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JOINT PROBABILITIES IN QUANTUM MECHANICS *)

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Abstract :

A critical analysis of the problem of existence of joint probability distributions for incompatible quantum random variables is given. All "non-existence" theorems are discussed on a common basis with conclusion, that none of them really precludes the existence of quantum joint distributions. From the discussion follows also that in order to include joint probabilities into quantum mechanics it is necessary to enlarge the later. An example of such generalization is constructed, leading to a non-trivial quantum probability theory with joint probability distributions for incompatible variables.

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1 . QUANTUM PROBABILITY THEORY .

1.1. The problem .

Writings on foundations of quantum mechanics are full of impossibilities , complementarities , uncommeasurabilities and dilemmas , often stated in a strangely dogmatic form . Almost all these statements are based on a simple fact of existence of non-commuting pairs of self-adjoint operators , and are manifestations of difficulties with a reconcilia-tion of this formal feature of quantum mechanics with the common probabilistic interpretation of its formalism . Hence the problem arises about the existence of joint probability distributions (JPDs) for quantum variables represented by non-commuting operators . The problem is not solved up to now and one of goals of this paper is to demonstrate that . We are going to show , namely , that :

(i) there is no convincing reasons to reject a possibility of existence of quantum JPDs (see Section 2) ; and

(ii) there is no example of a satisfactory extension of quantum mechanics up to define JPDs for incompatible var-iables (see Section 3) .

Moreover , it seems that (see Section 4) :

(iii) there are some (perhaps not quite doubtful) arguments which encourage attempts to look for such an extension of quantum mechanics ; and

(iv) there exist a simple but not trivial model of such an extension (perhaps not quite academic) .

1.2. A law of Nature ?

The prevailing attitude towards the problem of JPDs in quantum mechanics , more or less echoing ideas of the Copenhagen School , reduces to an a priori negative answer: there is no joint measurements and no JPDs for incompatible quantum variables . The basis for such solution of the problem form : thought experiments , like the Heisenberg microscope , and some unclear philosophical ideas , like complementarity . The resulting view is in some respects extremally conservative and claims essentially , that quantum mechanics is a finite theory , unable to develop . Particularly the difficulties in introducing quantum JPDs are treated as a manifestation of something like a law of Nature . I will not discuss the Copenhagen views here , see e.g. Popper (1967) and Bunge (1967) for a serious critique , confining myself to an obvious remark , that this paper contradicts some of ideas of the Copenhagen interpretation .

1.3. A task of experimenters .

Concerning "anti-JPDs" thought experiments , I will not study them here , mainly because they prove nothing , like thought experiments at all . They could have only a heuristic value , suggesting attempts to find a proof of non-existence of quantum JPDs , so it is rather more reasonable to analyse such proofs (Section 2) .

I do not worry myself about semantics of joint measurements as well , like e.g. quantum theorists do not worry themself about a possibility to build up instruments corresponding to the whole uncountable set of self-adjoint operators on the quantum Hilbert space . If it will appear possible to formulate an extended quantum theory with JPDs , then it will be a task of experimenters to find if the JPDs are measurable , and how to measure them .

1.4. A short exposition .

The rest of this Section is devoted to a short exposition of fundamentals of classical and quantum theories of probability (provided the later may be called a theory).

The classical probability theory in the traditional formulation rests on a triple , consisiting of a sample space , a Boolean algebra of subsets of this space , and a probability measure on it . It can be defended , however , that the sample space is essentially a redundant element , and that basic notions of the classical probability theory can be defined by means of an abstract Boolean σ-algebra and probability measures on it (Łoś 1954) . This approach, a little generalizing the traditional one , is especially appropriate for our considerations .

1.5. The classical probability theory .

So the classical probability theory is based on an abstract Boolean σ-algebra $\mathcal{L}_c$ of events , with elements a , b , ... Probability is a σ-additive homomorphism α (probability measure , state) from $\mathcal{L}_c$ into the real unit interval $\langle 0,1\rangle$, where both $\mathcal{L}_c$ and $\langle 0,1\rangle$ are considered in the natural way as ortho-modular σ-ortho-p.o.-sets . The family of all such homomorphisms is a σ-convex set , denoted by $\mathcal{S}_c$. A classical random variable (random function , observable) A , based on a measurable space $(\mathbb{S}, \mathcal{B}(\mathbb{S}))$ is defined as a σ-additive homomorphism from $\mathcal{B}(\mathbb{S})$ into $\mathcal{L}_c$; the family of all classical random variables will be denoted by $\mathcal{O}_c$. Obviously a composition $\alpha_A := \alpha \circ A$ of a random variable and a state is a probability measure on the measurable space corresponding to A .

The simplest random variables are based on the two-element subset $\{0,1\}$ of $\mathbb{R}^1$. They will be called elementary random variables , and are in a natural 1-1-correspondence with elements of $\mathcal{L}_c$. We will not distinguish between an elementary random variable and the corresponding element of $\mathcal{L}_c$.

1.6. Classical joint probability distributions .

Any two classical random variables , say $A_1 : \mathcal{B}(\mathbb{S}_1) \mapsto \mathcal{L}_c$ $A_2 : \mathcal{B}(\mathbb{S}_2) \mapsto \mathcal{L}_c$, are compatible in the sense , that there is a third random variable , $J(A_1, A_2)$, based on $\mathbb{S} = \mathbb{S}_1 \times \mathbb{S}_2$ such that $J(A_1, A_2)(X \times Y) = A_1(X) \wedge A_2(Y)$ for any $X \in \mathcal{B}(\mathbb{S}_1)$, $Y \in \mathcal{B}(\mathbb{S}_2)$. The random variable $J(A_1, A_2)$ will be called the classical joint random variable (joint observable , JRV) of A_1 and A_2 . It can be proved , that the $\mathcal{L}_c$-valued function on $\mathcal{B}(\mathbb{S}_1) \times \mathcal{B}(\mathbb{S}_2)$, given by $J(X \times Y) = A_1(X) \wedge A_2(Y)$, can be uniquely extended to the Boolean σ-algebra $\mathcal{B}(\mathbb{S}_1 \times \mathbb{S}_2)$ generated by $\mathcal{B}(\mathbb{S}_1) \times \mathcal{B}(\mathbb{S}_2)$, to form a σ-additive homomorphism into $\mathcal{L}_c$. Hence for any two classical random variables there exists one and only one classical joint random variable .

For a classical JRV , like any random variable , one can define $\alpha_{A_1, A_2} := \alpha \circ J(A_1, A_2)$. This composition (called the classical joint probability distribution , JPD, for A_1 and A_2 in α) is obviously a probability measure on $\mathbb{S}_1 \times \mathbb{S}_2$. Some of properties of classical JPDs are listed below .

