BULLETIN OF THE POLISH
ACADEMY OF SCIENCES
Mathematics
Vol. 31, No. 3-4, 1983

OPERATOR THEORY

Linear-topological Spaces of Operators Affiliated with a von Neumann Algebra

by

Leszek Jan CIACH

Presented by K. URBANIK on November 11, 1982

Summary. Non-commutative integration theory was initiated by Segal [6]. Here, we give the concept of a "subadditive measure" on projectors of a von Neumann algebra, which generalizes the regular gage in the sense of [6], and simple proofs of some theorems on the completeness of the non-commutative Lebesgue spaces and of the non-commutative Lorentz ones. The proofs of these theorems are based on the "subadditivity" of the gage, with its "additivity" disregarded. Theorems 2.3 and 2.4 are well known in the commutative case.

1. Preliminaries. Throughout, $\mathscr{A}$ stands for a von Neumann algebra acting in a complex Hilbert space $\mathscr{H}$. The centre of the algebra $\mathscr{A}$ is denoted by $\mathscr{Z}$. As is well known, there exist a locally compact Hausdorff space Ω, a Radon measure ν (unique up to equivalence of measures), a $*$-isomorphism $\Phi:\mathscr{Z}\to\mathscr{L}^{\infty}(\Omega,\nu)$ and a dimension function $d(\cdot)$ (unique up to multiplication by a positive real measurable function) mapping the projectors from $\mathscr{A}$ into ν-measurable non-negative extended real-valued functions defined on Ω and satisfying conditions 1–9 of ([6], definition 1.4). The set of all orthogonal projections from $\mathscr{A}$ is denoted by Proj. $(\mathscr{A})$.

DEFINITION 1.1. A subadditive measure on projections of a von Neumann algebra $\mathscr{A}$ is a function $m:\text{Proj.}(\mathscr{A})\to[0,\infty]$ such that

(i) $m(0)=0,\ m(p)=0\Rightarrow p=0;$

(ii) $p\leqslant q\Rightarrow m(p)\leqslant m(q),\ p,q\in\text{Proj.}(\mathscr{A});$

(iii) $p\sim q\Rightarrow m(p)=m(q),\ p,q\in\text{Proj.}(\mathscr{A});$

(iv) $m(p+q)\leqslant m(p)+m(q),\ p,q\in\text{Proj.}(\mathscr{A}),\ p\perp q;$

(v) $p_n\uparrow p\Rightarrow m(p_n)\uparrow m(p),\ p,p_n\in\text{Proj.}(\mathscr{A}).$

Note that

$$m(p\vee q)\leqslant m(p)+m(q),\ p,q\in\text{Proj.}(\mathscr{A}).$$

Throughout this paper, we denote by m_1 a regular gage in the sense of [6] and by m_0 a subadditive measure defined as follows

$$m_0(p) = \inf\{0 < \alpha \leqslant \infty : v[d(p) > \alpha] \leqslant \alpha\}.$$

We say that a densely defined closed operator a affiliated with $\mathscr{A}$ is m-measurable if (see [3], [1])

$$m(e_\lambda^\perp) \to 0 \quad \text{as } \lambda \to \infty,$$

where $|a| = \int_0^\infty \lambda de_\lambda$ is the spectral resolution of $|a|$. $\mathscr{L}_m(\mathscr{A}) = \mathscr{L}_m$ stands for a $*$-algebra of m-measurable operators.

The rearrangement $a(\alpha)$ of an operator $a \in \mathscr{L}_m(\mathscr{A})$ is a function $a(\alpha):(0, \infty) \to [0, \infty)$ defined by (see [9], [1])

$$a(\alpha) = \inf\{0 \leqslant \lambda < \infty : m(e_\lambda^\perp) \leqslant \alpha\},$$

where $|a| = \int_0^\infty \lambda de_\lambda$.

$\mathscr{L}_m(\mathscr{A})$ is a Fréchet space with the metric (see [1])

$$\varrho(a, b) = \inf\{\varepsilon > 0 : m(e_\varepsilon^\perp) \leqslant \varepsilon\} = \inf\{\varepsilon > 0 : (a-b)(\varepsilon) \leqslant \varepsilon\},$$

where

$$|a-b| = \int_0^\infty \lambda de_\lambda, \; a, b \in \mathscr{L}_m(\mathscr{A}).$$

We say that a sequence of m-measurable operators $\{a_n\}$ converges in measure m to an m-measurable operator a ($a_n \xrightarrow{m} a$) if a_n converges to a in the metric $\varrho(\cdot, \cdot)$ (see [7], [3], [1]).

PROPOSITION 1.1. ([1], proposition 2.7). *If* $a_n \xrightarrow{m} a$ ($a, a_n \in \mathscr{L}_m$) *then* $a_n(\alpha) \to a(\alpha)$ *at each point of continuity of the function* $a(\alpha)$.

