) if $\Psi = \Psi_{\alpha,\beta} = \frac{\beta}{\alpha}\|\cdot\|_\Upsilon^{1/\beta}$, then:

a) the conditions of (i) are satisfied;

b) if

$$\begin{cases} \Upsilon \in \mathrm{SC}([0,\Upsilon^{-1}(\frac{1}{2})]) \cap \mathrm{C}^1([0,\Upsilon^{-1}(1)]) & : \widetilde{\mathrm{type}}(\mathcal{N}) = I_\infty^{s.f.} \\ \Upsilon \in \mathrm{SC}(\mathbb{R}) \cap \mathrm{C}^1(\mathbb{R}) & : \widetilde{\mathrm{type}}(\mathcal{N}) = II_1 \\ \Upsilon \in \mathrm{SC}(\mathbb{R}) \cap \mathrm{C}^1(\mathbb{R}) & : \widetilde{\mathrm{type}}(\mathcal{N}) = II_\infty, \end{cases} \tag{64}$$

then the conditions of (ii) are satisfied;

c) if (64) holds, and, additionally, $\Upsilon^{\mathbf{Y}}$ is an Orlicz function such that

$$\begin{cases} \Upsilon^{\mathbf{Y}} \in \mathrm{SC}([0,\varpi_{\Upsilon^{\mathbf{Y}}}(1)]),\ \exists u > 0\ \Upsilon((\Upsilon^{\mathbf{Y}})'_+) \geq \frac{1}{2} & : \widetilde{\mathrm{type}}(\mathcal{N}) = I_\infty^{s.f.} \\ \Upsilon^{\mathbf{Y}} \in \mathrm{N} & : \widetilde{\mathrm{type}}(\mathcal{N}) = II, \end{cases} \tag{65}$$

then the conditions of (iii) are satisfied;

(v) if Ψ and Υ are as in (iv).c), and $\varnothing \neq K \subseteq L_\Upsilon(\mathcal{N},\tau)$ is convex and closed, then $\overleftarrow{\mathfrak{P}}_K^{D_\Psi}$ is norm-to-norm continuous on $L_\Upsilon(\mathcal{N},\tau)$.

Proof. (i)–(iii) follow from Propositions 3.2 and 4.14. (vi) and (v) follow from Propositions 3.5 and 2.9, combined with Proposition 4.14, and the fact [7, Thm. 3.9] that, if $(X, \|\cdot\|_X)$ is reflexive, then $(X, \|\cdot\|_X)$ is Fréchet differentiable iff $((X^\star, \|\cdot\|_{X^\star})$ is strictly convex and has the Radon–Riesz property), and with the fact that local uniform convexity implies both the Radon–Riesz property [137, Prop. (p. 352)] and strict convexity. In order to identify the conditions (64) as sufficient for the (iii) case, we use Proposition 4.13. □

Proposition 4.16. *Let Υ be an Orlicz function such that $\Upsilon(1) = 1$, $\lim_{u\to+0} \frac{\Upsilon(u)}{u} = 0$, $\lim_{u\to\infty} \frac{\Upsilon(u)}{u} = \infty$, and there exist $t, s \in \mathbb{R}^+$ such that $t < s$, $u \mapsto \frac{\Upsilon^{-1}(u)}{u^t}$ is nondecreasing, and $u \mapsto \frac{\Upsilon^{-1}(u)}{u^s}$ is non-increasing. Let $(\mathcal{X},\mu)$ be a measure space, such that one of the following conditions holds:*

a) $(\mathcal{X},\mu)$ is purely atomic, $\mu(\mathcal{X}) = \infty$, $\Upsilon \in \triangle_2^0 \cap \mathrm{SC}([0,\Upsilon^{-1}(\frac{1}{2})]) \cap \mathrm{C}^1([0,\Upsilon^{-1}(1)])$, $\liminf_{u\to 0} \frac{\Upsilon(2u)}{\Upsilon(u)} > 2$;

b) $(\mathcal{X},\mu)$ *is atomless,* $\mu(\mathcal{X})<\infty$*,* $\Upsilon\in\triangle_2^\infty\cap\mathrm{SC}(\mathbb{R})\cap\mathrm{C}^1(\mathbb{R})$*,* $\liminf_{u\to\infty}\frac{\Upsilon(2u)}{\Upsilon(u)}>2$*;*

c) $(\mathcal{X},\mu)$ *is atomless,* $\mu(\mathcal{X})=\infty$*,* $\Upsilon\in\triangle_2\cap\mathrm{SC}(\mathbb{R})\cap\mathrm{C}^1(\mathbb{R})$*,* $\liminf_{u\to 0}\frac{\Upsilon(2u)}{\Upsilon(u)}>2$*,* $\liminf_{u\to\infty}\frac{\Upsilon(2u)}{\Upsilon(u)}>2$*.*

Let $\Psi=\Psi_{\beta,\beta}=\|\cdot\|_\Upsilon^{1/\beta}:L_\Upsilon(\mathcal{X},\mu)\to\mathbb{R}^+$*,* $\beta\in\,]0,1[$*. Let* $\varnothing\neq C\subseteq B(L_1(\mathcal{X},\mu),\|\cdot\|_1)$ *be* ℓ_Υ*-convex and closed. Then:*

(i) $\overleftarrow{\mathfrak{P}}_C^{D_{\ell_\Upsilon,\Psi_{\beta,\beta}}}$ *satisfies (62), and is norm-to-norm continuous on* $B(L_1(\mathcal{X},\mu),\|\cdot\|_1)$ *and on* $S(L_1(\mathcal{X},\mu),\|\cdot\|_1)$*;*

(ii) $D_{\ell_\Upsilon,\Psi_{\varphi_{\beta,\beta}}}:(L_1(\mathcal{X},\mu))^+\times(L_1(\mathcal{X},\mu))^+\to\mathbb{R}^+$ *reads* $\forall\omega,\phi\in(L_1(\mathcal{X},\mu))^+$

$$D_{\ell_\Upsilon,\Psi_{\varphi_{\beta,\beta}}}(\omega,\phi)=\left\|\Upsilon^{-1}(\omega)\right\|_\Upsilon^{1/\beta}+\frac{1-\beta}{\beta}\left\|\Upsilon^{-1}(\phi)\right\|_\Upsilon^{1/\beta}-\frac{1}{\beta}\left\|\Upsilon^{-1}(\phi)\right\|_\Upsilon^{1/\beta-1}\frac{\int\mu\Upsilon^{-1}(\omega)\Upsilon'\left(\frac{\Upsilon^{-1}(\phi)}{\|\Upsilon^{-1}(\phi)\|_\Upsilon}\right)}{\int\mu\Upsilon^{-1}(\phi)\Upsilon'\left(\frac{\Upsilon^{-1}(\phi)}{\|\Upsilon^{-1}(\phi)\|_\Upsilon}\right)},\tag{66}$$

where $\Upsilon'(t):=\frac{\mathrm{d}\Upsilon(t)}{\mathrm{d}t}>0\ \forall t>0$*;*

