ISRAEL MATHEMATICAL CONFERENCE PROCEEDINGS, VOL. 5, 1992

SOME RECENT GENERAL RESULTS IN INTERPOLATION THEORY

YU. A. BRUDNYI

TECHNION, ISRAEL INSTITUTE OF TECHNOLOGY

1. INTRODUCTION

I think that most of the results which I will present here will be new for most participants, but the main aim of my talk is not so much to present new results, but rather to draw attention to some interesting research problems. All of these problems are related to the so called general theory of interpolation spaces, more specifically, to the construction and study of the properties of various interpolation methods.

The development of this theory is still far from complete. There are two important considerations which have motivated its development up till now and which should continue to do so in the future.

* First, the theory should give us a stable framework for calculations which before were obtained by good luck and ingenuity.
* Secondly, it should be able to give timely answers to new questions which constantly arise in applications.

At the same time, specific calculations and applications are a powerful stimulus for the development of the theory. Indeed, its successful development is impossible without the prior accumulation of a "critical mass" of concrete results. Such a "critical mass" had collected at the end of the fifties, and it led to the glorious period of formation and development of the theory during the years 1959-1966. Since then some important papers have appeared from time to time in this field but of course their quantity (not quality) cannot be compared with that powerful stream of papers dealing with calculations and applications. Let me express the hope that conferences like this one will give new impetus to the development of important parts of interpolation theory.

Most of the results which I will mention in this lecture have been obtained by M. Aizenstein, N. Krugljak and myself, during the time when Krugljak and I were writing our book [1]. All these results and their proofs can be found in Chapter II of the book, together with a number of other classical and new results.

2. SOME DEFINITIONS AND NOTATION

Let me quickly recall some standard notions:

(i) *Banach couples* $\vec{A} = (A_0, A_1)$

(ii) The *Banach space* $\mathcal{L}(\vec{A},\vec{B})$ *of linear operators* T which map $\vec{A}$ to $\vec{B}$ with norm

$$\|T\|_{\vec{A},\vec{B}} := \sup_{i=0,1} \left\{ \|T|_{A_i}\|_{A_i,B_i} \right\}$$

(iii) *Interpolation functors*, i.e. maps F from the category of Banach couples $\vec{\mathcal{B}}$ to the category of Banach spaces $\mathcal{B}$ such that $F(\vec{A})$ is an interpolation space with respect to $\vec{A}$ for all $\vec{A} \in \vec{\mathcal{B}}$.

(iv) If A is any intermediate space of the couple $\vec{A}$, then A^o denotes the closure of the intersection space $\Delta(\vec{A})$ in A . $\vec{A}^o$ is the couple (A_0^o, A_1^o) . We say that A is *regular* if $A^o = A$. Similarly, $\vec{A}$ is *regular* if $\vec{A}^o = \vec{A}$.

(v) If A is any intermediate space of $\vec{A}$, then $A^\sim$ denotes the *Gagliardo completion* of A with respect to the sum space $\Sigma(\vec{A})$.

(vi) *Triples*: For each couple $\vec{A}$ we refer to $(\vec{A}, A)$, where A is some intermediate space of $\vec{A}$, as a triple.

We now recall the Aronszajn-Gagliardo theorem:

Given any class $\mathcal{C}$ *of triples, there exist two interpolation functors* $F_{\mathcal{C}}$ *and* $G_{\mathcal{C}}$ *such that:*

$$G_{\mathcal{C}}(\vec{A}) \leq^1 A \leq^1 F_{\mathcal{C}}(\vec{A}) \qquad \textit{for all } (\vec{A}, A) \in \mathcal{C}$$

Moreover, the functor $G_{\mathcal{C}}$ *is maximal among all functors* G *having the property* $G(\vec{A}) \leq^1 A$ *for all* $(\vec{A}, A) \in \mathcal{C}$. *Similarly* $F_{\mathcal{C}}$ *has an analogous minimal property.*