A support of an operator $a \eta \mathscr{A}$ is the smallest projection $p \in \mathscr{A}$ such that $pa = a$.

2. Completeness of spaces $\mathscr{L}_m^{\delta\sigma}(\mathscr{A})$, $\mathscr{L}_{m,t}^{\delta}(\mathscr{A})$ of m-measurable operators.

DEFINITION 2.1. (cf. [9], [2], [5], [1]). For each $a \in \mathscr{L}_m(\mathscr{A})$ we set

(1) $$\|a\|_\delta = \left\{\int_0^\infty a^\delta(\alpha)\, d\alpha\right\}^{1/\delta}, \; 0 < \delta < \infty;$$

(2) $$\|a\|_\infty = \sup\{a(\alpha) : \alpha > 0\};$$

(3) $$\|a\|_0 = m(\operatorname{supp}|a|);$$

(4) $$\|a\|_{\delta\sigma} = \left\{\frac{\sigma}{\delta}\int_0^\infty \alpha^{\frac{\sigma-\delta}{\delta}} a^\sigma(\alpha)\, d\alpha\right\}^{1/\sigma}, \; 0 < \delta < \infty, \; 0 < \sigma < \infty;$$

(5) $$\|a\|_{\delta\infty} = \sup\{\alpha^{1/\delta} a(\alpha) : \alpha > 0\}, \; 0 < \delta \leqslant \infty.$$

Note that

$$\| a \|_{\delta\delta} = \| a \|_{\delta},\ 0 < \delta < \infty.$$

DEFINITION 2.2. We define the spaces $\mathscr{L}_m^{\delta}(\mathscr{A})$ (non-commutative Lebesgue space) and $\mathscr{L}_m^{\delta\sigma}(\mathscr{A})$ (non-commutative Lorentz space) of m-measurable operators a by the conditions

$$\mathscr{L}_m^{\delta}(\mathscr{A}) = \mathscr{L}_m^{\delta} = \{a \in \mathscr{L}_m : \| a \|_{\delta} < \infty,\ 0 \leq \delta \leq \infty\};$$

$$\mathscr{L}_m^{\delta\sigma}(\mathscr{A}) = \mathscr{L}_m^{\delta\sigma} = \{a \in \mathscr{L}_m : \| a \|_{\delta\sigma} < \infty\}.$$

Observe that $\| \cdot \|_{\delta\sigma}$, $0 < \delta \leq \infty$, $0 < \sigma \leq \infty$, is a quasi-norm (norm in special cases) in the space $\mathscr{L}_m^{\delta\sigma}$. Onviously, $\mathscr{L}_m^{0}(\mathscr{A})$ is a vector space and $\| \cdot \|_0$ is a 0-homogeneous norm. In other words $\mathscr{L}_m^0$ is a normed Abelian group in the sense of [4].

PROPOSITION 2.1. *$\mathscr{L}_m^0(\mathscr{A})$ is complete (with respect to the metric defined by the 0-homogeneous norm $\| \cdot \|_0$).*

Proof. Suppose $\| a_k - a_n \|_0 \to 0$ as $k, n \to \infty$, where $a_n \in \mathscr{L}_m^0(\mathscr{A})$, $n \geq 1$. It is immediate that the sequence $\{a_n\}$ is fundamental in measure and, hence, there exists an operator $a \in \mathscr{L}_m$, $a_n \xrightarrow{m} a$. Moreover, for $n \geq 1$, $a_k - a_n \underset{k}{\xrightarrow{m}} a - a_n$ and meas. $\{\text{supp}\,[(a_k - a_n)(\alpha)]\} = m\,(\text{supp}\,| a_k - a_n |) = \| a_k - a_n \|_0 \to 0$ as $k, n \to \to \infty$. In consequence, $\| a - a_n \|_0 = m\,(\text{supp}\,| a - a_n |) = \text{meas.}\,\{\text{supp}\,[(a - a_n)(\alpha)]\} \to 0$ since $(a_k - a_n)(\alpha) \underset{k}{\to} (a - a_n)(\alpha)$ at each point of continuity of the function $(a - a_n)(\alpha)$, q.e.d.

THEOREM 2.1. *For $\delta > 0$, $\mathscr{L}_m^{\delta\sigma}(\mathscr{A})$ is a complete quasi-normed space.*

Proof. It is not difficult to observe that

$$a(\alpha) = \inf\{\| a - b \|_{\infty} : \| b \|_0 \leq \alpha\},\ a \in \mathscr{L}_m.$$

Using theorem 5.10 of [4], it now follows that

$$(\mathscr{L}_m^0, \mathscr{L}_m^{\infty})_{\theta\gamma:k}^{[1\ \theta]} = \mathscr{L}_m^{\delta\sigma},$$

where $\theta = \delta/(\delta + 1)$, $\sigma = \theta\gamma$.