1.7. The marginal property .

There is a natural σ-embedding I_1 of $\mathcal{B}(\mathbb{S}_1)$ into $\mathcal{B}(\mathbb{S}_1 \times \mathbb{S}_2)$, taking elements of $\mathcal{B}(\mathbb{S}_1)$ onto vertical stripes on $\mathbb{S}_1 \times \mathbb{S}_2$, $I_1 : \mathcal{B}(\mathbb{S}_1) \ni X \longmapsto X \times \mathbb{S}_2 \in \mathcal{B}(\mathbb{S}_1 \times \mathbb{S}_2)$. It can be easily seen , that $\alpha \circ J(A_1, A_2) \circ I_1$ is a probability measure on $\mathbb{S}_1$, called a marginal distribution for α_{A_1, A_2} . Of course , $A_1 = J(A_1, A_2) \circ I_1$ and $\alpha_{A_1} = \alpha_{A_1, A_2} \circ I_1$. The same concerns the second random variable A_2 . So we have the full conformity between marginal distributions and individual ones . We will refer to this property as to the marginal property of JPDs .

1.8. A Functional closeness .

The marginal property is a special case of a more general one . If f is a measurable function from a measurable space $\mathbb{S}$ to a measurable space $\mathbb{T}$, then with any random variable A on $\mathbb{S}$ we can associate a random variable on $\mathbb{T}$ by $A(f^{-1}(X))$, $X \in \mathcal{B}(\mathbb{T})$. This random variable will be denoted by $f(A)$. This means , that the family $\mathcal{O}_c$ is functionally closed . If we consider now a JRV $J(A_1, A_2)$ instead of A above , and $f_1 : \mathbb{S}_1 \times \mathbb{S}_2 \longmapsto \mathbb{S}_1$ defined by $f_1(x_1, x_2) = x_1$ instead of a general f , then we obtain $f_1(J(A_1, A_2)) = A_1$, as $I_1 = f_1^{-1}$. We will use a shorter notation : $f(J(A_1, A_2)) = f(A_1, A_2)$.

1.9. A functional independence .

Let us take two measurable functions $f_1 : \mathbb{S}_1 \mapsto \mathbb{T}_1$ and

$f_2 : \mathbb{S}_2 \mapsto \mathbb{T}_2$. We can define a function $f_1 \times f_2$: $\mathbb{S}_1 \times \mathbb{S}_2 \mapsto \mathbb{T}_1 \times \mathbb{T}_2$ by $f_1 \times f_2\,(x,y) = \big(f_1(x),\ f_2(y)\big)$, $x \in \mathbb{S}_1$, $y \in \mathbb{S}_2$. It is also ~~s~~ measurable. From JRV $J\,(A_1, A_2)$ we can now obtain a new classical random variable $J\,\big(f_1(A_1),\ f_2(A_2)\big) := J\,(A_1, A_2) \circ \big(f_1^{-1} \times f_2^{-1}\big)$. This property of JRVs and JPDs will be called the functional independence.

1.10. Determining sets.

The functional closeness (1.8.) of $\mathcal{O}_c$ leads to the following problem : find a set of functions from $\mathbb{S}_1 \times \mathbb{S}_2$ into $\mathbb{T}$, such that the corresponding set of probability measures $\alpha \circ f(A_1, A_2)$ on $\mathbb{T}$ determines α_{A_1, A_2}.

The classical probability theory offers examples of solution of this problem in special cases. Two of them are the following :

(i) if $\mathbb{S}_1 = \mathbb{S}_2 = \mathbb{T} = \mathbb{R}^1$ with the algebra of Borel sets, then all linear functions from $\mathbb{R}^2$ to $\mathbb{R}^1$ form such determining set for any probability measure on $\mathbb{R}^2$ (the theorem of Cramér and Wold, comp. Urbanik 1961).

(ii) with measurable spaces as above, the knowledge of mean values only for all monomials of two variables suffices to determine a probability measure on $\mathbb{R}^2$ (the theorem of moments).

1.11. A striking analogy.

In the formal apparatus of the elementary quantum theory one can distinguish a structure in some respects similar to $\mathcal{L}_c$. It is the "quantum logic" $\mathcal{L}_Q^{(*)}$, i.e. an ortho-modular σ-ortho-lattice, being an abstraction of the lattice of all projections acting on the separable Hilbert space, which underlies the quantum description (Jauch, 1968 ; Mackey, 1963). There is a striking analogy between the basic structure of the classical probability theory and the fundamentals of quantum mechanics viewed from the "quantum - logical" point of view. The basic notions of the classical probability theory, reviewed in 1.5., can be almost literally transferred into the quantum theory, just substituting $\mathcal{L}_Q^{(1)}$ in the place of $\mathcal{L}_c$. Thus the quantum probability is described by a σ-additive homomorphism α (state, probability measure on $\mathcal{L}_Q^{(1)}$) from $\mathcal{L}_Q^{(1)}$ into the real interval $\langle 0,1 \rangle$. A quantum random variable (observable) A, based on a measurable space $(\mathbb{S}, \mathcal{B}(\mathbb{S}))$ is a σ-additive homomorphism from $\mathcal{B}(\mathbb{S})$ into $\mathcal{L}_Q^{(1)}$ (generalized spectral measure, $\mathcal{L}_Q^{(1)}$-valued measure on $\mathbb{S}$). The composition $\alpha_A = \alpha \circ A$ is a <u>classical probability measure</u> on $\mathbb{S}$ (Mackey, 1963 ; Jauch, 1974).

<u>1.12. The Hilbertian realization.</u>

If we take the standard realization of $\mathcal{L}_Q^{(1)}$ as the lattice of orthogonal projections acting on a separable Hilbert space $\mathcal{H}$, then the set of quantum states $\mathcal{S}_Q^{(1)}$ becomes identical (the Gleason theorem) with the set of all linear self-adjoint non-negative trace class operators on $\mathcal{H}$ (statistical operators, density matrices). On the other hand, any quantum random variable based on $\mathbb{R}^1$ defines a self-adjoint linear operator on $\mathcal{H}$ (the spectral theorem). Also the Borel functions of $\mathbb{R}^1$-based quantum random variables are well defined in the Hilbertian frame, leading to the von Neumann - Stone functional calculus for self-adjoint operators .

1.13. Serious obstacle.

The close analogy of the quantum and classical probabilities cannot be carried too far . Serious obstacle appears just when we try to define quantum JRVs and quantum JPDs. Indeed , if we believe , that $\mathcal{L}_Q^{(1)}$ plays exactly the same role in the quantum probability theory as $\mathcal{L}_c$ does in the classical one , then we should define quantum JRV $J(A_1, A_2)$ as a random variable based on $\mathbb{S}_1 \times \mathbb{S}_2$, and such that $J(A_1, A_2)(X \times Y) = A_1(X) \wedge A_2(Y)$, $X \in \mathcal{B}(\mathbb{S}_1)$, $Y \in \mathcal{B}(\mathbb{S}_2)$. This is , however , impossible , because the set function $X \times Y \longmapsto A_1(X) \wedge A_2(Y)$ defined on $\mathcal{B}(\mathbb{S}_1) \times \mathcal{B}(\mathbb{S}_2)$, cannot be , in a general case , extended to an $\mathcal{L}_Q^{(1)}$-valued measure (see 2.3) . Thus we come up against the problem of quantum JPDs .

