MIDPOINT LOCAL UNIFORM CONVEXITY, AND OTHER GEOMETRIC PROPERTIES OF BANACH SPACES

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TABLE OF CONTENTS

INTRODUCTION

In recent years, considerable progress has been made in the classification and characterization of Banach spaces according to various geometric properties of their unit spheres. Among the first and most important of the properties to be investigated were rotundity and smoothness. Then, various uniformity conditions were imposed on these two properties, with the result that each gave rise to a chain of properties, related according to their varying degrees of uniformity. Chapter I contains the definitions of these properties, along with the notations to be used throughout the paper.

In chapter II, we consider midpoint local uniform convexity (m.l.u.c.), and show that it is a link in the convexity chain between local uniform convexity (l.u.c.) and rotundity (R). This property has been known and considered for some time by other people, notably G. Lumer and M. M. Day, but to the writer's knowledge has not previously been investigated directly in its relation to other convexity and smoothness properties. The section ends with sufficiency conditions for m.l.u.c. in B and in B*.

The notion of m.l.u.c. was initially considered in the hope that it would be dual to strong differentiability of the norm (Str). Although this hope did not materialize, we were able to find another well-known property, (H), which, in the case of reflexive Banach spaces, fits into the convexity chain between l.u.c. and m.l.u.c., and which does in fact yield full duality with (Str). These are the principal results of chapter III.

Product spaces are considered in chapter IV. In particular, it is shown that the l_p product of m.l.u.c. spaces is m.l.u.c., and also that the l_p product of (H) spaces is (H).

Chapter V contains an assortment of isomorphism results, many of which were previously known, and included here only for the sake of completeness. The principal result of this chapter is an improvement on a theorem of Fan and Glicksberg, and states that if B* is separable, then B is isomorphic to an l.u.c. space.

CHAPTER I

Definitions and Notation

Throughout this paper, B will designate an arbitrary Banach space, $\|\cdots\|$ the norm in B, U the unit ball of B (the set of all points with norm ≤ 1), S the unit sphere of B (the set of all points in B of norm one), and B^* the conjugate or adjoint space of B.

The properties listed below depend on the norm and linear structure, and thus can be defined for arbitrary normed linear spaces (not necessarily complete). But, since our primary concern is with Banach spaces, we have phrased all the definitions in terms of a Banach space B. In some cases, we have listed two equivalent formulations of the same property, so that we may use whichever formulation best suits our needs in a given situation. Although most of the properties discussed below actually apply to the unit ball U of B, we shall use the convention that "B is (-)" means "B is a Banach space whose unit ball U is (-)".

In general, we shall write "$\lim x_n$" for "$\lim_{n \to \infty} x_n$" when the omission will not lead to confusion. We shall also use the conventional shorthand "iff" for "if and only if".

<u>Definition 1.1</u> B is <u>rotund</u> (R) iff $\|x + y\| = \|x\| + \|y\|$ implies $x = ty$, $t > 0$, whenever $x \neq 0$ and $y \neq 0$. [Note: In many papers, this property is called <u>strictly convex</u>.]

<u>Definition 1.2</u> B is <u>locally uniformly convex</u> (l.u.c.) iff given $\epsilon > 0$ and an element x with $\|x\| = 1$, there exists $\delta(\epsilon, x) > 0$

such that $\frac{\|x + y\|}{2} \leqq 1 - \delta(\epsilon, x)$ whenever $\|x - y\| \geqq \epsilon$ and $\|y\| = 1$.

Since much of our work in this paper will deal with sequences, we list the following equivalent formulation of l.u.c.

Definition 1.2a B is locally uniformly convex (l.u.c.) iff $\|x_n\| = \|x_o\| = 1$ and $\lim \|x_n + x_o\| = 2$ implies $\lim \|x_n - x_o\| = 0$.

Definition 1.3 B is uniformly convex (u.c.) iff given $\epsilon > 0$, there exists $\delta(\epsilon) > 0$ such that $\frac{\|x + y\|}{2} \leqq 1 - \delta(\epsilon)$ whenever $\|x - y\| \geqq \epsilon$, and $\|x\| = \|y\| = 1$.

Definition 1.4 B is smooth (S) iff at every point of S there is only one supporting hyperplane of U.

An equivalent formulation of smoothness is

Definition 1.4a B is smooth (S) iff for each x_o in S, $\lim_{h \to o} (\|x_o + hx\| - \|x_o\|)/h$ exists for every x in B.

Definition 1.5 The norm in B is strongly differentiable at the point x_o in S [written: B is (Str) at x_o] iff the limit in Definition 1.4a is attained uniformly over x in S. We say that B is (Str) iff B is (Str) at every point x_o in S.

It is evident from the above definitions that u.c. implies l.u.c. implies (R), and (Str) implies (S).

Definition 1.6 The norms $\|\cdots\|$ and $\|\cdots\|_1$ are equivalent iff for any sequence $\{x_n\}$, it follows that $\lim \|x_n\| = 0$ iff $\lim \|x_n\|_1 = 0$.

It can be shown that a necessary and sufficient condition for $\|\cdots\|$ and $\|\cdots\|_1$ to be equivalent is that there exist numbers a,b with $0 < a \leqq b < \infty$ such that

$a \|x\| \leq \|x\|_1 \leq b \|x\|$ for all x.

We shall also need Minkowski's inequality:

If $p > 1$, and if $\{a_n\}$, $\{b_n\}$ are sequences of real numbers for which $\sum_{n=1}^{\infty} |a_n|^p$ and $\sum_{n=1}^{\infty} |b_n|^p$ are finite, then

$$\{ \sum_{n=1}^{\infty} |a_n + b_n|^p\}^{1/p} \leq \{ \sum_{n=1}^{\infty} |a_n|^p\}^{1/p} + \{ \sum_{n=1}^{\infty} |b_n|^p\}^{1/p}$$

and equality holds iff, for some $t > 0$, $a_n = tb_n$ for all n.

We list below two theorems due to V. Smulian [10] which will be used extensively throughout this paper.

Theorem 1.1 B^* is (Str) at f_o, $\|f_o\| = 1$, iff from $\lim f_o(x_n) = \|f_o\|$ where $\|x_n\| = 1$, it follows that $\lim_{m,n \to \infty} \|x_m - x_n\| = 0$.

Theorem 1.2 B is (Str) at x_o, $\|x_o\| = 1$, iff from $\lim f_n(x_o) = \|x_o\|$ where f_n in B^* and $\|f_n\| = 1$, it follows that $\lim_{m,n \to \infty} \|f_m - f_n\| = 0$.

Several well-known Banach spaces are mentioned in this paper, and for convenience we define them:

m : the space of bounded sequences $x = (x^1, x^2, \cdots)$ with $\|x\| = \sup_i |x^i|$

c_o : the subspace of m consisting of those sequences which converge to zero.

l_p for $p \geqq 1$: the space of sequences $x = (x^1, x^2, x^3, \cdots)$ for which $\sum_{i=1}^{\infty} |x^i|^p$ is finite, with $\|x\| = \{ \sum_{i=1}^{\infty} |x^i|^p\}^{1/p}$

C [0,1] : the space of continuous functions x on the interval [0,1], with $\|x\| = \sup \{|x(t)| : t \text{ in } [0,1]\}$.

CHAPTER II

Midpoint Local Uniform Convexity

Geometrically, Lovaglia's l.u.c. states that if the midpoint of a variable chord having one end point fixed on the unit sphere approaches the unit sphere, then the length of the chord approaches zero. We now wish to investigate another type of convexity, which, for lack of any other appropriate name, we choose to call midpoint local uniform convexity. Geometrically, the latter states that if the midpoint of a variable chord in the unit sphere approaches a fixed point on the unit sphere, then the length of the chord approaches zero. Formally, it is defined as follows:

<u>Definition 2.1</u> B is <u>midpoint</u> <u>locally</u> <u>uniformly</u> <u>convex</u> (m.l.u.c.) iff given $\epsilon > 0$ and an element x_o with $\|x_o\| = 1$, there exists $\delta(\epsilon, x_o) > 0$ such that $\|x + y - 2x_o\| \geqq \delta$ whenever $\|x - y\| \geqq \epsilon$ and $\|x\| = \|y\| = 1$.

Again, since much of our work in this paper will deal with sequences, we list the following equivalent formulation of m.l.u.c.

<u>Definition 2.1a</u> B is <u>midpoint</u> <u>locally</u> <u>uniformly</u> <u>convex</u> (m.l.u.c) iff $\|x_n\| = \|y_n\| = \|x_o\| = 1$ and $\lim\|x_n + y_n - 2x_o\| = 0$ implies that $\lim\|x_n - y_n\| = 0$.

<u>Lemma 2.1</u> <u>The final implication in Definition 2.1a may be replaced by either</u> $\lim\|x_n - x_o\| = 0$ <u>or</u> $\lim\|y_n - x_o\| = 0$.

<u>Proof</u>: First note that $\lim\|x_n - x_o\| = 0$ iff $\lim\|y_n - x_o\| = 0$ under the hypothesis $\lim\|x_n + y_n - 2x_o\| = 0$, which follows immediately from the following two inequalities:

$$\|y_n - x_o\| = \|x_n + y_n - 2x_o + x_o - x_n\|$$
$$\leq \|x_n + y_n - 2x_o\| + \|x_n - x_o\|$$
$$\|x_n - x_o\| = \|x_n + y_n - 2x_o + x_o - y_n\|$$
$$\leq \|x_n + y_n - 2x_o\| + \|y_n - x_o\|$$

The fact that $\lim\|x_n - x_o\| = 0$ implies $\lim \|x_n - y_n\| = 0$ follows from the above, and

$$\|x_n - y_n\| = \|x_n - x_o + x_o - y_n\|$$
$$\leq \|x_n - x_o\| + \|y_n - x_o\|$$

That $\lim\|x_n - y_n\| = 0$ implies $\lim\|x_n - x_o\| = 0$ follows from the hypothesis, and the inequality

$$\|x_n - x_o\| \leq 2\|x_n - x_o\| = \|2x_n - 2x_o\|$$
$$= \|x_n + y_n - 2x_o + x_n - y_n\|$$
$$\leq \|x_n + y_n - 2x_o\| + \|x_n - y_n\| . \quad \text{QED}$$

<u>Theorem 2.1</u> <u>If B is l.u.c., then B is m.l.u.c.</u>

<u>Proof</u>: Suppose B is l.u.c. Choose $\{x_n\}$, $\{y_n\}$, and x_o in B such that $\|x_n\| = \|y_n\| = \|x_o\| = 1$ and $\lim\|x_n + y_n - 2x_o\| = 0$. For each n,

$$\|x_n + y_n - 2x_o\| \geq \left|\|x_n + y_n\| - \|2x_o\|\right|$$
$$= 2 - \|x_n + y_n\| \geq 0$$

Since the limit of the left side of the above inequality is zero by hypothesis, we thus get

(1) $\lim\|x_n + y_n\| = 2$

For a fixed n, consider the plane P determined by the points O, x_n, and y_n. Let z_1, z_2, and z_3 be the points on the chord from

x_n to y_n defined by $(3x_n + y_n)/4$, $(x_n + y_n)/2$, and $(x_n + 3y_n)/4$, respectively. Let $\bar{z}_1$, $\bar{z}_2$, and $\bar{z}_3$ be the corresponding points on $P \cap S$ (the unit sphere in the plane P) determined by the rays r_1, r_2, and r_3 from O through z_1, z_2, and z_3, respectively. Let L be the line through $\bar{z}_2$ parallel to the chord from x_n to y_n, and let u_1 and u_3 be the intersections of L with r_1 and r_3, respectively.