(iii) in particular, for $\bar{\Upsilon}(\omega,\phi):=\int\mu\Upsilon^{-1}(\omega)\Upsilon'(\Upsilon^{-1}(\phi))$*,*

$$D_{\ell_\Upsilon,\Psi_{\varphi_{\beta,\beta}}}(\omega,\phi)=\frac{1}{\beta}\left(1-\frac{\bar{\Upsilon}(\omega,\phi)}{\bar{\Upsilon}(\phi,\phi)}\right)\quad\forall\omega,\phi\in(S(L_1(\mathcal{X},\mu)),\|\cdot\|_1)^+.\tag{67}$$

Proof. (i) Let $\Upsilon:\mathbb{R}\to\mathbb{R}^+$ be even, strictly convex, continuously differentiable, with $\Upsilon(u)=0$ iff $u=0$, $\lim_{u\to+0}\frac{\Upsilon(u)}{u}=0$, $\lim_{u\to\infty}\frac{\Upsilon(u)}{u}=\infty$. Then $\Upsilon^{\mathbf{Y}}\in\triangle_2^\infty$ (resp., $\Upsilon^{\mathbf{Y}}\in\triangle_2^0$) iff $\liminf_{u\to\infty}\frac{\Upsilon(2u)}{\Upsilon(u)}>2$ (resp., $\liminf_{u\to 0}\frac{\Upsilon(2u)}{\Upsilon(u)}>2$) [94, Eqn. (5)] [85, Thm. 4.2]. Under additional conditions of $\Upsilon(1)=1$, and existence of $t,s\in\mathbb{R}^+$, such that $t<s$, $u\mapsto\frac{\Upsilon^{-1}(u)}{u^t}$ is nondecreasing, and $u\mapsto\frac{\Upsilon^{-1}(u)}{u^s}$ is nonincreasing, the uniform homeomorphy of Kaczmarz map between unit balls (resp., unit spheres) of $(L_1(\mathcal{X},\mu),\|\cdot\|_1)$ and $(L_\Upsilon(\mathcal{X},\mu),\|\cdot\|_\Upsilon)$ has been proved in [48, Thm. 2.4] (resp., [48, Cor. 2.5]) (= [47, Thm 4.5] (resp., [47, Cor. 4.6])). The rest follows by a conjunction of Propositions 3.5.(vi), 4.14, and 4.15.

(ii) By [67, Lemm. 2] (cf. also [85, Eqn. (18.29)], [95, §3], [116, Eqn. (10)]), if $\|\cdot\|_\Upsilon$ is Gateaux differentiable, then

$$\mathfrak{D}^{\mathrm{G}}\|x\|_\Upsilon=\frac{\Upsilon'\left(\frac{x}{\|x\|_\Upsilon}\right)}{\int\mu\frac{x}{\|x\|_\Upsilon}\Upsilon'\left(\frac{x}{\|x\|_\Upsilon}\right)}\quad\forall x\in L_\Upsilon(\mathcal{X},\mu)\setminus\{0\}.\tag{68}$$

(iii) Follows from (66) by a direct calculation.

□

Remark 4.17. Propositions 4.14 and 4.15 avoid consideration of the noncommutative Orlicz spaces over type I_n W*-algebras with finite n. This is due to a priori different behaviour of Orlicz spaces over finite atomic measure spaces, as compared with Orlicz spaces over infinite atomic measure spaces, combined with the deficit of results characterising the linear norm-geometric properties of the finite atomic case (with a notable exception of [72, Thm. 2.2, Thm. 2.3]). Cf. also [66, Thm. 3].

4.3 Generalised spin factors

Proposition 4.18. *Let* $(V, \|\cdot\|_V)$ *be a generalised spin factor* [21, Def. 4]*, i.e.* $V = X \oplus \mathbb{R}$*, where* $(X, \|\cdot\|_X)$ *is a reflexive Banach space, and*

$$\forall\phi = (x, \lambda) \in V \quad \begin{cases} \phi \geq 0 \;:\Longleftrightarrow\; \lambda \geq \|x\|_X \\ \|\phi\|_V := \max\{|\lambda|, \|x\|_X\}. \end{cases} \tag{69}$$

Let $\beta \in]0, 1[$*,* $\alpha \in]0, \infty[$*,* $V_1^+ := \{x \in V^+ \mid \|\cdot\|_V = 1\}$*, and define*

$$\ell_{/\mathbb{R}} : V_1^+ \ni \phi =: (x, 1) \mapsto x \in B(X, \|\cdot\|_X). \tag{70}$$

Then:

(i) $(V, \|\cdot\|_V)$ *satisfies spectral duality condition of* [4, Def. (p. 55)] *iff* $\Psi_{\alpha,\beta} : X \to \mathbb{R}^+$ *is Euler–Legendre with respect to* $\|\cdot\|_X$*;*

(ii) $D_{\ell_{/\mathbb{R}}, \Psi_{\alpha,\beta}} : V_1^+ \times V_1^+ \to \mathbb{R}^+$ *is an information on* V_1^+*;*