1.14. Compatibility .

We call two elements a, b of $\mathcal{L}_Q^{(1)}$ compatible , and denote $a \sim b$, if there exist three elements c , a_1 , b_1 such that : $c \leqslant a_1'$, $c \leqslant b_1'$, $a_1 \leqslant b_1'$, $a = a_1 \vee c$, $b = b_1 \vee c$. Two quantum random variables A_1 and A_2 are compatible iff every element of $A_1(\mathcal{B}(\mathbb{S}_1))$ is compatible with every element of $A_2(\mathcal{B}(\mathbb{S}_2))$. This definition generalizes the commutativity of operators on $\mathcal{H}$. Two compatible quantum random ~~variables are~~ compatible in the sense of 1.6. and vice-versa, so the problem of quantum JPDs arises because of the appearance of incompatible quantum random variables , i.e. because of the non-Boolean character of $\mathcal{L}_Q^{(1)}$.

The common attitude towards the problem of quantum JPDs accepts the impossibility to define quantum JRVs in the classical sense of 1.6. From this point of view the lattice structure of $\mathcal{L}_Q^{(1)}$ is unessential : to define all the basic notions it suffices to take an ortho-modular σ-ortho-p.o.-set , say $\mathcal{L}_Q^{(2)}$ (compare Mackey , 1963), instead of $\mathcal{L}_Q^{(1)}$. The approach based on $\mathcal{L}_Q^{(2)}$ generalizes the proceeding one , for there are ortho-modular σ-ortho-p.o.-sets which are not lattices .

2. ARE QUANTUM JOINT PROBABILITIES POSSIBLE ?

2.1. Is the quantum mechanics possible ?

There is no theorem to state that quantum JPDs for incompatible quantum random variables do exist . But there is a lot ones claiming that quantum JPDs cannot exist . A general scheme of such a theorem is the following : one assumes some of the listed above (see 1.6 to 1.10) properties of classical JPDs to hold for hypothetical quantum JPDs, and then demonstrates that the assumed properties cannot be satisfied in the frame of standard (Hilbertian) quantum mechanics . Such arguments hardly tell something concerning the very question of existence of quantum JPDs . They tell merely which classical properties cannot be ascribed to quantum JPDs . In the same manner one could argue , that the quantum mechanics is impossible , starting from observation , that some of properties of classical mechanics (e.g. orthogonality of any two pure states) do not hold therein .

2.2. Two properties .

Throughout this Section we assume the two following properties of quantum JPDs :

(i) α_{A_1,A_2} is a probability measure on the corresponding product space $\mathbb{S}_1 \times \mathbb{S}_2$.

(ii) $\alpha_{A_1,A_2}(X \times \mathbb{S}_2) = \alpha_{A_1}(X)$,
$\alpha_{A_1,A_2}(\mathbb{S}_1 \times Y) = \alpha_{A_2}(Y)$, $X \in \mathcal{B}(\mathbb{S}_1)$, $Y \in \mathcal{B}(\mathbb{S}_2)$
(the marginal property of 1.7).

All the considerations below concern $\mathbb{R}^1$-based random variables , many of them can be proved only for the Hilbertian realization of $\mathcal{L}_Q^{(4)}$ (or $\mathcal{L}_Q^{(3)}$) . This restriction do not limit essentially a generality of results , because for our purposes it suffices to demonstrate that some of classical properties cannot be stisfied by quantum JPDs at least for one triple A_1 , A_2 , α in one concrete realization of $\mathcal{L}_Q^{(4)}$.

2.3. Can quantum JRVs exist at all ?

Let us begin with a closer examination of the question if quantum JRVs can exist at all .

Returning to the remark in 1.13 , it is indeed impossible to find $J(A_1,A_2) \in \mathcal{O}_Q^{(4)}$ with property : $J(A_1,A_2)(X \times Y) = A_1(X) \wedge A_2(Y)$ for $X , Y \in \mathcal{B}(\mathbb{R}^1)$, when A_1 , A_2 are incompatible . It would imply , namely , that $[A_1(X) \wedge A_2(Y)] \vee [A_1(X) \wedge A_2(\mathbb{R}^1 \setminus Y)] \vee [A_1(\mathbb{R}^1 \setminus X) \wedge A_2(Y)] \vee [A_1(\mathbb{R}^1 \setminus X) \wedge A_2(\mathbb{R}^1 \setminus Y)] = e$ (the maximal element of $\mathcal{L}_Q^{(4)}$) , what can be achieved only if $A_1(X) \sim A_2(Y)$ (for a proof see e.g. Piron, 1976 , p. 25) . Thus if there exist a quantum random variable $J(A_1,A_2)$ on $\mathbb{R}^2$ such that $J(A_1,A_2)(X \times Y) = A_1(X) \wedge A_2(Y)$

for any $X, Y \in \mathcal{B}(\mathbb{R}^1)$, then $A_1 \sim A_2$ (compare e.g. Gudder 1968a, Theorem 3.1 or Jauch 1974).

So let us reject the considered property of quantum JRVs and try to find $J(A_1, A_2)$ just among $\mathbb{R}^2$-based quantum random variables. Unfortunately, we fall at once into conflict with the rather natural marginal property, since it is easy to demonstrate, that the assumption : $J(A_1, A_2) \in \mathcal{O}_Q^{(1)}$ together with the marginal property imply $A_1 \sim A_2$ (see e.g. Gudder 1968a, Lemma 2.1. and Varadarajan 1962, Theorem 3.4.) Thus we see, that quantum JRVs cannot be quantum variables (in the sense of 1.11).

2.4. Extending $\mathcal{O}_Q^{(1)}$.

It becomes clear from now, that any further attempt to disprove the hypothesis of existence of quantum JPDs, as well as any attempt to find them, must initially assume a generalization of the standard structure of quantum mechanics, at least by extending $\mathcal{O}_Q^{(1)}$, because JPDs without JRVs look rather queer.

Thus ~~e.g.~~ one can hope to avoid the contradiction of 2.3 relaxing assumptions concerning $J(A_1, A_2)$ (so going beyond $\mathcal{O}_Q^{(1)}$). If we take, however, into account the general assumption of this Section, that $\alpha \circ J(A_1, A_2)$ is a probability measure on $\mathbb{R}^2$ for any $\alpha \in \mathcal{S}_Q^{(1)}$, then it appears that we are still uncomfortably close to the case $J(A_1, A_2) \in \mathcal{O}_Q^{(1)}$. For example, if we assume, that $J(A_1, A_2)$ defines an $\mathcal{L}_Q^{(1)}$-valued function on Borel rectangles on $\mathbb{R}^2$, such that $\alpha \circ J(A_1, A_2)$ generates a probability measure on $\mathbb{R}^2$, then $J(A_1, A_2)$ must essentially belong to $\mathcal{O}_Q^{(1)}$. This observation together with the marginal property for $J(A_1, A_2)$ can serve as a basis for further "impossibility proofs" for quantum JPDs .

2.5. Correspondence rules .

The main idea of another kind of theorems concerning quantum JPDs originates from the known property of classical JPDs , mentioned in 1.10. Thus , if we only could guess $f(A_1, A_2)$ for all functions from some determinig set and for every pair A_1 , A_2 of quantum random variables , then the problem of JPDs would be solved .