If $\mathcal{C}$ consists of a single triple $(\vec{A}, A)$ then, for well known reasons, $F_{\mathcal{C}}$ is denoted by $\mathrm{Orb}_A(\vec{A};\circ)$ *(orbit functor)* and $G_{\mathcal{C}}$ by $\mathrm{Corb}_A(\circ;\vec{A})$ *(coorbit functor)*. It is surprising that although orbits and coorbits were defined in 1965, it was not until 1981 that it was shown (by Brudnyi-Krugljak and by Janson) that the "classical" interpolation functors such as the real and complex methods are orbits or coorbits of certain simple couples such as $\vec{\ell}_\infty := \left(\ell_\infty, \ell_\infty(\{2^{-n}\}_{n\in\mathbb{Z}})\right)$, $\vec{\ell}_1 := \left(\ell_1, \ell_1(\{2^{-n}\}_{n\in\mathbb{Z}})\right)$ and $\left(FL_1, FL_1(\{2^{-n}\}_{n\in\mathbb{Z}})\right)$. Why do these particular couples play such an important role, but others apparently do not? (Question: Do orbit or coorbit functors generated by some other nice couple such as $\left(\ell_2, \ell_2(\{2^{-n}\}_{n\in\mathbb{Z}})\right)$ have good properties like reiteration?)

3. INTERPOLATION OF FINITE DIMENSIONAL SPACES

In 1965 Aronszajn and Gagliardo stated that there are many different interpolation methods. They were thus strongly motivated to develop a general theory encompassing all such methods. Now, a quarter of a century later, we can observe that so far only three essentially different interpolation methods have been created. (All known methods are either the real method, complex method, abstract versions of the Calderón-Lozanovskii construction, or other closely related methods.) Why are there so few methods? I think that this is because we have not yet carefully studied the case of finite dimensional couples.

Here are two simple examples:

Example (i): Let $\vec{A}$ be a regular finite dimensional couple. The regularity means that $\vec{A}$ can be regarded as the space $\mathbb{R}^n$ equipped with two norms. Obviously in this case all exact interpolation spaces of $\vec{A}$ can be obtained by the (general) real method, but only to within equivalence of norms. Let $\gamma(\vec{A})$ be the least constant of equivalence for all exact interpolation spaces of $\vec{A}$. Let γ_n be the supremum of $\gamma(\vec{A})$ as $\vec{A}$ ranges over all regular n-dimensional couples. It can be proved (using compactness and a theorem which I obtained with Krugljak) that $\gamma_n < \infty$. However $\lim_{n\to\infty} \gamma_n = \infty$. But how fast does γ_n tend to ∞? This is an open question.

In order to construct new interpolation methods we need to know the answer to a similar question: Consider a quantity $\gamma_n'(\vec{B})$ similar to γ_n, obtained by letting $\vec{A}$ range over all n-dimensional subcouples of a fixed infinite dimensional couple $\vec{B}$. How fast does $\gamma_n'(\vec{B})$ tend to ∞ ?

Example (ii): We shall obtain a sequence of couples $\vec{A}_n$ by equipping $\mathbb{R}^n$ with the two (semi) norms:

$$\|x\|_{\ell_\infty^n} = \max_{1\le i\le n} |x_i| , \qquad \|x\|_{\mathrm{Lip}^n} = \max_{1\le i\le n-1} |x_{i+1}-x_i|$$

For these couples $\gamma(\vec{A}_n) \to \infty$. But can we characterize all the exact interpolation spaces of $\vec{A}_n$? This is an open problem.

We recall that if the second norm is replaced by $\|x\|_{\ell_1^n}$ then the famous Calderón-Mityagin theorem gives a complete description of all exact interpolation spaces.

4. FUNCTORS DEFINED BY FINITE DIMENSIONAL APPROXIMATION

We shall now describe two important classes of interpolation functors of such a type. We will denote them by $\mathcal{FG}$ (finitely generated) and $\mathcal{C}omp$ (computable). Each functor in the class $\mathcal{FG}$ is completely determined by its "trace" on the class $\vec{\mathcal{B}}_{fin}$ of finite dimensional regular couples. In fact this sentence can be taken as the definition of $\mathcal{FG}$. We shall present a result which establishes the connection between functors in $\mathcal{FG}$ and Janson's approximation condition.

