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LINEAR FUNCTIONALS ON ORLICZ SPACES

BY

TSUYOSHI ANDÔ

§ 1. *Introduction.* The main purpose of this paper is to obtain the general form of bounded linear functionals on Orlicz spaces. As is pointed out in [4, p. 263], until now the general form has not been found except under certain conditions. As every Orlicz space is a *modulared semi-ordered linear space* in the sense of H. NAKANO [7, § 35] and the structure theory of modulared spaces was discussed fully by him [7, Ch. VII–X], the theory may provide a powerful tool for our research. In the present paper, following the idea of the preceding paper [1], we shall succeed in representating each bounded linear functional by a pair of a function and a finitely additive measure, and in describing the functional norms.

The body of this paper is devided into six sections. In § 2, the definition of Orlicz spaces and some known results are stated. In § 3, each bounded linear functional is represented as the sum of two linear functionals, one of which is of function-type and the other is singular (Theorem 1). In § 4, singular linear functionals are put in correspondence to finitely additive measures, and the functional norms are evaluated (Theorem 4). In § 5, the principal theorem is obtained. In § 6, singular linear functionals are characterized in terms of their functional norms (Theorem 6).

§ 2. *Preliminaries.* If $M(\xi)$ is a real valued convex function such that $M(0) = 0$, $M(\xi) = M(-\xi)$ and $M(\xi)/\xi \to \infty$ as $\xi \to \infty$, then it is called an *N-function*. The *complementary function* $\overline{M}(\xi)$ of an N-function $M(\xi)$ is defined by the relation:

$$\overline{M}(\xi) = \sup_{-\infty<\eta<\infty} \{\xi\eta - M(\eta)\}.$$

Then it is known (cf. [4, § 2]) that $\overline{M}(\xi)$ is also an N-function and $M(\xi)$ is the complementary function of $\overline{M}(\xi)$. By definition, $M(\xi)$ and $\overline{M}(\eta)$ together satisfy the so-called Young's inequality:

$$|\xi\eta| \le M(\xi) + \overline{M}(\eta) \qquad \text{for all } \xi, \eta. \tag{1}$$

Let Δ be an abstract set and μ_0 be the (non-negative) countably additive measure defined on the (infinite) σ-algebra B of subsets of Δ. We assume that $0 < \mu_0(\Delta) < \infty$ and μ_0 is complete, i.e. $\mu_0(E) = 0$, $F \subset E$ implies $F \in \mathsf{B}$. $E, F, G, \ldots$ denote elements of B. $\mathfrak{M}$ denotes the set of all real valued finitely additive measures (neither necessarily countably additive nor non-negative) on B such that $\sup_{E \in \mathsf{B}} |\nu(E)| < \infty$ and $\nu(F) = 0$, if $\mu_0(F) = 0$. $\mathfrak{M}$ is a linear space with the usual addition and scalar multiplication. It is known (cf. [9]) that $\mathfrak{M}$ is a lattice with the usual ordering: $\nu_1 \geq \nu_2$, if and only if $\nu_1(E) \geq \nu_2(E)$ for all $E \in \mathsf{B}$. In fact, for any $\nu \in \mathfrak{M}$ its *positive part* ν^+ and the *negative part* ν^- are given respectively by

$$\nu^+(E) = \sup_{\mathsf{B} \ni F \subset E} \nu(F) \quad \text{and} \quad \nu^-(E) = -\inf_{\mathsf{B} \in F \subset E} \nu(F).$$

We have $\nu = \nu^+ - \nu^-$ and the *absolute* $|\nu| = \nu^+ + \nu^-$. $\mathfrak{M}$ is a Banach space with the norm: $||\nu|| = |\nu| (\Delta)$. Throughout the paper, elements of $\mathfrak{M}$ and real valued B-measurable functions on Δ are called simply *measures* and *functions* respectively.

Now we consider the functionals defined by

$$\mathsf{M}(f) = \int M(f)\, d\mu_0 \quad \text{and} \quad \overline{\mathsf{M}}(f) = \int \overline{M}(f)\, d\mu_0,$$

where $\int$ denotes the integral on Δ. The *Orlicz space* L^*_M defined by the N-function $M(\xi)$ is the set of all functions f such that $\mathsf{M}(\alpha f) < \infty$ for some $\alpha = \alpha(f) > 0$, (note that our notation differs from that in [5] and [6]). L^*_M is a linear space with the usual addition and scalar multiplication. By Young's inequality (1) we have

$$\int |f.g|\, d\mu_0 \leq \overline{\mathsf{M}}(f) + \mathsf{M}(g) \qquad \text{for all } f, g. \tag{2}$$

Furthermore corresponding to mutual complementarity between $M(\xi)$ and $\overline{M}(\xi)$, it is known (cf. [4, § 14]) that

$$\overline{\mathsf{M}}(g) = \sup_{f \in L^*_M} \{\int f.g\, d\mu_0 - \mathsf{M}(f)\} \quad \text{for all } g \in L^*_{\overline{M}}, \tag{2'}$$

and $L^*_{\overline{M}}$ is exactly the set of all functions g such that

$$\int |f.g|\, d\mu_0 < \infty \qquad \text{for all } f \in L^*_M.$$

By means of the functionals $\mathsf{M}(f)$ and $\overline{\mathsf{M}}(g)$ we introduce two norms on L^*_M by the relations:

$$||f||_M = \sup_{\overline{\mathsf{M}}(g) \leq 1} \int f.g\, d\mu_0 \qquad (\textit{the first norm}), \tag{3}$$

$$|||f|||_M = \inf_{\mathsf{M}(\xi f) \leq 1} |\xi|^{-1} \qquad (\textit{the second norm}). \tag{4}$$

The first norm is also defined intrinsically by Amemiya's formula ([8, § 83], also [6] and [4, § 10])

$$||f||_M = \inf_{\xi > 0} \frac{1 + \mathsf{M}(\xi f)}{\xi}. \tag{3'}$$

Then with each of the two norms L^*_M is a Banach space, and they are equivalent, i.e. $|||f|||_M \leq ||f||_M \leq 2\, |||f|||_M$ (cf. [5, Ch. 2, § 2], [7, § 40]). Furthermore L^*_M is a lattice with the usual ordering (cf. [7, App.]). The positive part f^+ and the negative part f^- of $f \in L^*_M$ are defined respectivily by

$$f^+(t) = \begin{cases} f(t) & \text{for } f(t) \geq 0, \\ 0 & \text{elsewhere,} \end{cases} \quad \text{and} \quad f^-(t) = \begin{cases} -f(t) & \text{for } f(t) \leq 0, \\ 0 & \text{elsewhere.} \end{cases}$$

Then it is clear that $f = f^+ - f^-$ and $|f| = f^+ + f^-$.