Day ([2], page 112) has shown that B is (R) iff every two-dimensional subspace of B is (R). Therefore, B is l.u.c. implies that B is (R) implies that $P \cap S$ is (R). Hence, L either is tangent to U at $\bar{z}_2$, or it cuts U on one side or the other of $\bar{z}_2$. In either case, at least one of the points u_1, u_3 is outside U.

Suppose $\|u_1\| > 1$. Consider the similar triangles with vertices at O, z_1, z_2, and O, u_1, $\bar{z}_2$. Then,

$$\|z_2\| \,/\, \|\bar{z}_2\| = \|z_1\| \,/\, \|u_1\| \text{ , or}$$

$$\|z_1\| = (\|u_1\| \cdot \|z_2\|)/\|\bar{z}_2\| = \|u_1\|\cdot\|z_2\| > \|z_2\|$$

which, by definition of the z's, says

$\|(3x_n + y_n)/4\| > \|(x_n + y_n)/2\|$. Thus we have

$$(2) \qquad \|(3x_n + y_n)/2\| > \|x_n + y_n\| \qquad \text{for each } n.$$

We now have for each n,

$$2 = \|x_n\| + \|x_o\| \geq \|x_n + x_o\|$$

$$= \|x_n + (x_n + y_n)/2 - (x_n + y_n)/2 + x_o\|$$

$$= \|x_n + (x_n + y_n)/2 - \{(x_n + y_n)/2 - x_o\}\|$$

$$\geq \|x_n + (x_n + y_n)/2\| - \|(x_n + y_n)/2 - x_o\|$$

$$= \|(3x_n + y_n)/2\| - \|(x_n + y_n - 2x_o)/2\|$$

$$> \|x_n + y_n\| - \|(x_n + y_n - 2x_o)/2\|$$

As n becomes infinite, both limits on the right exist, the first by (1) and the second by hypothesis, so

$$2 \geqq \lim\|x_n + x_o\| \geqq \lim \{\|x_n + y_n\| - \|(x_n + y_n - 2x_o)2\|\}$$

$$= \lim\|x_n + y_n\| - \lim\|(x_n + y_n - 2x_o)/2\|$$

$$= 2$$

Therefore, $\lim \|x_n + x_o\| = 2$. But B is l.u.c., so this implies that $\lim\|x_n - x_o\| = 0$.

In the other case, if $\|u_3\| > 1$, we could show by a similar procedure that $\|z_3\| > \|z_2\|$ and thus that $\lim\|y_n + x_o\| = 2$. Then, B is l.u.c. would imply that $\lim\|y_n - x_o\| = 0$.

Thus, in either case, we find that B is m.l.u.c., by Lemma 2.1. QED

It is evident from the definition of m.l.u.c. that m.l.u.c. implies (R). We now give an example to show that m.l.u.c. is stronger than (R).

Example 2.1 Let c_o be the Banach space of sequences converging to zero; that is, if $x = (x^1, x^2, x^3, \cdots)$ is an element of c_o, then $\|x\| = \sup|x^i|$. In c_o, we define a new norm $\|\cdots\|_1$ as follows:

$$\|x\|_1 = \|x\| + \{\sum_{i=1}^{\infty}(|x^i|/2^{i-1})^2\}^{1/2}$$

Then we have, for each x,

$$\|x\| \leqq \|x\|_1 = \|x\| + \{\sum_{i=1}^{\infty}(|x^i|/2^{i-1})^2\}^{1/2}$$

$$\leqq \|x\| + \{\sum_{i=1}^{\infty}(\sup|x^i|/2^{i-1})^2\}^{1/2}$$

$$= \|x\| + \{\textstyle\sum_{i=1}^{\infty}(\|x\|/2^{i-1})^2\}^{1/2}$$

$$= \|x\| + \|x\| \cdot \{\textstyle\sum_{i=1}^{\infty} 1/4^{i-1}\}^{1/2}$$

$$= (1 + 2/\sqrt{3})\|x\| .$$

Therefore, $\|\cdots\|_1$ and $\|\cdots\|$ are equivalent norms. Let B be the space c_0 renormed with $\|\cdots\|_1$. We first show that B is (R). Let x and y be any two elements of B such that $\|x\|_1 = \|y\|_1 = 1$ and $\|x + y\|_1 = \|x\|_1 + \|y\|_1$. Using the definition of the norm and the Minkowski inequality, we have

$$\|x\|_1 + \|y\|_1 = \|x + y\|_1$$

$$= \|x + y\| + \{\textstyle\sum_{i=1}^{\infty}(|x^i + y^i|/2^{i-1})^2\}^{1/2}$$

$$\leq \|x + y\| + \{\textstyle\sum_{i=1}^{\infty}\{(|x^i| + |y^i|)/2^{i-1}\}^2\}^{1/2}$$

$$\leq \|x + y\| + \{\textstyle\sum_{i=1}^{\infty}(|x^i|/2^{i-1})^2\}^{1/2} + \{\textstyle\sum_{i=1}^{\infty}(|y^i|/2^{i-1})^2\}^{1/2}$$

$$\leq \|x\| + \|y\| + \{\textstyle\sum_{i=1}^{\infty}(|x^i|/2^{i-1})^2\}^{1/2} + \{\textstyle\sum_{i=1}^{\infty}(|y^i|/2^{i-1})^2\}^{1/2}$$

$$= \|x\|_1 + \|y\|_1$$

Equality throughout forces equality in the Minkowski inequality, which in turn implies that x = y. Hence, B is (R). We now show that B is not m.l.u.c. Let $x_0 = (1/2, 0, 0, 0, \cdots)$, and for each $n \geq 2$, let $x_n = (a_n, 0, 0, \cdots, 0, 1/2, 0, 0, \cdots)$ and $y_n = (a_n, 0, 0, \cdots, 0, -1/2, 0, 0, \cdots)$, where 1/2 and -1/2 are the nth coordinates, and where a_n is determined by the relation $a_n^2 + (1/2^n)^2 = 1/4$, so that $a_n = 1/2(1 - 1/4^{n-1})^{1/2}$. For each n, $a_n \leq 1/2$, so it follows that $\|x_n\| = \|y_n\| = 1/2$. Also, it is easy to verify that for all $n \geq 2$, we have $\|x_n\|_1 = \|y_n\|_1 = \|x_o\|_1 = 1$. From the definition of a_n above, we see that $\lim a_n = 1/2$. For each n, $x_n + y_n - 2x_o = (2a_n - 1, 0, 0, 0, \cdots)$, from which it follows that $\lim \|x_n + y_n - 2x_o\|_1 = 0$. But for each n, $x_n - y_n = (0, 0, 0, \cdots, 0, 1, 0, 0, \cdots)$, and thus we see that for each n, $\|x_n - y_n\| = 1$. Therefore, $\|x_n - y_n\|_1 \geq 1$, so B is not m.l.u.c. This concludes Example 2.1.

Next, we give an example to show that l.u.c. is stronger than m.l.u.c.

Example 2.2 Let l_2 be the space of sequences $x = (x^1, x^2, x^3, \cdots)$ such that $\sum_{i=1}^{\infty} (x^i)^2$ is finite, and with the norm defined as usual by $\|x\| = \{\sum_{i=1}^{\infty}(x^i)^2\}^{1/2}$. We want to determine an equivalent norm $\|\cdots\|_1$ in the space such that this new norm is m.l.u.c. but not l.u.c. The plan is to pare down the original unit ball in the direction of the first coordinate axis to get a "hyper-ellipsoidal" convex set symmetric about zero, then use this set to define a new norm. The set is best defined in terms of its sections with the coordinate planes and the hyperplanes x^1 = constant.

Let U be the unit ball of l_2, and let $x = (x^1, x^2, x^3, \cdots)$ be any element in U. In each coordinate plane $x^1 \times x^j$, $j \geq 2$, use the l_p norm, where $p = (2j - 2)/(2j - 3)$; that is, if x is of the form $x = (x^1, 0, 0, \cdots, 0, x^j, 0, 0, \cdots)$, then define the functional m by

(1) $m(x) = \|x\|_p = (|x^1|^p + |x^j|^p)^{1/p}$, where p is defined as above. [Note: Throughout the remainder of this example, p

will always be defined as above. The value of p depends on the coordinate j, and thus should be written as p_j or p(j); however, the notational dependence of p on j will be omitted in an effort to simplify an already cumbersome notation.] In each coordinate plane $x^i \times x^j$, where $2 \leq i < j$, use the l_2 norm; that is, if x is of the form $x = (0, 0, \cdots, 0, x^i, 0, 0, \cdots, 0, x^j, 0, 0, \cdots)$, then define m by

(2) $\quad m(x) = \| x \|$.

Note from (1) and (2) that if x consists of just one coordinate, then $m(x) = \| x \|$.

Now, if x does not lie in a coordinate plane, we know that $|x^1| < 1$, since x is in U, so for each j, $j \geq 2$, choose $t_j > 0$ such that

(3) $\quad |x^1|^p + t_j^{-p} = 1.$

In each hyperplane x^1 = constant, we can take $\pm\, 1/t_j$, $j = 2, 3, 4, \cdots$ as the ends of the semi-axes and determine the "hyper-ellipsoid"

$$\left[\frac{x^2}{1/t_2}\right]^2 + \left[\frac{x^3}{1/t_3}\right]^2 + \cdots + \left[\frac{x^j}{1/t_j}\right]^2 + \cdots = 1,$$

or $\quad \sum_{j=2}^{\infty} t_j^2 (x^j)^2 = 1.$ Thus, in this case, define m by

(4) $\quad m(x) = \{ \sum_{j=2}^{\infty} t_j^2 (x^j)^2 \}^{1/2}$

Note that if $x^1 = 0$, then by (3), each $t_j = 1$, so by (4), $m(x) = \| x \|$; also, (2) is just a special case of this last result. Since (1) and (4) are the necessary defining relations for m, we combine them into one definition for easy reference. If x is in U, define

$$(5) \quad m(x) = \begin{cases} \|x\|_p & \text{if } x \text{ is in the plane } x^1 \times x^j \\ \{\sum_{j=2}^{\infty} t_j^2 (x^j)^2\}^{1/2} & \text{otherwise} \end{cases}$$

where $t_j = (1 - |x^1|^p)^{-1/p}$ for each j

and $p = (2j - 2)/(2j - 3)$

The functional m has the following properties:

(a) $m(x) \geqq 0$, and $m(x) = 0$ iff $x = 0$.

This follows immediately from the definition.

(b) m is symmetric; i.e., $m(-x) = m(x)$.

This also follows immediately from the definition.

(c) For each x in U, $m(x) \leqq \sqrt{2}$.

To prove this, we first need the following lemma.

