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Decomposition Theorems in Banach Spaces

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Abstract. In this paper, we present decompositions of arbitrary elements of uniformly convex and uniformly smooth Banach spaces in the form of two d-orthogonal projections on convex closed cones and subspaces. Earlier corresponding results were known only for Hilbert spaces. We also establish new properties of the metric and generalized projection operators in Banach spaces.

1 Preliminaries

It is well known (see, for instance, [12]) that an arbitrary element x of a Hilbert space H admits the Beppo Levi decomposition in the shape of sum of two mutually orthogonal (metric) projections $P_M x$ and $P_{M^\perp} x$ of this element on a subspace M and its orthogonal complement $M^\perp$, i. e.,

$$x = P_M x + P_{M^\perp} x, \tag{1.1}$$

where

$$(P_{M^\perp} x, v) = 0, \quad \forall v \in M. \tag{1.2}$$

Here (x, y) denotes the inner (scalar) product of x and y.

The representation (1.1) and (1.2) shows that $P_M x$ is the best approximation of x among all the elements of the subspace M. This is a basis for many deep results in various areas of mathematics such as geometry of spaces, functional analysis, differential equations, approximation, and optimization.

Moreau has extended in [13] this result to convex closed cones in Hilbert spaces in the form

$$x = P_K x + P_{K^0} x, \quad (P_K x, P_{K^0} x) = 0, \tag{1.3}$$

where K is an original cone and K^0 is its polar cone in H.

It turned out that the transition from Hilbert to Banach spaces is not so simple, and it has required to attract both metric and generalized projection operators in Banach spaces (see [1]). Below we show how it is doing.

Let B be a real uniformly convex and uniformly smooth Banach space [8], B^* its conjugate (dual) space, $\|\cdot\|, \|\cdot\|_*, \|\cdot\|_H$ norms in the spaces B, B^* and H,

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respectively, θ_B and θ_{B^*} be origins of B and B^*. Let Ω be a convex closed set in the space B. As usually, we denote the duality pairing of B^* and B by $\langle \varphi, x\rangle$, where $\varphi \in B^*$ and $x \in B$ (in other words, $\langle \varphi, x\rangle$ is a dual product of φ and x).

Let $J : B \to B^*$ be the normalized duality mapping determined by the equalities:

$$\langle Jx, x\rangle = \|Jx\|_* \|x\| = \|x\|^2. \tag{1.4}$$

It is known that J is a homogeneous, continuous and strictly monotone operator in uniformly convex and uniformly smooth Banach spaces. It is also bijective operator, therefore for any $x \in B$ there exists a unique element $\phi \in B^*$ such that $\phi = Jx$. In our case, $Jx = grad\|x\|^2/2$. Similarly, the normalized duality mapping from B^* to B which is denoted by J^* has the same properties, and then $x = J^*\phi$ for all $\varphi \in B^*$. Obviously, $J\theta_B = \theta_{B^*}$. In a Hilbert space, the duality mapping J is the identity operator I_H. The examples of duality mappings in the spaces l^p, L^P and Sobolev spaces $W^{p,m}$, $p \in (1, \infty)$, can be found in [1].

Each uniformly convex and uniformly smooth Banach space B is reflexive together with its conjugate space B^*. In these spaces (and, more generally, in any reflexive strictly convex Banach space with the strictly convex dual), $J^* = J^{-1}$, where J^{-1} is the inverse operator to J. Therefore, the equalities $JJ^* = I_{B^*}$ and $J^*J = I_B$ hold.

Let us recall the definitions of the metric and generalized projection operators.

Definition 1.1 The operator $P_\Omega : B \to \Omega \subset B$ is called metric projection operator if it assigns to each $x \in B$ its nearest point $\bar{x} \in \Omega$, i.e., the solution $\bar{x}$ for the minimization problem

$$\|x - \bar{x}\| = \inf_{\xi \in \Omega} \|x - \xi\|. \tag{1.5}$$

Under our conditions, the metric projection operator is well defined, i.e., there exists a unique *projection* $\bar{x}$ for each $x \in B$ called the best approximation [9].

The operator P_Ω can be effectively used in Hilbert spaces due to the following properties:

(a) Each point $\xi \in \Omega$ is fixed point of P_Ω, i.e., $P_\Omega \xi = \xi$.

(b) P_Ω is monotone (accretive) in H :

$$(\bar{x} - \bar{y}, x - y) \geq 0.$$

(c) A point $\bar{x}$ is the metric projection of x on $\Omega \subset H$ if and only if the inequality

$$(x - \bar{x}, \bar{x} - \xi) \geq 0, \qquad \forall \xi \in \Omega,$$

holds. The property (c) will be called *the basic variational principle* for P_Ω in H.

(d) The operator P_Ω produces the absolutely best approximation of each $x \in H$ with respect to the functional $V_1(x, \xi) = \|x - \xi\|_H^2$. This means that

$$\|\bar{x} - \xi\|_H^2 \leq \|x - \xi\|_H^2 - \|x - \bar{x}\|_H^2, \qquad \forall \xi \in \Omega.$$

Consequently, P_Ω is *the conditionally non-expansive* operator in a Hilbert space, i.e.,

(e) $\|\bar{x} - \xi\|_H \leq \|x - \xi\|_H, \qquad \forall \xi \in \Omega.$

Actually, metric projection operator P_Ω in Hilbert (and only in Hilbert) spaces has a stronger property of non-expansiveness:

$$\|\bar{x} - \bar{y}\|_H \leq \|x - y\|_H, \qquad \forall x, y \in H.$$

It is important to emphasize that the properties (b), (d) and (e) of the operator P_Ω do not hold in Banach spaces [3,5]. However, the basic variational principle (c) is satisfied in the form (see, for instance, [1])

$$\langle J(x-\bar{x}), \bar{x}-\xi\rangle \geq 0, \quad \forall \xi \in \Omega. \tag{1.6}$$

The construction of generalized projection operators Π_Ω in Banach spaces was introduced in [1] by analogy with the metric projections in Hilbert space.