(iii) if $\varnothing \neq C \subseteq V_1^+$ *is* $\ell_{/\mathbb{R}}$*-convex* $\ell_{/\mathbb{R}}$*-closed, then* C *is left* $D_{\ell_{/\mathbb{R}}, \Psi_{\alpha,\beta}}$*-Chebyshëv, and* $D_{\ell_{/\mathbb{R}}, \Psi_{\alpha,\beta}}$ *satisfies*

$$D_{\ell_{/\mathbb{R}}, \Psi_{\alpha,\beta}}(\phi, \overleftarrow{\mathfrak{P}}_C^{D_{\ell_{/\mathbb{R}}, \Psi_{\alpha,\beta}}}(\psi)) + D_{\ell_{/\mathbb{R}}, \Psi_{\alpha,\beta}}(\overleftarrow{\mathfrak{P}}_C^{D_{\ell_{/\mathbb{R}}, \Psi_{\alpha,\beta}}}(\psi), \psi) \leq D_{\ell_{/\mathbb{R}}, \Psi_{\alpha,\beta}}(\phi, \psi) \; \forall(\phi, \psi) \in C \times V_1^+, \tag{71}$$

with $\leq$ *replaced by* $=$ *if* C *is* $\ell_{/\mathbb{R}}$*-affine.*

Proof. According to [21, Thm. 1] (recently independently rediscovered in [77, Thm. 6.6]), a generalised spin factor $(V = X \oplus \mathbb{R}, \|\cdot\|_V)$ satisfies spectral duality condition iff $(X, \|\cdot\|_X)$ is Gateaux differentiable and strictly convex. Combining this with Proposition 2.7 and Remark 3.4 gives (i). The rest then follows from Proposition 3.5. □

Acknowledgements

I thank: Lucien Hardy, Ravi Kunjwal, Jerzy Lewandowski, and Marcin Marciniak for hosting me as a scientific visitor; Francesco Buscemi, Paolo Gibilisco, and Anna Jenčová for hospitality and discussions; Michał Eckstein and Karol Horodecki for help. This research was supported in part by Polish National Science Centre (NCN grants 2015/18/E/ST2/00327 and 2021/42/A/ST2/00356) and by Perimeter Institute for Theoretical Physics. Research at Perimeter Institute is supported by the Government of Canada through Industry Canada and by the Province of Ontario through the Ministry of Research and Innovation.

References

Cyrillic names and titles were transliterated from original using the system: ц = c, ч = ch, х = kh, ж = zh, ш = sh, щ = š, и = i, й = ĭ, і = ī, ы = y, ю = yu, я = ya, ë = ë, э = è, ъ = ‘, ь = ’, and analogously for capitalised letters, with an exception of Х = H at the beginnings of words (which is bijective due to the lack of ыа and ыу combinations). Whenever possible, Chinese Mandarin (resp., Cantonese) names and titles were nonbijectively romanised from original, using pīnyīn (resp., toneless Yale).

[1] Abdullaev R.Z., 1984, *Prostranstva* L_p *dlya polukonechnykh JBW-algebr*, Ph.D. thesis, Institut Matematiki, Akademiya nauk Uzbekskoĭ SSR, Tashkent. www.fuw.edu.pl/∼kostecki/scans/abdullaev1984.pdf. ↑ 2, 11, 12.

[2] Al’ber Ya.I., 1993, *Generalized projection operators in Banach spaces: properties and applications*, Funct. Diff. Equat. **1**, 1–21. arXiv:funct-an/9311002 (also: www.ariel.ac.il/wp/fde/wp-content/uploads/sites/97/2020/02/1993-1.pdf). ↑ 5.

[3] Al'ber Ya.I., Butnariu D., 1997, *Convergence of Bregman projection methods for solving consistent convex feasibility problems in reflexive Banach spaces*, J. Optim. Theor. Appl. **92**, 33–61. ↑ 4.
[4] Alfsen E.M., Shultz F.W., 1976, *Non-commutative spectral theory for affine function spaces on convex sets*, Mem. Amer. Math. Soc. **172**, American Mathematical Society, Providence. ↑ 12, 18.
[5] Amari S. (甘利俊一), 1985, *Differential–geometrical methods in statistics*, Lecture Notes in Statistics **28**, Springer, Berlin. ↑ 5.
[6] Amari S. (甘利俊一), Nagaoka H. (長岡浩司), 1993, *Jōhō kika no hōhō (情報幾何の方法)*, Iwanami Shoten, Tōkyō (Engl. transl. rev. ed.: 2000, *Methods of information geometry*, Transl. Math. Monogr. **191**, American Mathematical Society, Providence). ↑ 6, 7.
[7] Anderson K.W., 1960, *Midpoint local uniform convexity, and other geometric properties of Banach spaces*, Ph.D. thesis, University of Illinois, Urbana. www.fuw.edu.pl/∼kostecki/scans/anderson1960.pdf. ↑ 16.
[8] Arazy J., 1981, *More on convergence in unitary matrix spaces*, Proc. Amer. Math. Soc. **83**, 44–48. ↑ 15.
[9] Arazy J., 1981, *On the geometry of the unit ball of unitary matrix spaces*, Integr. Eq. Oper. Th. **4**, 151–171. ↑ 15.
[10] Ascoli G., 1932, *Sugli spazi lineari metrici e loro varietà lineari*, Ann. Mat. Pura Appl. Ser. IV **10**, 33–81, 203. ↑ 4.
[11] Asplund E., 1967, *Positivity of duality mappings*, Bull. Amer. Math. Soc. **73**, 200–203. euclid:bams/1183528777. ↑ 9.
[12] Ayupov Sh.A., 1982, *Extension of traces and type criterions for Jordan algebras of self-adjoint operators*, Math. Z. **181**, 253–268. ↑ 11.
[13] Ayupov Sh.A., 1986, *Klassifikaciya i predstavlenie uporyadochennykh ĭordanovykh algebr*, Fan, Tashkent. www.fuw.edu.pl/∼kostecki/scans/ayupov1986.pdf. ↑ 2, 11, 12.
[14] Banach S., 1922, *Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales*, Fund. Math. **3**, 133–181. matwbn.icm.edu.pl/ksiazki/fm/fm3/fm3120.pdf. ↑ 3.
[15] Bauschke H.H., Borwein J.M., Combettes P.L., 2001, *Essential smoothness, essential strict convexity, and Legendre functions in Banach spaces*, Commun. Contemp. Math. **3**, 615–647. people.ok.ubc.ca/bauschke/Research/18.pdf. ↑ 4, 5.
[16] Bauschke H.H., Borwein J.M., Combettes P.L., 2003, *Bregman monotone optimization algorithms*, Soc. Industr. Appl. Math. J. Contr. Optim. **42**, 596–636. people.ok.ubc.ca/bauschke/Research/28.pdf. ↑ 1, 5.
[17] Bauschke H.H., Macklem M.S., Wáng X.F. (王宪福), 2011, *Chebyshev sets, Klee Sets, and Chebyshev centers with respect to Bregman distances: recent results and open problems*, in: Bauschke H.H., Burachik R.S., Combettes P.L., Elser V., Luke D.R., Wolkowicz H. (eds.), *Fixed-point algorithms for inverse problems in science and engineering*, Springer, Berlin, pp. 1–21. arXiv:1003.3127. ↑ 5.
[18] Bauschke H.H., Noll D., 2002, *The method of forward projections*, J. Nonlin. Conv. Anal. **3**, 191–205. people.ok.ubc.ca/bauschke/Research/20.pdf. ↑ 5.
[19] Berdikulov M.A., 2005, *Ponyatie sleda na prostranstvakh c poryadkovoĭ ediniceĭ i prostranstvo integriruemykh èlementov*, Mat. trudy **8**, 39–48. mathnet.ru:mt61 (Engl. transl.: 2006, *The notion of a trace on order-unit spaces and the space of integrable elements*, Siber. Adv. Math. **16**:2, 34–42). ↑ 12.
[20] Berdikulov M.A., 2006, *Order-unit spaces which are Banach dual spaces*, Zh. mat. fiz., anal., geom. **2**, 130–137. mathnet.ru:jmag27. ↑ 12.
[21] Berdikulov M.A., Odilov S.T., 1995, *Obobšennye spin-faktory*, Uzbek. mat. zhurn. **1995**:1, 10–15. www.fuw.edu.pl/∼kostecki/scans/berdikulovodilov1995.pdf. ↑ 18.
[22] Birnbaum Z.W., Orlicz W., 1931, *Über die Verallgemeinerung des Begriffes der zueinander konjugierten Potenzen*, Stud. Math. **3**, 1–67. matwbn.icm.edu.pl/ksiazki/sm/sm3/sm311.pdf. ↑ 13.
[23] Brègman L.M., 1966, *Relaksacionnyĭ metod nahozhdeniya obšeĭ tochki vypuklykh mnozhestv i ego primenenie dlya zadach optimizacii*, Dokl. Akad. nauk SSSR **171**, 1019–1022. mathnet.ru:dan32741 (Engl. transl.: 1966, *A relaxation method of finding a common point of convex sets and its application to problems of optimization*, Soviet Math. Dokl. **7**, 1578–1581). ↑ 1, 2, 3, 5.
[24] Brøndsted A., 1964, *Conjugate convex functions in topological vector spaces*, Kong. Danske Vidensk. Selsk. Mat.-fys. Medd. **34**, 1–26. ↑ 4.
[25] Browder F.E., 1965, *Multi-valued monotone nonlinear mappings and duality mappings in Banach spaces*, Trans. Amer. Math. Soc. **118**, 338–351. ↑ 9.
[26] Brunk H.D., Ewing G.M., Utz W.R., 1957, *Minimizing integrals in certain classes of monotone functions*, Pacific J. Math. **7**, 833–847. euclid:pjm/1103043663. ↑ 2.
[27] Burkill J.C., 1928, *Strong and weak convergence of functions of general type*, Proc. London Math. Soc.