<u>Lemma 2.2</u> <u>If</u> a <u>and</u> p <u>are</u> <u>real</u> <u>numbers</u> <u>such</u> <u>that</u> $0 < a < 1$ <u>and</u> $1 < p \leqq 2$, <u>then</u> $(1 - a^p)^{2/p} \geqq 1 - a$.

<u>Proof</u>: From the restrictions on a and p, we have $a^p \leqq a$, so $1 - a^p \geqq 1 - a$. Hence, $\log (1 - a^p) \geqq \log (1 - a)$. Since $1 < p \leqq 2$, we have $1/2 \leqq 1/p < 1$, so $1/p \log (1 - a^p) \geqq 1/p \log (1 - a) \geqq 1/2 \log (1 - a)$, or $\log (1 - a^p)^{1/p} \geqq \log (1 - a)^{1/2}$, which in turn implies that $(1 - a^p)^{1/p} \geqq (1 - a)^{1/2}$, and squaring both sides then yields the desired result. QED.

Now, to prove (c), we have for each x in U,

$$m(x) = \{\sum_{j=2}^{\infty} t_j^2 (x^j)^2\}^{1/2} = \{\sum_{j=2}^{\infty} \frac{(x^j)^2}{(1 - |x^1|^p)^{2/p}}\}^{1/2}$$

$$\leqq \{\sum_{j=2}^{\infty} \frac{(x^j)^2}{1 - |x^1|}\}^{1/2} \qquad \text{by Lemma 2.2}$$

$$\leqq \left\{ \sum_{j=2}^{\infty} \frac{(x^j)^2}{\|x\| - |x^1|} \right\}^{1/2} \qquad \text{since } x \text{ is in } U$$

$$= \left\{ \frac{\sum_{j=2}^{\infty} (x^j)^2}{\|x\| - |x^1|} \right\}^{1/2} = \left\{ \frac{\|x\|^2 - |x^1|^2}{\|x\| - |x^1|} \right\}^{1/2}$$

$$= (\|x\| + |x^1|)^{1/2} \leqq (2\|x\|)^{1/2} \leqq 2^{1/2}.$$

(d) m is continuous on U

Let $\{x_n\}$ be a sequence of elements in U and x an element of U such that $\lim x_n = x$. For each $j \geqq 2$, choose $t_j > 0$ such that $|x^1|^p + t_j^{-p} = 1$, and choose $t_{n,j} > 0$ such that $|x_n^1|^p + t_{n,j}^{-p} = 1$, $n = 1, 2, 3, \cdots$. Convergence in l_2 implies coordinatewise convergence, so it follows that $\lim x_n^i = x^i$ for each $i = 1, 2, \cdots$. Since for each n and each $j \geqq 2$, we have $t_{n,j} = (1 - |x_n^1|^p)^{-1/p}$, it follows that for each $j \geqq 2$, $\lim\ t_{n,j} = \lim (1 - |x_n^1|^p)^{-1/p} = (1 - |x^1|^p)^{-1/p} = t_j$. Now, for each n, we have $0 \leqq m^2(x_n) \leqq 2$ by (a) and (c), so $\{m^2(x_n)\}$ is a bounded sequence of real numbers, and hence must have at least one cluster point. Let y be an arbitrary cluster point of $\{m^2(x_n)\}$, and let $\{m^2(x_{n_v})\}$ be a subsequence of $\{m^2(x_n)\}$ which converges to y. Then, since $\{x_{n_v}\}$ is a subsequence of $\{x_n\}$, it follows that $\lim x_{n_v}^i = x^i$ for each $i = 1, 2, 3, \cdots$, and also, $\lim\ t_{n_v,j} = t_j$ for each $j \geqq 2$. Thus,

$$y = \lim m^2(x_{n_v}) = \lim \sum_{j=2}^{\infty} t_{n_v,j}^2 (x_{n_v}^j)^2$$

$$\leqq \sum_{j=2}^{\infty} \lim \{t_{n_v,j}^2 (x_{n_v}^j)^2\}$$

$$= \sum_{j=2}^{\infty} \{\lim t^2_{n_v, j}\} \{\lim (x^j_{n_v})^2\}$$

$$= \sum_{j=2}^{\infty} t_j^2 (x^j)^2$$

$$= m^2(x)$$

We want to show that equality must hold in the above inequality. Suppose by way of contradiction that $y < m^2(x)$. Let $\epsilon = m^2(x) - y > 0$. Choose a subsequence of n_v's such that $m^2(x) > m^2(x_{n_v}) + \epsilon/2$. Choose J so large that $\sum_{j=2}^{J} t_j^2(x^j)^2 > m^2(x) - \epsilon/4$. For each j, we have $\lim t^2_{n_v,j}(x^j_{n_v})^2 = \{\lim t^2_{n_v,j}\} \{\lim (x^j_{n_v})^2\} = t_j^2(x^j)^2$

Hence, for each $j = 2, 3, \cdots, J$, we can choose N_j such that for all $n_v \geqq N_j$,

$$t^2_{n_v,j}(x^j_{n_v})^2 > t_j^2(x^j)^2 - \epsilon/5^j$$

Then, for all n_v such that $n_v \geqq \max \{N_j : j \leqq J\}$,

$$m^2(x) > m^2(x_{n_v}) + \epsilon/2 = \sum_{j=2}^{\infty} t^2_{n_v,j}(x^j_{n_v})^2 + \epsilon/2$$

$$\geqq \sum_{j=2}^{J} t^2_{n_v,j}(x^j_{n_v})^2 + \epsilon/2$$

$$> \sum_{j=2}^{J} \{t_j^2(x^j)^2 - \epsilon/5^j\} + \epsilon/2$$

$$= \sum_{j=2}^{J} t_j^2(x^j)^2 - \sum_{j=2}^{J} \{\epsilon/5^j\} + \epsilon/2$$

$$> m^2(x) - \epsilon/4 - \sum_{j=2}^{J} \{\epsilon/5^j\} + \epsilon/2$$

$$\geqq m^2(x) + \epsilon/4 - \sum_{j=2}^{\infty} \epsilon/5^j$$

$$= m^2(x) + \epsilon/4 - \epsilon/4$$

$$= m^2(x)$$

Since the strict inequalities in the above are impossible, we thus get a contradiction. Therefore $y = m^2(x)$. But y was originally chosen as an arbitrary cluster point of $\{m^2(x_n)\}$. Hence, $\{m^2(x_n)\}$ has a unique cluster point, and $\lim m^2(x_n) = m^2(x)$, or $\lim m(x_n) = m(x)$. Therefore, m is continuous on U.

(e) For each x such that $\|x\| < 1/2$, $m(x) < 1$. Note that $\|x\| < 1/2$ implies that $|x^1| < 1/2$. Choose $t_j > 0$, $j \geq 2$, such that $|x^1|^p + t_j^{-p} = 1$. Then, for each j we have $t_j^{-p} = 1 - |x^1|^p > 1 - |x^1| > 1 - 1/2 = 1/2$, so $t_j^p < 2$. But since $t_j \geq 1$ for each j, and since $1 < p \leq 2$, this implies that $t_j < 2$ for each j. Thus,

$$m(x) = \{\sum_{j=2}^{\infty} t_j^2 (x^j)^2\}^{1/2} < \{\sum_{j=2}^{\infty} 4 (x^j)^2\}^{1/2}$$

$$= 2 \cdot \{\sum_{j=2}^{\infty} (x^j)^2\}^{1/2} \leq 2 \cdot \|x\| < 2 \cdot 1/2 = 1.$$

(f) m is a rotund function; that is, for $x = (x^1, x^2, x^3, \cdots)$ and $y = (y^1, y^2, y^3, \cdots)$ in U, $m\{(x + y)/2\} < 1/2\{m(x) + m(y)\}$.

The result follows immediately in case x and y lie in the same coordinate plane, since m is then an l_p norm for some $p > 1$, which is (R): also for $x^1 = y^1 = 0$, since m is then the l_2 norm, which is (R). The remaining possibilities are considered in two cases.

<u>Case 1.</u> $|x^1| = |y^1|$

If $x^1 = y^1$, then $(x^1 + y^1)/2 = x^1$, so x, y, and $(x + y)/2$ lie in the same hyperplane x^1 = constant. The t_j's in (3) depend only on the first coordinate; thus x, y, and $(x + y)/2$ have the same t_j's. Using the Minkowski inequality, we thus get

$$m\{(x+y)/2\} = \{\sum_{j=2}^{\infty} t_j^2((x^j+y^j)/2)^2\}^{1/2}$$
$$= 1/2\ \{\sum_{j=2}^{\infty} (t_jx^j + t_jy^j)^2\}^{1/2}$$
$$\leq 1/2\ (\{\sum_{j=2}^{\infty} t_j^2(x^j)^2\}^{1/2} + \{\sum_{j=2}^{\infty} t_j^2(y^j)^2\}^{1/2})$$
$$= 1/2\ \{m(x) + m(y)\}$$

If $x^1 = -y^1$, then $(x^1+y^1)/2 = 0$. Thus, x and y have the same t_j's, and the corresponding sequence for $(x+y)/2$ takes the value one for each j. Using the fact that for each j, $t_j \geq 1$, and the Minkowski inequality, we get

$$m\{(x+y)/2\} = \{\sum_{j=2}^{\infty}((x^j+y^j)/2)^2\}^{1/2}$$
$$= 1/2\ \{\sum_{j=2}^{\infty}(x^j+y^j)^2\}^{1/2}$$
$$\leq 1/2\ \{\sum_{j=2}^{\infty} t_j^2(x^j+y^j)^2\}^{1/2}$$
$$= 1/2\ \{\sum_{j=2}^{\infty}(t_jx^j + t_jy^j)^2\}^{1/2}$$
$$\leq 1/2\ \{\sum_{j=2}^{\infty}(|t_jx^j| + |t_jy^j|)^2\}^{1/2}$$
$$\leq 1/2\ (\{\sum_{j=2}^{\infty} |t_jx^j|^2\}^{1/2} + \{\sum_{j=2}^{\infty} |t_jy^j|^2\}^{1/2})$$
$$= 1/2\ (\{\sum_{j=2}^{\infty} t_j^2(x^j)^2\}^{1/2} + \{\sum_{j=2}^{\infty} t_j^2(y^j)^2\}^{1/2})$$
$$= 1/2\ \{m(x) + m(y)\}.$$

Note that equality in either of the above inequalities would force equality in the Minkowski inequality, and this in turn would force $x = y$. Thus, if x and y are distinct, then the strict inequalities must hold.

Case 2. $|x^1| \neq |y^1|$

Without loss of generality, we may assume that $|y^1| > |x^1|$. Choose $t_j > 0$ such that $|x^1|^p + t_j^{-p} = 1$, $s_j > 0$ such that $|y^1|^p + s_j^{-p} = 1$, and $r_j > 0$ such that $|(x^1+y^1)/2|^p + r_j^{-p} = 1$. Since in each plane $x^1 \times x^j$ we are using an l_p norm with $p > 1$, which is (R), we have for each j, $1/r_j > 1/2(1/t_j + 1/s_j)$, or

(6) $r_j < 2s_j t_j/(s_j + t_j)$ for each j.