Any postulate , which defines quantum random variables $f(A_1, A_2) : \mathcal{B}(\mathbb{R}^1) \longmapsto \mathcal{L}_Q^{(1)}$ for a given pair A_1 , A_2 of $\mathbb{R}^1$-based quantum random variables and for a determining set of Borel functions from $\mathbb{R}^2$ to $\mathbb{R}^1$, is called traditionally a correspondence rule . As an example of such a rule can serve the known von Neumann's additivity postulate (von Neumann 1955 , see also Park and Margenau , 1968 , for a critical analysis), other examples are provided in numerous literature on this subject (see e.g. Shewell,1959, Margenau and Cohen, 1967, and others) .

2.6. The von Neumann's rule .

There is a nice sequence of theorems by von Neumann (1955) , Urbanik (1961) , Varadarajan (1962 , Proposition 4.2) and Nelson (1967) based on the von Neumann's correspondence rule . This rule , motivated presumably by a suggestive but misleading analogy , identifies the additivity (in the sense of 1.8) of quantum random variables with the additivity of corresponding operators in the Hilbertian representation . Starting from this assumption one can hope to deduce α_{A_1,A_2} from all $\alpha\circ(\lambda A_1 + \mu A_2)$ with $\alpha\in\mathcal{S}_Q^{(1)}$, $\lambda,\mu\in\mathbb{R}^1$, the addition corresponding to the operator one. It is not surprising, that such obtained α_{A_1,A_2} is a probability measure for any $\alpha\in\mathcal{S}_Q^{(1)}$ only if $A_1\sim A_2$. In the other case it is overdefined, as can be checked on elementary examples, and does not exist.

A proof of this theorem is simple for A_1, A_2 with purely discrete spectra, and is based on observation (Urbanik, 1961), that in this case one can always find a linear function from $\mathbb{R}^2$ to $\mathbb{R}^1$ which separates all points of the direct product of spectra of A_1 and A_2. Once such a function is done, we find easily, that there must exist a mapping of $\mathcal{B}(\mathbb{R}^2)$ into $\mathcal{L}_Q^{(1)}$, and the argument of 2.4 ends the proof. In the case of A_1, A_2 with continuous spectra a proof is more complicated (see e.g. Varadarajan, 1962) but in some sense superfluous, as the self-contradictory character of the von Neumann's rule (together with the two assumptions of 2.2) is already evident.

2.7. Correspondence rules are impossible.

The most known correspondence rules concern position and momentum variables, and take into account the class $\{x^n y^m ;\ n,m = 0,1,2,\ldots\}$ of Borel functions. This choice of quantum random variables is caused by a tendency to find a phase-space description of quantum phenomena, whereas the specific class of functions follows just from the classical theorem of moments. In order to determine JPDs it is necessary, however, to have a correspondence rule working for any pair of quantum random variables, at least for $\mathbb{R}^1$-based ones. We will prove, that any such "universal" correspondence rule is impossible .

For this purpose let us take two incompatible elementary random variables $a, b \in \mathcal{L}_Q^{(1)}$, such that the corresponding projectors P_a and P_b on the Hilbert space $\mathcal{H}$ project on one-dimensional subspaces P_a^1 and P_b^1 respectively . Any correspondence rule of the mentioned type assumes then , that there is a linear operator , say G , such that $\alpha_{a,b}(1,1) = \mathrm{Tr}(\varrho_\alpha G)$ for any $\alpha \in \mathcal{S}_Q^{(1)}$, where ϱ_α is the statistical operator corresponding to α . We will demonstrate , that restrictions imposed on G by the two conditions of 2.2 are strong enough to make the existence of such G impossible .

Because of $0 \leqslant \alpha_{a,b}(1,1) \leqslant 1$, the operator G is self-adjoint , with spectrum inside of the unit interval $\langle 0,1 \rangle$. Moreover , (i) and (ii) of 2.2 imply that if $\alpha(a) = 0$ or $\alpha(b) = 0$ then $\alpha_{a,b}(1,1) = 0$, so the point 0 belongs to the discrete part of the spectrum of G, and P_a^0 , P_b^0 are contained in G^0 (eigen-subspaces corresponding to the eigen-value 0). Of course , the closed linear span $P_a^0 \oplus P_b^0$ of P_a^0 and P_b^0 is also contained in G^0 . On the other hand , the two-dimensional subspace $P_a^1 \oplus P_b^1$ contains one-dimensional subspaces $P_a^0 \cap (P_a^1 \oplus P_b^1)$ and $P_b^0 \cap (P_a^1 \oplus P_b^1)$. These subspaces differ from P_b^1 and P_a^1 as a and b are incompatible , nevertheless $P_a^1 \oplus P_b^1 = \left[P_a^0 \cap (P_a^1 \oplus P_b^1)\right] \oplus \left[P_b^0 \cap (P_a^1 \oplus P_b^1)\right]$. This means , that $P_a^1 \oplus P_b^1$ is also contained in G^0 and we obtain a contradiction , because the two assumptions of 2.2 imply that if e.g. $\alpha(a) = 1$ then $\alpha_{a,b}(1,1) = \alpha(b)$.

2.8. What exactly has been proved ?

What exactly has been proved in 2.7 is : any correspondence rule concerning a function $f(x,y)$ such that $f(0,0) = f(0,1) = f(1,0) = 0$, and applicable to elementary random variables , contradicts (i) and (ii) of 2.2.

Observe also , that the argument of 2.7 applies to the von Neumann's correspondence rule . It applies also to any conjecture , that quantum JPDs , satisfying (i) and (ii) of 2.2 could be obtained from linear operators , hence provides a simplified version of the Wigner's theorem on quantum JPDs (Wigner , 1971) . And it is also closely connected with the known theorem of Cohen (Margenau and Cohen , 1967; Srinivas and Wolf , 1975 ; Cartwright , 1974 ; Cohen , 1966). Let us stress once more , that results of this kind do not prove the alleged "impossibility" to define quantum JPDs , but demonstrate merely , that the properties listed in 2.2 are incompatible with correspondence rules .

By the way let us remark , that correspondence rules are by no means such "natural" and plausible , as it is often stated on the ground , that "quantum mechanics is committed to representing such quantities by operators" (Cartwright , 1974 , p.131) . Actually just this assumption, i.e. the correspondence between quantum measurable quantities and $\mathcal{L}_Q^{(1)}$-valued measures is to be broken when we are going to speak about correspondence rules . Indeed , if we want to proceed exactly along orthodox lines , we must assume the existence of quantum JRVs , what contradicts (i) and (ii) of 2.2 (see 2.3) . The whole idea of correspondence rules was invented to overcome this trouble and to find quantum JPDs in an indirect way. If we try to built up something like the quantum JRV on the basis of any correspondence rule, we find a semi-spectral measure, which is beyond the standard formalism. Actually, the proof above as well as the theorem of Cohen, state that this generalization of the notion of quantum random variable also contradicts (i) and (ii) of 2.2.

2.9. Another obstacle.

Another obstacle we meet if we assume the "fuctional independence" (1.9) of classical JPDs to hold in the quantum case. We cannot formulate this property by means of a JRV because of the proceeding results, but only as a property of JPDs : $\alpha_{f_1(A_1),f_2(A_2)}(X \times Y) = \alpha_{A_1,A_2}(f_1^{-1}(X) \times f_2^{-1}(Y))$ for any Borel functions f_1 and f_2, any $\mathbb{R}^1$-based quantum random variables A_1 and A_2, and for any X, $Y \in \mathcal{B}(\mathbb{R}^1)$. This property is especially plausible from the point of view of the standard quantum mechanics, as functions of self-adjoint operators are well defined.