The minimization problem (1.5) is equivalent to

$$P_\Omega x = \bar{x}; \;\; \bar{x} : V(x,\bar{x}) = \inf_{\xi\in\Omega} V(x,\xi), \;\; V(x,\xi) = \|x-\xi\|^2.$$

Now we notice that $V(x,\xi)$ can be considered not only as the square of the distance between an arbitrary point ξ and the fixed point x, but also as a Lyapunov functional with respect to ξ.

In Hilbert (and only in Hilbert) space

$$V(x,\xi) = \|x\|_H^2 - 2(x,\xi) + \|\xi\|_H^2.$$

We have shown in [1] that the similar functionals in Banach spaces can be constructed by using the Young-Fenchel transformation of the conjugate functions $f(\xi)$ and $f^*(\varphi)$:

$$f^*(\varphi) = \sup_{\xi\in B}\{\langle\varphi,\xi\rangle - f(\xi)\}.$$

Introduce the functional $W(x,\xi) : B \times B \to R$ by the formula:

$$W(x,\xi) = \|x\|^2 - 2\langle Jx,\xi\rangle + \|\xi\|^2. \tag{1.7}$$

It is easy to verify that

$$(\|x\| + \|\xi\|)^2 \geq W(x,\xi) \geq (\|x\| - \|\xi\|)^2 \geq 0,$$

i.e., the functional $W(x,\xi)$ is non-negative and finite on any bounded set. Besides, $\forall x, y \in B$

$$\langle Jx - Jy, y\rangle \leq \langle Jx, x-y\rangle.$$

Let ξ be fixed. It was proved in [1] that $W(x,\xi)$ is a continuous functional with respect to x, it is convex and differentiable with respect to $\varphi = Jx$ and $grad_\varphi W(x,\xi) = 2(x-\xi)$ is a monotone operator in B [15]. Besides, $W(x,\xi) \to \infty$ if and only if $\|x\| \to \infty$. Finally, $W(x,\xi) = 0$ if $x = \xi$.

Now we can present a generalized projection operator Π_Ω in Banach space.

Definition 1.2 Operator $\Pi_\Omega : B \to \Omega \subset B$ is called the generalized projection operator if it associates to an arbitrary fixed point $x \in B$ the minimum point of the functional $W(x,\xi)$, i.e. the solution of the minimization problem

$$\Pi_\Omega x = \hat{x}; \quad \hat{x} : W(x,\hat{x}) = \inf_{\xi\in\Omega} W(x,\xi).$$

The element $\hat{x} \in \Omega \subset B$ is called the generalized projection of the point x.

Existence and uniqueness of the operator Π_Ω follow from the above given properties of the functional $W(x,\xi)$ and the strict monotonicity of the duality mapping.

We describe below the properties of the operator Π_Ω similar to (a)–(e) which make this operator essentially effective in uniformly convex and uniformly smooth Banach spaces (see also [2]).

(f) Each point $\xi \in \Omega$ is fixed point of Π_Ω, i.e. $\hat{\xi} = \xi$. This also means that $\Pi_\Omega = I_B$ if $\Omega = B$.

(g) Π_Ω is d-accretive operator in B (see [6]):

$$\langle Jx - Jy, \hat{x} - \hat{y}\rangle \geq 0, \ \forall x, y \in B.$$

(h) The point $\Pi_\Omega x = \hat{x}$ is the generalized projection of x on $\Omega \subset B$ if and only if the following inequality is satisfied:

$$\langle Jx - J\hat{x}, \hat{x} - \xi\rangle \geq 0, \ \forall \xi \in \Omega. \tag{1.8}$$

The property (h) will be called *the basic variational principle* for Π_Ω in B.

(i) The operator Π_Ω gives the absolutely best approximation of $x \in B$ with respect to the functional $W(x,\xi)$:

$$W(\hat{x},\xi) \leq W(x,\xi) - W(x,\hat{x}), \ \forall \xi \in \Omega.$$

Consequently, Π_Ω is a conditionally non-expansive operator with respect to the functional $W(x,\xi)$ in the Banach space B, i.e., $W(\hat{x},\xi) \leq W(x,\xi)$, $\forall \xi \in \Omega$.

(j) In Hilbert space, $W(x,\xi) = \|x-\xi\|_H^2$, and the operator Π_Ω coincides with the metric projection operator P_Ω.

Proposition 1.3 *Let M_α be an one-dimensional subspace of B spanned upon the element e_α with a unit norm, i.e., $\|e_\alpha\| = 1$. Then the generalized projection $\Pi_{M_\alpha} x$ of an arbitrary element $x \in B$ on M_α is $\langle Jx, e_\alpha\rangle e_\alpha$, where $\langle Jx, e_\alpha\rangle$ is the generalized Fourier coefficient.*

In fact, $\hat{x} = \Pi_{M_\alpha} x = \lambda e_\alpha$ with some λ, $-\infty < \lambda < +\infty$. On the other hand,

$$\hat{x} = \arg \min_{\xi \in M_\alpha} W(x,\xi) =$$

$$\arg \min_{-\infty<\lambda<+\infty} (\|x\|^2 - 2\langle Jx, \lambda e_\alpha\rangle + \|\lambda e_\alpha\|^2).$$

This gives the equation for λ : $-2\langle Jx, e_\alpha\rangle + 2\lambda = 0$, i.e. $\lambda = \langle Jx, e_\alpha\rangle$. □

It has been shown in [1] that $\pi_\Omega = \Pi_\Omega J^*$ is also the generalized projection operator from B^* to $\Omega \subset B$ with the same properties as Π_Ω. For instance, the property (h) has now the form:

(k) The point $\pi_\Omega \phi = \tilde{\phi}$ is the generalized projection of $\phi \in B^*$ on $\Omega \subset B$ if and only if the inequality

$$\langle \phi - J\tilde{\phi}, \tilde{\phi} - \psi\rangle \geq 0, \ \forall \psi \in \Omega, \tag{1.9}$$

is satisfied. The property (1.9) is widely used in the theory of nonlinear variational inequalities and optimization problems [1,4].