To finish the proof, it suffices to use theorem 5.4 in [4] and proposition 2.1, q.e.d.

REMARK 2.1. Fix a non-negative countably additive measure μ defined on the σ-algebra of all Borel sets of the half-line $(0, \infty)$ with the properties: $0 < \mu\,[(0, \beta)] < \infty$, $0 < \beta < \infty$, $\mu\,[(0, \infty)] = \infty$, $\int_0^{\infty} a(\alpha/2)\, d\mu \leq \text{const.} \int_0^{\infty} a(\alpha)\, d\mu$. We define the space $\mathscr{L}_{m,\mu}^{\delta}(\mathscr{A})$ by the condition

$$\| a \|_{\delta} = \{\int_0^{\infty} a^{\delta}(\alpha)\, d\mu\}^{1/\delta},\ 0 < \delta < \infty.$$

$\mathscr{L}_{m,\mu}^{\delta}$ is a quasi-Banach space for each $0 < \delta < \infty$ (Banach space in special cases). Indeed, let $\| a_k - a_n \|_\delta \to 0$ as $k, n \to \infty$ and $a_n \overset{m}{\Rightarrow} a \in \mathcal{L}_m(\mathcal{A})$. Then

$$\| a - a_n \|_\delta = \| |a - a_n|^\delta \|_1^{1/\delta} \leq 10 \frac{\lim}{k} \| |a_k - a_n|^\delta \|_1^{1/\delta} \to 0,$$

as $n \to \infty$ since $|a_k - a_n|^\delta \overset{m}{\Rightarrow} |a - a_n|^\delta$ as $k \to \infty$ (see [1], corollary 5.1).

Completeness of weighted $\mathcal{L}_m^\delta$ $(\mathcal{A})$ spaces. Fix a non-negative m-measurable (or gross m-measurable in the sense of [1]) operator t. Let t^{-1} exist and let $t^{-1} \in \mathcal{L}_m$, that is, $m(e_{\frac{1}{\lambda}}) \to 0$ as $\lambda \to \infty$ and $m(e_\lambda) \to 0$ as $\lambda \to 0$. Let $0 < \delta < \infty$. We define the space $\mathcal{L}_{m,t}^\delta(\mathcal{A}) = \mathcal{L}_{m,t}^\delta$ (non-commutative weighted $\mathcal{L}_m^\delta$ space) of measurable operators a by the condition

$$\| a \|_{\delta, m, t} = \| a \|_{\delta, t} = \| t^{1/2\delta} a t^{1/2\delta} \|_\delta < \infty.$$

Obviously, $\mathcal{L}_{m,t}^\delta(\mathcal{A})$ is a vector space and $\| \cdot \|_{\delta, t}$ is a quasi-norm (norm in special cases).

THEOREM 2.2. *For an arbitrary* $0 < \delta < \infty$, $\mathcal{L}_{m,t}^\delta(\mathcal{A})$ *is a quasi-Banach space.*

Proof. Let $\{a_n\} \subset \mathcal{L}_{m,t}^\delta$ be a Cauchy sequence. Hence, by theorem 2.1. (see also [1], proposition 3.2), $t^{1/2\delta} a_n t^{1/2\delta} \overset{m}{\Rightarrow} a' = t^{1/2\delta} (t^{-1/2\delta} a' t^{-1/2\delta}) t^{1/2\delta} = t^{1/2\delta} a t^{1/2\delta} \in \mathcal{L}_m(\mathcal{A})$, that is, $a_n \overset{m}{\Rightarrow} a \in \mathcal{L}_m$ and $t^{1/2\delta}(a_k - a_n) t^{1/2\delta} \overset{m}{\Rightarrow} t^{1/2\delta}(a - a_n) t$ as $k \to \infty$. So, in virtue of Proposition 1.1,

$$\| a - a_n \|_{\delta, t} = \| t^{1/2\delta}(a - a_n) t^{1/2\delta} \|_\delta \leq \frac{\lim}{k} \| t^{1/2\delta}(a_k - a_n) t^{1/2\delta} \|_\delta =$$

$$= \frac{\lim}{k} \| a_k - a_n \|_{\delta, t} \to 0 \text{ as } n \to \infty,$$

which ends the proof of Theorem 2.2, q.e.d.