Note that $|y^1| > |x^1|$ implies that $s_j > t_j$ for each j. We first prove the following lemma.

Lemma 2.3 Let a, b, c, d, be four real numbers such that $0 \leq a < 1$, $0 < b < 1$, $c \geq 1$, $d > 1$; also, a = b iff c = d and a < b iff c < d. Then $(a + b)/(c + d) \leq 1/2(a/d + b/c)$.

Proof: We consider three separate cases.

(i) c = d. Then, $(a + b)/(c + d) = (a + b)/2d$

$$= 1/2(a/d + b/d) = 1/2(a/d + b/c)$$

(ii) c < d. Then, a < b, so $a/d < a/c < b/c$. Since $d - c > 0$,

$$(d - c)\, a/d < (d - c)\, b/c$$

$$(1 - c/d)\, a < (d/c - 1)\, b$$

$$(d/c - 1)\, b - (1 - c/d)\, a > 0$$

$$db/c - b - a + ca/d > 0$$

$$a + b < db/c + ca/d$$

$$2(a + b) < a + b + db/c + ca/d$$

$$2(a + b) < (c + d)\,(a/d + b/c)$$

so, $(a + b)/(c + d) < 1/2(a/d + b/c)$

(iii) c > d. Then, a > b, so $a/d > b/d > b/c$. Since $c - d > 0$,

$$(c - d)\, a/d > (c - d)\, b/c$$

$$(c/d - 1)\, a > (1 - d/c)\, b$$

$$(c/d - 1)\, a - (1 - d/c)\, b > 0$$

$ca/d - a - b + db/c > 0$ same as step 4 in (ii). QED

Now, returning to the proof of Case 2 of (f),

$$m\{(x + y)/2\} = \{\sum_{j=2}^{\infty} r_j^2((x^j + y^j)/2)^2\}^{1/2}$$

$$\leq \{\sum_{j=2}^{\infty}(\{4s_j^2 t^2{}_j/(s_j + t_j)^2\}\{(x^j + y^j)/2\}^2)\}^{1/2}$$

by (6)

$$= \{\sum_{j=2}^{\infty} s_j^2 t_j^2 \{(x^j + y^j)/(s_j + t_j)\}^2\}^{1/2}$$

$$\leq \{\sum_{j=2}^{\infty} \{s_j^2 t_j^2 |x^j + y^j| /(s_j + t_j)\}^2\}^{1/2}$$

$$\leq \{\sum_{j=2}^{\infty} s_j^2 t_j^2 \{(|x^j| + |y^j|)/(s_j + t_j)\}^2\}^{1/2}$$

$$< \{\sum_{j=2}^{\infty} s_j^2 t_j^2 \{1/2(|x^j|/s_j + |y^j|/t_j)\}^2\}^{1/2}$$

by Lemma 2.3

$$= \sum_{j=2}^{\infty} \{1/4 \; s_j^2 t_j^2 (|x^j|^2/s_j^2 + 2|x^j y^j|/(s_j t_j) + |y^j|^2/t_j^2)\} \; 1/2$$

$$= 1/2 \; \{\sum_{j=2}^{\infty} (t_j^2 |x^j|^2 + 2 s_j t_j |x^j \bar{y}^j| + s_j^2 |y^j|^2)\}^{1/2}$$

$$= 1/2 \; \{\sum_{j=2}^{\infty} (t_j |x^j| + s_j |y^j|)^2\}^{1/2}$$

$$\leq 1/2 \left[\{\sum_{j=2}^{\infty} (t_j |x^j|)^2\}^{1/2} + \{\sum_{j=2}^{\infty} (s_j |y^j|)^2\}^{1/2} \right]$$

$$= 1/2 \left[\{\sum_{j=2}^{\infty} t_j^2 (x^j)^2\}^{1/2} + \{\sum_{j=2}^{\infty} s_j^2 (y^j)^2\}^{1/2} \right]$$

$$= 1/2 \; \{m(x) + m(y)\}.$$

This completes the proof of (f).

We thus see as a result of the above properties that m is a continuous, symmetric, convex function defined on the open unit ball U^o. Therefore, $E = m^{-1}((-1, 1))$ is open, symmetric, convex, and also, $E \subseteq U^o$. Using this last result and (e), we see that

(g) $1/2 \; U^o \subseteq E \subseteq U^o$.

Hence, the Minkowski functional of E, $f_E(x)$, is a norm, so for each x in l_2, define

(7) $||x||_1 = f_E(x) = \inf \{r: x/r \text{ is in } E \text{ and } r > 0\}$.

We next note that the following relationships hold:

(h) $||x||_1 \leq 1$ iff $m(x) \leq 1$ and $||x||_1 = 1$ iff

$m(x) = 1$. This follows immediately from the manner in which E was determined from m, and $||\cdots||_1$ from E.

(i) For each x in l_2, $||x|| \leq ||x||_1 \leq 2||x||$.

This follows immediately from (g).

(j) For each x in E, $m(x) \leq ||x||_1$

Proof of (j)

Let $x = (x^1, x^2, x^3, \cdots)$ be any non-zero element in E. Choose $k > 0$ such that $||kx||_1 = 1$. Then, $||x||_1 = 1/k$, where $k \geq 1$, and $kx = (kx^1, kx^2, kx^3, \cdots)$. Choose $t_j > 0$ and $s_j > 0$ such that $|x^1|^p + t_j^{-p} = 1$ and $|kx^1|^p + s_j^{-p} = 1$, respectively. Now, $|kx^1| \geq |x^1|$ implies $s_j \geq t_j$, so

$$\begin{aligned} m(x) &= \{\textstyle\sum_{j=2}^{\infty} t_j^2(x^j)^2\}^{1/2} \leq \{\textstyle\sum_{j=2}^{\infty} s_j^2(x^j)^2\}^{1/2} \\ &= 1/k\ \{k^2 \textstyle\sum_{j=2}^{\infty} s_j^2(x^j)^2\}^{1/2} \\ &= 1/k\ \{\textstyle\sum_{j=2}^{\infty} s_j^2(kx^j)^2\}^{1/2} = 1/k\ \{m(kx)\} \\ &= 1/k = ||x||_1. \end{aligned}$$

As a result of (i) above, we see that the norms $||\cdots||$ and $||\cdots||_1$ are equivalent. Thus, let B be the space l_2 renormed with $||\cdots||_1$. Since B is isomorphic to l_2, B is reflexive ([2], Theorem 1, page 56). Also, B is (R), by (f) above. In order to show that B is m.l.u.c., we shall first show that B is (A) [See Definition 3.1 below]. To accomplish this, we need two properties of lim sups of sets of real numbers, which we list below as lemmas without proofs.

<u>Lemma 2.4</u> <u>If $\{a_n\}$ and $\{b_n\}$ are two bounded sequences of real numbers, then</u>

$\limsup (a_n + b_n) \leq \limsup a_n + \limsup b_n$.

__Lemma 2.5__ __If__ $\{c_n\}$ __and__ $\{d_n\}$ __are two bounded sequences of real numbers, and if__ $c_n \leq d_n$ __for each__ n, __then__ $\limsup c_n \leq \limsup d_n$.

Now, let $\{x_n\}$ be a sequence of elements of B and x_o an element of B such that w-lim $x_n = x_o$ and $\lim \|x_n\|_1 = \|x_o\|_1 = 1$. Then, by Definition 3.1, we want to show that $\lim \|x_n - x_o\|_1 = 0$. Without loss of generality, we may assume that $\|x_n\|_1 = 1$ for each n, since otherwise we could first normalize each element without affecting the weak convergence. For each $j \geq 2$, choose

$$(8)\quad \begin{aligned} &t_j \geq 1 \text{ such that } |x_o^1|^p + t_j^{-p} = 1, \text{ and} \\ &t_{n,j} \geq 1 \text{ such that } |x_n^1|^p + t_{n,j}^{-p} = 1,\ n = 1, 2, \cdots . \end{aligned}$$

Let $\epsilon > 0$ be given. Then, since

$$1 = \|x_o\|_1 = \|x_o\|_1^2 = m^2(x_o) = \sum_{j=2}^{\infty} t_j^2 (x_o^j)^2,$$

we can choose the integer J so large that

$$(9)\quad \sum_{j=2}^{J} t_j^2 (x_o^j)^2 > 1 - \epsilon$$

Since w-lim $x_n = x_o$, we have

$$(10)\quad \lim x_n^i = x_o^i \quad \text{for } i = 1, 2, 3, \cdots , J.$$

Using (10) and (8), we get

$$(11)\quad \begin{aligned} \lim t_{n,j} &= \lim (1 - |x_n^1|^p)^{-1/p} \\ &= (1 - |x_o^1|^p)^{-1/p} = t_j \quad \text{for each } j. \end{aligned}$$

Then, by (10) and (11), we have

$$(12)\quad \begin{aligned} \lim \sum_{j=2}^{J} t_{n,j}^2 (x_n^j)^2 &= \sum_{j=2}^{J} \lim \{t_{n,j}^2 (x_n^j)^2\} \\ &= \sum_{j=2}^{J} \{\lim t_{n,j}^2\} \{\lim (x_n^j)^2\} \end{aligned}$$

$$= \sum_{j=2}^{J} t_j^2 (x_o^j)^2$$

Since for each n,

$$1 = \|x_n\|_1^2 = m^2(x_n) = \sum_{j=2}^{\infty} t_{n,j}^2 (x_n^j)^2$$

$$= \sum_{j=2}^{J} t_{n,j}^2 (x_n^j)^2 + \sum_{j=J+1}^{\infty} t_{n,j}^2 (x_n^j)^2,$$

we have for each n,

$$(13) \quad \sum_{j=J+1}^{\infty} t_{n,j}^2 (x_n^j)^2 = 1 - \sum_{j=2}^{J} t_{n,j}^2 (x_n^j)^2$$

Now, as n becomes infinite, the limit on the right exists, by (12), so the limit on the left exists, and

$$(14) \quad \lim \sum_{j=J+1}^{\infty} t_{n,j}^2 (x_n^j)^2 = \lim\{1 - \sum_{j=2}^{J} t_{n,j}^2 (x_n^j)^2\}$$

$$= 1 - \lim \sum_{j=2}^{J} t_{n,j}^2 (x_n^j)^2$$

$$= 1 - \sum_{j=2}^{J} t_j^2 (x_o^j)^2 \quad \text{by (12)}$$

$$< 1 - (1 - \epsilon) \quad \text{by (9)}$$

$$= \epsilon$$

For each n and j, $(x_n^j)^2 \leq t_{n,j}^2 (x_n^j)^2$, so

$\sum_{j=J+1}^{\infty} (x_n^j)^2 \leq \sum_{j=J+1}^{\infty} t_{n,j}^2 (x_n^j)^2$ for each n. Thus,

$$(15) \quad \limsup \sum_{j=J+1}^{\infty} (x_n^j)^2 \leq \limsup \sum_{j=J+1}^{\infty} t_{n,j}^2 (x_n^j)^2$$

by Lemma 2.5

$$= \lim \sum_{j=J+1}^{\infty} t_{n,j}^2 (x_n^j)^2$$

since lim exists

$$< \epsilon \quad \text{by (14)}$$

Also, $\lim \sum_{j=1}^{J} (x_n^j)^2 = \sum_{j=1}^{J} \lim (x_n^j)^2 = \sum_{j=1}^{J} (x_o^j)^2$

exists, so

$$(16)\quad \limsup \sum_{j=1}^{J} (x_n^j)^2 = \lim \sum_{j=1}^{J} (x_n^j)^2$$

$$= \sum_{j=1}^{J} (x_o^j)^2 \leq ||x_o||^2$$

Hence,

$$\limsup ||x_n||^2 = \limsup \{ \sum_{j=1}^{J}(x_n^j)^2 + \sum_{j=J+1}^{\infty}(x_n^j)^2 \}$$

$$\leq \limsup \sum_{j=1}^{J}(x_n^j)^2 + \limsup \sum_{j=J+1}^{\infty}(x_n^j)^2$$

by Lemma 2.4

$$< ||x_o||^2 + \epsilon \qquad \text{by (15) and (16).}$$

Since this must hold for all $\epsilon > 0$, we thus have $\limsup ||x_n|| \leq ||x_o||$. Therefore, using the fact that x_o is the weak limit of x_n, we get $||x_o|| \leq \liminf ||x_n|| \leq \limsup ||x_n|| \leq ||x_o||$, so $\lim ||x_n|| = ||x_o||$. But l_2 is (A), which implies that $\lim ||x_n - x_o|| = 0$, and hence, $\lim ||x_n - x_o||_1 = 0$ by the equivalence of norms. QED.