From definitions of the projection operators P_Ω and Π_Ω, and from the basic variational principles (c) and (h), the following assertions can be easily obtained.

Lemma 1.4 *Let Ω be a non-empty closed convex set in the Banach space B. Then*

$$\Pi_\Omega \theta_B = P_\Omega \theta_B.$$

Lemma 1.5 *Let Ω be an a non-empty closed convex set in the Banach space B and $\theta_B \in \Omega$. Then:*
(1) $\|\Pi_\Omega x\|^2 \leq \langle Jx, \Pi_\Omega x\rangle \leq \|x\|^2$ *and* $\|\Pi_\Omega x\| \leq \|x\|$,
(2) $\langle Jx - J\Pi_\Omega x, \Pi_\Omega x\rangle \geq 0$,
(3) $\langle Jx, x - \Pi_\Omega x\rangle \geq 0$.

We now establish "the dual oddness property" of the operators P_Ω.

Lemma 1.6 *Let Ω be a non-empty closed convex set in the Banach space B. Let x be an arbitrary element of B. Then*

$$-P_\Omega x = P_{(-\Omega)}(-x). \tag{1.10}$$

Proof. By (1.6), estimate the dual product

$$D = \langle J((-x) - (-\bar{x})), (-\bar{x}) - (-\xi)\rangle$$

for all $(-\xi) \in -\Omega$, where $\bar{x} = P_\Omega x$. Since the inclusion $(-\xi) \in -\Omega$ is equivalent to $\xi \in \Omega$ and the operator J is odd, one has

$$D = \langle J(x - \bar{x}), \bar{x} - \xi\rangle \geq 0, \quad \forall \xi \in \Omega.$$

Hence,

$$\langle J((-x) - (-\bar{x})), (-\bar{x}) - (-\xi)\rangle \geq 0, \quad \forall\, (-\xi) \in -\Omega.$$

This means that $-\bar{x} = P_{(-\Omega)}(-x)$. □

Further, one follows the representation of projection operators on "shifted" sets.

Lemma 1.7 *Let Ω be a non-empty closed convex set in the Banach space B, and let x and m be arbitrary elements of B. Then*

$$P_{\Omega+m} x = m + P_\Omega(x - m). \tag{1.11}$$

Proof. From the basic variational principle (1.6) for the projections P_Ω in B, we have

$$\langle J(x - m - P_\Omega(x - m)), P_\Omega(x - m) - \xi\rangle \geq 0, \quad \forall \xi \in \Omega$$

This is equivalent to the following inequality

$$\langle J(x - (m + P_\Omega(x - m))), P_\Omega(x - m) + m - \eta\rangle \geq 0,$$

$$\forall \eta = \xi + m \in \Omega + m.$$

Applying the same principle once again implies (1.11). □

Remark 1.8 If $m \in \Omega$ then the equality

$$P_\Omega(m + x) = P_\Omega m + P_{\Omega - m} x$$

follows from (1.11).

Denote modulus of smoothness and modulus of convexity of the space B by $\rho_B(\tau)$ and $\delta_B(\varepsilon)$, respectively [8]. If $\|x\| \leq R$ and $\|\xi\| \leq R$ then (see [1])

$$(2L)^{-1} R^2 \delta_B(\|x - \xi\|/2R) \leq \langle Jx - J\xi, x - \xi\rangle \leq 2L^{-1} R^2 \rho_B(4\|x - \xi\|/R), \tag{1.12}$$

where L $(1 < L < 3.18)$ is the Figiel's constant ([8], p. 128). We apply (1.12) in order to obtain the estimate (1.13).

Lemma 1.9 *Suppose that Ω is a non-empty closed convex set in Banach space B, $\theta_B \in \Omega$, $\hat{x}$ is the generalized projection of $x \in B$ on Ω, $g_B(\varepsilon) = \delta(\varepsilon)/\varepsilon$, and $g_B^{-1}(\cdot)$ is the inverse function. If $\|x\| \le R$ then*

$$\|Jx - J\hat{x}\|_* \le 2CR, \quad C = g_{B^*}^{-1}(4L) = const. \tag{1.13}$$

The proof is carried out according to scheme of Corollary 2.4 of [2] if to account into consideration that

$$0 \le \langle Jx - J\hat{x}, \hat{x}\rangle = \langle Jx - J\hat{x}, x\rangle - \langle Jx - J\hat{x}, x - \hat{x}\rangle,$$

i.e.,

$$\langle Jx - J\hat{x}, x\rangle \ge \langle Jx - J\hat{x}, x - \hat{x}\rangle.$$

We use now closed convex cones and subspaces as the sets Ω.

Definition 1.10 A set $K \subset B$ is said to be a cone if it contains the elements λx, $\lambda > 0$, together with elements x.

Any cone $K \subset B$ induces the following two cones: dual cone K^* and polar cone K^0.

Definition 1.11 The dual cone is given by the formula

$$K^* = \{\phi \in B^* : \ \langle \phi, x\rangle \ge 0, \ \ \forall x \in K\}.$$

Definition 1.12 The polar cone is given by the formula

$$K^0 = \{\phi \in B^* : \ \langle \phi, x\rangle \le 0, \ \ \forall x \in K\}.$$

The cones K^* and K^0 are convex and closed in the space B^*. It is obvious that $K^* = -K^0$. If $x \in intK$ then $\langle \phi, x\rangle > 0$ for all $\phi \in K^*$ and $\langle \phi, x\rangle < 0$ for all $\phi \in K^0$. Therefore, $\langle \phi, x\rangle = 0$ only if $x \in \partial K$, where ∂K is the boundary of K.

If K is a subspace $M \subset B$ then $K^* = K^0 = M^\perp$, where $M^\perp = \{\phi \in B^* : \langle \phi, x\rangle = 0, \ \ \forall x \in M\}$ is the annihilator in the space B^*.

The operators P_K and Π_K are nonlinear in general. However, they are homogeneous for any $\lambda > 0$ (for any $-\infty < \lambda < +\infty$ if K is a closed subspace of B).