(A couple $\vec{A}$ satisfies Janson's approximation condition if for each $a \in \Sigma(\vec{A})$ and $\varepsilon > 0$ there exists a finite rank linear operator $P = P_{a,\varepsilon}$ mapping $\Sigma(\vec{A})$ into $\Delta(\vec{A})$ such that

$$\|P\|_{\vec{A}} \le 1 + \varepsilon \quad \text{and} \quad \|a - Pa\|_{\Delta(\vec{A})} \le \varepsilon .)$$

Theorem (Aizenstein-Brudnyi) $F \in \mathcal{FG}$ *if and only if*

$$F = \mathrm{Orb}_A(\vec{A};\circ)$$

for some $\vec{A}$ *which satisfies the approximation condition and for some intermediate space* A *such that* $\Delta(\vec{A})$ *is dense in* A.

Examples: For any interpolation functor F, let F^o denote the functor defined by taking $F^o(\vec{A}) := F(\vec{A})^o$. Then F^o is in the

class $\mathcal{FG}$ when F is any of the following functors:
$\mathcal{K}_\Phi$ (general K-method) for non degenerate parameters Φ ,
$\mathcal{J}_\Phi$ (general J-method) ,
C_θ (the first Calderón method) and of course also C^θ (the second Calderón method) since by Bergh's theorem $(C^\theta)^o = C_\theta$,
The functors $\langle\cdot\rangle_\varphi$ (Peetre) , φ_u and φ_ℓ (Ovchinnikov) $\langle\cdot,\varphi\rangle$ (Gustavsson-Peetre) for all non-degenerate concave functions $\varphi:\mathbb{R}_+ \to \mathbb{R}_+$.

Let $\vec{\mathcal{B}}_{fin}(\vec{A})$ be the set of all finite dimensional regular subcouples of $\vec{A}$.

<u>Definition:</u> An interpolation functor F is said to be *computable* on some couple $\vec{A}$ if

$$\|a\|_{F(\vec{A})} = \inf \{\|a\|_{F(\vec{B})} : \vec{B} \in \vec{\mathcal{B}}_{fin}(\vec{A}) \}$$

for all elements a .

We say that F is *computable* if it is computable on <u>all</u> couples. The class of all computable couples is denoted by $\mathcal{C}omp$.

The concept of computable functor was introduced in the category of Banach spaces by Herz and Wick-Pelletier and generalized to the category of so-called "doolittle diagrams" by Kaijser and Wick-Pelletier. I would like to point out that this concept is closely related to the following condition on orbit functors $F = \mathrm{Orb}_A(\vec{A};\cdot)$ introduced and used by Janson:

(*) For all couples $\vec{B}$ and all $b \in \Delta(\vec{B})$
$\|b\|_{F(\vec{B})} = \inf \{ \Sigma \|T_k\|_{\vec{A},\vec{B}} \|a_k\| \}$
where the infimum is taken over all *finite* families $\{T_k\} \subset \mathcal{L}(\vec{A},\vec{B})$ and $\{a_k\} \subset \Delta(\vec{A})$ such that $b = \Sigma T_k a_k$.

We will refer to all orbit functors F satisfying (*) as *correct* .

<u>Theorem</u> (Aizenstein-Brudnyi) *The functor F is computable if and only if it is finitely generated and correct.*

Most functors F^o in the preceding list of examples are computable. The only exception is φ_u^o; we do not know yet whether it is computable.

The preceding theorem shows that $\mathcal{FG} \subset \mathcal{C}omp$ but we do not know if the inclusion is strict. (Perhaps φ_u^o might be the missing counterexample?)

It is very significant that there exists a subcategory of $\vec{\mathcal{B}}$ on which all interpolation functors F^o are computable.

Theorem (Brudnyi) *F^o is computable on every couple satisfying the Janson approximation condition.*

Let us now mention a very useful criterion for correctness of orbit functors which are generated by a single element. (Recall that several "naturally occuring" functors turn out to be of this type.)