We have now shown that B is both (R) and (A); i.e., B is (H), according to Definition 3.2 below. Since B is also reflexive, it follows that B is m.l.u.c. by Theorem 3.3 below.

It remains to show that B is not l.u.c. Let $x = (1, 0, 0, 0, \cdots)$ and $x_n = (0, 0, \cdots, 0, 1, 0, \cdots)$ where the nth coordinate is one. Then, for each n, $m(x) = m(x_n) = 1$, so $||x_n||_1 = ||x||_1 = 1$ for each n. Also, for each n, we have, by (j)

$||x_n + x||_1 \geq m(x_n + x) = (|x^1|^p + |x^n|^p)^{1/p} = 2^{1/p}$ where $p = (2n - 2)/(2n - 3)$ for each n. Thus, since $\lim p = \lim (2n - 2)/(2n - 3) = 1$, we see that

$\lim \|x_n + x\|_1 = 2$. But

$x_n - x = (-1, 0, 0, \cdots, 0, 1, 0, 0, \cdots)$, so

$\|x_n - x\|_1 \geqq \|x_n - x\| = \{(-1)^2 + (1)^2\}^{1/2} = 2^{1/2}$.

Therefore B is not l.u.c.

This concludes Example 2.2.

Combining the results of Theorem 2.1 with Examples 2.1 and 2.2, we have

Theorem 2.2 *For any Banach space, the following implications hold:*

$$\text{u.c.} \Longrightarrow \text{l.u.c.} \Longrightarrow \text{m.l.u.c.} \Longrightarrow (R)$$

Furthermore, none of these implications can be reversed.

We now state some sufficient conditions for m.l.u.c.

Theorem 2.3 *If B is (Str) and if linear functionals attain their maximum on the unit sphere of B, then B* is m.l.u.c.*

Proof: Let g_o be an element of B* and $\{f_n\}$, $\{g_n\}$ two sequences of elements of B* such that

$\|f_n\| = \|g_n\| = \|g_o\| = 1$, and

$\lim \|f_n + g_n - 2g_o\| = 0$. Choose x_o in B such that $\|x_o\| = 1$ and $g_o(x_o) = 1$. Then, for each n,

$$\begin{aligned}
\|f_n + g_n - 2g_o\| &= \sup \{|(f_n + g_n - 2g_o)(x)| : x \text{ in } U\} \\
&= \sup \{|f_n(x) + g_n(x) - 2g_o(x)| : x \text{ in } U\} \\
&\geqq |f_n(x_o) + g_n(x_o) - 2g_o(x_o)| \\
&= |f_n(x_o) + g_n(x_o) - 2| \\
&\geqq 2 - |f_n(x_o) + g_n(x_o)|
\end{aligned}$$

$$\geq 2 - (|f_n(x_o)| + |g_n(x_o)|)$$

$$\geq 0.$$

Since by hypothesis the left side of the above inequality approaches zero as n becomes infinite, equality must hold at each step, and hence we have

$\lim (f_n(x_o) + g_n(x_o)) = 2$. But this implies that $\lim f_n(x_o) = 1$ and $\lim g_n(x_o) = 1$. Since B is (Str), Theorem 1.2 yields the fact that $\lim_{m,n\to\infty} ||f_m - f_n|| = 0$. By completeness, there exists f_o in B* such that $||f_o|| = 1$ and $\lim ||f_n - f_o|| = 0$. Hence, $\lim f_n(x_o) = f_o(x_o)$, so $f_o(x_o) = 1$. Since B is (S), we must have $f_o = g_o$. Therefore $\lim ||f_n - g_o|| = 0$, and hence B* is m.l.u.c. by Lemma 2.1.

<u>Theorem 2.4</u> <u>If B* is (Str), then B is m.l.u.c.</u>

<u>Proof</u>: Let x_o be an element of B and $\{x_n\}$, $\{y_n\}$ two sequences of elements of B such that

$||x_n|| = ||y_n|| = ||x_o|| = 1$, and

$\lim ||x_n + y_n - 2x_o|| = 0$. Choose f_o in B* such that $||f_o|| = 1$ and $f_o(x_o) = 1$. Then for each n,

$$\begin{aligned}
||x_n + y_n - 2x_o|| &= \sup \{ |f(x_n + y_n - 2x_o)| : ||f|| \leq 1 \} \\
&\geq |f_o(x_n + y_n - 2x_o)| \\
&= |f_o(x_n) + f_o(y_n) - 2f_o(x_o)| \\
&= |f_o(x_n) + f_o(y_n) - 2| \\
&\geq 2 - |f_o(x_n) + f_o(y_n)|
\end{aligned}$$

$$\geqq 2 - (|f_o(x_n)| + |f_o(y_n)|)$$

$$\geqq 0$$

Since by hypothesis the left side of the above inequality approaches zero as n becomes infinite, equality must hold at each step, and hence we have $\lim (f_o(x_n) + f_o(y_n)) = 2$. But this implies that $\lim f_o(x_n) = 1$ and $\lim f_o(y_n) = 1$. B* is (Str), so by Theorem 1.1 we get

$\lim_{m,n \to \infty} \|x_m - x_n\| = 0$. By completeness, there exists y_o in B such that $\|y_o\| = 1$ and $\lim \|x_n - y_o\| = 0$. By continuity of f_o, $\lim f_o(x_n) = f_o(y_o)$, so $f_o(y_o) = 1$. Since B* is (S), B is (R), and hence we must have $y_o = x_o$. Thus,

$\lim \|x_n - x_o\| = 0$, so B is m.l.u.c. by Lemma 2.1. QED

CHAPTER III

Duality

We have known for some time ([3], page 518) that in reflexive Banach spaces, (R) and (S) are dual properties. With the introduction of the stronger properties of smoothness and convexity defined in chapters I and II, namely, (Str), l.u.c., and m.l.u.c., it would seem desirable and reasonable to hope for some sort of duality to exist between (Str) and l.u.c., or between (Str) and m.l.u.c. Using the results of Lovaglia's Theorems 2.2 and 2.3, we see that, for a reflexive space B, if either B or B* is l.u.c., then the other is (Str). However, his Theorems 2.4 and 2.6, which reverse the implication, require an additional hypothesis, which is called <u>weak</u> <u>l.u.c</u>. in B (or <u>weak</u>* <u>l.u.c</u>. in B*). That some such additional hypothesis is necessary is confirmed by Corollary 2 to Theorem 3.2 below, which answers the question of duality between (Str) and l.u.c. in the negative. On the other hand, we see from Theorems 2.3 and 2.4 above that for a reflexive B, if either B or B* is (Str), then the other is m.l.u.c. To date, all attempts to reverse this implication have been unsuccessful, but it is apparent that full duality is not present. Termed loosely, l.u.c. is "too strong" to yield full duality with (Str), and, on the basis of present knowledge, m.l.u.c. appears to be "too weak". Therefore, we might hope to find some property "between" l.u.c. and m.l.u.c. which will yield the desired duality.

<u>Definition 3.1</u> <u>B</u> <u>has</u> <u>property</u> (<u>A</u>) [written: <u>B</u> <u>is</u> (<u>A</u>)] iff the

following condition is satisfied: If a sequence $\{x_n\}$ of elements of B converges weakly to the element x in B, and if

$\lim \|x_n\| = \|x\|$, then $\lim \|x_n - x\| = 0$.

This is a well-known property of u.c. spaces, and can also be shown to be a property of l.u.c. spaces. It has been investigated by many people, but we are primarily interested in the results of Kadec [6], and Fan and Glicksberg [5]. Following the notation used in the latter paper, we will find it convenient to have the following definition:

Definition 3.2 *B is (H)* if B is both (A) and (R). It is evident from the above definition that l.u.c. implies (H). Fan and Glicksberg have proved the following ([5], Theorem 3):

Theorem 3.1 *If B^* is (Str), then B is reflexive.*

The next theorem is also due to Fan and Glicksberg, but since they omit a direct proof, we furnish it here.

Theorem 3.2 *If B is reflexive and (H), then B^* is (Str).*

Proof: Let f_o be an arbitrary element of B^* such that $\|f_o\| = 1$, and let $\{x_n\}$ be a sequence of elements of B such that $\|x_n\| = 1$ and $\lim f_o(x_n) = 1$. Since B is reflexive, we can choose x_o in B such that $\|x_o\| = 1$ and $f_o(x_o) = 1$. Also by reflexivity, $\{x_n\}$ has at least one weak cluster point. Let y_o be an arbitrary weak cluster point of $\{x_n\}$ and let $\{x_{v_n}\}$ be a subsequence of $\{x_n\}$ converging weakly to y_o. Then,

$f_o(y_o) = \lim f_o(x_{v_n}) = \lim f_o(x_n) = 1$. Thus it follows that y_o has norm 1. But since B is reflexive and (R), we have by duality that B^* is (S), and hence $y_o = x_o$. Thus x_o is the unique weak cluster point of $\{x_n\}$, so w-lim $x_n = x_o$. Then, by (H), we get $\lim \|x_n - x_o\| = 0$. Therefore B* is (Str), by Theorem 1.1. QED.

<u>Corollary 1</u> <u>If B is reflexive and (H), then B is m.l.u.c.</u>

<u>Proof</u>: B* is (Str), by Theorem 3.2, and hence B is m.l.u.c., by Theorem 2.4. QED

The question naturally arises here as to whether or not (H) implies m.l.u.c. in the case of a non-reflexive Banach space. This is an open question at the moment, though a negative answer seems likely.

<u>Corollary 2</u> <u>There is a (reflexive) non-l.u.c. Banach space B with the property that B* is (Str).</u>

<u>Proof</u>: We first remark that if B* is (Str), then B is necessarily reflexive, by Theorem 3.1. Let B be the space constructed in Example 2.2. We showed that B is not l.u.c.; however, it is reflexive and (H), and hence B* is (Str), by Theorem 3.2. QED.