A bounded linear functional φ on L^*_M is said to be *positive* and is denoted by $\varphi \geq 0$, if $\varphi(f) \geq 0$ for all $0 \leq f \in L^*_M$. $\varphi \geq \psi$ means that $\varphi - \psi$ is positive. From the theory of semi-ordered linear space [7, Ch. IV] it is known that with this ordering the set of all bounded linear functionals on L^*_M is a lattice, and any bounded linear functional φ can be represented as the sum $\varphi = \varphi^+ - \varphi^-$, where the *positive part* φ^+ and the *negative part* φ^- are defined respectively by the relations:

$$\varphi^+(f) = \sup_{0 \leq g \leq f} \varphi(g) \text{ and } \varphi^-(f) = -\inf_{0 \leq g \leq f} \varphi(g) \quad \text{for all } f \geq 0.$$

The sum $\varphi^+ + \varphi^-$ denoted by $|\varphi|$. From the definition, it is easy to see that

$$|\varphi|\, (|f|) = \sup_{|g| \leq |f|} \varphi(g) \qquad \text{for all } f \in L^*_M. \tag{5}$$

Bounded linear functionals are denoted by $\varphi, \psi, \ldots$

From (2) it follows that each function $g \in L^*_{\overline{M}}$ can be considered as a bounded linear functional on L^*_M under the interpretation:

$$\varphi(f) = \int f.g\, d\mu_0. \tag{6}$$

It is known [5, Ch. 2, § 2] that the functional norms of φ in (6) are given by the first and second norms of g in $L^*_{\overline{M}}$, i.e.

$$\sup \varphi(f)/||f||_M = |||g|||_{\overline{M}} \quad \text{and} \quad \sup \varphi(f)/|||f|||_M = ||g||_{\overline{M}}. \tag{7}$$

By this reason we use the following notations for the functional norms of a general bounded linear functional φ

$$||\varphi||_{\overline{M}} = \sup \varphi(f)/|||f|||_M \quad \text{and} \quad |||\varphi|||_{\overline{M}} = \sup \varphi(f)/||f||_M. \tag{8}$$

We conclude this section in describing some properties of the functional $\mathsf{M}(f)$ and the norms, which we shall use without any mention: $|f| \leq |g|$ implies

$$\mathsf{M}(f) \leq \mathsf{M}(g) \quad \text{and} \quad ||f||_M \leq ||g||_M \ (|||f|||_M \leq |||g|||_M).$$

The similar inequality holds for the functional norms.

§ 3. *Decomposition.* For any function f and $E \in \mathsf{B}$ we denote by f_E the function $f . \chi_E$ where χ_E is the characteristic function of E,

$$f_E(t) = \begin{cases} f(t) & \text{for } t \in E \\ 0 & \text{elsewhere.} \end{cases} \tag{9}$$

E_M denotes the closure (with respect to the norm topology) of the set of all essentially bounded functions in L^*_M. E_M is a linear space such that $|g| \leq |f|$, $f \in \mathsf{E}_M$ implies $g \in \mathsf{E}_M$. Functions f in E_M can be characterized by the condition that $||f_{E_k}||_M \to 0$ as $\mu_0(E_k) \to 0$. Also it is known (cf. [5, Ch. 2, § 3] and [4, § 10]) that f is in E_M if and only if $\mathsf{M}(\alpha f) < \infty$ for all $\alpha > 0$. A bounded linear functional φ is said to be of *function-type,* if it is expressed in the form (6). φ is said to be *singular,* if $\varphi(f) = 0$ for all $f \in \mathsf{E}_M$.

Corresponding to the decomposition of a measure into the countably additive part and the purely finitely additive part (see [9]), we can state (cf. [5, Ch. 1, § 1, Th. 2]):

Theorem 1. *Any bounded linear functional φ can be expressed uniquely in the form $\varphi = \varphi_c + \varphi_S$, where φ_c is of function-type and φ_S is singular.*

Proof. The real valued function $\nu(E) = \varphi(\chi_E)$ on B is a measure, absolutely continuous with respect to μ_0, because

$$|\varphi(\chi_{E_k})| \leq ||\varphi||_{\overline{M}} \cdot |||\chi_{E_k}|||_M \to 0 \quad \text{as} \quad \mu_0(E_k) \to 0.$$

Then by the Radon-Nikodym's theorem [2, § 31] there exists uniquely an integrable function g such that $\varphi(\chi_E) = \int_E g \, d\mu_0$ for all $E \in \mathsf{B}$. Defining φ_c by the relation $\varphi_c(f) = \int f . g \, d\mu_0$, as in [5, Ch. 1, § 1, Th. 2], we can prove that φ_c is a bounded linear functional on L^*_M and $\varphi(f) = \varphi_c(f)$ for all $f \in \mathsf{E}_M$. Writing $\varphi_S = \varphi - \varphi_c$, from the definition of φ_c it follows that $\varphi_S(f) = 0$ for all $f \in \mathsf{E}_M$, that is φ_S is singular. Thus a decomposition is obtained. If $\varphi = \varphi_1 + \varphi_2$, where φ_1 is of function-type and φ_2 is singular, then $\varphi_c - \varphi_1$ $(= \varphi_2 - \varphi_S)$ is of function-type and singular at the same time. Let h be its representation function, then $\int_E h \, d\mu_0 = (\varphi_c - \varphi_1)(h) = 0$ for all $E \in \mathsf{B}$ because $\varphi_c - \varphi_1$ is singular, hence $h = 0$, i.e. $\varphi_c = \varphi_1$. Thus the uniqueness of the decomposition is established.

By this theorem, we can characterize bounded linear functionals of function-type.