Ser. 2, **28**, 493–500. ↑ 13.
[28] Butnariu D., Censor Y., Reich S., 1997, *Iterative averaging of entropic projections for solving stochastic convex feasibility problems*, Comput. Optim. Appl. **8**, 21–39. math.haifa.ac.il/yair/BCRcoap97.pdf. ↑ 4.
[29] Butnariu D., Iusem A.N., 1997, *Local moduli of convexity and their application to finding almost common fixed points of measurable families of operators*, in: Censor Y., Reich S. (eds.), *Recent developments in optimization theory and nonlinear analysis. AMS/IMU special session on optimization and nonlinear analysis, May 24–26, 1996, Jerusalem, Israel*, Contemp. Math. **204**, American Mathematical Society, Providence, pp. 61–91. ↑ 4.
[30] Butnariu D., Iusem A.N., 2000, *Totally convex functions for fixed point computation and infinite dimensional optimization*, Kluwer, Dordrecht. ↑ 4.
[31] Butnariu D., Iusem A.N., Zălinescu C., 2003, *On uniform convexity, total convexity and convergence of the proximal point and outer Bregman projection algorithms in Banach spaces*, J. Conv. Anal. **10**, 35–61. ↑ 5.
[32] Chén S.T. (陈述涛), 1986, *Some rotundities of Orlicz spaces with Orlicz norm*, Bull. Pol. Acad. Sci. Math. **34**, 585–596. www.fuw.edu.pl/∼kostecki/scans/chen1986.pdf. ↑ 15.
[33] Chencov N.N., 1968, *Nesimmetrichnoe rasstoyanie mezhdu raspredeleniyami veroyatnosteĭ, entropiya i teorema Pifagora*, Mat. zametki **4**, 323–332. mathnet.ru:mz9452 (Engl. transl.: 1968, *Nonsymmetrical distance between probability distributions, entropy and the theorem of Pythagoras*, Math. Notes Acad. Sci. USSR **4**, 686–691). ↑ 1, 5.
[34] Chencov N.N., 1972, *Statisticheskie reshayuŝie pravila i optimal'nye vyvody*, Nauka, Moskva (Engl. transl.: 1982, *Statistical decision rules and optimal inference*, Transl. Math. Monogr. **53**, American Mathematical Society, Providence). ↑ 5.
[35] Chilin V.I., Dodds P.G., Sukochev F.A., 1997, *The Kadec–Klee property in symmetric spaces of measurable operators*, Israel J. Math. **97**, 203–219. ↑ 15.
[36] Chilin V.I., Krygin A.W., Sukochev F.A., 1992, *Extreme points of convex fully symmetric sets of measurable operators*, Integr. Eq. Oper. Th. **15**, 186–226. ↑ 15.
[37] Chilin V.I., Krygin A.W., Sukochev F.A., 1992, *Uniform and local uniform convexity of non-commutative symmetric spaces of measurable operators*, Math. Proc. Cambridge Phil. Soc. **111**, 355–368. ↑ 15.
[38] Ciorănescu I., 1990, *Geometry of Banach spaces, duality mappings, and nonlinear problems*, Kluwer, Dordrecht. ↑ 5.
[39] Clarkson J.A., 1936, *Uniformly convex spaces*, Trans. Amer. Math. Soc. **40**, 396–414. ↑ 3.
[40] Csiszár I., 1995, *Generalized projections for non-negative functions*, Acta Math. Hung. **68**, 161–185. ↑ 3.
[41] Csiszár I., Matúš F., 2012, *Generalized minimizers of convex integral functions, Bregman distance, pythagorean identities*, Kybernetika **48**, 637–689. arXiv:1202.0666. ↑ 2.
[42] Ştrătilă S., 1981, *Modular theory in operator algebras*, Editura Academiei/Abacus Press, Bucureşti/Tunbridge Wells. ↑ 2.
[43] Cuī Y.Ā. (崔云安), Hudzik H., Nowak M., Płuciennik R., 1999, *Some geometric properties in Orlicz sequence spaces equipped with Orlicz norm*, J. Conv. Anal. **6**, 91–113. ↑ 15.
[44] Czerwińska M.M., Kamińska A., 2017, *Geometric properties of noncommutative symmetric spaces of measurable operators and unitary matrix ideals*, Comment. Math. (Prace Mat.) **57**, 45–122. arXiv:1704.02033. ↑ 15.
[45] Czerwińska M.M., Kamińska A., Kubiak D., 2012, *Smooth and strongly smooth points in symmetric spaces of measurable operators*, Positivity **16**, 29–51. ↑ 15.
[46] Day M.M., 1944, *Uniform convexity in factor and conjugate spaces*, Ann. Math. **45**, 375–385. ↑ 3.
[47] Delpech S., 2005, *Approximations höldériennes de fonctions entre espaces d'Orlicz. Modules asymptotiques uniformes*, Ph.D. thesis, Université Bordeaux I, Bordeaux. ↑ 17.
[48] Delpech S., 2005, *Modulus of continuity of the Mazur map between unit balls of Orlicz spaces and approximation by Hölder mappings*, Illinois J. Math. **49**, 195–216. ↑ 17.
[49] Dodds P.G., de Pagter B., 2014, *Normed Köthe spaces: a non-commutative viewpoint*, Indag. Math. N.S. **25**, 206–249. ↑ 14, 15.
[50] Dodds P.G., de Pagter B., Sukochev F.A., 2023, *Noncommutative integration and operator theory*, Springer, Cham. ↑ 14, 15.
[51] Dodds P.G., Dodds T.K.-Y., 1995, *Some aspects of the theory of symmetric operator spaces*, Quaest. Math. **18**, 47–89. ↑ 15.
[52] Dodds P.G., Dodds T.K.-Y., de Pagter B., 1993, *Noncommutative Köthe duality*, Trans. Amer. Math.