Let us take an elementary random variable $a \in \mathcal{L}_Q^{(1)}$ and a state $\alpha \in \mathcal{S}_Q^{(1)}$. To any $b \in \mathcal{L}_Q^{(1)}$ corresponds now a number $\alpha(a)^{-1}\alpha_{a,b}(1,1) \in \langle 0,1 \rangle$, and this correspondence is a probability measure on $\mathcal{L}_Q^{(1)}$, as can be easily demonstrated. Indeed, let a_1, a_2, ... be any countable sequence of pairwise orthogonal elements of $\mathcal{L}_Q^{(1)}$. There is a quantum random variable, say A, such that this sequence belongs to $\tilde{A}(\mathcal{B}(\mathbb{R}^1))$. From the assumed functional independence we conclude that if e.g. $A(X) = a_1$, then $\alpha_{a,A}(\{1\} \times X) = \alpha_{a,a_1}(1,1)$ and so on. Thus if we take into account that quantum JPDs should be classical probability measures , we see that $\alpha(a)^{-1}\alpha_{a,b}(1,1)$ does belong to $\mathcal{S}_Q^{(1)}$. Let us denote this state by $\alpha_{(a)}$. It is easy to check , that if $\alpha(b) = 1$ then $\alpha_{(a)}(b) = 1$. Now if we take the special case of the Hilbertian representation of $\mathcal{L}_Q^{(1)}$, then it becomes evident , that if α is a pure state then $\alpha_{(a)} = \alpha$, and $\alpha_{a,b}(1,1) = \alpha(a)\,\alpha(b)$. Applying again the functional independence we reach the conclusion that the quantum JPD for a pure state should be a product distribution (Bugajski , 1976). The same conclusion could be obtained also in more general schemas , if only $\mathcal{L}_Q^{(1)}$ and $\mathcal{S}_Q^{(1)}$ would be equipped in some additional properties .

So we have obtained the following result : the properties (i) and (ii) of quantum JPDs together with the "independence" property eliminate all possibilities for JPDs except the trivial product distributions .

2.10. Trivial solutions .

We see , that no one of simple properties of classical JPDs can be added to the assumed in 2.2 ones . The properties of 2.2 alone can be satisfied in plenty of manners , for example by the trivial product distributions , or by equally trivial formula : α_{A_1,A_2} is the product distribution for incompatible quantum random variables , and the standard distribution of 1.14 for compatible ones . If we realize , that even these trivial solutions of the problem of quantum JPDs demand an extension of the quantum formalism (see 2.4) , then they become unacceptable , for one can hope that going beyond the standard formalism one can find more interesting solutions .

2.11. A tempting idea .

So we should frankly recognize , that $\mathcal{L}_Q^{(1)}$ and the whole standard formalism are too narrow to embody a quantum probability theory with JPDs (this was in fact assumed in 2.4 - 2.9). A tempting idea is to enlarge $\mathcal{L}_Q^{(2)}$ (we will use $\mathcal{L}_Q^{(2)}$ instead of $\mathcal{L}_Q^{(1)}$ because the lattice properties of the later are now unessential) up to a Boolean σ-algebra and try to define quantum JPDs in this context . So let us assume , that quantum theory is in fact described by means of some Boolean σ-lattice , say $\mathcal{L}_Q^{(3)}$, which plays the role of $\mathcal{L}_Q^{(2)}$. In order to assure a correspondence with the previous description we should have a mapping ω of $\mathcal{L}_Q^{(2)}$ into $\mathcal{L}_Q^{(3)}$. With such a mapping we can define an $\mathcal{L}_Q^{(3)}$-valued function on a measurable space $(\mathbb{S},\mathcal{B}(\mathbb{S}))$ related to an $\mathbb{S}$-based quantum random variable $A^{(2)}$, by: $A^{(3)} = \omega \circ A^{(2)}$, as well as a family of real functions on $\mathcal{L}_Q^{(3)}$ corresponding to a quantum state $\alpha^{(2)}$, by : $\alpha^{(2)} = \alpha^{(3)} \circ \omega$.

Observe , that $\alpha^{(2)}_{A^{(2)}} = \alpha^{(3)}_{A^{(3)}}$, and $f(A^{(3)}) = \omega \circ f(A^{(2)})$ with any measurable function f , so the necessary conditions for a "smooth" extension of the standard quantum formalism are fulfilled . It means , that what we look for , is not a new theory , but only an extended formalism , fully compatible with the standard one . Both above relations appear to be also necessary conditions of Kochen and Specker (1967) for a successful introduction of a hidden variable theory . With one reservation : it is still not decided if $A^{(3)}$ and $\alpha^{(3)}$ belong to $\mathcal{O}_c^{(3)}, \mathcal{S}_c^{(3)}$ respectively .

2.12. 0,1-states .

If we want to have a kind of a hidden variables theory we have to assume $\omega \circ A^{(2)} \in \mathcal{O}_c^{(3)}$ for any $A^{(2)} \in \mathcal{O}_Q^{(2)}$, i.e. that $A^{(3)} = \omega \circ A^{(2)}$ is a classical random variable on $\mathcal{L}_Q^{(3)}$. It is impossible , however , as can be easily demonstrated , because then ω is forced to be a σ-additive homomorphism of ortho-p.o.-sets , and a restriction to $\mathcal{L}_Q^{(2)}$ of any state $\alpha^{(3)} \in \mathcal{S}_c^{(3)}$, defined by : $\alpha^{(2)} = \alpha^{(3)} \circ \omega$, is a state on $\mathcal{L}_Q^{(2)}$. Any Boolean σ-lattice has a separating set of 0,1-states (σ-additive homomorphisms from the lattice to the two--element Boolean lattice), so $\mathcal{L}_Q^{(2)}$ must have such a set as well . But it contradicts the Gleason theorem , so we conclude , that $\omega \circ A^{(2)}$ cannot generally be a classical random variable . Some "anti-hidden variables" theorems rest on these simple facts and follow this line of reasoning (Kochen and Specker , 1967 ; Bugajska and Bugajski , 1972). Some other such theorems make more or less implicit assumption that ω is a σ-additive homomorphism and demonstrate then that $\mathcal{L}_Q^{(2)}$ cannot possess "too many" 0,1-states . It concerns e.g. results of Jauch and Piron (1967) , Gudder (1968b) , Zierler and Schlessinger (1965).

The failure of introducing the hidden variables theory with ω being a σ-additive homomorphism can be also viewed as an impossibility of solving the problem of quantum JPDs in purely classical terms . Proposed models of hidden variables violate in an essential way the requirements of 2.11 and cannot help to solve our problem . Thus in the model of Ochs (1970 , 1971) there is no mapping as the correspondence $A^{(2)} \mapsto A^{(3)}$ is one-to-many , whereas in the model of Bohm and Bub (1966) there is no $\mathcal{L}_Q^{(3)}$ at all (Gudder , 1970).

2.13. Concluding remarks .

We have discussed all theorems which allegedly preclude quantum JPDs from the existence . Any of such theorems complements properties of 2.2 with another ones and obtains a contradiction . It restricts seriously possible properties of quantum JPDs , leading to trivial solutions .