Theorem (Krugljak-Mastylo) *If $\vec{A}$ is a couple of Banach lattices both having the Fatou property, then every orbit functor $Orb_A(\vec{A};\cdot)$ which is generated by a single element is correct.*

Computable functors have some remarkable properties. The first of them is of course Janson's famous duality theorem. This result will appear below as part of a more general framework. But now I will describe some other properties, which deal with the stability of such functors under some natural operations, and also with interpolation of nonlinear operators:

(i) Stability under superposition:

Theorem (Aizenstein-Brudnyi) *Let F_0, F_1 and F be computable on some couple $\vec{A}$ and suppose that $\Delta(\vec{A})$ is dense in $F_0(\vec{A}) \cap F_1(\vec{A})$. Then the functor $F(F_0,F_1)$ is computable on $\vec{A}$.*

(This functor is defined by $F(F_0,F_1)(\vec{B}) := F(F_0(\vec{B}),F_1(\vec{B}))$ for all $\vec{B}$.) Corollary: (Weak reiteration theorem) *Suppose that the functors F_0, F_1, F and G are computable and that $F(F_0,F_1)(\vec{B}) = G(\vec{B})$ for all regular finite-dimensional couples $\vec{B}$. Then $F(F_0,F_1)$ coincides with G on all couples $\vec{A}$ for which $\Delta(\vec{A})$ is dense in $F_0(\vec{A}) \cap F_1(\vec{A})$.*

Example: As proved by Janson, the functor C_θ is both an orbit and coorbit on $\vec{\mathcal{B}}_{fin}$. So the preceding corollary gives the Calderón reiteration formula without any calculation.

We do not know whether it is possible to do without the density condition in the preceding theorem. If it were possible then our theorem would also yield the stronger reiteration result of Cwikel-Janson for C_θ.

(ii) Interpolation of nonlinear operators:
Let $Lip(\vec{A},\vec{B})$ denote the space of all continous maps T from $\Sigma(\vec{A})$ into $\Sigma(\vec{B})$ which are Lipschitz from A_i into B_i, $i=0,1$, and satisfy $T(0) = 0$. The norm of T in this space is the maximum of the Lipschitz constants of $T|_{A_i}$.

Theorem (Aizenstein-Brudnyi) *The estimate*

$$\|T(a_1) - T(a_2)\|_{F(\vec{B})} \le \|T\|_{Lip(\vec{A},\vec{B})}\|a_1 - a_2\|_{F(\vec{A})}$$

holds for each computable F, *each* $T \in Lip(\vec{A},\vec{B})$ *and each* a_1, a_2 *in* $F(\vec{A})$.

Corollary (generalization of a theorem of Felix Browder): *If* $\vec{A}$ *has the Janson approximation property, then for every functor* F^o *the above estimate holds for all* $T \in Lip(\vec{A})$.

Example (a problem of Lions): Is the functor C_θ stable under Lipschitz mappings?

M. Cwikel constructed a very interesting example of an operator T and a couple $\vec{A}$ such that $T:\Sigma(\vec{A}) \to \Sigma(\vec{A})$ and $T|_{A_0} \in Lip(A_o)$ but $T|_{A_1}$ is only bounded. He proved that T does not map $C_\theta(\vec{A})$ to itself. So in this context the answer is no. But from our theorem we see that for $T \in Lip(\vec{A},\vec{B})$ the answer is yes!

5. DUALITY

Let us recall that if A is an intermediate space of $\vec{A}$ then the *dual space* A' of A is the linear subspace of the Banach conjugate space $\Delta(\vec{A})^*$ defined by the finiteness of the (Banach) norm

$$\|a'\|_{A'} := \sup\{\langle a',a\rangle : \|a\|_A \le 1 , a \in \Delta(\vec{A})\}$$

The *dual couple* of $\vec{A}$ is the couple $\vec{A}' = (A_0',A_1')$. It is natural to expect that if A is an interpolation space of $\vec{A}$ then A′ is an interpolation space of $\vec{A}'$. But life can be hard: There exists a counterexample to this assertion, constructed by N. Krugljak. So the mapping F′ from the subcategory $\vec{\mathcal{B}}'$ of dual couples to $\mathcal{B}$ given by:

$$(**) \qquad F'(\vec{A}') := F(\vec{A})' \qquad \text{for all } \vec{A} \in \vec{\mathcal{B}}$$

in general is not an interpolation functor. This leads us to define the *dual functor* DF of a given interpolation functor F in a way which is slightly different from (**) .