Soc. **339**, 717–750. www.ams.org/journals/tran/1993-339-02/S0002-9947-1993-1113694-3/. ↑ 15.

[53] Edgar G.A., Sucheston L., 1992, *Stopping times and directed processes*, Cambridge University Press, Cambridge. ↑ 14.

[54] Eguchi S., 1983, *Second order efficiency of minimum contrast estimators in a curved exponential family*, Ann. Statist. **11**, 793–803. euclid:aos/1176346246. ↑ 3, 6.

[55] Eguchi S., 1985, *A differential geometric approach to statistical inference on the basis of contrast functionals*, Hiroshima Math. J. **15**, 341–391. euclid:hmj/1206130775. ↑ 6.

[56] Euler L., 1770, *Institutionum calculi integralis*, Vol. **3**, Academia Scientiarum Imperialis, Sankt-Peterburg. pbc.gda.pl/dlibra/publication/20282/edition/16413 (Russ. transl.: 1958, *Integral'noe ischislenie*, Vol. **3**, Gosudarstvennoe izdatel'stvo fiziko-matematicheskoĭ literatury, Moskva; Engl. transl.: 2010, *Foundations of integral calculus*, Vol. **3**, www.17centurymaths.com/contents/integralcalculusvol3.htm). ↑ 4.

[57] Fack T., Kosaki H. (幸崎秀樹), 1986, *Generalized s-numbers of τ-measurable operators*, Pacific J. Math. **123**, 269–300. euclid:pjm/1102701004. ↑ 10.

[58] Falcone A.J., Takesaki M. (竹崎正道), 2001, *The non-commutative flow of weights on a von Neumann algebra*, J. Funct. Anal. **182**, 170–206. www.math.ucla.edu/~mt/papers/QFlow-Final.tex.pdf. ↑ 2, 9, 10.

[59] Fenchel W., 1949, *On conjugate convex functions*, Canadian J. Math. **1**, 73–77. www.cs.cmu.edu/~suvrit/teach/papers/1949_fenchel_conjugate_convex_functions.pdf. ↑ 4.

[60] Fennich R., 1980, *Stricte convexité de la norme modulaire des espaces intégraux de type Orlicz et $\triangle_2$-condition*, Travaux du Séminaire d'Analyse Convexe **10**, fascicule 1, exposé n° 9, Université des sciences et techniques du Languedoc, U.E.R. de mathématiques, Montpellier–Perpignan. www.fuw.edu.pl/~kostecki/scans/fennich1980.pdf. ↑ 15.

[61] Fréchet M., 1925, *Les espaces abstraits topologiquement affines*, Acta Math. **47**, 25–52. ↑ 3.

[62] Gateaux M.R., 1914, *Sur la représentation des fonctionnelles continues*, Atti Real. Accad. Lincei. Rend. Class. sci. fis., mat. nat. Ser. 5 **23**, 310–315. www.fuw.edu.pl/~kostecki/scans/gateaux1914.pdf. ↑ 4.

[63] Gibilisco P., Pistone G., 1998, *Connections on non-parametric statistical manifolds by Orlicz space geometry*, Inf. Dim. Anal. Quant. Prob. Relat. Top. **1**, 325–347. art.torvergata.it/retrieve/handle/2108/49737/18230/IDAQP1998.pdf. ↑ 8.

[64] Goldstein S., Labuschagne L.E., 2020, *Notes on noncommutative L^p and Orlicz spaces*, Wydawnictwo Uniwersytetu Łódzkiego, Łódź. hdl.handle.net/11089/33192. ↑ 14.

[65] Grothendieck A., 1955, *Réarrangements de fonctions et inégalités de convexité dans les algèbres de von Neumann munies d'une trace*, Sém. Bourbaki **3**, 127–139. numdam:B_1954-1956__3__127_0. ↑ 14.

[66] Grząślewicz R., 1985, *Finite dimensional Orlicz spaces*, Bull. Pol. Acad. Sci. Math. **33**, 277–283. www.fuw.edu.pl/~kostecki/scans/grzaslewicz1985.pdf. ↑ 17.