On the other hand , such properties as e.g. the existence of quantum JRVs (2.3) or the functional independence of quantum random variables (2.9) are so natural , that their absence could cause a desperate abandonment of quantum JPDs at all . Observe , however , that in all contradictory sets of properties discussed above appears the common part : properties (i) and (ii) of 2.2. It becomes evident that one of them is responsible for all these failures , contrary to often expressed views , that the guilty is one of correspondence principles (Park and Margenau , 1968). Indeed , in none attempt to introduce quantum JPDs these two properties hold, what is demonstrated in the next Section .

3. HAVE BEEN QUANTUM JOINT PROBABILITIES FOUND ?

3.1. Two groups .

There is a good amount of proposals to complement quantum mechanics with something , which could be called a quantum JPD . Any one of them rejects one of assumptions of 2.2. , except the conjecture of Margenau and Hill (1961) that all quantum JPDs are the product ones . This trivial solution cannot be , however , seriously taken for granted . Other proposals can be schematically divided into the first and the second groups , rejecting the first or the second property of 2.2. respectively . Let us make some remarks on them .

~~3.2. Inner measures .~~

The simplest non-trivial proposal was done by Jauch , who at the end of his paper (Jauch, 1974) remarked , that the function $\alpha\circ\left(A_1(X)\wedge A_2(Y)\right)$ for $X , Y \in \mathcal{B}(\mathbb{R}^1)$, $A_1 , A_2 \in \mathcal{O}_Q^{(1)}$ and $\alpha \in \mathcal{S}_Q^{(1)}$ "represents in fact the probability that in a given state the variable A_1 assumes values in the set X while at the same time the variable A_2 assumes values in the set Y" . This supposition enables us to define merely an $\mathcal{L}_Q^{(1)}$-valued function on Borel rectangles on $\mathbb{R}^2$, which cannot be (if A_1 , A_2 are incompatible) extended to a measure (see 2.3.) . This function can be , however , considered as an $\mathcal{L}_Q^{(1)}$-valued inner measure on $\mathbb{R}^2$, restricted to Borel rectangles . So we see , that the Jauch proposal implies a broadening of $\mathcal{O}_Q^{(1)}$ to the set of all inner $\mathcal{L}_Q^{(1)}$-valued measures , say $\mathcal{O}_Q^{(1)}$ (the sets $\mathcal{L}_Q^{(1)}$ and $\mathcal{S}_Q^{(1)}$ remain unchanged). The resulting JPDs for incompatible random variables are normed inner measures , whereas for compatible ones conform the standard quantum rule (1.14.) .

Let us observe , that if one is convinced , that $\mathcal{L}_Q^{(1)}$ with its lattice structure is indeed the quantum counterpart of $\mathcal{L}_C$, then the Jauch's approach appears to be the only possible .

Because the usual measures are just a special kind of inner measures , the set $\mathcal{O}_Q^{(1)}$ of standard random variables is contained in $\mathcal{O}_Q^{(4)}$. This means that the Jauch's broadening of the standard formalism does not affect any numerical results of it .

3.3. Too empty .

Instead of analysing further the resulting generalization of classical probability theory , we point out some its feature , which makes the whole approach doubtful .

It is rather easy to see , that the quantum JPDs appearing in the Jauch's theory are in some sense "too empty" , or that they map "too many" Borel rectangles on 0 . Jauch himself was aware of it , and demonstrated it on an example of two random variables forming a Weyl pair (e.g. quantum position and momentum variables for a free particle). For such a pair A_1 , A_2 Jauch proves , that $A_1(X) \wedge A_2(Y) = o$ (the least element of $\mathcal{L}_Q^{(4)}$) for any X , Y $\in \mathcal{B}(\mathbb{R}^1)$ with finite Lebesgue measure . It is hard to reconcile this fact with the Jauch's statement quoted in 3.2. , for it would mean , loosely speaking , that quantum particles have no position and momentum at all .

3.4. A new condition .

This unplausible feature of the Jauch's proposal can be demonstrated in a more persuasive manner . Let us take a pure state $\alpha \in \mathcal{S}_Q^{(4)}$ and assume , that there is one and only one atom of $\mathcal{L}_Q^{(4)}$, say a , with property : $\alpha(a) = 1$. (This is evidently true in the Hilbertian representation).

~~Then~~

Then we consider a quantum random variable A_1 such that $a \in A(\mathcal{B}(\mathbb{R}^1))$. It can be proved, that $A_1^{-1}(a)$ is a point of $\mathbb{R}^1$, say x. Thus if we produce the state α in a laboratory, we know a priori that the value of A_1 in α is exactly x, or that the probability measure α_{A_1} is concentrated in x. Having the state α produced we can measure any other quantum random variable, say A_2, with resulting probability measure α_{A_2} on $\mathbb{R}^1$. This a little trivial example can be viewed as a joint measurement of A_1 and A_2 in α with resulting quantum JPD $\alpha_{A_1 A_2}$ concentrated on $\{x\} \times \mathbb{R}^1 \in \mathcal{B}(\mathbb{R}^2)$ and equal α_{A_2} on it. This scheme can be evidently generalized to any state α and any measureble subset X of $\mathbb{R}^1$. Hence it is natural to assume the following restriction on quantum JPDs:

if α_{A_1} is concentrated on $X \in \mathbb{R}^1$, then $\alpha_{A_1 A_2}(X \times Y) = \alpha_{A_2}(Y)$ for every $Y \in \mathcal{B}(\mathbb{R}^1)$.

This condition is not satisfied by the Jauch's quantum JPDs. Indeed, it leads to an implication:

$$\alpha(a) = 1 \implies \alpha(a \wedge b) = \alpha(b) \text{ for any } \alpha \in \mathcal{S}_Q^{(1)},\ a, b \in \mathcal{L}_Q^{(1)}.$$

In the Hilbertian representation it means that $(a \wedge b)' \wedge b$ is orthogonal to a, so a and b are compatible.

It can be argued, that the above scheme does not provide a joint measurement of A_1 and A_2, because of the time assymetry of the laboratory operations corresponding to the measurements of A_1 and A_2. Similar arguments are set forth since years against the time-of-flight joint measurements (compare a discussion of Park and Margenau, 1968, ~~in [20]~~ pp. 242-244). This kind of arguments does not destroy our example, as essentially the same experiment could be realized by a correlation-type arrangement , with exactly simultaneous measurement of B_1 (a random variable correlated to A_1) on one of correlated subsystems , and A_2 on the other . Indeed , one of the "paradoxical" aspects of the Einstein , Podolsky and Rosen experiment is the possibility to obtain simultaneous values of two incompatible quantum random variables in a state beeing a mixture of eigen-states of one of them (see e.g. (Park and Margenau, 1968, ~~[25]~~ pp. 244-245).

Thus the above condition seems to be substantiated , what is an argument against the Jauch's proposal .