Lemma 1.13 *Let K be a non-empty closed convex cone in the Banach space B with the vertex at the origin θ_B, and let x be an arbitrary element of B. If $\Pi_K x = \hat{x}$ (resp. $P_K x = \bar{x}$) then the equality $\Pi_K(\lambda x) = \lambda\hat{x}$ (resp. $P_K(\lambda x) = \lambda\bar{x}$) holds for any $\lambda > 0$ (for all λ if K is a subspace).*

The proof follows from the properties (h) and (1.6) and Definition 1.10. This lemma has many applications, in particular, it gives the property of "conditional linearity" to the projection operator P_M.

Proposition 1.14 *Let x be an arbitrary element of the Banach space B and y be an arbitrary element of the subspace $M \subset B$. Then*

$$P_M(\alpha x + \beta y) = \alpha P_M x + \beta P_M y, \ \ \forall \alpha, \beta : \ -\infty < \alpha, \beta < +\infty.$$

The following dual assertions are immediate consequences of the basic variational principles (1.6), (1.8) and fundamental relation $(K^0)^0 = K$, see [17]:

Corollary 1.15 *If $K \subset B$ is a closed convex cone with the vertex at θ_B, $K^0 \subset B^*$ is the polar cone, x is an arbitrary element of the Banach space B, then:*
(1) $x \in K$ if and only if $P_{K^0} Jx = \theta_{B^*}$; $Jx \in K^0$ if and only if $P_K x = \theta_B$,
(2) $x \in K$ if and only if $\Pi_{K^0} Jx = \theta_{B^*}$; $Jx \in K^0$ if and only if $\Pi_K x = \theta_B$.

Introduce the set C_K by the formula:

$$C_K = \{x \in B : \ P_K x = \theta_B\},$$

which is a kernel of the map P_K. It is easy to check that C_K is a cone. In fact, by Lemma 1.13, if $P_K x = \theta_B$ then $P_K(\lambda x) = \theta_B$, for $\forall \lambda > 0$, i.e. Definition 1.10 is satisfied. Moreover, C_K is a closed cone because it is the inverse image of a closed set under a continuous map. In other words, let $x_n \in C_K$ and $x_n \to x$. It is clear that $Jx_n \in K^0$. Since J is a continuous operator in the space B therefore $Jx_n \to Jx$. Consequently, $Jx \in K^0$ because K^0 is the closed cone. The claim is stated now from the item (1) of Corollary 1.15.

By Lemma 1.13, Π_K is a homogeneous operator too. Therefore, by using item (2) of Corollary 1.15, one can define the closed cone C'_K in the space B similarly to the cone C_K :

$$C'_K = \{x \in B : \ \Pi_K x = \theta_B\}.$$

The closed cones C_{K^0} and C'_{K^0} are constructed in dual space B^* analogously:

$$C_{K^0} = \{\phi \in B^* : \ P_{K^0}\phi = \theta_{B^*}\}$$

and

$$C'_{K^0} = \{\phi \in B^* : \ \Pi_{K^0}\phi = \theta_{B^*}\}.$$

In this case, we can reformulate Corollary 1.15 for these cones.

Corollary 1.16 *If $K \subset B$ is a closed convex cone with the vertex at θ_B, $K^0 \subset B^*$ is the polar cone, x is an arbitrary element of the Banach space B, then*
(1) $Jx \in K^0$ if and only if $x \in C_K$; $x \in K$ if and only if $Jx \in C_{K^0}$,
(2) $x \in K$ if and only if $Jx \in C'_{K^0}$; $Jx \in K^0$ if and only if $x \in C'_K$.

In [2] we have proved the following important statement.

Lemma 1.17 *Let K be a non-empty closed convex cone in the Banach space B with the vertex at θ_B. An element $\hat{x} \in K$ is the generalized projection of $x \in B$ on K if and only if*
(1) $\langle Jx - J\hat{x}, \hat{x}\rangle = 0$,
(2) $\langle Jx - J\hat{x}, v\rangle \leq 0, \ \ \forall v \in K$.

This lemma and the property (h) were used in [2] to obtain the decomposition of an arbitrary element x of the space B in the form: there exists $w \in B$ such that

$$x = J^*\Pi_{K^0} Jx + w, \tag{1.14}$$

where

$$\langle \Pi_{K^0} Jx, w\rangle = 0. \tag{1.15}$$

By using (1.6) and repeating the arguments in [2] one can prove the analogous assertion for the metric projections (see also [14]).

Lemma 1.18 *Let K be a non-empty closed convex cone in the Banach space B with the vertex in θ_B. An element $\bar{x} \in K$ is the metric projection of $x \in B$ on K if and only if*
(1) $\langle J(x - \bar{x}), \bar{x}\rangle = 0$,
(2) $\langle J(x - \bar{x}), v\rangle \leq 0, \ \ \forall v \in K$.

This lemma gives the representation of another type, namely, there exists $v \in B$ such that

$$x = P_K x + v, \tag{1.16}$$

where

$$\langle Jv, P_K x\rangle = 0. \tag{1.17}$$

Definition 1.19 (see [11], p. 115). We say that elements $\phi \in B^*$ and $x \in B$ is mutually d-orthogonal and write $\phi \perp^d x$ (or $x \perp^d \phi$), if $\langle \phi, x\rangle = \langle x, \phi\rangle = 0$. An element is d-orthogonal to a subset if and only if it is d-orthogonal to each element of the subset.

The relations (1.15) and (1.17) mean that $\Pi_{K^0} Jx \perp^d w$ and $P_K x \perp^d Jv$. Observe that d-orthogonality is a symmetric characterization of Banach spaces. In Hilbert space, it specializes to the usual definition of orthogonality: $x \perp^d y \iff (x, y) = 0$.