Corollary. *A bounded linear functional φ is of function-type, if and only if the measure $\nu(E) = \varphi(f_E)$ on B is absolutely continuous with respect to μ_0 for all $f \in L^*_M$.*

Proof. According to Theorem 1, φ is of function-type, if and only if $\varphi = \varphi_c$. Let φ be of function-type, corresponding to $g \in L^*_M$. Since $f . g$ is integrable, its indefinite integral $\varphi(f_E) = \int_E f . g \, d\mu_0$ is absolutely continuous with respect to μ_0. Conversely let the measure $\varphi(f_E)$ is absolutely continuous with respect to μ_0 for all $f \in L^*_M$. For any f, writing $E_k = \{t; |f(t)| \leq k\}$, we have $|f_{E_k}| \leq |f|$ $(k = 1, 2, \ldots)$ and $\mu_0(\Delta - E_k) \to 0$, hence $\lim_{k\to\infty} f_{E_k}(t) = f(t)$ almost everywhere. Then by the Lebesgue's theorem [2, § 26] we have $\int_{E_k} f . g \, d\mu_0 \xrightarrow[k\to\infty]{} \int f . g \, d\mu_0 = \varphi_c(f)$. On the other hand, since $\varphi(f) = \varphi(f_{E_k}) + \varphi(f_{\Delta - E_k}) = \int_{E_k} f . g \, d\mu_0 + \varphi(f_{\Delta - E_k})$ $(k = 1, 2, \ldots)$ and $\varphi(f_{\Delta - E_k}) \to 0$ by hypothesis, we have $\varphi(f) = \varphi_c(f)$.

This corollary means that the mapping $\varphi \to \varphi_c$ is the projection onto the set of all *continuous* linear functionals in the sense of H. Nakano [7, § 19], hence it is a lattice-homomorphism (cf. [7, § 5]). The same holds for the mapping $\varphi \to \varphi_S$. In the following, we shall use only the facts $|\varphi_c| = |\varphi|_c$ and $|\varphi_S| = |\varphi|_S$.

Between the set of all bounded linear functionals of function-type and L^*_M the mapping $\varphi \to g$ in (6) is linear, lattice-isomorphic and the norm preserving. Thus bounded linear functionals of function-type can be fully described by their representation functions. On what condition is every bounded linear functional of function-type? The necessary and sufficient condition is that $M(\xi)$ satisfies the so-called (Δ_2) condition: $\overline{\lim}_{\xi\to\infty} M(2\xi)/M(\xi) < \infty$, or equivalently $L^*_M = \mathsf{E}_M$ (cf. [5, Ch. 2, § 3]). As to singular linear functionals, circumstances are more complicated. The essential difficulties originate in the fact that, contrary to the above corollary, for singular φ, $\varphi(f_E)$ is generally neither absolutely continuous nor countably additive, so the Radon-Nikodym's theorem can not be applied. This is the reason to consider finitely additive measures.

§ 4. *Singular linear functionals.* First of all we describe some properties of functionals $\mathsf{M}(f)$ which facilitate the proofs of this section.

Lemma 1. (*a*) *For any $\varepsilon > 0$ and $0 \leq f \in L^*_M$ such that*

$\mathsf{M}(f) < \infty$ *there exists* $E \in \mathsf{B}$ *such that* $f_E \in \mathsf{E}_M$ *and* $\mathsf{M}(f - f_E) < \varepsilon$.

(*b*) *For any sequence of non-negative functions* $\{f_k\}_1^\infty$ *such that* $\sum_1^\infty M(f_k) < \infty$, *there exists* g *such that* $g \geq f_k$ $(k = 1, 2, \ldots)$ *and* $\mathsf{M}(g) \leq \sum_1^\infty \mathsf{M}(f_k)$.

Proof. (*a*) Writing $E_k = \{t; f(t) \leq k\}$, we have $\lim_{k\to\infty} f_{E_k}(t) = f(t)$ almost everywhere and $|f - f_{E_k}| \leq |f|$ $(k = 1, 2, \ldots)$. The Lebesgue's theorem [2, § 26] guarantees $\mathsf{M}(f - f_{E_k}) \to 0$. Thus for some n $\mathsf{M}(f - f_{E_n}) < \varepsilon$.

(*b*) Writing $g_k(t) = \operatorname{Max}\{f_1(t), \ldots, f_k(t)\}$, we have

$$0 \leq g_1 \leq g_2 \leq \ldots$$

By induction, we can prove that $\mathsf{M}(g_k) \leq \sum_1^k \mathsf{M}(f_n)$ $(k = 1, 2, \ldots)$, hence from the Lebesgue's theorem it follows that

$$\mathsf{M}(g) = \lim_{k\to\infty} \mathsf{M}(g_k) \leq \sum_1^\infty \mathsf{M}(f_k),$$

where $g(t) = \sup_{k=1,2,\ldots} g_k(t)$.

We begin with deriving some important properties of functional norms of singular linear functionals.

Lemma 2 (cf. [1]). *Let* φ *and* ψ *be arbitrary singular positive linear functionals. Then*

(*a*) $|||\varphi|||_{\overline{M}} = ||\varphi||_{\overline{M}} = \sup_{\mathsf{M}(f)<\infty} \varphi(f)$.

(*b*) *For any* $\varepsilon > 0$ *there exists* $g \geq 0$ *such that*

$$\mathsf{M}(g) \leq \varepsilon \quad \textit{and} \quad ||\varphi||_{\overline{M}} = \varphi(g).$$

(*c*) $||\varphi + \psi||_{\overline{M}} = ||\varphi||_{\overline{M}} + ||\psi||_{\overline{M}}$.

Proof. (*a*) From the definition (8) it follows that

$$|||\varphi|||_{\overline{M}} \leq ||\varphi||_{\overline{M}} = \sup_{\mathsf{M}(f)\leq 1} \varphi(f) \leq \sup_{\mathsf{M}(f)<\infty} \varphi(f).$$

Thus it suffices to prove that $\sup_{\mathsf{M}(f)<\infty} \varphi(f) \leq |||\varphi|||_{\overline{M}}$. Let $f \geq 0$ be arbitrarily chosen such that $\mathsf{M}(f) < \infty$. According to Lemma 1, there exists $0 \leq h \in \mathsf{E}_M$ such that $0 \leq h \leq f$ and $\mathsf{M}(f - h) < \varepsilon$. Then by Amemiya's formula (3′) $||f - h||_{\overline{M}} \leq 1 + \mathsf{M}(f - h) \leq 1 + \varepsilon$. On the other hand, $\varphi(f) = \varphi(f - h) + \varphi(h) = \varphi(f - h)$, because φ is singular by assumption and $h \in \mathsf{E}_M$, hence

$$\varphi(f) = \varphi(f - h) \leq (1 + \varepsilon) . \varphi(f - h)/||f - h||_M \leq (1 + \varepsilon)\ |||\varphi|||_{\overline{M}}.$$

Since $\varepsilon > 0$ and $f \geq 0$ are arbitrary, this establishes the desired inequality. In the process of the proof we have seen that for any $\varepsilon > 0$ $||\varphi||_{\overline{M}} = \sup_{\mathsf{M}(f)<\varepsilon} \varphi(f)$.