<u>Definition</u>: DF is the maximal functor among all interpolation functors G which satisfy $G(\vec{A}) \le^1 F(\vec{A})'$ for all regular couples $\vec{A}$.

Of course there are other possible variants of this definition: E.g. take the maximal extension using finite dimensional instead of regular couples. (This functor is denoted by $D_{fin}F$). But I think that the next two results show that our choice of definition is the correct one.

Intuitively orbits and coorbits seem to be dual to eachother. But here is a precise statement:

<u>Theorem</u> (Brudnyi) *Let the couple* $\vec{A}$ *and its intermediate space* A *both be regular. Let* $F := Orb_A(\vec{A};\cdot)$ *and* $G := Corb_{A'}(\cdot;\vec{A}')$. *Then the norms of* $DF(\vec{B})$ *and* $G(\vec{B})$ *coincide on* $\Delta(\vec{B})$ *for all* $\vec{B} \in \vec{\mathcal{B}}$.

In particular, if $G(\vec{B}) \le^1 (G^o(\vec{B}))^{\sim}$ *then* $DF(\vec{B}) = G(\vec{B})$.

The second result further confirms our choice of definition:

<u>Theorem</u> (Aizenstein-Brudnyi) *If* F *is computable then* $DF = D_{fin}F = F'$.

<u>Corollary</u> (slight generalization of Janson's first duality theorem): *If* $Orb_A(\vec{A};\cdot)$ *is computable, then*

$$Orb_A(\vec{A};\cdot)' = Corb_{A'}(\cdot;\vec{A}') \qquad .$$

It is very natural to seek an analogous duality result for coorbits. Janson's second duality theorem requires very strong assumptions on the coorbit. We can understand why this is so in view of the following result:

Theorem (Brudnyi-Krugljak) *Let $\vec{A}$ be a regular couple and A an intermediate space of $\vec{A}$ satisfying* $A \leq^1 (A^o)^\sim$. *Then the equality*

$$\mathrm{Corb}_{A'}(\vec{B};\vec{A}')' = \mathrm{Orb}_A(\vec{A},\vec{B}')$$

holds for all regular couples $\vec{B}$ if and only if the closed unit ball of $\mathrm{Orb}_A(\vec{A},\vec{B}')$ is weak-star closed in the space $\Delta(\vec{B})^*$.
(In fact this result can be made into part of an alternative proof of the second Janson duality theorem.)

6. REAL INTERPOLATION OF WEAKLY COMPACT OPERATORS

Finally here is one more example of an application of the general theory.

Definition: An operator $T \in \mathcal{L}(\vec{A},\vec{B})$ is said to be *weakly compact* if $T|_{\Delta(\vec{A})}$ is weakly compact as an operator from $\Delta(\vec{A})$ into $\Sigma(\vec{B})$.

Beauzamy posed and solved the question of whether a weakly compact operator $T \in \mathcal{L}(\vec{A},\vec{B})$ is also weakly compact from $\vec{A}_{\theta,p}$ into $\vec{B}_{\theta,p}$.

Question: For which parameters Φ does the real method functor $\mathcal{K}_\Phi$ interpolate weak compactness, i.e. have the property that $T|_{\mathcal{K}_\Phi(\vec{A})} : \mathcal{K}_\Phi(\vec{A}) \to \mathcal{K}_\Phi(\vec{B})$ is weakly compact for all weakly compact $T \in \mathcal{L}(\vec{A},\vec{B})$?

Here is the complete answer:

Theorem (Aizenstein-Brudnyi) *The functor $\mathcal{K}_\Phi$ interpolates weak compactness if and only if the space $\mathcal{K}_\Phi(\ell_\infty,\ell_\infty\{(2^{-n})\})$ is reflexive.*

Example (generalizing Beauzamy's theorem):
Let Φ be a quasipower parameter of the $\mathcal{K}$ method and suppose that it is also reflexive. Then $\mathcal{K}_\Phi(\vec{A})$ is reflexive if and only if the closed unit ball of $\Delta(\vec{A})$ is weakly precompact in $\Sigma(\vec{A})$.

REFERENCE

[1] Y. Brudnyi and N. Krugljak, Interpolation Functors and Interpolation Spaces I, North Holland, Amsterdam, 1991.