Introduce the coefficient

$$k = \frac{\langle Jx, y\rangle}{\|x\|\|y\|},\ \forall x, y \neq 0.$$

Applying the Cauchy-Schwarz inequality for $\langle Jx, y\rangle$ and the definition (1.4), it is not difficult to see that $-1 \le k \le 1$ and $k = 0$ if $Jx \perp^d y$. From Lemma 1.17 and Lemma 1.18, we have $\forall x \in B$

$$\|\hat{x}\| = k_1\|x\| \le \|x\|,$$

where

$$k_1 = \frac{\langle Jx, \hat{x}\rangle}{\|x\|\|\hat{x}\|},$$

and

$$\|x - \bar{x}\| = k_2\|x\| \le \|x\|,$$

where

$$k_2 = \frac{\langle J(x - \bar{x}), x\rangle}{\|x - \bar{x}\|\|x\|}.$$

The decompositions (1.14) and (1.16) are "semi-definite" because the elements w and v are unknown. The goal of this paper is to obtain the completely determined decompositions of arbitrary elements from both spaces B and B^* in the form of the sum of two d-orthogonal projections on convex closed cones and subspaces.

2 Main results

First of all, we will reformulate Lemma 1.6 and Lemma 1.7 for the cones K, K^*, K^0.

Lemma 2.1 *Let K be a non-empty closed convex cone in the Banach space B with the vertex at the origin θ_B, $K^* \subset B^*$ and $K^0 \subset B^*$ be dual and polar cones, respectively. Let ϕ be an arbitrary element of the Banach space B^*. Then*

$$P_{K^*}(-\phi) = -P_{K^0}\phi.$$

Lemma 2.2 *The following relations are valid for arbitrary elements* $x \in B$, $m \in B$, $\psi \in B^*$, *and* $\zeta \in B^*$:

$$P_{K+m}x = m + P_K(x - m), \tag{2.1}$$

$$P_{K^*+\zeta}\psi = \zeta + P_{K^*}(\psi - \zeta), \tag{2.2}$$

$$P_{K^0+\zeta}\psi = \zeta + P_{K^0}(\psi - \zeta). \tag{2.3}$$

In particular,
(i) $P_{K+m}(x + m) = m + P_K x$.
(ii) If $m \in M$ then

$$P_M x = m + P_M(x - m).$$

(iii) Setting in (2.3) $\psi = \theta_{B^*}$ and $\zeta = Jx$ we get

$$P_{K^0+Jx}\theta_{B^*} = P_{K^0}(-Jx) + Jx. \tag{2.4}$$

The following statement is the Banach space analogue of the Kurokawa's representation which was established for Hilbert spaces.

Lemma 2.3 *Let* $K \subset B$ *be a closed convex cone with the vertex at* θ_B, $K^* \subset B^*$ *be a dual cone,* x *be an arbitrary element of the Banach space* B. *Then*

$$\Pi_K x = J^* \Pi_{K^*+Jx} J\theta_B. \tag{2.5}$$

Proof. Let $\hat{x} = \Pi_K x$. By Lemma 1.17
(1) $\langle Jx - J\hat{x}, \hat{x}\rangle = 0$,
(2) $\langle Jx - J\hat{x}, v\rangle \le 0, \ \ \forall v \in K$.
By Definition 1.11 and the inequality (2) we obtain the inclusion

$$J\hat{x} - Jx \in K^*$$

Therefore,

$$J\hat{x} \in K^* + Jx.$$

Let us choose an arbitrary $\psi \in K^* + Jx$. Then

$$\langle \psi - J\hat{x}, \hat{x}\rangle = \langle (\psi - Jx) - (J\hat{x} - Jx), \hat{x}\rangle$$

$$= \langle \psi - Jx, \hat{x}\rangle - \langle J\hat{x} - Jx, \hat{x}\rangle = \langle \psi - Jx, \hat{x}\rangle \ge 0.$$

Here we used the equality (1) and the inclusions $\psi - Jx \in K^*$ and $\hat{x} \in K$. Since $J^*J = J^{-1}J = I_B$, we can rewrite the previous inequality as follows

$$\langle J^*J\theta_B - J^*J\hat{x}, J\hat{x} - \psi\rangle \ge 0, \ \ \forall \psi \in K^* + Jx.$$

From the basic variational principle (h) for the projection operator Π_Ω, we have then equality

$$J\hat{x} = \Pi_{K^*+Jx} J\theta_B,$$

which is equivalent to (2.5). □

In Hilbert spaces, (2.5) implies

$$P_K x = P_{K^*+x}\theta_H, \ \ \forall x \in H, \tag{2.6}$$

which is the fact of [10].

The main theorem of the paper is as follows.

Theorem 2.4 *If $K \subset B$ is a closed convex cone with the vertex at θ_B, $K^0 \subset B^*$ is the polar cone, x and ϕ are arbitrary elements of the Banach space B and B^*, respectively, then*

$$x = P_K x + J^* \Pi_{K^0} Jx, \tag{2.7}$$

where

$$\Pi_{K^0} Jx \perp^d P_K x, \quad i.e., \quad \langle \Pi_{K^0} Jx, P_K x\rangle = 0, \tag{2.8}$$

and

$$\phi = P_{K^0}\phi + J\Pi_K J^*\phi, \tag{2.9}$$

where

$$\Pi_K J^*\phi \perp^d P_{K^0}\phi, \quad i.e., \quad \langle \Pi_K J^*\phi, P_{K^0}\phi\rangle = 0. \tag{2.10}$$

Proof. We are starting to prove (2.9) and (2.10). By Lemma 2.1 one has

$$\phi - P_{K^0}\phi = \phi + P_{K^*}(-\phi).$$

If $\psi = \theta_{B^*}$ and $\zeta = \phi$, then (2.2) yields

$$\phi + P_{K^*}(-\phi) = P_{K^*+\phi}\theta_{B^*}.$$

By Lemma 1.4 we obtain

$$P_{K^*+\phi}\theta_{B^*} = \Pi_{K^*+\phi}\theta_{B^*}.$$

Further, Lemma 2.3 and the properties of J imply:

$$\Pi_{K^*+\phi}\theta_{B^*} = J\Pi_K J^*\phi.$$

The chain of these equalities gives (2.9). The equality (2.10) arises from Lemma 1.18 (1) because

$$\langle J^*(\phi - \bar{\phi}), \bar{\phi}\rangle = 0.$$

Now we consider the dual space B^* and the cone $K^0 \subset B^*$. In this case

$$x - P_K x = x - P_{(K^0)^0} x = x + P_{(K^0)^*}(-x)$$

because $(K^0)^0 = K$. The decomposition (2.7), (2.8) can be obtained from (2.1) with $x = \theta_B$, $m = x$ and (2.5) written in the form

$$\Pi_{K^0} Jx = J\Pi_{(K^0)^*+x}\theta_B = J\Pi_{(K^0)^*+J^*(Jx)} J^*\theta_{B^*}. \tag{2.11}$$