(*b*) As above, there exists a sequence of non-negative functions $\{f_k\}_1^\infty$ such that $\mathsf{M}(f_k) \leq \varepsilon/2^k$ and $||\varphi||_{\overline{M}} \leq \varphi(f_k) + 1/k$ $(k = 1, 2, \ldots)$. By Lemma 1 there exists g such that $\mathsf{M}(g) \leq \sum_1^\infty \mathsf{M}(f_k) \leq \varepsilon$ and $g \geq f_k$ $(k = 1, 2, \ldots)$. Then we have

$$||\varphi||_{\overline{M}} \leq \varphi(g) + 1/k \leq ||\varphi||_{\overline{M}} + 1/k \qquad (k = 1, 2, \ldots),$$

from this it follows that $\varphi(g) = ||\varphi||_{\overline{M}}$.

(*c*) According to (*b*), there exists $g \geq 0$ such that $\mathsf{M}(g) < \infty$ and $||\varphi + \psi||_{\overline{M}} = \varphi(g) + \psi(g)$. If $\varphi(g) + \psi(g) < ||\varphi||_{\overline{M}} + ||\psi||_{\overline{M}}$ (say $\varphi(g) < ||\varphi||_{\overline{M}}$), again by (*b*) we can find $h \geq 0$ such that $\mathsf{M}(h) < \infty$ and $||\varphi||_{\overline{M}} = \varphi(h)$. Then by Lemma 1 there exists $f \geq 0$ such that $f \geq g$, $f \geq h$ and $\mathsf{M}(f) \leq \mathsf{M}(g) + \mathsf{M}(h) < \infty$. For such f we have

$$||\varphi + \psi||_{\overline{M}} \geq (\varphi + \psi)(f) > \varphi(g) + \psi(g) = ||\varphi + \psi||_{\overline{M}}.$$

This contradiction establishes the assertion (*c*).

The set of all singular linear functionals constitutes a lattice under the induced ordering. Lemma 2 (*c*) shows that the set of all singular linear functionals is an *abstract* (*L*)-*space* in the sense of S. Kakutani [3].

For any function f and $E \in \mathsf{B}$, f_E is the result of restricting f on E. The notion of restriction can be extended in the natural way over all bounded linear functionals of function-type. In fact, if φ is of function-type, corresponding to g, then its restriction φ_E is defined as the functional, corresponding to g_E. In what way can we extend the notion to general bounded linear functionals? The answer is quite simple. The restriction φ_E is defined by the relation:

$$\varphi_E(f) = \varphi(f_E) \quad \text{for} \quad f \in L^*_M. \tag{10}$$

When φ is of function-type, this definition is in conformity with the old one. If φ is singular and $E \cap F = 0$, then by Lemma 2 we have

$$||\varphi_{E \cup F}||_{\overline{M}} = ||\varphi_E + \varphi_F||_{\overline{M}} = ||\varphi_E||_{\overline{M}} + ||\varphi_F||_{\overline{M}}.$$

Thus we have obtained a measure.

Lemma 3. *If* φ *is a singular positive linear functional, the function* ν_φ *defined on* B *by*

$$\nu_\varphi(E) = ||\varphi_E||_{\overline{M}} \tag{11}$$

is a (non-negative) measure such that $\nu_\varphi(\Delta) = ||\varphi||_{\overline{M}}$.

In this manner each singular positive linear functional is put in correspondence to a (non-negative) measure. Our next task is to see how we can reconstruct the original functional by means of the corresponding measure. For this purpose, we introduce the new functional $\rho(f)$ on L^*_M by

$$\rho(f) = \inf_{\mathsf{M}(\xi f)<\infty} |\xi|^{-1}. \tag{12}$$

Lemma 4. *The functional $\rho(f)$ has the following properties:*

(*a*) $\rho(\alpha f) = |\alpha|\,\rho(f)$ *for all real* α,
(*b*) $|g| \leq |f|$ *implies* $\rho(g) \leq \rho(f)$,
(*c*) $f.g = 0$ *implies* $\rho(f+g) = \mathrm{Max}\,\{\rho(f), \rho(g)\}$,
(*d*) $\rho(f) \leq 1$, *if and only if* $\mathsf{M}((1-\varepsilon)f) < \infty$ *for all* $1 > \varepsilon > 0$,
(*e*) $\rho(f) \geq 1$, *if and only if* $\mathsf{M}((1+\varepsilon)f) = \infty$ *for all* $\varepsilon > 0$,
(*f*) $\rho(f) = 0$, *if and only if* $f \in \mathsf{E}_M$.

These are immediate consequences of the definition (12), and we omit the proof. By the way, we remark that $\rho(f)$ is nothing but the norm in the quotient space L^*_M/E_M (the quotient norms produced both by $\|f\|_M$ and $|||f|||_M$ coincide with each other). Since, as mentioned in § 2, the existence of non-trivial singular functionals results from $L^*_M \neq \mathsf{E}_M$, to introduce the functional $\rho(f)$ is natural to treat singular linear functionals. We shall use the following immediate consequence of Lemmas 2–3:

$$\|\varphi\|_{\overline{M}} = \sup \varphi(f)/\rho(f) \qquad \text{for all singular } \varphi. \tag{13}$$

For any non-negative $\nu \in \mathfrak{M}$ we define the functional $\varphi_\nu(f)$ on the set of all non-negative functions by

$$\varphi_\nu(f) = \inf \textstyle\sum_1^n \rho(f_{E_k}).\nu(E_k) \qquad \text{for } f \geq 0, \tag{14}$$

where the infimum is taken over all finite disjoint partitions $\{E_k\}_1^n$ of $\varDelta$. (In the following, any partition is composed of elements in B).

Lemma 5. *The functional $\varphi_\nu(f)$ has the following properties:*

(*a*) $\varphi_\nu(\alpha f) = \alpha\varphi_\nu(f)$ *for all non-negative real* α,
(*b*) $0 \leq f \leq g$ *implies* $\varphi_\nu(f) \leq \varphi_\nu(g)$,
(*c*) $\varphi_\nu(f+g) = \varphi_\nu(f) + \varphi_\nu(g)$,
(*d*) $0 \leq \varphi_\nu(f) \leq \rho(f).\nu(\varDelta)$.