3.5. Negative probabilities .

There is a class of proposals which violate (i) of 2.2. in a much more drastic manner then the Jauch's approach . I mean this quantum probability theories , which assume JPDs to be signed measures , as for example the known Wigner-Moyal approach ((Moyal, 1949) ~~[24]~~ , generalized to a whole family of quantum phase-space descriptions ((Srinivas and Wolf, 1975, Margenau and Cohen, 1967) ~~[36, 24]~~ . The lack of positive semidefiniteness cannot be reconciled , however , with fundamental intuitions about the notion of probability . It is a sufficient reason to disregard them as a possible candidates for a quantum probability theory with JPDs . Nevertheless we do not deny their practical usefulness .

On the same shortcoming suffers the Prugovečki (1966)) idea ~~[32]~~ to introduce a complex probability calculus , generalizing the standard quantum theory . The real parts of his complex measures , intended to represent quantum JPDs , take on negative values as well ((Prugovečki, 1967) ~~[33]~~ . This makes the whole approach unacceptable , despite its other interesting features .

3.6. Coherent states .

It is sometimes stated [34] , that the standard quantum mechanics is not able to describe joint measurements . Similar conclusion could be drawn out from considerations of Section 2 . There is , however , an original attempt to formulate a theory of joint measurements (and JPDs) of incompatible random variables in a manner strictly paralell to the orthodox theory of single measurements and in terms of the standard formalism . It is a papper of She and Heffner [34] (1966) . Surely one of properties listed in 2.2. have to be rejected , and in this case it is the marginal property which does not hold .

Like the phase-space descriptions , She and Heffner concentrate their attention on a Weyl pair of quantum random variables , say position q and momentum p of a free particle , and want to find their JPDs for any state α. It appears that there is many JPDs corresponding to a given $\alpha \in \mathcal{S}_Q^{(1)}$ and fixed quantum variables q and p , all having a physical interpretation . The simplest among them is a result of an "ideal measurement" and is defined by the density function $P(x,y) = \frac{1}{\hbar} (\psi_{x,y}, \varrho_\alpha \psi_{x,y})$ on $\mathbb{R}^2$, with ϱ_α - the statistical operator corresponding to α in the Hilbertian representation , $\psi_{x,y}$ - coherent state of Glauber . Now it can be seen on simple examples , that if α_q is concentrated on an interval $\langle x_1 , x_2 \rangle \subset \mathbb{R}^1$ then $P(x,y)$ fails to be concentrated on the stripe $\langle x_1, x_2 \rangle \times \mathbb{R}^1$, contrary to the marginal property . The property discussed in 3.4. also does not hold , as a consequence of it .

3.7. Conditional distributions .

Every of the above approaches could be probably developed up to form a generalized probability theory , including the quantum probability theory described in Section 1 and assuming JPDs for uncompatible random variables . However the mentioned unplausible features of them discourage from this .

It should be also noted that the term "JPD" appears in the "operational" approach to quantum probability (Davies and Lewis, 1970, Davies, 1970, Ingarden) ~~[9, 8, 11]~~ . This very elegant and full-fledged theory generalizes in a "smooth" manner the standard quantum mechanics , both making precise and solving problems related to quantum conditioning . From the formal point of view the JPDs in this theory are probability measures with the marginal property satisfied only for one of involved variables . It is easy to find , that they result from manifestly sequential operations , hence it should be better to call them conditional probability distributions .

4. A PROPOSAL .

4.1. A question , and an answer .

I hope , that points (i) and (ii) of 1.1. are now fully demonstrated , and the field for speculations about quantum JPDs and a full quantum probability theory is free.

Perhaps it is not too late for a reflection over the basic question : why to seek a full quantum probability theory which could include JPDs for incompatible variables ?

The most natural answer is : just because such a theory is not forbidden . More precisely : because there is no convincing argument against the possibility of such a theory .

4.2. Elementary cases .

The lack of conclusive arguments against is not the only support of the tendency to formulate a full quantum probability theory . There are also at hand some other reasons to do this .

To begin with , let us observe , that the prevailing quantum dogma : incompatibility means that the joint measurement is impossible , is impaired by elementary cases of incompatible quantum random variables having jet comm proper states , hence well defined JPDs in these states .

Less obvious examples of well defined JPDs for incompatible variables are provided by the Wigner-Moyal distributions when they are non-negative . Combining results of Urbanik ~~[37]~~ (1964) and Hudson ~~[15]~~ (1974) we see , that it is achieved only when they are product distributions (in accordance with the general result of 2.9.) .

4.3. Thought experiments .

There is many non-formal "anti-JPDs" arguments based on various thought experiments . Their only value is that they encourage attempts to find a proof of the impossibility of quantum JPDs . On the other hand , agreeing that such experiments prove nothing , one could invent "pro-JPDs" thought experiments and find them encouraging to look for a full quantum probability theory . Such experiments has been still invented (compare e.g. Ballentine, 1970, Park and Margenau, 1968, Prugovečki, 1969 ~~[1, 29, 33]~~). The most famous of them are : the Einstein , Podolsky and Rosen - type experiments (see 3.4.) and the time-of-flight measurements . They two have at least one advantage over the celebrated Heisenberg γ-ray microscope and Bohr moving screens : they are just since years realized in laboratories .

4.5. Aesthetic reasons .

The next reasons to seek a new extended version of the quantum probability theory are of an aesthetic nature. Simply , a probability theory (even a generalized one) without joint probabilities looks uncomely , as well as a propositional calculus without conjunction .

More seriously : the development of quantum probability theory was stopped almost at the starting point , because such notions , as quantum JPDs and quantum conditioning , necessary for any further progress , could not be defined . The generalization of quantum mechanics made by the operational approach provides an encouraging example of overcoming these obstacles . What prevents us to make a next step and look for a generalization admitting JPDs?

My last argument for quantum JPDs is the most doubtful one . It is based on two unpopular premises : quantum mechanics defines not only a probability calculus of its own , but also a propositional calculus of its own ; and: there is a strong connection between probability and logic with one of connecting bridges provided by the known from classical theories interrelation between JPDs and conjunction (compare also the Jauch's proposal , 3.2.) . It is not an appropriate place here to defend these statements , let us only remark , that the first thesis gains more and more adherents among logicians and philosophers , while the second is evidently true in the classical case . Once we are convinced in them , the need for quantum JPDs becomes obvious .

4.6. A more cautious manner .

Let us now sketch briefly how a new quantum probability theory could be constructed . We start from $\mathcal{L}_Q^{(2)}$, $\mathcal{O}_Q^{(2)}$ and $\mathcal{S}_Q^{(2)}$. Considerations of Section 2 suggest that a successful introduction of JPDs into the quantum theory can be achiewed only by extending of $\mathcal{L}_Q^{(2)}$ to an ortho-modular lattice , say $\mathcal{L}_Q^{(5)}$. The extension , $\omega : \mathcal{L}_Q^{(2)} \longmapsto \mathcal{L}_Q^{(5)}$, may not lose identity of elements of $\mathcal{L}_Q^{(2)}$, so we must assume that $\omega(a) = \omega(b)$ implies $a = b$ for any $a, b \in \mathcal{L}_Q^{(2)}$.