Indeed,

$$\begin{aligned} x + P_{(K^0)^*}(-x) &= P_{(K^0)^*+x}\theta_B \\ &= \Pi_{(K^0)^*+J^*(Jx)} J^*\theta_{B^*} = J^*\Pi_{K^0} Jx. \end{aligned}$$

Thus, (2.7) is true. Finally, as before, (2.8) arises from Lemma 1.18 (1). Uniqueness of (2.7) and (2.9) is proved like in [18], p. 256. The theorem is accomplished. □

Introduce the following definition.

Definition 2.5 (1) An element $\psi \in B^*$ is called to be J-co-ordinate sum of the elements $y \in B$ and $\phi \in B^*$ if

$$\psi = Jy + \phi,$$

(2) an element $x \in B$ is called to be J^*-co-ordinate sum of the elements $y \in B$ and $\phi \in B^*$ if

$$x = y + J^*\phi.$$

The relations (2.7) - (2.10) show that an arbitrary element $x \in B$ is the J^*-co-ordinate sum of two mutually d-orthogonal projections $P_K x$ and $\Pi_{K^0} Jx$, and an arbitrary element $\phi \in B^*$ is the J-co-ordinate sum of two mutually d-orthogonal projections $P_{K^0}\phi$ and $\Pi_K J^*\phi$.

It is not difficult to see that along with P_K, the operator $J(I - P_K)$ is also projection operator from B to $K^0 \subset B^*$ for all $x \in B$. Indeed, from (2.7) one has

$$J(I - P_K) = \Pi_{K^0} J = \pi_{K^0}.$$

Furthermore,

$$\pi_{K^0}(x - P_K x) = \pi_{K^0} x,$$

and

$$\|\Pi_{K^0} Jx\|_* \leq \|x - \Pi_K x\|.$$

In addition, for all $\psi \in B^*$

$$\pi_K(\psi - P_{K^0}\psi) = \pi_K \psi,$$

in particular, for all $x \in B$

$$\pi_K(Jx - P_{K^0} Jx) = \Pi_K x$$

and

$$\|\Pi_K x\| \leq \|Jx - \Pi_{K^0} Jx\|_*.$$

Remark 2.6 We have established that (2.9) follows from (2.5). The inverse assertion is also valid: (2.5) follows from (2.9). The same conclusion can be done for the couple (2.7) and (2.11).

Remark 2.7 Corollaries 1.15 and 1.16 can be also obtained proceeding from the decompositions (2.7) and (2.9). However, the open problem is to prove whether the cone C_K is convex and $J^*\Pi_{K^0} Jx$ belongs to C_K. The same problems arise for the cones C_{K^0}, C'_K and C'_{K^0}.

Denote $T = P_K + J^*\Pi_{K^0} J,\ T : B \to B$. It is easy to check (see [1]) that

$$2C^2\delta_B(\|x - y\|/2C) \leq \langle Jx - Jy, Tx - Ty\rangle \leq 2C^2\rho_B(4\|x - y\|/C), \tag{2.12}$$

where $\delta_B(\epsilon)$ is the modulus of convexity, $\rho_B(\tau)$ is the modulus of smoothness of the space B, and

$$C = \sqrt{(\|x\|^2 + \|y\|^2)/2}.\square$$

From Theorem 2.4 we deduce the corollary.

Corollary 2.8 *The following relations are valid for arbitrary elements $x \in B$ and $\psi \in B^*$:*

$$\langle \Pi_{K^0} Jx, x\rangle = \|\Pi_{K^0} Jx\|_*^2 \leq \|x\|^2, \tag{2.13}$$

$$\langle \psi, \Pi_K J^*\psi\rangle = \|\Pi_K J^*\psi\|^2 \leq \|\psi\|_*^2. \tag{2.14}$$

This gives in a Hilbert space (cf [18]):

$$(x, P_{K^0} x) = \|P_{K^0} x\|_H^2, \quad (x, P_K x) = \|P_K x\|_H^2.$$

Proof. By Theorem 2.4, for all $x \in B$

$$\langle \Pi_{K^0} Jx, x\rangle = \langle J(x - P_K x), x\rangle$$

$$= \langle J(x - P_K x), x - P_K x\rangle + \langle J(x - P_K x), P_K x\rangle.$$

It follows from Lemma 1.18 and the definition (1.4) that

$$\langle J(x - P_K x), P_K x\rangle = 0$$

and

$$\langle J(x - P_K x), x - P_K x\rangle = \|x - P_K x\|^2 = \|\Pi_{K^0} Jx\|_*^2.$$

Observe by Corollary 1.15 that $\Pi_{K^0} Jx = \theta_{B^*}$ if $x \in K$. Thus, (2.13) is satisfied, and (2.14) is obtained in the same manner. □

Return now to subspaces of Banach spaces.

Corollary 2.9 *If $M \subset B$ is a closed subspace, $M^\perp \subset B^*$ is its annihilator, then every element x of the Banach space B has one and only one decomposition*

$$x = P_M x + J^* \Pi_{M^\perp} Jx, \tag{2.15}$$

where

$$\langle \Pi_{M^\perp} Jx, v\rangle = 0, \quad \forall v \in M. \tag{2.16}$$

Proof. The equality (2.15) is obtained similarly to the previous theorem, and (2.16) is a direct consequence of the annihilator definition.