Proof. (*a*), (*b*) and (*d*) are evident from Lemma 3. In order to prove (*c*), we first treat the case $f.g = 0$. Writing $E = \{t;\, f(t) > 0\}$ and $F = \{t;\, g(t) > 0\}$, we have $E \cap F = 0$. Let $\{E_k\}_1^n$ be an arbitrary finite disjoint partition of $\varDelta$. Writing

$$C_k = \begin{cases} E \cap E_k & (k = 1, 2, \ldots, n) \\ F \cap E_{k-n} & (k = n+1, \ldots, 2n), \end{cases}$$

we obtain a finite disjoint partition such that

$$(f+g)_{E_k} = f_{C_k} + g_{C_{k+n}} \qquad (k = 1, 2, \ldots, n).$$

Since $f_{C_k}.g_{C_{k+n}} = 0$, by Lemma 4 we have

$$\textstyle\sum_1^n \rho((f+g)_{E_k}).\nu(E_k) = \sum_1^n \mathrm{Max}\,\{\rho(f_{C_k}), \rho(g_{C_{k+n}})\}.\nu(E_k) \geq$$
$$\geq \textstyle\sum_1^{2n} \rho(f_{C_k}).\nu(C_k) + \sum_1^{2n} \rho(g_{C_k}).\nu(C_k).$$

Since the partition $\{E_k\}_1^n$ is arbitrary, this establishes the inequality:

$$\varphi_\nu(f+g) \geq \varphi_\nu(f) + \varphi_\nu(g).$$

Now let $\{F_k\}_1^m$ be another arbitrary finite disjoint partition of $\varDelta$. Then, as above, we have

$$\textstyle\sum_1^n \rho(f_{E_k}).\nu(E_k) + \sum_1^m \rho(g_{F_k}).\nu(F_k) \geq$$
$$\geq \textstyle\sum_1^n \rho(f_{E\frown E_k}).\nu(E \cap E_k) + \sum_1^m \rho(g_{F\frown F_k}).\nu(F \cap F_k) =$$
$$= \textstyle\sum_1^{n+m} \rho((f+g)_{D_k}).\nu(D_k),$$

where $D_k = E \cap E_k$ $(k = 1, 2, \ldots, n)$ and $= F \cap F_{k-n}$ $(k = n+1, \ldots, n+m)$, consequently we have

$$\varphi_\nu(f+g) \leq \varphi_\nu(f) + \varphi_\nu(g).$$

Thus we have proved that (*c*) is valid for f, g such that $f.g = 0$. Coupled with (*b*), this yields that for any finite disjoint partition $\{E_k\}_1^n$ of $\varDelta$ and non-negative numbers $\{\alpha_k\}_1^n$ we have

$$\varphi_\nu(\textstyle\sum_1^n \alpha_k f_{E_k}) = \sum_1^n \alpha_k \varphi_\nu(f_{E_k}).$$

Next we shall treat the case that $0 \leq g \leq f$. Since g/f is an essentially bounded function, for any $\varepsilon > 0$ there can be found a finite disjoint partition $\{E_k\}_1^n$ of $\varDelta$ and non-negative numbers $\{\alpha_k\}_1^n$ such that

$$\textstyle\sum_1^n \alpha_k \chi_{E_k} \leq g/f \leq \sum_1^n (\alpha_k + \varepsilon)\chi_{E_k},$$

i.e.

$$\textstyle\sum_1^n \alpha_k f_{E_k} \leq g \leq \sum_1^n (\alpha_k + \varepsilon) f_{E_k}.$$

Then by the above result and (*b*) we have

$$\varphi_\nu(f+g) \leq \varphi_\nu(f + \textstyle\sum_1^n (\alpha_k + \varepsilon) f_{E_k}) =$$
$$= \varphi_\nu(\textstyle\sum_1^n (1 + \alpha_k + \varepsilon) f_{E_k}) = \sum_1^n (1 + \alpha_k + \varepsilon)\varphi_\nu(f_{E_k}) =$$
$$= (1+\varepsilon)\varphi_\nu(f) + \varphi_\nu(\textstyle\sum_1^n \alpha_k f_{E_k}) \leq (1+\varepsilon)\varphi_\nu(f) + \varphi_\nu(g),$$

and similarly

$$\varphi_\nu(f) + \varphi_\nu(g) \leq (1+\varepsilon)\varphi_\nu(f+g).$$

Since $\varepsilon > 0$ is arbitrary, we get to the conclusion that (*c*) is valid for f, g such that $0 \leq g \leq f$. Finally we prove (*c*) for general f, g.

Writing

$$E = \{t; f(t) \geq g(t)\} \quad \text{and} \quad F = \{t; f(t) < g(t)\},$$

we have $E \cap F = 0$, hence $f + g = (f_E + g_E) + (f_F + g_F)$. Since $f_E \geq g_E$ and $f_F \leq g_F$ by the definition of E and F, the foregoing results can be applied

$$\begin{aligned}\varphi_\nu(f + g) &= \varphi_\nu(f_E + g_E) + \varphi_\nu(f_F + g_F) = \\ &= \varphi_\nu(f_E) + \varphi_\nu(g_E) + \varphi_\nu(f_F) + \varphi_\nu(g_F) = \\ &= \varphi_\nu(f) + \varphi_\nu(g).\end{aligned}$$

This completes the proof.

The functional φ_ν is defined only for non-negative functions. It can be extended in the natural way over all L^*_M.

Lemma 6. *The functional φ_ν defined by*

$$\varphi_\nu(f) = \varphi_\nu(f^+) - \varphi_\nu(f^-) \qquad \text{for } f \in L^*_M$$

is a singular positive linear functional such that $||\varphi_\nu||_{\overline{M}} \leq \nu(\Delta)$.

Proof. It is easy to see that $\varphi_\nu(\alpha f) = \alpha\varphi_\nu(f)$ for all real α. To prove additivity, remark that

$$(f + g)^+ - (f + g)^- = f + g = f^+ - f^- + g^+ - g^-,$$

hence $(f + g)^+ + f^- + g^- = (f + g)^- + f^+ + g^+$. Applying Lemma 5, we obtain

$$\varphi_\nu((f + g)^+) + \varphi_\nu(f^-) + \varphi_\nu(g^-) = \varphi_\nu((f + g)^-) + \varphi_\nu(f^+) + \varphi_\nu(g^+),$$

accordingly

$$\varphi_\nu((f + g)^+) - \varphi_\nu((f + g)^-) = \varphi_\nu(f^+) - \varphi_\nu(f^-) + \varphi_\nu(g^+) - \varphi_\nu(g^-),$$

hence $\varphi_\nu(f + g) = \varphi_\nu(f) + \varphi_\nu(g)$ by definition. On the other hand, evidently $\varphi_\nu(f) = 0$ for all $f \in \mathsf{E}_M$, so φ_ν is singular. Finally by Lemma 4 we have

$$|\varphi_\nu(f)| \leq \varphi_\nu(f^+) + \varphi_\nu(f^-) = \varphi_\nu(|f|) \leq \rho(f) . \nu(\Delta),$$

so by (13) $||\varphi_\nu||_{\overline{M}} \leq \nu(\Delta)$.