REMARK 2.2. Let $\mathcal{A}$ be a finite and σ-finite von Neumann algebra and ψ a faithful normal state on $\mathcal{A}$ $(\psi(\mathbf{1}) = 1)$. As is well known [6], there exists a regular gage on Proj. $(\mathcal{A})$, $at \geq 0$, $t \in \mathcal{L}_{m_1}^1$, besides, there exists a unique linear functional $\tilde{m}_1(\cdot)$ on the space $\mathcal{L}_{m_1}^1$ that extends m_1 and

$$\psi(a) = \tilde{m}_1(at) = \tilde{m}_1(ta), \; a \in \mathcal{A}.$$

It is clear that t^{-1} exists and $t^{-1} \in \mathcal{L}_{m_1}$. We define (see [8]) the space $\mathcal{L}_\psi^\delta(\mathcal{A})$ (non-commutative $\mathcal{L}^\delta$ space associated with a faithful normal state ψ) of m_1-measurable operators a by the condition

$$\| a \|_{\delta, \psi} = \| a \|_{\delta, m_1, t} = \| t^{1/2\delta} a t^{1/2\delta} \|_{\delta, m_1} < \infty.$$

This definition does not depend on the regular gage m_1 (see [8]).

More generally, let $\mathcal{A}$ be a semifinite von Neumann algebra and ψ a faithful normal semifinite weight on the positive cone $\mathcal{A}_+$ of $\mathcal{A}$

$$\psi(a) = \sup_n \tilde{m}_1(t_n^{\frac{1}{2}} a t_n^{\frac{1}{2}}), \; a \in \mathcal{A}_+,$$

where $t = \int_0^\infty \lambda de_\lambda$, $t_n = \int_0^n \lambda de_\lambda$, $t, t^{-1} \in \mathcal{L}_{m_1}$ (or t, t^{-1}-gross m_1-measurable in the sense of [1]). We define $\mathcal{L}_\psi^\delta = \mathcal{L}_{m_1, t}^\delta$.

Using theorems 6.2 and 6.4 of [1] we have

THEOREM 2.3. *Let $\mathscr{A}$ be a finite and σ-finite von Neumann algebra. Suppose that $\mathscr{A}$ contains no minimal non-zero projections. Then $\mathscr{L}^{\delta}_{m_i, t}$ has the trivial dual for $0 < \delta < {}^1/_{2-i}$, $i = 0, 1$.*

To finish, let us notice the verity of the following theorem [1].

DEFINITION 2.3. A set $A \subset \mathscr{L}^{\delta}_{m}(\mathscr{A})$ is said to be uniformly *m*-integrable of order δ if it satisfies the following conditions:

(i) For any $\varepsilon > 0$, there exists some $\sigma > 0$ such that, for any $p \in \text{Proj.}(\mathscr{A})$, $m(p) < \sigma$ and any $a \in A$

$$\| ap \|_{\delta} = \| |a| p \|_{\delta} < \varepsilon.$$

(ii) For any $\varepsilon > 0$, there exists some $q \in \text{Proj.}(\mathscr{A})$, $m(q) < \infty$, such that, for any $a \in A$

$$\| aq^{\perp} \|_{\delta} = \| |a| q^{\perp} \|_{\delta} < \varepsilon.$$

THEOREM 2.4. ([1], theorem 3.2). *A sequence $\{a_n\}$ of operators from $\mathscr{L}^{\delta}_{m}(\mathscr{A})$ converges in $\mathscr{L}^{\delta}_{m}(\mathscr{A})$ to an operator $a \in \mathscr{L}^{\delta}_{m}(\mathscr{A})$ if and only if $a_n \xrightarrow{m} a$ and $\{a_n\}$ is uniformly m-integrable of order δ, $o < \delta < \infty$.*

COROLLARY 2.1. *If $\| a - a_n \|_{\delta} \to 0$ $\left(a, a_n \in \mathscr{L}^{\delta}_{m}(\mathscr{A})\right)$, then $|a_n|^{\delta} \to |a|^{\delta}$ in $\mathscr{L}^{1}_{m}(\mathscr{A})$ for $\delta \geq 1$.*

INSTITUTE OF APPLIED MECHANICS, HOLY CROSS TECHNICAL UNIVERSITY, ALEJA TYSIĄCLECIA PP 7, 25-314 KIELCE
(INSTYTUT MECHANIKI STOSOWANEJ, POLITECHNIKA ŚWIĘTOKRZYSKA)

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Л. Я. Циах, **Линейно-топологические пространства операторов, присоединенных к алгебре фон Неймана**

Некоммутативная теория интегрирования была впервые построена И. Е. Сигалом в [6]. В этой заметке вводится понятие „субаддитивной меры” на проекторах алгебры фон Неймана, которое обогащает понятие некоммутативной меры Сигала. Здесь же приводятся прямые доказательства теорем о полноте некоммутативных пространств Лебега и Лорентца. Доказательства этих теорем опираются только на субаддитивность меры без использования ее аддитивности. Теоремы 2.3 и 2.4 хорошо известны в коммутативном случае.