<u>Theorem 3.3</u> <u>If B is reflexive, then the following implications hold</u>:

$$\text{u.c.} \Longrightarrow \text{l.u.c.} \Longrightarrow (H) \Longrightarrow \text{m.l.u.c.} \Longrightarrow (R)$$

<u>Proof</u>: Follows immediately from Theorem 2.2, the remark following Definition 3.2, and Corollary 1 to Theorem 3.2. QED.

As noted previously, the implication on the left side of (H) in Theorem 3.3 cannot be reversed, as shown by Example 2.2. The best we can do at the moment by way of a converse is stated in Theorem 3.4 below. First, however, we need a definition. The following is a well-known property of weak limits: If w-lim $x_n = x$, then $\|x\| \leq \liminf \|x_n\|$. This suggests the following property:

(P) If w-lim $x_n = x$, and if lim $\|x_n\|$ exists, then $\|x\| = \lim \|x_n\|$.

<u>Definition 3.3</u> We will say <u>B is (P)</u> iff B has property (P).

We might note here that the space l_1 is (P), since in l_1 weak and norm convergence of a sequence to an element are equivalent ([2], Cor. 1, page 33). However, the space l_2 is not (P), since the sequence of unit vectors converges weakly to zero, whereas the sequence of their norms converges to one. In view of the hypothesis of Theorem 3.4, an interesting open question here is whether or not there is an infinite-dimensional, reflexive Banach space which is (P).

<u>Theorem 3.4</u> <u>If a reflexive space B is (H) and (P), then B is l.u.c.</u>

<u>Proof</u>: Let $\{x_n\}$ be a sequence of elements of B and x_o an element of B with $\|x_n\| = \|x_o\| = 1$ and $\lim \|x_n + x_o\| = 2$. Since B is reflexive, $\{x_n\}$ has at least one weak cluster point. Thus, let y_o be an arbitrary weak cluster point of $\{x_n\}$, and let $\{x_{v_n}\}$ be a subsequence of $\{x_n\}$ which converges weakly to y_o. Since $\|x_{v_n}\| = 1$ for each n, we have $\|y_o\| = 1$ by (P). Then, since B is (H), we have $\lim \|x_{v_n} - y_o\| = 0$. So,

$$\begin{aligned} 2 &= \lim \|x_{v_n} + x_o\| = \lim \|x_{v_n} - y_o + y_o + x_o\| \\ &\leqq \lim (\|x_{v_n} - y_o\| + \|y_o + x_o\|) \\ &= \lim \|x_{v_n} - y_o\| + \|y_o + x_o\| \\ &= \|y_o + x_o\| \leqq \|y_o\| + \|x_o\| = 2. \end{aligned}$$

Equality throughout implies that $||y_o + x_o|| = 2$, so $||x_o|| = ||y_o|| = ||(x_o + y_o)/2|| = 1$. But B is (R) and hence we must have $y_o = x_o$. Thus we see that x_o is the unique weak cluster point of $\{x_n\}$, and from the reflexivity of B it follows that $\{x_n\}$ converges weakly to x_o. Using (H) again, we finally get $\lim ||x_n - x_o|| = 0$. Therefore, B is l.u.c. QED

Whether or not the implication on the right side of (H) in Theorem 3.3 can be reversed is not known at this time. However, a negative answer seems reasonable in view of the discussion at the beginning of this section, and the fact that the property (H), which we now have "between" l.u.c. and m.l.u.c., does in fact yield the desired duality, as shown in Theorem 3.9 below. We now continue with the results on duality.

Theorem 3.5 If B is reflexive and B* is (H), then B is (Str).

Proof: B reflexive implies that B* is reflexive, so by Theorem 3.2, B** is (Str), which in turn implies that B is (Str). QED

Theorem 3.6 If B* is (Str), then B is (H).

Proof: Since (Str) implies (S), it follows that B is (R) by duality. To show that B is also (A), let $\{x_n\}$ be a sequence of elements of B such that

$$\text{w-lim } x_n = x_o, \quad ||x_o|| = 1, \text{ and } \quad \lim ||x_n|| = ||x_o||.$$

We want to show that $\{x_n\}$ converges in norm to x_o as n becomes infinite. Using the notation of Fan and Glicksberg, we thus want to show that $\lim x_n = x_o$. Choose f_o in B* such that $||f_o|| = 1$ and $f_o(x_o) = 1$. Then, since x_o is the weak limit of

x_n, we have $\lim f_o(x_n) = f_o(x_o) = 1 = \|f_o\|$. Since $\lim \|x_n\| = 1$, there can be at most a finite number of x_n's such that $\|x_n\| = 0$, so without loss of generality we may assume that for each n, $\|x_n\| > 0$. For each n, let $y_n = x_n/\|x_n\|$. Then for each n, $\|y_n\| = 1$. Furthermore,

$$\lim f_o(y_n) = \lim f_o(x_n/\|x_n\|) = \lim (1/\|x_n\|)\, f_o(x_n)$$
$$= (\lim 1/\|x_n\|)\,(\lim f_o(x_n)) = \|f_o\|.$$

B* is (Str) by hypothesis, so $\lim \|y_m - y_n\| = 0$ by Theorem 1.1. By completeness, there exists y_o in B such that $\|y_o\| = 1$ and $\lim y_n = y_o$. For each n, $x_n = \|x_n\|\cdot y_n$, so $\lim x_n = \lim \|x_n\|\cdot y_n = (\lim \|x_n\|)(\lim y_n) = y_o$. Since norm convergence implies weak convergence, we thus have w-lim $x_n = y_o$. Therefore, $y_o = x_o$, by uniqueness of weak limits, and hence we have $\lim x_n = x_o$. QED.

Combining Theorems 3.1 and 3.6, we get

<u>Theorem 3.7</u> <u>If B* is (Str), then B is reflexive and (H).</u>

<u>Theorem 3.8</u> <u>If B is reflexive and (Str), then B* is (H).</u>

<u>Proof</u>: B reflexive and (Str) implies that B** is (Str), and hence B* is (H) by Theorem 3.4. QED.

Combining the results of Theorems 3.2, 3.5, 3.6, and 3.8, we see that we have full duality between the properties (Str) and (H), which we now state in the following

<u>Theorem 3.9</u> <u>If B is reflexive and B [B*] has one of the properties (Str) or (H), then B* [B] has the other.</u>

Since, in a reflexive B, an isomorphism of either B or B* determines an isomorphism of the other, we also have

Theorem 3.10 If B is reflexive, and if B [B*] is isomorphic to a space which has either of the properties (Str) or (H), then B* [B] is isomorphic to a space which has the other.

CHAPTER IV

Product Spaces

Let $\{B_i\}$ be a sequence of Banach spaces. Denote by $\|\cdots\|_i$ the norm in B_i. Let $P_p(B_i)$ be the space of sequences $x = \{x^i\}$, x^i in B_i, for which $\sum_{i=1}^{\infty} \|x^i\|_i^p$ is convergent, where $i \leq p < \infty$. Let $\|\cdots\|$ be the norm in $P_p(B_i)$, where $\|x\| = (\sum_{i=1}^{\infty} \|x^i\|_i^p)^{1/p}$. It is readily verified that $P_p(B_i)$ is a Banach space.

__Theorem 4.1__ __For__ $p \geq 1$, $P_p(B_i)$ __is__ (__A__) __if__ __each__ B_i __is__ (__A__).

__Proof__. Let $\{x_n\}$ be a sequence of elements of $P_p(B_i)$ such that w-lim $x_n = x$ and $\lim \|x_n\| = \|x\|$. We want to show that $\lim \|x_n - x\| = 0$. Without loss of generality, we may assume that $\|x_n\| = \|x\| = 1$, for otherwise we could normalize each element and proceed as in the proof of Theorem 3.4. Then, since $\|x_n\|^p = 1$, we have $\|x_n^i\|_i^p \leq 1$ and hence $\|x_n^i\|_i \leq 1$ for each i. Thus, for each i, $\{\|x_n^i\|_i\}$ is a bounded sequence of real numbers. Therefore, by diagonalizing, we can determine a subsequence of n's such that $\lim \|x_n^i\|_i$ exists for each i. Now, since x is the weak limit of the subsequence x_n, it follows that w-lim $x_n^i = x^i$ for each i, and thus we have

(1) $\|x^i\|_i \leq \liminf \|x_n^i\|_i = \lim \|x_n^i\|_i$ for each i.

We want to show that equality must hold in (1) for each i. Suppose by way of contradiction that for some index j, $\|x^j\|_j < \lim \|x_n^j\|_j$. Then, since p is finite, it follows that

$\|x^j\|_j^p < \lim \|x_n^j\|_j^p$. Let $\epsilon = (\lim \|x_n^j\|_j^p) - \|x^j\|_j^p > 0$. Choose a finite set A of indices i such that

$\sum_{i \in A} \|x^i\|_i^p > 1 - \epsilon/4$. Without loss of generality, we may assume that j is in A, for otherwise we could replace A by $A' = A \cup \{j\}$. For each $i \neq j$ in A, choose N_i such that $\|x_n^i\|_i^p > \|x^i\|_i^p - \epsilon/5^i$ for $n \geq N_i$, and choose N_j such that $\|x_n^j\|_j^p > \|x^j\|_j^p + \epsilon/2$ for $n \geq N_j$. Then, for each n such that $n \geq \max_{\substack{i,j \in A \\ i \neq j}} (N_i, N_j)$, we have

$$1 = \|x_n\|^p = \sum_{i=1}^{\infty} \|x_n^i\|_i^p \geq \sum_{i \in A} \|x_n^i\|_i^p = \|x_n^j\|_j^p + \sum_{\substack{i \in A \\ i \neq j}} \|x_n^i\|_i^p$$

$$> \|x^j\|_j^p + \epsilon/2 + \sum_{\substack{i \in A \\ i \neq j}} (\|x^i\|_i^p - \epsilon/5^i)$$

$$= \|x^j\|_j^p + \epsilon/2 + \sum_{\substack{i \in A \\ i \neq j}} \|x^i\|_i^p - \sum_{\substack{i \in A \\ i \neq j}} \epsilon/5^i$$

$$= \sum_{i \in A} \|x^i\|_i^p + \epsilon/2 - \sum_{\substack{i \in A \\ i \neq j}} \epsilon/5^i$$

$$> \sum_{i \in A} \|x^i\|_i^p + \epsilon/2 - \sum_{i=1}^{\infty} \epsilon/5^i$$

$$= \sum_{i \in A} \|x^i\|_i^p + \epsilon/2 - \epsilon/4 = \sum_{i \in A} \|x^i\|_i^p + \epsilon/4$$

$$> 1 - \epsilon/4 + \epsilon/4 = 1$$

Since the strict inequalities in the above are impossible, we thus get a contradiction. Therefore, equality holds in (1) for each i. Each B_i is (A) by hypothesis, so for each i, $\lim \|x_n^i - x^i\|_i = 0$. Hence,

$$\lim \|x_n - x\| = \lim \left(\sum_{i=1}^{\infty} \|(x_n - x)^i\|_i^p \right)^{1/p}$$

$= \lim \left(\sum_{i=1}^{\infty} \|x_n^i - x^i\|_i^p \right)^{1/p}$

$= (\lim \sum_{i=1}^{\infty} \|x_n^i - x^i\|_i^p)^{1/p} \leq (\sum_{i=1}^{\infty} \lim \|x_n^i - x^i\|_i^p)^{1/p} = 0.$

QED.