[67] Grząślewicz R., Hudzik H., 1992, *Smooth points of Orlicz spaces equipped with Luxemburg norm*, Math. Nachr. **155**, 31–45. Erratum: 1994, Math. Nachr. **167**, 330. ↑ 15, 17.

[68] Hahn H., 1927, *Über lineare Gleichungssysteme in linearen Räumen*, J. reine angew. Math. **157**, 214–229. gdz.sub.uni-goettingen.de/dms/load/img/?PPN=GDZPPN002170183&IDDOC=252029. ↑ 3.

[69] Harremoës P., 2017, *Quantum information on spectral sets*, in: Bossert M., Hanly S., ten Brink S., Ulukus S. (eds.), *2017 IEEE International Symposium on Information Theory*, IEEE, Piscataway, pp. 1549–1553. arXiv:1701.06688. ↑ 2.

[70] Helly E., 1921, *Über Systeme linearer Gleichungen mit unendlich vielen Unbekannten*, Monatsh. Math. Phys. **31**, 60–91. ↑ 3.

[71] Hilsum M., 1981, *Les espaces L^p d'une algèbre de von Neumann définies par la dérivée spatiale*, J. Funct. Anal. **40**, 151–169. ↑ 10.

[72] Hudzik H., Pallaschke D., 1997, *On some convexity properties of Orlicz sequence spaces equipped with the Luxemburg norm*, Math. Nachr. **186**, 167–185. ↑ 15, 17.

[73] Iochum B., 1984, *Cônes autopolaires et algèbres de Jordan*, Springer, Berlin. ↑ 2, 11, 12.

[74] Iochum B., 1986, *Non-associative L^p-spaces*, Pacific J. Math. **122**, 417–433. euclid:pjm/1102701894. ↑ 2, 12.

[75] Iusem A., Gárciga Otero R., 2001, *Inexact versions of proximal point and augmented lagrangian algorithms in Banach spaces*, Numer. Funct. Anal. Optim. **22**, 609–640. ↑ 8.

[76] Jenčová A., 2005, *Quantum information geometry and non-commutative L_p spaces*, Inf. Dim. Anal. Quant. Prob. Relat. Top. **8**, 215–233. www.mat.savba.sk/~jencova/pdf/lpspaces.pdf (early version: arXiv:math-ph/0311004). ↑ 2, 8, 10.

[77] Jenčová A., Pulmannová S., 2021, *Geometric and algebraic aspects of spectrality in order unit spaces: a comparison*, J. Math. Anal. Appl. **504**:125360, 1–32. arXiv:2102.01628. ↑ 18.

[78] Kaczmarz S., 1933, *O homeomorfji pewnych przestrzeni. — The homeomorphy of certain spaces*, Bull. Internat. Acad. Polon. Sci. Lett., Class. Sci. Math. Natur.: Sér. A, Sci. Math. **1933**:2, 145–148. www.fuw.edu.pl/~kostecki/scans/kaczmarz1933.pdf. ↑ 8.

[79] Kamińska A., 1981, *Rotundity of Orlicz–Musielak sequence spaces*, Bull. Acad. Polon. Sci., Sér. sci. math. **29**, 137–144. ↑ 15.

[80] Klee V., 1953, *Convex bodies and periodic homeomorphisms in Hilbert space*, Trans. Amer. Math. Soc. **74**, 10–43. ↑ 5.

[81] Kosaki H. (幸崎秀樹), 1980, *Canonical L^p-spaces associated with an arbitrary abstract von Neumann algebra*, Ph.D. thesis, University of California, Los Angeles. dmitripavlov.org/scans/kosaki-thesis.pdf. ↑ 10.

[82] Kosaki H. (幸崎秀樹), 1984, *Applications of uniform convexity of noncommutative L^p-spaces*, Trans. Amer. Math. Soc. **283**, 265–282. www.ams.org/journals/tran/1984-283-01/S0002-9947-1984-0735421-6. ↑ 10.

[83] Kostecki R.P., 2011, *The general form of γ-family of quantum relative entropies*, Open Sys. Inf. Dyn. **18**, 191–221. arXiv:1106.2225. ↑ 2, 8, 10.

[84] Kostecki R.P., 2013, *W^*-algebras and noncommutative integration*, arXiv:1307.4818. ↑ 2.

[85] Krasnosel'skiĭ M.A., Rutickiĭ Ya.B., 1958, *Vypuklye funkcii i prostranstva Orlicha*, Gosudarstvennoe izdatel'stvo fiziko-matematicheskoĭ literatury, Moskva (Engl. transl.: 1961, *Convex functions and Orlicz spaces*, Nordhoff, Groningen). ↑ 17.

[86] Kreĭn S.G., Petunin Yu.I., Semënov E.M., 1978, *Interpolyaciya lineĭnykh operatorov*, Nauka, Moskva (Engl. rev. transl.: 1982, *Interpolation of linear operators*, Transl. Math. Monogr. **54**, American Mathematical Society, Providence). ↑ 14, 15.

[87] Krygin A.V., Sukochev F.A., Chilin V.I., 1991, *Ravnomernaya vypuklost' i lokal'naya ravnomernaya vypuklost' simmetrichnykh prostranstv izmerimykh operatorov*, Dokl. Akad. nauk SSSR **318**, 555–558. mathnet.ru:dan6127 (Engl. transl.: 1991, *Uniform convexity and local uniform convexity of symmetric spaces of measurable operators*, Soviet Math. Dokl. **43**, 445–448). ↑ 15.

[88] Kullback S., 1959, *Information theory and statistics*, Wiley, New York (2nd rev. ed.: 1968, Dover, New York). ↑ 1, 5.

[89] Kunze W., 1990, *Noncommutative Orlicz spaces and generalized Arens algebras*, Math. Nachr. **147**, 123–138. ↑ 13.

[90] Labuschagne L.E., Majewski W.A., 2011, *Maps on noncommutative Orlicz spaces*, Illinois J. Math. **55**, 1053–1081. arXiv:0902.3078. ↑ 13.

[91] Lauritzen S.L., 1987, *Statistical manifolds*, in: Amari S. (甘利俊一), Barndorff-Nielsen O.E., Kass R.E., Lauritzen S.L., Rao C.R., *Differential geometry in statistical inference*, Institute of Mathematical Statistics, Hayward, pp. 163–216. ↑ 6.

[92] Legendre A.-M., 1787, *Mémoire sur l'intégration de quelques équations aux différences partielles*, Mém. Académ. Royale Sci. **1787**, 309–351. thibaut.horel.org/convex/legendre-1787.pdf. ↑ 4.