Now if $A^{(2)}$ is a random variable , then $\omega \circ A^{(2)} = A^{(5)}$ should be a random variable too . We cannot , however , define random variables related to $\mathcal{L}_Q^{(5)}$ as a σ-additive homomorphisms , because then ω is forced to a σ-embedding and we fall into the discussed case of $\mathcal{L}_Q^{(4)}$, $\mathcal{O}_Q^{(4)}$. Hence we must define random variables on $\mathcal{L}_Q^{(5)}$ in a more cautious manner , as a homomorphisms from a measurable space into

$\mathcal{L}_Q^{(5)}$. In other words, a random variable $A^{(5)}$ based on $\mathbb{S}$ appears to be an inner measure on $\mathbb{S}$ with values in $\mathcal{L}_Q^{(5)}$. It implies that ω is an embedding.

Once ω is only an embedding (and not a σ-additive embedding), any state $\alpha^{(2)} \in \mathcal{S}_Q^{(2)}$ defines a homomorphism of $\omega(\mathcal{L}_Q^{(2)})$ into $\langle 0,1\rangle$. We assume that it can be extended into an outer measure on $\mathcal{L}_Q^{(5)}$. Thus the set $\mathcal{S}_Q^{(5)}$ will consist of normed outer measures on $\mathcal{L}_Q^{(5)}$ such that $\alpha^{(5)} \circ \omega \in \mathcal{S}_Q^{(2)}$ for any $\alpha^{(5)} \in \mathcal{S}_Q^{(5)}$.

Like 2.11. we have : $(\alpha^{(5)} \circ \omega)_{A^{(2)}} = \alpha_{\omega \circ A^{(2)}}$, and : $\omega \circ f(A^{(2)}) = f(\omega \circ A^{(2)})$. Thus, in spite of the unusual features of random variables and states we have defined on $\mathcal{L}_Q^{(5)}$, there is a proper connection with the standard theory, based on $\mathcal{L}_Q^{(2)}$.

4.7. Outer measures.

As concerns JPDs we assume, that for any two $A_1^{(5)}$, $A_2^{(5)} \in \mathcal{O}_Q^{(5)}$, the JRV $J^{(5)}(A_1^{(5)}, A_2^{(5)})$ is the inner $\mathcal{L}_Q^{(5)}$-valued measure on $\mathbb{S}_1 \times \mathbb{S}_2$ generated by the set function which takes $X \times Y$ into $A_1^{(5)}(X) \wedge A_2^{(5)}(Y)$ for any $X \in \mathcal{B}(\mathbb{S}_1)$, $Y \in \mathcal{B}(\mathbb{S}_2)$. The JPD related to this JRV is defined in the usual way : $\alpha^{(5)}_{A_1^{(5)} A_2^{(5)}} = \alpha^{(5)} \circ J^{(5)}(A_1^{(5)}, A_2^{(5)})$ and appears to be an outer measure on $\mathbb{S}_1 \times \mathbb{S}_2$. Thus we see that the first assumption of 2.2. is broken. It is not surprising, for our theory is not classical, so it should be expected that some of classical properties will not be preserved.

The possibility of defining JRV for any pair of variables was achiewed owing to the seemingly mild relaxation of the restrictions placed on ω . If ω were a σ-additive embedding , then we should obtain the Jauch's theory of 3.2. The natural question arises : does our proposal not suffer the shortage of Jauch's theory ?

4.8. Minimal Boolean extension .

Obviously it suffices to prove , that $\alpha^{(5)}(a) = 1$ implies $\alpha^{(5)}(a \wedge b) = \alpha^{(5)}(b)$ for any $a, b \in \mathcal{L}_Q^{(5)}$, $\alpha^{(5)} \in \mathcal{S}_Q^{(5)}$. Let us observe , that $\alpha^{(5)}(a' \wedge b) = 0$ in this case . We cannot , however , proceed further till we assume $\mathcal{L}_Q^{(5)}$ to be a Boolean σ-algebra . Then we have : $b = (a \wedge b) \vee (a' \wedge b)$ and $\alpha^{(5)}(b) \leq \alpha^{(5)}(a \wedge b) + \alpha^{(5)}(a' \wedge b) = \alpha^{(5)}(a \wedge b)$. On the other hand, $b \geq a \wedge b$ what implies that $\alpha^{(5)}(b) \geq \alpha^{(5)}(a \wedge b)$.

Thus the property of 3.4. holds owing to the assumed distributivity of $\mathcal{L}_Q^{(5)}$. As Zierler and Schlesinger (~~[10]~~ 1965), see also (Mielnik, 1968 ~~[23]~~) have demonstrated the existence of minimal Boolean extension for any ortho-modular σ-ortho-p.o.-set , we can choose $\mathcal{L}_Q^{(5)}$ to be just the minimal Boolean extension of $\mathcal{L}_Q^{(2)}$. Thus our theory bears a resemblance to the (empty) variant of hidden variables theory mentioned in 2.12., with one vital difference : our embedding ω is not additive .

There is no trouble with defining functions of JRVs in our scheme , because the set $\mathcal{O}_Q^{(5)}$ is functionally closed . Also the desired independence property of 2.9. is satisfied . Thus we see , that our proposal has some nice features , making it worth of further studies .

4.9. Main features .

Let us recapitulate main features of the scheme outlined above :

(i) instead of $\mathcal{L}_Q^{(2)}$ we take its minimal Boolean extension $\mathcal{L}_Q^{(5)}$ with the canonical embedding ω ;

(ii) we define $\mathcal{O}_Q^{(5)}$ as the set of all inner $\mathcal{L}_Q^{(5)}$-valued measures on measurable spaces , and embed $\mathcal{O}_Q^{(2)}$ into $\mathcal{O}_Q^{(5)}$ by ω ;

(iii) we define $\mathcal{S}_Q^{(5)}$ as the set of all normed outer measures on $\mathcal{L}_Q^{(5)}$ such that $\alpha^{(5)} \circ \omega \in \mathcal{S}_Q^{(2)}$ for any $\alpha^{(5)} \in \mathcal{S}_Q^{(5)}$;

(iv) we define the JRV $J^{(5)}(A_1^{(5)}, A_2^{(5)})$ for any $A_1^{(5)}$, $A_2^{(5)} \in \mathcal{O}_Q^{(5)}$ as the inner $\mathcal{L}_Q^{(5)}$-valued measure on $\mathbb{S}_1 \times \mathbb{S}_2$, generated by the set function taking all $X \times Y$ into $A_1^{(5)}(X) \wedge A_2^{(5)}(Y)$ with $X \in \mathcal{B}(\mathbb{S}_1)$, $Y \in \mathcal{B}(\mathbb{S}_2)$;

(v) the JPD $\alpha^{(5)}_{A_1^{(5)} A_2^{(5)}} = \alpha^{(5)} \circ J^{(5)}(A_1^{(5)}, A_2^{(5)})$ is an outer measure on $\mathcal{B}(\mathbb{S}_1 \times \mathbb{S}_2)$ satisfying the marginal property , the property of 3.4. and other discussed in Section 2 , except the conformity with the standard JPDs for compatible standard variables ;

(vi) the arising theory , based on the minimal Boolean extension of $\mathcal{L}_Q^{(2)}$ and on unconventional $\mathcal{O}_Q^{(5)}, \mathcal{S}_Q^{(5)}$ can serve as an example of a generalized full quantum theory of probability .

A lot of further problems arise about this scheme , both formal and physical , including the basic one : if it has a kind of physical interpretation . Perhaps we reach answers in the future .

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