Day ([3], Theorem 6) proved that for $p > 1$, $P_p(B_i)$ is (R) if each B_i is (R), so we immediately have the following

<u>Corollary</u> <u>For</u> $p > 1$, $P_p(B_i)$ <u>is</u> (<u>H</u>) <u>if</u> <u>each</u> B_i <u>is</u> (<u>H</u>).

<u>Theorem 4.2</u> <u>For</u> $p > 1$, $P_p(B_i)$ <u>is</u> <u>m.l.u.c.</u> <u>if</u> <u>each</u> B_i <u>is</u> <u>m.l.u.c.</u>

<u>Proof.</u> For $k \geqq 1$, and $n = 1, 2, 3, \cdots$, define

$$x_o = (x_o^1, x_o^2, \cdots, x_o^k, x_o^{k+1}, x_o^{k+2}, \cdots)$$

$$x_{o,k} = (0, 0, \cdots, 0, x_o^{k+1}, x_o^{k+2}, \cdots)$$

$$x_n = (x_n^1, x_n^2, \cdots, x_n^k, x_n^{k+1}, x_n^{k+2}, \cdots)$$

$$x_{n,k} = (0, 0, \cdots, 0, x_n^{k+1}, x_n^{k+2}, \cdots)$$

Let $\|x_n\| = \|y_n\| = \|x_o\| = 1$ and $\lim \|x_n + y_n - 2x_o\| = 0$. Then we want to show that $\lim \|x_n - y_n\| = 0$. For each n, $\|x_n + y_n\| \leqq \|x_n\| + \|y_n\| = 2$, so we have for each n that $\|x_n + y_n - 2x_o\| \geqq \left| \|x_n + y_n\| - 2\|x_o\| \right| = 2 - \|x_n + y_n\| \geqq 0.$ But since $\lim \|x_n + y_n - 2x_o\| = 0$ by hypothesis, we thus get

(1) $\lim \|x_n + y_n\| = 2$

Using the definition of the norm and the Minkowski inequality, we have for each n,

$$\|x_n + y_n\| = (\sum_{i=1}^{\infty} \|x_n^i + y_n^i\|_i^p)^{1/p}$$

$$= (\sum_{i=1}^{k} \|x_n^i + y_n^i\|_i^p + \sum_{i=k+1}^{\infty} \|x_n^i + y_n^i\|_i^p)^{1/p}$$

$$= (\sum_{i=1}^{k} \|x_n^i + y_n^i\|_i^p + \|x_{n,k} + y_{n,k}\|^p)^{1/p}$$

$$\leq \{ \sum_{i=1}^{k} (\|x_n^i\|_i + \|y_n^i\|_i)^p + (\|x_{n,k}\| + \|y_{n,k}\|)^p \}^{1/p}$$

$$\leq (\sum_{i=1}^{k} \|x_n^i\|_i^p + \|x_{n,k}\|^p)^{1/p} + (\sum_{i=1}^{k} \|y_n^i\|_i^p + \|y_{n,k}\|^p)^{1/p}$$

$$= \|x_n\| + \|y_n\| = 2.$$

From this result and (1), we thus get

$$(2) \quad \lim \ (\sum_{i=1}^{k} \{\|x_n^i\|_i + \|y_n^i\|_i\}^p + \{\|x_{n,k}\| + \|y_{n,k}\|\}^p)^{1/p} = 2$$

For each i, $\|x_n^i\|_i \leqq \|x_n\| = 1$, so $\{\|x_n^i\|_i\}$ is a bounded sequence of real numbers; hence, by diagonalizing, we can determine a sequence of n's for which $\lim \|x_n^i\|_i$ exists for each i. For each i, let $a_i = \lim \|x_n^i\|_i$. Since $1 = \|x_n\|^p = \sum_{i=1}^{k} \|x_n^i\|_i^p + \|x_{n,k}\|^p$, we also have

$$\lim \|x_{n,k}\|^p = 1 - \lim \sum_{i=1}^{k} \|x_n^i\|_i^p = 1 - \sum_{i=1}^{k} \lim \|x_n^i\|_i^p$$

$$= 1 - \sum_{i=1}^{k} a_i^p = A_k^p$$

Now, for each n in the sequence determined above, $\{\|y_n^i\|_i\}$ is a bounded sequence of real numbers, so by diagonalizing, we can determine a subsequence of n's for which $\lim \|y_n^i\|_i$ exists for each i. For each i, let $b_i = \lim \|y_n^i\|_i$. Since $1 = \|y_n\|^p = \sum_{i=1}^{k} \|y_n^i\|_i^p + \|y_{n,k}\|^p$, we have

$$\lim \|y_{n,k}\|^p = 1 - \lim \sum_{i=1}^{k} \|y_n^i\|_i^p = 1 - \sum_{i=1}^{k} \lim \|y_n^i\|_i^p$$

$$= 1 - \sum_{i=1}^{k} b_i^p = B_k^p$$

Using these results and (2), we thus get

$$2 = \lim \{ \sum_{i=1}^{k} (\|x_n^i\|_i + \|y_n^i\|_i)^p + (\|x_{n,k}\| + \|y_{n,k}\|)^p \}^{1/p}$$

$$= \{ \sum_{i=1}^{k}(a_i + b_i)^p + (A_k + B_k)^p \}^{1/p}$$

$$\leq (\sum_{i=1}^{k} a_i^p + A_k^p)^{1/p} + (\sum_{i=1}^{k} b_i^p + B_k^p) = 1 + 1 = 2.$$

This forces equality in the Minkowski inequality, from which it follows that $a_i = b_i$ for each i, and $A_k = B_k$ for each k, or

$$(3) \quad \lim \|x_n^i\|_i = \lim \|y_n^i\|_i \text{ for each i, and}$$

$$\lim \|x_{n,k}\| = \lim \|y_{n,k}\| \text{ for each k.}$$

For each n, $\|x_n + y_n - 2x_o\| = (\sum_{i=1}^{\infty} \|x_n^i + y_n^i - 2x_o^i\|_i^p)^{1/p}$

$$\geq (\sum_{i=1}^{\infty} \Big| \|x_n^i + y_n^i\|_i - 2\|x_o^i\|_i \Big|^p)^{1/p}$$

$$\geq 0.$$

Since $\lim \|x_n + y_n - 2x_o\| = 0$ by hypothesis, we have $\lim (\sum_{i=1}^{\infty} \Big| \|x_n^i + y_n^i\|_i - 2\|x_o^i\|_i \Big|^p)^{1/p} = 0$. But the limit of the sum of a series of non-negative terms can be zero only if the limit of each term is zero. Thus,

$$(4) \quad \lim \|x_n^i + y_n^i\|_i = 2\|x_o^i\|_i \text{ for each i.}$$

From (3) and (4), we get, for each i,

$$2\|x_o^i\|_i = \lim \|x_n^i + y_n^i\|_i \leq \lim (\|x_n^i\|_i + \|y_n^i\|_i)$$

$$= \lim \|x_n^i\|_i + \lim \|y_n^i\|_i = 2 \lim \|x_n^i\|_i, \text{ or}$$

$$(5) \quad \operatorname{Lim} \|x_n^i\|_i \geq \|x_o^i\|_i \text{ for each i.}$$

We want to show that equality must hold in (5). Suppose by way of contradiction that for some index j, $\lim \|x_n^j\|_j > \|x_o^j\|_j$. Let $\epsilon = \lim \|x_n^j\|_j^p - \|x_o^j\|_j^p > 0$. Choose a subsequence of n's such that

$$(6) \quad \|x_n^j\|_j^p > \|x_o^j\|_j^p + \epsilon/2 \quad \text{for each n.}$$

Since $\|x_o\| = 1$, we can choose a finite set J of indices i such that

(7) $\sum_{i \in J} \|x_o^i\|_i^p > 1 - \epsilon/4$

Without loss of generality, we can assume j is in J, for otherwise we could replace J by $J' = J \cup \{j\}$. By (5), for each i in J we can choose N_i such that for all $n \geq N_i$,

(8) $\|x_n^i\|_i^p \geq \|x_o^i\|_i^p - \epsilon/5^i$

Let $N = \max \{N_i : i \text{ in } J\}$. Then, for all $n \geq N$ in the above subsequence, we have

$$1 = \|x_n\|^p = \sum_{i=1}^{\infty} \|x_n^i\|_i^p \geq \sum_{i \in J} \|x_n^i\|_i^p = \|x_n^j\|_j^p + \sum_{\substack{i \in J \\ i \neq j}} \|x_n^i\|_i^p$$

$$> \|x_o^j\|_j^p + \epsilon/2 + \sum_{\substack{i \in J \\ i \neq j}} \|x_n^i\|_i^p \qquad \text{by (6)}$$

$$\geq \|x_o^j\|_j^p + \epsilon/2 + \sum_{\substack{i \in J \\ i \neq j}} (\|x_o^i\|_i^p - \epsilon/5^i) \qquad \text{by (8)}$$

$$= \|x_o^j\|_j^p + \epsilon/2 + \sum_{\substack{i \in J \\ i \neq j}} \|x_o^i\|_i^p - \sum_{\substack{i \in J \\ i \neq j}} \epsilon/5^i$$

$$= \sum_{i \in J} \|x_o^i\|_i^p + \epsilon/2 - \sum_{\substack{i \in J \\ i \neq j}} \epsilon/5^i$$

$$> 1 - \epsilon/4 + \epsilon/2 - \sum_{i=1}^{\infty} \epsilon/5^i \qquad \text{by (7)}$$

$$= 1 + \epsilon/4 - \epsilon/4 = 1.$$

We thus get a contradiction. Therefore equality must hold in (5). At this point we have shown that from the hypothesis $\lim \|x_n + y_n - 2x_o\| = 0$, it follows that

(9) $\lim \|x_n^i\|_i = \lim \|y_n^i\|_i = \|x_o^i\|_i$ for each i.

Also, it follows directly from (9) and the second equation in (3) that

(10) $\lim \|x_{n,k}\| = \lim \|y_{n,k}\| = \|x_{o,k}\|$ for each k.