We will show simpler than in [18] that the decomposition (2.15) is unique in the class of J^* - co-ordinate couples. Suppose that there are two representation of this kind:

$$x = u_1 + J^*\psi_1, \quad u_1 \in M, \quad \psi_1 \in M^\perp$$

and

$$x = u_2 + J^*\psi_2, \quad u_2 \in M, \quad \psi_2 \in M^\perp.$$

Then the equality

$$u_2 - u_1 = J^*\psi_1 - J^*\psi_2, \tag{2.17}$$

is valid, where

$$u_2 - u_1 \in M, \quad \psi_1 - \psi_2 \in M^\perp.$$

From this we have

$$0 = \langle \psi_1 - \psi_2, u_2 - u_1\rangle = \langle \psi_1 - \psi_2, J^*\psi_1 - J^*\psi_2\rangle.$$

It is known (see [1]) that

$$\langle \psi_1 - \psi_2, J^*\psi_1 - J^*\psi_2\rangle \geq (2L)^{-1} R^2 \delta_B(\|J^*\psi_1 - J^*\psi_2\|/2R),$$

where $\delta_B(\epsilon)$ is the modulus of convexity of the space B, L $(1 < L < 3.18)$ is the Figiel's constant, and

$$R = \sqrt{(\|\psi_1\|^2 + \|\psi_2\|^2)/2}.$$

Since $\delta_B(0) = 0$ and J^* is bijective operator, we obtain: $J^*\psi_1 = J^*\psi_2$ and $\psi_1 = \psi_2$. It follows from (2.17) that $u_1 = u_2$. □

Remark 2.10 By (2.16), $\Pi_{M^\perp} Jx$ is d-orthogonal to the subspace M. In particular,

$$\langle \Pi_{M^\perp} Jx, P_M x\rangle = 0.$$

Therefore, x is the J^*-co-ordinate sum of two mutually d-orthoganal projections $P_M x$ and $\Pi_{M^\perp} Jx$. In a Hilbert space, (2.15) and (2.16) together coincide with (1.1) and (1.2), also (2.7) and (2.8) coincide with (1.3).

Emphasize that if $M \subset B$ is closed subspace and $M^\perp \subset B^*$ is its annihilator then:
(i) an element $\hat{x} \in M$ is the generalized projection of $x \in B$ on M if and only if

$$\langle Jx - J\hat{x}, v\rangle = 0, \quad \forall v \in M.$$

This means that $Jx - J\hat{x} \in M^\perp$;
(ii) an element $\bar{x} \in M$ is the metric projection of $x \in B$ on M if and only if

$$\langle J(x - \bar{x}), v\rangle = 0, \quad \forall v \in M.$$

This means that $J(x - \bar{x}) \in M^\perp$ (cf. [7]);
(iii) Proposition 1.14 implies the following:

$$P_{M^\perp}(Jx + tJ(x - P_M x)) = P_{M^\perp} Jx + t\Pi_{M^\perp} Jx, \quad -\infty < t < +\infty.\square$$

It is clear that an arbitrary linear functional $f(x)$ in Banach spaces B is written in the form

$$f(x) = \langle \psi, x\rangle, \quad \forall x \in B, \tag{2.18}$$

where ψ is the fixed element of the space B^*. The following proposition gives the way for a construction of this element.

Proposition 2.11 *Suppose that $f : B \to \Re$ is a linear continuous functional on Banach space B,*

$$N = \{u \in B : \ f(u) = 0\}$$

is the null subspace (kernel) of this functional, $N^\perp$ is its annihilator. Let an arbitrary $x_0 \notin N$. The representation (2.18) holds, where the element $\psi \in B^*$ has the form

$$\psi = f(J^*\zeta_0)\frac{\zeta_0}{\|\zeta_0\|_*^2}, \quad \zeta_0 = \Pi_{N^\perp} Jx_0, \tag{2.19}$$

and it is uniquely determined by the functional f.

Proof. There follows from Theorem 2.9 that

$$x_0 = P_N x + J^*\Pi_{N^\perp} Jx_0, \quad \langle \Pi_{N^\perp} Jx, v\rangle = 0, \quad \forall v \in N. \tag{2.20}$$

By Corollary 1.16, $\Pi_{N^\perp} Jx_0 \neq 0$, consequently, $f(J^*\Pi_{N^\perp} Jx_0) = c \neq 0$. Let $x_1 = c^{-1}x_0$. Then by the homogeneous property of the normalized duality mapping and Lemma 1.13, we have

$$f(x - f(x)J^*\Pi_{N^\perp} Jx_1) = f(x) - c^{-1}f(x)f(J^*\Pi_{N^\perp} Jx_0) = 0.$$

Therefore, $x - c^{-1}f(x)J^*\Pi_{N^\perp} Jx_0 \in N$ and there exists $y \in N$ such that

$$x = y + c^{-1}f(x)J^*\Pi_{N^\perp} Jx_0.$$

Denote $f(x) = \gamma$ and $\zeta_0 = \Pi_{N^\perp} Jx_0$. We can rewrite the previous equality as

$$x = y + c^{-1}\gamma J^*\zeta_0.$$

This means that the space B is the J^* - co-ordinate sum of d–orthogonal subspace N and one-dimensional subspace generated by the element ζ_0. Now a calculation of the dual product

$$\langle \zeta_0, x\rangle = \langle \zeta_0, y\rangle + c^{-1} f(x)\|\zeta_0\|^2$$

leads to (2.18) and (2.19), because $\langle \zeta_0, y\rangle = 0$.