[93] Lovaglia A.R., 1955, *Locally uniformly convex Banach spaces*, Trans. Amer. Math. Soc. **78**, 225–238. ↑ 3.

[94] Lozinskiĭ S.M., 1946, *On convergence in mean of Fourier series*, Dokl. Akad. nauk SSSR **51**, 7–10. ↑ 17.

[95] Lumer G., 1963, *On the isometries of reflexive Orlicz spaces*, Ann. Inst. Fourier Grenoble **13**, 99–109. ↑ 17.

[96] Luxemburg W.A.J., 1955, *Banach function spaces*, Ph.D. thesis, Technische Hogeschool te Delft, Delft. repository.tudelft.nl/assets/uuid:252868f8-d63f-42e4-934c-20956b86783f/71308.pdf. ↑ 14, 15.

[97] Mandelbrojt S., 1939, *Sur les fonctions convexes*, Compt. Rend. Acad. Sci. Paris **209**, 977–978. thibaut.horel.org/convex/mandelbrot-39.pdf. ↑ 13.

[98] Martín-Márquez V., Reich S., Sabach S., 2012, *Right Bregman nonexpansive operators in Banach spaces*, Nonlin. Anal. Theor. Meth. Appl. **75**, 5448–5465. ssabach.net.technion.ac.il/files/2015/12/MRS2012-1.pdf. ↑ 1, 5.

[99] Masuda T. (増田哲也), 1983, *L_p-spaces for von Neumann algebra with reference to a faithful normal semifinite weight*, Publ. Res. Inst. Math. Sci. Kyōto Univ. **19**, 673–727. doi:10.2977/prims/1195182447. ↑ 10.

[100] Mazur S.M., 1929, *Une remarque sur l'homéomorphie des champs fonctionnels*, Stud. Math. **1**, 83–85. matwbn.icm.edu.pl/ksiazki/sm/sm1/sm114.pdf. ↑ 2, 8.

[101] Mazur S.M., 1933, *Über konvexe Mengen in linearen normierten Räumen*, Stud. Math. **4**, 70–84.

matwbn.icm.edu.pl/ksiazki/sm/sm4/sm4113.pdf. ↑ 4.
[102] Mazur S.M., 1933, *Über schwache Konvergenz in den Räumen (L^p)*, Stud. Math. **4**, 128–133. matwbn.icm.edu.pl/ksiazki/sm/sm4/sm4119.pdf. ↑ 4.
[103] Medzhitov A.M., 1987, *Simmetrichnye prostranstva na polukonechnykh algebrakh von Neĭmana*, Dokl. Akad. nauk Uzb. SSR **1987**:4, 10–12. www.fuw.edu.pl/∼kostecki/scans/medzhitov1987.pdf. ↑ 14.
[104] Moreau J.-J., 1962, *Fonctions convexes en dualité*, Séminaires de mathématiques, Faculté des sciences de Montpellier, Montpellier. thibaut.horel.org/convex/moreau-62.pdf. ↑ 4.
[105] Muratov M.A., 1979, *Ideal'nye podprostranstva v kol'ce izmerimykh operatorov*, Ph.D. thesis, Tashkentskiĭ Gosudarstvennyĭ Universitet, Tashkent. www.fuw.edu.pl/∼kostecki/scans/muratov1979phd.pdf. ↑ 13.
[106] Muratov M.A., 1979, *Norma Lyuksemburga v prostranstve Orlicha izmerimykh operatorov*, Dokl. Akad. nauk Uzb. SSR **1979**:1, 5–6. www.fuw.edu.pl/∼kostecki/scans/muratov1979.pdf. ↑ 13.
[107] Nagaoka H. (長岡浩司), Amari S. (甘利俊一), 1982, *Differential geometry of smooth families of probability distributions*, Technical report METR **82-7**, University of Tōkyō, Tōkyō. www.fuw.edu.pl/∼kostecki/scans/nagaokaamari1982.pdf. ↑ 8.
[108] Nelson E., 1974, *Notes on non-commutative integration*, J. Funct. Anal. **15**, 103–116. ↑ 13.
[109] Níng Z. (宁哲), 2010, *Orlicz-Lorentz kōngjiān de yīzhì tú xìng yǔ júbù yīzhì tú xìng* (Orlicz-Lorentz空间的一致凸性与局部一致凸性), Ph.D. thesis, Sūzhōu dàxué (苏州大学), Sūzhōu (苏州). www.fuw.edu.pl/∼kostecki/scans/ning2010.pdf. ↑ 15.
[110] Norden A.P., 1937, *Über Paare konjugierter Parallerübertragungen*, Trudy semin. vekt. tenzorn. anal. **4**, 205–255. www.fuw.edu.pl/∼kostecki/scans/norden1937.pdf. ↑ 6.
[111] Ojima I. (小嶋泉), 2004, *Temperature as order parameter of broken scale invariance*, Publ. Res. Inst. Math. Sci. Kyōto Univ. **40**, 731–756. arXiv:math-ph/0311025. ↑ 10.
[112] Orlicz W., 1932, *O pewnej klasie przestrzeni typu B. — Über eine gewisse Klasse von Räumen vom Typus B*, Bull. Internat. Acad. Polon. Sci. Lett., Class. Sci. Math. Natur.: Sér. A, Sci. Math. **1932**:2, 207–220. www.fuw.edu.pl/∼kostecki/scans/orlicz1932.pdf. ↑ 13.
[113] Ovchinnikov V.I., 1970, *O s-chislakh izmerimykh operatorov*, Funkc. anal. pril. **4**, 78–85. mathnet.ru:faa2668 (Engl. transl.: 1970, *s-numbers of measurable operators*, Funct. Anal. Appl. **4**, 236–242). ↑ 14.
[114] Ovchinnikov V.I., 1970, *Simmetrichnye prostranstva izmerimykh operatorov*, Dokl. Akad. nauk SSSR **191**, 769–771. mathnet.ru:dan35316 (Engl. transl.: 1970, *Symmetric spaces of measurable operators*, Soviet Math. Dokl. **11**, 448–451). ↑ 14.
[115] Petz D., 2007, *Bregman divergence as relative operator entropy*, Acta Math. Hungar. **116**, 127–131. web.archive.org/web/20170705131857/http://www.renyi.hu/∼petz/pdf/112bregman.pdf. ↑ 2.
[116] Rao M.M., 1965, *Smoothness of Orlicz spaces. I*, Proc. Konink. Nederl. Akad. Wetensch. Amsterdam Ser. A **68** (= Indag. Math. **27**), 671–680. ↑ 17.
[117] Rao M.M., 1971, *Approximately tame algebras of operators*, Bull. Acad. Polon. Sci., Sér. Sci. Math., Astr. Phys. **19**, 43–47. www.fuw.edu.pl/∼kostecki/scans/rao1971.pdf. ↑ 13.
[118] Raynaud Y., 2002, *On ultrapowers of non commutative L_p spaces*, J. Oper. Th. **48**, 41–68. ↑ 9, 10.
[119] Reich S., Sabach S., 2009, *A strong convergence theorem for a proximal-type algorithm in reflexive Banach spaces*, J. Nonlin. Conv. Anal. **10**, 471–485. ssabach.net.technion.ac.il/files/2015/12/RS2009.pdf. ↑ 4.
[120] Resmerita E., 2004, *On total convexity, Bregman projections and stability in Banach spaces*, J. Conv. Anal. **11**, 1–16. www.heldermann-verlag.de/jca/jca11/jca0379.pdf. ↑ 5.
[121] Ricard É., 2015, *Hölder estimates for the noncommutative Mazur maps*, Arch. Math. **104**, 37–45. arXiv:1407.8334. ↑ 11.
[122] Rockafellar R.T., 1963, *Convex functions and dual extremum problems*, Ph.D. thesis, Harvard University, Cambridge. sites.math.washington.edu/∼rtr/papers/rtr001-PhDThesis.pdf. ↑ 4.
[123] Rockafellar R.T., 1967, *Conjugates and Legendre transforms of convex functions*, Canad. J. Math. **19**, 200–205. sites.math.washington.edu/∼rtr/papers/rtr014-LegendreTransform.pdf. ↑ 4.
[124] Sanov I.N., 1957, *O veroyatnosti bol'shikh otkloneniĭ sluchaĭnykh velichin*, Matem. sb. **84** (nov. ser. **42**), 11–42. mathnet.ru:msb5043 (Engl. transl.: 1961, *On the probability of large deviations of random variables*, Sel. Transl. Math. Statist. Probab. **1**, 213–244). ↑ 1, 5.
[125] Semënov E.M., 1964, *Teoremy vlozheniya dlya banakhovykh prostranstv izmerimykh funkciĭ*, Dokl. Akad. nauk SSSR **156**, 1292–1295. mathnet.ru:dan29751 (Engl. transl.: 1964, *Imbedding theorems for Banach spaces of measurable functions*, Soviet Math. Dokl. **5**, 831–834). ↑ 14.
[126] Sen R.N., 1948, *Parallel displacement and scalar product of vectors*, Proc. Nat. Inst. Sci. India **14**, 45–52. ↑ 6.