We now want to show that $\lim \|x_n - y_n\| = 0$, and we shall do this by showing that its negation leads to a contradiction. To this end, suppose there exists a subsequence of n's and a number $r > 0$ such that $\|x_n - y_n\| \geq r$. Then

$$0 < r \leq \|x_n - y_n\| = (\sum_{i=1}^{\infty} \|x_n^i - y_n^i\|_i^p)^{1/p}$$

$$= (\sum_{i=1}^{k} \|x_n^i - y_n^i\|_i^p + \sum_{i=k+1}^{\infty} \|x_n^i - y_n^i\|_i^p)^{1/p}$$

$$= (\sum_{i=1}^{k} \|x_n^i - y_n^i\|_i^p + \|x_{n,k} - y_{n,k}\|^p)^{1/p}$$

$$\leq (\sum_{i=1}^{k} \|x_n^i - y_n^i\|_i^p)^{1/p} + \|x_{n,k} - y_{n,k}\|$$

$$\leq (\sum_{i=1}^{k} \|x_n^i - y_n^i\|_i^p)^{1/p} + \|x_{n,k}\| + \|y_{n,k}\|$$

Therefore, $(\sum_{i=1}^{k} \|x_n^i - y_n^i\|_i^p)^{1/p} \geq r - (\|x_{n,k}\| + \|y_{n,k}\|)$

Since $\lim \|x_{n,k}\| = \lim \|y_{n,k}\| = \|x_{o,k}\|$ and $\lim_{k \to \infty} \|x_{o,k}\| = 0$, there exist k and n_o such that $\|x_{n,k}\| + \|y_{n,k}\| < r$ for all $n \geq n_o$. Then, for this choice of k and all $n \geq n_o$, we have

$$(\sum_{i=1}^{k} \|x_n^i - y_n^i\|_i^p)^{1/p} \geq s > 0$$

Hence there exists an i_o with $1 \leq i_o \leq k$ and a subsequence of n's for which

$$\|x_n^{i_o} - y_n^{i_o}\|_{i_o} \geq t > 0$$

Now, $\|x_n^{i_o} - y_n^{i_o}\|_{i_o} \leq \|x_n^{i_o}\|_{i_o} + \|y_n^{i_o}\|_{i_o}$ and $\lim (\|x_n^{i_o}\|_{i_o} + \|y_n^{i_o}\|_{i_o}) = 2\,\|x_o^{i_o}\|_{i_o}$. Therefore,

$\|x_o^{i_o}\|_{i_o} \neq 0$. So there exists a subsequence of n's for which $\|x_n^{i_o}\|_{i_o} \neq 0$ and $\|y_n^{i_o}\|_{i_o} \neq 0$. Then,

$$\liminf \left\| \frac{x_n^{i_o}}{\|x_n^{i_o}\|_{i_o}} - \frac{y_n^{i_o}}{\|y_n^{i_o}\|_{i_o}} \right\|_{i_o} = \liminf \left\| \frac{x_n^{i_o}}{\|x_o^{i_o}\|_{i_o}} - \frac{y_n^{i_o}}{\|x_o^{i_o}\|_{i_o}} \right\|_{i_o}$$

$$= (1/\|x_o^{i_o}\|_{i_o}) \liminf \|x_n^{i_o} - y_n^{i_o}\|_{i_o}$$

$$\geq t/\|x_o^{i_o}\|$$

$$\geq t$$

Since B_{i_o} is m.l.u.c., there exists $\delta = \delta(t, x_o^{i_o}) > 0$ such that

$$\liminf \left\| \frac{x_n^{i_o}}{\|x_n^{i_o}\|_{i_o}} + \frac{y_n^{i_o}}{\|y_n^{i_o}\|_{i_o}} - 2\frac{x_o^{i_o}}{\|x_o^{i_o}\|_{i_o}} \right\|_{i_o} \geq \delta$$

Therefore, $\liminf \|x_n^{i_o} + y_n^{i_o} - 2x_o^{i_o}\|_{i_o} \geq \delta \|x_o^{i_o}\|_{i_o}$

But then,

$$\liminf \|x_n + y_n - 2x_o\| = \liminf \left(\sum_{i=1}^{\infty} \|x_n^i + y_n^i - 2x_o^i\|_i^p \right)^{1/p}$$

$$\geq \liminf \|x_n^{i_o} + y_n^{i_o} - 2x_o^{i_o}\|_{i_o}$$

$$\geq \delta \|x_o^{i_o}\|_{i_o}$$

$$> 0$$

which contradicts the fact that $\lim \|x_n + y_n - 2x_o\| = 0$.

Therefore, $\lim \|x_n - y_n\| = 0$, and thus $P_p(B_i)$ is m.l.u.c.

QED.

CHAPTER V

Some Isomorphism Results

Clarkson [1] proved that any separable Banach space is isomorphic to a space which is (R). Klee ([7], Theorem A1.11) showed that every separable reflexive Banach space is isomorphic to a space which is (R) and also to a space which is (S). Day ([3], Theorem 4) improved on both of these by proving that any separable Banach space is isomorphic to a space which is (RS), that is, simultaneously (R) and (S). Kadec ([6], Theorem 2) proved the following:

Theorem 5.1 *Any separable Banach space is isomorphic to a space which is (A).*

Actually, Kadec asserts a stronger result. Immediately after the proof of Theorem 2, he displays a new norm for C, the space of continuous functions on the interval [0,1], which he states is (H). Once this is verified, we can use the fact that C is the universal separable space to extend the result to all separable spaces. Since we plan to use this result below, we shall prove it formally, and use a somewhat different method. We start with two lemmas, the first of which is obvious.

Lemma 5.1 *If B is (A), then every linear subspace of B is (A).*

Lemma 5.2 *If B is separable and (A), then B can be renormed with an equivalent norm such that, under the new norm, B is (H).*

Proof: Following the method of Clarkson [1] or Day ([3], page 518), for each x in B, define $Tx = \{f_i(x)/2^i\}$, where $\{f_i\}$ is a bounded sequence of elements of B* which is total over

B. Then T is a one-one continuous linear map from B into l_2 . Define a new norm $\|\cdots\|_1$ in B as follows:
For each x in B, $\|x\|_1 = \|x\| + \|Tx\|_{l_2}$, where $\|\cdots\|$ is the original norm in B. This new norm is equivalent to the old norm, for if K is a bound for the f_i, we have for each x in B that

$$\begin{aligned}\|x\| \leq \|x\|_1 &= \|x\| + \|Tx\|_{l_2} \\ &= \|x\| + \{\textstyle\sum_{i=1}^{\infty} (f_i(x)/2^i)^2\}^{1/2} \\ &\leq \|x\| + \{\textstyle\sum_{i=1}^{\infty} (\|f_i\| \cdot \|x\|/2^i)^2\}^{1/2} \\ &\leq \|x\| + \{\textstyle\sum_{i=1}^{\infty} (K \cdot \|x\|/2^i)^2\}^{1/2} \\ &= \|x\| + K\|x\| (\textstyle\sum_{i=1}^{\infty} 4^{-i})^{1/2} \\ &\leq \|x\| + K\|x\| \\ &= (1 + K)\|x\|\end{aligned}$$

To show that $\|\cdots\|_1$ is (R), suppose that $\|x\|_1 = \|y\|_1 = 1$ and $\|x + y\|_1 = \|x\|_1 + \|y\|_1$.
Then, using the triangle inequality, we get

$$\begin{aligned}\|x\|_1 + \|y\|_1 = \|x + y\|_1 &= \|x + y\| + \|T(x + y)\|_{l_2} \\ &= \|x + y\| + \|Tx + Ty\|_{l_2} \\ &\leq \|x + y\| + \|Tx\|_{l_2} + \|Ty\|_{l_2} \\ &\leq \|x\| + \|y\| + \|Tx\|_{l_2} + \|Ty\|_{l_2} \\ &= \|x\|_1 + \|y\|_1\end{aligned}$$

Equality throughout implies that

$\|Tx + Ty\|_{l_2} = \|Tx\|_{l_2} + \|Ty\|_{l_2}$. But l_2 is (R), so we must have Tx = Ty. Therefore, x = y, since T is one-one, and hence $\|\cdots\|$ is (R). Let us write (B, $\|\cdots\|_1$) for the space B with norm $\|\cdots\|_1$. Then, to show that (B, $\|\cdots\|_1$) is (A), we first note that

$$(B, \|\cdots\|_1) = (B, \|\cdots\|) \underset{l_1}{x} (T(B), \|\cdots\|_{l_2})$$

$$\subseteq (B, \|\cdots\|) \underset{l_1}{x} l_2 .$$

l_2 is u.c. and thus is (A). Also, (B, $\|\cdots\|$) is (A) by hypothesis. Therefore, (B, $\|\cdots\|$) $\underset{l_1}{x}$ l_2 is (A), by Theorem 4.1, and hence (B, $\|\cdots\|_1$) is (A) by Lemma 5.1. QED

<u>Theorem 5.2</u> (<u>Kadec</u>) <u>Any separable Banach space is isomorphic to a space which is</u> (<u>H</u>).

<u>Proof</u>: If B is separable, then by Theorem 5.1, B can be renormed so as to satisfy (A). This new space can then be again renormed so as to satisfy (H) by Lemma 5.2. QED

Fan and Glicksberg have proved the following theorem ([5], Theorem 6):

<u>Theorem 5.3</u> <u>If a normed linear space X satisfies</u> (<u>H</u>) <u>and if X* is separable</u>, <u>then X is isomorphic to a space which is l.u.c</u>.

We are now in a position to improve on this theorem, as follows:

<u>Theorem 5.4</u> <u>If B* is separable</u>, <u>then B is isomorphic to an l.u.c. space</u>.

<u>Proof</u>: B* separable implies that B is separable,

([11], Theorem 4.3-E, page 187), so by Theorem 5.2, B can be renormed with an equivalent norm such that, under the new norm, B is (H). The result then follows from Theorem 5.3. QED

As an immediate consequence of the above theorem, we have the following

Corollary Every separable reflexive Banach space is isomorphic to an l.u.c. space.

That separability of B* is not a necessary condition for B to be isomorphic to an l.u.c. space is evident from an example by Phelps ([9], page 447) wherein he renormed l_1 with an equivalent norm which is l.u.c., and of course, l_1^* is equivalent to m, which is not separable.

In general, an isomorphism of B* may not be determined by an isomorphism of B, but Klee has observed ([7], Theorem A1.2) that if B* is renormed with an equivalent norm and if the new unit ball is w*-closed, then this new norm is the conjugate norm of a new equivalent norm in B. Using this fact plus Lovaglia's theorem ([8], Theorem 2.3) which states that if B* is l.u.c. then B is (Str), we have

Theorem 5.5 If B* is isomorphic to a space which is l.u.c. and if the new unit ball is w*-closed, then B is isomorphic to a space which is (Str).

As an immediate consequence of the duality shown earlier, we also have

Theorem 5.6 If B* is isomorphic to a space which is (Str), then B is isomorphic to a space which is m.l.u.c.

<u>Proof</u>: By Theorem 3.1, if B* is (Str), then B is reflexive. Since reflexivity is preserved under isomorphism, we thus have that B is reflexive. Then by Theorem 3.8, B is isomorphic to a space which is (H), and hence also m.l.u.c., by Theorem 3.2. QED

One final result is perhaps worthy of mention in this chapter. Dixmier ([4], Theorem 20') has shown that if the fourth conjugate space of B is (R), then B is reflexive. We can draw a parallel result from this chapter regarding l.u.c. Phelps' example following the corollary to Theorem 5.4 exhibits a non-reflexive space B such that B* is l.u.c. However, using Lovaglia's theorem stated immediately before Theorem 5.5, we have the following

<u>Theorem 5.7</u> <u>If B** is l.u.c., then B is reflexive</u>.

<u>Proof</u>: B** is l.u.c. implies that B* is (Str), which in turn implies that B is reflexive. QED

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