Now that we are in possession of a method to construct a positive linear functional from a measure, we can reconstruct the original linear functional by means of the measure defined by (11).

Theorem 2. *Let φ be a singular positive linear functional and ν be the measure defined by* (11). *Then we have $\varphi = \varphi_\nu$.*

Proof. Let $\{E_k\}_1^n$ be an arbitrary finite disjoint partition of Δ.

Then

$$\begin{aligned}\varphi(f) &= \Sigma_1^n \varphi(f_{E_k}) = \Sigma_1^n \varphi_{E_k}(f_{E_k}) \leq \\ &\leq \Sigma_1^n \rho(f_{E_k}) . ||\varphi_{E_k}||_{\overline{M}} = \Sigma_1^n \rho(f_{E_k}) . \nu(E_k),\end{aligned}$$

hence according to the definition (14), $\varphi(f) \leq \varphi_\nu(f)$ for all $f \geq 0$, i.e. $\varphi \leq \varphi_\nu$. On the other hand, by Lemma 6 we have $||\varphi_\nu||_{\overline{M}} \leq \nu(\Delta) = ||\varphi||_{\overline{M}}$, hence $||\varphi_\nu||_{\overline{M}} = ||\varphi||_{\overline{M}}$. By Lemma 2 we have

$$||\varphi_\nu - \varphi||_{\overline{M}} = ||\varphi_\nu||_{\overline{M}} - ||\varphi||_{\overline{M}} = 0,$$

because $\varphi_\nu - \varphi \geq 0$, $\varphi \geq 0$ and $||\varphi_\nu||_{\overline{M}} = ||\varphi||_{\overline{M}}$, hence $\varphi = \varphi_\nu$.

The corresponding reflexivity does not hold for the mapping $\nu \to \varphi_\nu$, combined with the mapping $\varphi \to \nu_\varphi$. In order to obtain the reflexivity, we must restrict the domain of the mapping to the proper subset of $\mathfrak{M}$. For this purpose, we introduce a new notion: $\nu \in \mathfrak{M}$ is said to be *in the class M*, if there exists a disjoint sequence $\{G_k\}_1^\infty$ such that $|\nu|\ (\Delta - \mathsf{U}_1^\infty G_k) = 0$ and

$$\Sigma_1^\infty M(k) . \mu_0(G_k) < \infty, \tag{15}$$

$$\Sigma_1^\infty M((1 + 1/n)k) . \mu_0(G_k \cap E) = \infty \qquad (n = 1, 2, \ldots) \tag{16}$$

$$\text{for all } E \in \mathsf{B} \text{ such that } |\nu|\ (E) \neq 0.$$

From (16) it follows that $|\nu|\ (G_k) = 0$ $(k = 1, 2, \ldots)$, accordingly ν is *purely finitely additive* in the sense of K. Yosida and E. Hewitt [9]. The set of all measures in the class M constitutes a linear sublattice of $\mathfrak{M}$. To introduce this notion is justified by the following:

Lemma 7. *If φ is a singular positive linear functional, then the measure defined by* (11) *is in the class M.*

Proof. From the proof of Lemma 2, it is easily seen that $||\varphi_E||_{\overline{M}} = \varphi(f_E)$ for all $E \in \mathsf{B}$, where $f \geq 0$ satisfies the condition that $\mathsf{M}(f) < \infty$ and $||\varphi||_{\overline{M}} = \varphi(f)$. Writing $G_k = \{t; k + 1 > f(t) \geq k\}$ $(k = 1, 2, \ldots)$, we have a disjoint sequence. Now we shall see that $\{G_k\}_1^\infty$ satisfies (15) and (16). In fact, first $\nu_\varphi(\Delta - \mathsf{U}_1^\infty G_k) = \varphi(f_{\Delta - \mathsf{U}_1^\infty G_k}) \leq \varphi(\chi_\Delta) = 0$, because φ is singular by assumption. Next

$$\Sigma_1^\infty M(k) . \mu_0(G_k) \leq \Sigma_1^\infty \int_{G_k} M(f)\, d\mu_0 \leq \mathsf{M}(f) < \infty,$$

hence (15) is satisfied. Finally if $\nu_\varphi(E) = ||\varphi_E||_{\overline{M}} \neq 0$, by (13) $||\varphi_E||_{\overline{M}} = \varphi(f_E) \leq \rho(f_E) . ||\varphi_E||_{\overline{M}}$, consequently $\rho(f_E) = 1$. By Lemma 4 this in turn implies that $\mathsf{M}((1 + \varepsilon)f_E) = \infty$ for all $\varepsilon > 0$. Thus we have

$$\Sigma_{4n}^{\infty} M((1 + 1/n)k) \,.\, \mu_0(G_k \cap E) \geq$$
$$\geq \Sigma_{4n}^{\infty} M((1+1/2n)(k+1)) \,.\, \mu_0(G_k \cap E) \geq \Sigma_{4n}^{\infty} \int_{G_k} M((1+1/2n)f)\, d\mu_0 =$$
$$= \mathsf{M}((1 + 1/2n)f) - \Sigma_1^{4n-1} \int_{G_k} M((1 + 1/2n)f)\, d\mu_0 = \infty,$$

because $(1 + 1/n)k \geq (1 + 1/2n)(k + 1)$ for $k \geq 4n$. Thus we have (16).

For the measures in the class M the desired reflexivity is guaranteed.