Let us prove a uniqueness. Suppose that along with (2.18), the representation

$$f(x) = \langle \psi_1, x\rangle, \quad \forall x \in B$$

holds. Then

$$\langle \psi - \psi_1, x\rangle = 0, \quad \forall x \in B$$

and

$$\langle \psi - \psi_1, J^*\psi - J^*\psi_1\rangle = 0.$$

Similarly to Corollary 2.9, $\psi = \psi_1$. It is obvious that $\|f\|_* = \|\psi\|_*$. □

Observe that $dim\ N^\perp = 1$ always. By using Proposition 1.3, one can simplify the expression for ψ some more. Suppose that $N^\perp$ is the subspace of B^* spanned upon the element e^* with a unit norm. Then

$$\Pi_{N^\perp} Jx_0 = \langle e^*, J^*(Jx_0)\rangle e^* = \langle e^*, x_0\rangle e^*,$$

and this gives

$$\psi = f(J^*e^*)e^*. \square$$

Next we establish the equality of Pethagorean type for the Lyapunov functional (1.7) (cf. [2]). Denote $z_1 = P_K x$ and $z_2 = P_K x - x$. Then

$$W(z_2, z_1) = \|J^*\Pi_{K^o} Jx\|^2 + 2\langle \Pi_{K^o} Jx, P_K x\rangle + \|P_K x\|^2$$

$$= \|P_K x\|^2 + \|\Pi_{K^o} Jx\|_*^2,$$

i.e.,

$$W(z_2, z_1) = \|z_1\|^2 + \|z_2\|^2. \tag{2.21}$$

In a Hilbert space, (2.21) corresponds to the well known fact:

$$\|x\|_H^2 = \|\bar{x}\|_H^2 + \|x - \bar{x}\|_H^2.$$

In addition, by (1.4)

$$\langle Jx, x\rangle = \langle Jx, \hat{x}\rangle + \langle Jx, J^*\Pi_{K^o} Jx\rangle =$$

$$\|\hat{x}\|^2 + \langle Jx, J^*\Pi_{K^o} Jx\rangle, \ \forall x \in B.$$

Hence,

$$\|x\|^2 = \|\hat{x}\|^2 + \langle Jx, J^*\Pi_{K^o} Jx\rangle. \square$$

In conclusion, we will obtain some decomposition results for "shifted cones".

Corollary 2.12 *Let be arbitrary elements $m \in B$ and $\zeta \in B^*$. Then under the assumptions of Theorem 2.4, the following representations are satisfied:*

$$\phi = P_{K^o + \zeta}\phi + J\Pi_K J^*(\phi - \zeta) \tag{2.22}$$

and

$$x = P_{K+m} x + J^*\Pi_{K^o} J(x - m). \tag{2.23}$$

Proof. Rewrite (2.9) for the element $\phi - \zeta$:

$$\phi - \zeta = P_{K^0}(\phi - \zeta) + J\Pi_K J^*(\phi - \zeta).$$

Substitution $\psi = \phi$ for (2.3) and the previous equality give together (2.22). The second claim (2.23) can be proved by the similar way. □

Corollary 2.13 *Let ϕ be an arbitrary element of B^*. The operator*

$$T = P_{K^*}\phi - P_{K^*+\phi}\theta_{B^*}$$

is "quasi-invariant" in the sense that

$$P_{K^*}\phi - P_{K^*+\phi}\theta_{B^*} = P_{K^0}\phi - P_{K^0+\phi}\theta_{B^*}. \tag{2.24}$$

Proof. The representation (2.4) implies

$$P_{K^0+\phi}\theta_{B^*} = P_{K^0}(-\phi) + \phi.$$

Since for all $\phi \in B^*$

$$\phi - P_{K^*}\phi = \phi + P_{K^0}(-\phi) = P_{K^0+\phi}\theta_{B^*},$$

we have

$$\phi = P_{K^*}\phi + P_{K^0+\phi}\theta_{B^*}. \tag{2.25}$$

On the other hand,

$$\phi - P_{K^0}(\phi) = \phi + P_{K^*}(-\phi) = P_{K^*+\phi}\theta_{B^*}$$

Hence,

$$\phi = P_{K^0}\phi + P_{K^*+\phi}\theta_{B^*} \tag{2.26}$$

Comparing (2.25) and (2.26) we conclude that (2.24) is satisfied. □

In particular, if $\phi = Jx$ one has

$$P_{K^*}Jx - P_{K^*+Jx}J\theta_B = P_{K^0}Jx - P_{K^0+Jx}J\theta_B.$$

Our final theorem concerns Hilbert spaces.

Theorem 2.14 *Suppose that H is the Hilbert space, and arbitrary elements $x \in H$ and $m \in H$. The following decompositions hold:*
(1) $x = P_{K+m}x + P_{K^0+m}x - m$,
(2) $x = P_K(x - m) + P_{K^0+m}x$,
(3) $x = P_{K+m}x + P_{K^0}(x - m)$.

Proof. Let us recall that the Moreau decomposition is being written in the form

$$x = P_K x + P_{K^0}x, \quad \forall x \in H.$$

Then

$$x - m = P_K(x - m) + P_{K^0}(x - m). \tag{2.27}$$

Further we need (2.1) for K and K^0 which are

$$P_{K+m}x = m + P_K(x - m)$$

and

$$P_{K^0+m}x = m + P_{K^0}(x - m).$$

Substituting

$$P_K(x - m) = P_{K+m}x - m$$

and

$$P_{K^0}(x-m) = P_{K^0+m}x - m$$

to (2.27), we obtain (1). Using these substitutions separately, we see that the claims (2) and (3) follow from (2.27). □

Remark 2.15 We have to conclude from the decomposition theorems that the generalized projection operator Π_Ω is *canonical* in Banach spaces.

Remark 2.16 The elements m and ψ in (2.1)–(2.3) can be replaced by the operators $m(y) : B \to B$ and $\psi(u) : B^* \to B^*$.

Remark 2.17 (see [1]). All results of this paper are valid for dual mappings J^μ with the gauge function $\mu(t)$, defined by the relations

$$\|J^\mu x\|_{B^*} = \mu(\|x\|), \qquad \langle J^\mu x, x\rangle = \mu(\|x\|)\|x\|.$$

Remark 2.18 Sakai in [16] has noticed that (1.3) and (2.6) are equivalent. By this fact, he has considered the non-negative Radon measures on open sets of R^s and shown that they are closely related to the theory of Cartan-Deny. He applied this result to the obstacle problems in Sobolev space $W_0^{2,m}(K)$. We also proved in Theorem 2.4 that (2.9) and (2.5) are equivalent, and this allows to hope that the studies of [16] can be continued in the spaces $W_0^{p,m}(K)$, $p \in (1,\infty)$.

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