[127] Shima H. (志磨裕彦), 1976, *On certain locally flat homogeneous manifolds of solvable Lie groups*, Osaka J. Math. **13**, 213–229. ↑ 6.
[128] Shima H. (志磨裕彦), 2007, *The geometry of hessian structures*, World Scientific, Singapore. ↑ 7.
[129] Shmul'yan V.L., 1939, *On some geometrical properties of the sphere in a space of the type (B)*, Dokl. Akad. nauk SSSR **24**, 648–652. www.fuw.edu.pl/∼kostecki/scans/shmulyan1939.pdf. ↑ 3.
[130] Sukochev F.A., 1987, *Simmetrichnye prostranstva izmerimykh operatorov na konechnykh algebrakh fon Neĭmana*, Tashkentskiĭ Gosudarstvennyĭ Universitet, Tashkent. www.fuw.edu.pl/∼kostecki/scans/sukochev1987.pdf. ↑ 15.
[131] Tadzhibaev B.R., 1986, *Neassociativnye prostranstva Orlicha v ĭordanovykh algebrakh i ikh abstraktnaya kharakterizaciya*, Ph.D. thesis, Tashkentskiĭ Gosudarstvennyĭ Universitet, Tashkent. www.fuw.edu.pl/∼kostecki/scans/tadzhibaev1986.pdf. ↑ 13.
[132] Takesaki M. (竹崎正道), 1979, 2003, *Theory of operator algebras*, Vol. **1-3**, Springer, Berlin. ↑ 2.
[133] Terp M., 1981, *L^p-spaces associated with von Neumann algebras*, Københavns Univ. Math. Inst. Rapp. No. **3a+3b**, Matematisk Institut, Københavns Universitet, København. www.fuw.edu.pl/∼kostecki/scans/terp1981.pdf. ↑ 10.
[134] Tikhonov O.E., 1993, *Trace inequalities for spaces in spectral duality*, Stud. Math. **104**, 99–110. matwbn.icm.edu.pl/ksiazki/sm/sm104/sm10416.pdf. ↑ 12.
[135] Tsuda K., Rätsch G., Warmuth M.K., 2005, *Matrix exponentiated gradient updates for on-line learning and Bregman projection*, J. Mach. Learn. Res. **6**, 995–1018. ↑ 2.
[136] Turett B., 1976, *Rotundity of Orlicz spaces*, Proc. Konink. Nederl. Akad. Wetensch. Amsterdam Ser. A **79** (= Indag. Math. **38**), 462–469. ↑ 15.
[137] Výborný V., 1956, *O slabé konvergenci v prostorech lokálně stejnoměrně konvexních*, Časopis pěstov. matem. **81**, 352–353. ↑ 16.
[138] Wáng T.F. (王廷辅), Cuī Y.Ā. (崔云安), Zhāng T. (弢张), 1998, *Kadec–Klee property in Musielak–Orlicz function spaces equipped with the Luxemburg norm*, Scient. Math. **1**, 339–345. www.jams.or.jp/scm/contents/Vol-1-3/1-3-11.pdf. ↑ 15.
[139] Wiener N., 1948, *Cybernetics or control and communication in the animal and the machine*, Hermann/Technology Press/Wiley, Paris/Cambridge/New York (2nd rev. ed.: 1961). ↑ 3.
[140] Yeadon F.J., 1975, *Non-commutative L^p-spaces*, Math. Proc. Cambridge Phil. Soc. **77**, 91–102. ↑ 13, 14.
[141] Young W.H., 1912, *On classes of summable functions and their Fourier series*, Proc. Royal Soc. London Ser. A **87**, 225–229. ↑ 13.