Theorem 3. *If ν is a non-negative measure in the class M, then $\|(\varphi_\nu)_E\|_{\overline{M}} = \nu(E)$ for all $E \in \mathsf{B}$.*

Proof. First we prove the case $E = \Delta$. Let $\{G_k\}_1^\infty$ be the sequence satisfying the conditions (15) and (16), and the function f be defined by $f(t) = \Sigma_1^\infty M(k) \,.\, \chi_{G_k}$. The condition (15) guarantees $f \in L^*_M$, because $\mathsf{M}(f) = \Sigma_1^\infty M(k) \,.\, \mu_0(G_k) < \infty$. On the other hand, from the condition (16) it follows that

$$M((1 + 1/n)f_E) = \Sigma_1^\infty M((1 + 1/n)k) \,.\, \mu_0(G_k \cap E) = \infty \; (k = 1,2,\ldots)$$

for all $E \in \mathsf{B}$ such that $\nu(E) \neq 0$, hence by Lemma 4 $\rho(f_E) = 1$ for such E. Then for any finite disjoint partition $\{E_k\}_1^n$ of Δ we have $\Sigma_1^\infty \rho(f_{E_k}) \,.\, \nu(E_k) = \nu(\Delta)$, consequently $\varphi_\nu(f) = \nu(\Delta)$, hence by (13) $\|\varphi_\nu\|_{\overline{M}} = \nu(\Delta)$. Now we return to the general case. For a fixed $E \in \mathsf{B}$ defining the measure $\nu_1(F) = \nu(E \cap F)$ for $F \in \mathsf{B}$, we can easily prove that $\varphi_{\nu_1} = (\varphi_\nu)_E$, hence by the above result we have $\|(\varphi_\nu)_E\|_{\overline{M}} = \|\varphi_{\nu_1}\|_{\overline{M}} = \nu_1(\Delta) = \nu(E)$.

Up to this point we have been concerned with positive linear functionals and non-negative measures and established the correspondence between singular positive linear functionals and non-negative measures in the class M. Now we shall extend this correspondence in the natural way over all singular bounded linear functionals by the relation

$$\nu_\varphi = \nu_{\varphi^+} - \nu_{\varphi^-} \tag{17}$$

where $\nu_{\varphi^+}(E) = \|\varphi_E{}^+\|_{\overline{M}}$ and $\nu_{\varphi^-}(E) = \|\varphi_E{}^-\|_{\overline{M}}$ for $E \in \mathsf{B}$. By Lemma 2 it is easily seen that $\nu_{\varphi+\psi} = \nu_\varphi + \nu_\psi$ for $\varphi, \psi \geq 0$. By the arguments similar to that in the proof of Lemma 6 we can see that the mapping defined by (17) is linear and monotone, i.e. $\varphi \geq \psi$ implies $\nu_\varphi \geq \nu_\psi$. Similarly we extend the correspondence in Lemma 6 over all measures in the class M by the relation

$$\varphi_\nu = \varphi_{\nu^+} - \varphi_{\nu^-}. \tag{18}$$

Also the mapping defined by (18) is linear and monotone. From Theorem 2–3 it follows that the mappings (17) and (18) are mutually inverse to each other, consequently $\varphi \geq \psi$ if and only if $\nu_\varphi \geq \nu_\psi$, hence the mapping (17) is a lattice isomorphism, so in particular $|\varphi_\nu| = |\varphi|_\nu$. As to the norms it follows that

$$\|\varphi\|_{\overline{M}} = \| \, |\varphi| \, \|_{\overline{M}} = \nu_{|\varphi|}(\Delta) = |\nu_\varphi|(\Delta).$$

Summing up the results, we obtain

Theorem 4. *There exists the linear isomorphism $\varphi \leftrightarrow \nu_\varphi$ between the set of all singular linear functionals and the set of all measures in the class M such that $\|\varphi\|_{\overline{M}} = |\nu_\varphi|(\Delta)$, and $\varphi \geq \psi$ if and only if $\nu_\varphi \geq \nu_\psi$.*

§ 5. *The general form of bounded linear functionals.* Now we can describe the desired general form. For this purpose, taking into consideration the analogy between the definition (14) and that of the abstract integral, we propose to denote the value $\varphi_\nu(f)$ by $\mathsf{M}\int f\, d\nu$.

Theorem 5. *Any bounded linear functional φ on L^*_M can be represented uniquely in the form:*

$$\varphi(f) = \int f.g \, d\mu_0 + \mathsf{M}\int f\, d\nu \tag{19}$$

where g is a function in $L^*_{\overline{M}}$ and ν is a measure in the class M. The functional norms of φ are given by

$$\|\varphi\|_{\overline{M}} = \|g\|_{\overline{M}} + |\nu|(\Delta) = \inf_{\xi>0} \frac{1 + \overline{M}(\xi g)}{\xi} + |\nu|(\Delta), \tag{20}$$

$$\||\varphi|\|_{\overline{M}} = \inf \xi^{-1} \quad \textit{for all } \xi > 0 \tag{21}$$

such that $\overline{M}(\xi g) + \xi . |\nu|(\Delta) \leq 1$.

Proof. According to Theorem 1, φ may be expressed uniquely in the form $\varphi = \varphi_c + \varphi_S$, where φ_c is of function-type and φ_S is singular. Hence by Theorem 4 there exist $g \in L^*_{\overline{M}}$ and a measure ν in the class M such that

$$\varphi_c(f) = \int f.g\, d\mu_0 \quad \text{and} \quad \varphi_S(f) = \mathsf{M}\int f\, d\nu.$$

Thus a representation (19) is obtained. Uniqueness is evident. To prove (20) and (21), we may assume that $\varphi \geq 0$, because, as remarked before, all the correspondences in question are lattice homomorphic and the norms are invariant under the change of φ

into $|\varphi|$. For any $\varepsilon > 0$ by Lemma 2 and (8) there exist $f \geq 0$ and $h \geq 0$ such that

$$\mathsf{M}(f) < 1, \qquad ||\varphi_c||_{\overline{M}} \leq \varphi_c(f) + \varepsilon,$$

and $\mathsf{M}(h) \leq 1 - M(f)$, $||\varphi_S||_{\overline{M}} \leq \varphi_S(h) + \varepsilon$.
Again by Lemma 1 we can find $h_1 \geq 0$ such that $h_1 \geq f$, $h_1 \geq h$ and $\mathsf{M}(h_1) \leq \mathsf{M}(h) + \mathsf{M}(f) \leq 1$, hence

$$||\varphi_c||_{\overline{M}} + ||\varphi_S||_{\overline{M}} \leq \varphi_c(f) + \varphi_S(h) + 2\varepsilon \leq$$
$$\leq \varphi_c(h_1) + \varphi_S(h_1) + 2\varepsilon \leq ||\varphi_c + \varphi_S||_{\overline{M}} + 2\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we obtain

$$||\varphi_c||_{\overline{M}} + ||\varphi_S||_{\overline{M}} \leq ||\varphi||_{\overline{M}}, \text{ i.e. } ||\varphi_c||_{\overline{M}} + ||\varphi_S||_{\overline{M}} = ||\varphi||_{\overline{M}}.$$

On the other hand, from (7) and Theorem 4 it follows that

$$||g||_{\overline{M}} + |\nu|\,(\varDelta) = ||\varphi_c||_{\overline{M}} + ||\varphi_S||_{\overline{M}} = ||\varphi||_{\overline{M}}.$$

The second equality in (20) follows from Amemiya's formula (3′), applied to $||g||_{\overline{M}}$. Next we prove (21). Let $\xi > 0$, $\mathsf{M}(\xi g) + \xi|\nu|\,(\varDelta) \leq 1$, and $f \geq 0$ be arbitrarily chosen such that $||f||_M \leq 1$. Then by Amemiya's formula (3′) for any $\varepsilon > 0$ there exists $\eta \geq 1$ such that $1 + \mathsf{M}(\eta f) \leq (1 + \varepsilon)\eta$. From (2) and Lemma 2 it follows that

$$\xi\varphi(\eta f) = \xi\varphi_c(\eta f) + \xi\varphi_S(\eta f) \leq \mathsf{M}(\eta f) + \overline{\mathsf{M}}(\xi g) + \xi\,|\nu|\,(\varDelta) \leq$$
$$\leq \mathsf{M}(\eta f) + 1 \leq (1 + \varepsilon)\eta, \quad \text{i.e.} \quad \varphi(f) \leq (1 + \varepsilon)/\xi.$$

Since $\varepsilon > 0$ is arbitrary, from the definition (8) we have $|||\varphi|||_{\overline{M}} \leq \inf \xi^{-1}$ for all such ξ. To prove the converse inequality, we may assume that $|||\varphi|||_{\overline{M}}=1$. Suppose that $\overline{\mathsf{M}}(g) + |\nu|\,(\varDelta) > 1 + \delta$ for some $\delta > 0$. Then by (2′) and Lemma 2 for any ε such that $0 < \varepsilon < \delta/3$ there exist $f \geq 0$ and $h \geq 0$ such that

$$\mathsf{M}(f) < \infty, \quad \overline{\mathsf{M}}(g) < \varphi_c(f) - \mathsf{M}(f) + \varepsilon$$

and

$$\mathsf{M}(h) < \varepsilon, \quad |\nu|\,(\varDelta) < \varphi_S(h) + \varepsilon.$$

Again by Lemma 1 there can be found h_1 such that $h_1 \geq f$, $h_1 \geq h$ and $\mathsf{M}(h_1) \leq \mathsf{M}(f) + \mathsf{M}(h) \leq \mathsf{M}(f) + \varepsilon$. For such h_1 we have

$$\varphi(h_1) - \mathsf{M}(h_1) \geq \varphi_c(f) + \varphi_S(h) - \mathsf{M}(f) - \varepsilon \geq$$
$$\geq \overline{\mathsf{M}}(g) + |\nu|\,(\varDelta) - 3\varepsilon > 1 + \delta - 3\varepsilon.$$

hence $1 + \mathsf{M}(\xi h_2) < \xi$, where $\xi = \varphi(h_1) - \delta + 3\varepsilon$ and $h_2 = h_1/\xi$, accordingly by Amemiya's formula (3′) we have $||h_2||_M < 1$, hence $||h_1||_M < \varphi(h_1) - \delta + 3\varepsilon \leq ||h_1||_M - \delta + 3\varepsilon$. This contradiction establishes $\overline{\mathsf{M}}(g) + |\nu|\,(\varDelta) \leq 1$, consequently by definition the right side of (21) is not greater than the left.

§ 6. *Inner characterization of singular functionals.* Singular linear functionals were defined in connection with the subspace E_M. The question arises whether they are characterized in terms of their functional norms. Already we know that $||\varphi||_{\overline{M}} = |||\varphi|||_{\overline{M}}$ for all singular φ. The converse is also valid.

Theorem 6. *A bounded linear functional φ on L^*_M is singular, if and only if $||\varphi||_{\overline{M}} = |||\varphi|||_{\overline{M}}$.*

Proof. Let $||\varphi||_{\overline{M}} = |||\varphi|||_{\overline{M}}$. To see the assertion, it suffices to prove that $\varphi_c = 0$. Supposing the contrary, we assume that $||\varphi_c||_{\overline{M}} = 1$. From Theorem 5 it follows that

$$|||\varphi|||_{\overline{M}} \leq |||\varphi_c|||_{\overline{M}} + |||\varphi_S|||_{\overline{M}} \leq ||\varphi_c||_{\overline{M}} + ||\varphi_S||_{\overline{M}} = ||\varphi||_{\overline{M}},$$

hence $|||\varphi_c|||_{\overline{M}} = ||\varphi_c||_{\overline{M}} = 1$, because $|||\varphi|||_{\overline{M}} = ||\varphi||_{\overline{M}}$ by hypothesis. Let g be the function, corresponding to φ_c in (6), then by the above result and (7) we have $||g||_{\overline{M}} = |||g|||_{\overline{M}} = 1$. Since $\overline{M}(\xi)$ is an N-function, it is easy to see that $\overline{\mathsf{M}}(\xi g)/\xi \to \infty$ as $\xi \to \infty$, accordingly in Amemiya's formula (3′), applied to $||g||_{\overline{M}}$, the infimum may be replaced by the minimum, i.e. $1 + \overline{\mathsf{M}}(\xi_0 g) = \xi_0\,||g||_{\overline{M}} = \xi_0$ for some $\xi_0 \geq 1$. On the other hand, from $|||g|||_{\overline{M}} = 1$ it follows that $\overline{\mathsf{M}}(\xi g) \geq \xi$ for $\xi > 1$, because of convexity of $\overline{\mathsf{M}}(\eta g)$ (with respect to η) and the definition (4), hence we have $\xi_0 = 1$ and $\overline{\mathsf{M}}(\xi_0 g) = \overline{\mathsf{M}}(g) = 0$. Writing $\xi_1 = \sup_{\overline{M}(\xi)=0} \xi$, we have $|g| \leq \xi_1 \chi_\varDelta$ i.e. g is a bounded function. Then since $\overline{\mathsf{M}}(\xi g) < \infty$ for all $\xi > 0$ and $\overline{\mathsf{M}}(\xi g) \to \infty$ as $\xi \to \infty$, there exists $\xi_2 > 0$ such that $\overline{\mathsf{M}}(\xi_2 g) = 1$ and by the definition (4)

$$\xi_2 = \xi_2\,|||g|||_{\overline{M}} = |||\xi_2 g|||_{\overline{M}} = 1, \quad \text{i.e.} \quad \overline{\mathsf{M}}(g) = 1$$

contradicting $\overline{\mathsf{M}}(g) = 0$. This contradiction establishes the assertion $\varphi_c = 0$.

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Research Institute of Applied Electricity,
Hokkaidô University, Sapporo, Japan