BULLETIN OF THE POLISH
ACADEMY OF SCIENCES
MATHEMATICS
Vol. 34, No. 11–12, 1986

FUNCTIONAL ANALYSIS

Some Spaces in which Martingale Difference Sequences Are Unconditional

by

Fernando COBOS

Presented by W. ORLICZ on January 29, 1986

Summary. We study whether or not the spaces occurring in classical harmonic analysis have the unconditionality property for martingale differences. We also show some operator spaces having that property.

1. Introduction. During the last few years, the class of Banach spaces with the unconditionality property for martingale differences has been extensively studied (see, e.g., the papers by B. Maurey [15], D. J. Aldous [1], D. L. Burkholder [7], [8] and J. Bourgain [4], [5]). The main reason for the interest in this new class of spaces is that the analogues of several classical results on martingales and singular integrals are also true for a Banach space belonging to this class.

Some examples of spaces containing the unconditionality property for martingale differences (UMD-property for short) are **R** (see [6]), the Lebesgue spaces l_p, $L_p(\mathbf{T})$ for $1 < p < \infty$ (see [8]) and the Schatten classes $S_p(H, K)$ of compact operators between two separable Hilbert spaces [5]. On the other hand, since every UMD space is reflexive [15], [1], the spaces l_1 and l_∞ do not belong to that class. But there are few more known examples of concrete spaces with the UMD-property.

The purpose of this note is to study whether or not the spaces occurring in classical harmonic analysis have the UMD-property. Our attention will mainly be focussed on the Zygmund spaces $L_p(\log L)^\gamma$, the O'Neil spaces $K^p(\log^+ K)^\alpha$ and the Zygmund spaces Z^α of functions whose $1/\alpha$-th powers are exponentially integrable (see [21], [18] and [2]).

We also show some operator spaces that have the UMD-property. At this point, we shall work with the Lorentz-Marcinkiewicz operator spaces $S_{\varphi,q}(H, K)$, introduced and studied by the author in [9]. They are extensions of the Schatten classes: For $\varphi(t) = t^{1/q}$, $S_{\varphi,q}(H, K) = S_q(H, K)$.

To work out this programme, an essential tool will be the real interpolation method with function parameter developed by J. Peetre [19], T. F. Kalugina [12], J. Gustavsson [10], C. Merucci [16], [17] and others. In fact, we shall show that, concerning the UMD-property, the (φ, q)-method is stable if $1 < q < \infty$, while for the cases $q = 1$ and $q = \infty$ we shall derive a negative result. This observation will allow us to get a large part of our results.

2. Preliminaries. Let E be a (real or complex) Banach space and let (Ω, μ) be a measure space, with μ a positive σ-finite measure. For $1 \leqslant p < \infty$, we denote by $L_p(E) = L_p(E; \Omega)$ the usual vector-valued L_p-space in the sense of the Bochner integral. The cases $\Omega = [0, 1]$ and $\Omega = \mathbf{R}$, with μ the Lebesgue measure, will be of special interest for us.

Let $1 < p < \infty$. A Banach space E is said to have the unconditionality property for martingale differences (UMD-property, for short) if E-valued martingale difference sequences are unconditional in $L_p(E; [0, 1])$. For properties of UMD spaces we refer to the papers by D. L. Burkholder [7], [8] and J. Bourgain [4], [5].

Although the class of UMD spaces appears to depend on the choice of p, this is not the case [15], [7].

Let $f \in L_p(E; \mathbf{R})$ and $\varepsilon > 0$. The truncated Hilbert transform of f is defined by

$$\mathscr{H}_\varepsilon f(x) = \frac{1}{\pi} \int\limits_{|t| > \varepsilon} \frac{f(x-t)}{t}\, dt.$$

The following characterization of Banach spaces with the UMD-property in terms of the vector-valued Hilbert transform, due to D. L. Burkholder [8] and J. Bourgain [4], will be very useful for our considerations.

THEOREM A. *Let $1 < p < \infty$. A necessary and sufficient condition for a Banach space E to have the UMD-property is that the limit $\mathscr{H}f = \lim_{\varepsilon \to 0} \mathscr{H}_\varepsilon f$ exists almost everywhere for all $f \in L_p(E; \mathbf{R})$ and that there is a constant M_p such that*

$$\| \mathscr{H}f \|_{L_p(E)} \leqslant M_p \| f \|_{L_p(E)}.$$

Let $\mathbf{T}$ be the unit circle and let f be a scalar-valued measurable function on $\mathbf{T}$. The distribution function of f is defined by

$$D_f(y) = m\{x: |f(e^{ix})| > y\}, \quad 0 < y < \infty, \quad dm = dx/2\pi$$

and the non-increasing rearrangement of f by

$$f^*(t) = \inf\{y: D_f(y) \leqslant t\}, \quad 0 < t < 1.$$

The function f is said to belong to:

— The Zygmund space $L_p(\log L)^\gamma$, $1 \leqslant p < \infty$, $-\infty < \gamma < +\infty$ if

$$\int_0^{2\pi} [|f(e^{ix})| \log^\gamma (2 + |f(e^{ix})|)]^p \, dx < \infty.$$

— The O'Neil space $K^p(\log^+ K)^\alpha$, $1 \leqslant p < \infty$, $0 < \alpha < \infty$ if

$$\int_1^\infty D_f(y)^{1/p} (\log y)^{\alpha/p} \, dy < \infty.$$

— The Zygmund space Z^α, $0 < \alpha < \infty$ if

$$\int_0^{2\pi} \exp(\lambda |f(e^{ix})|^{1/\alpha}) \, dx < \infty$$

for some positive constant $\lambda = \lambda(f)$.

— The Lorentz-Zygmund space $L_{p,q}(\log L)^\gamma$, $1 \leqslant p \leqslant \infty$, $1 \leqslant q \leqslant \infty$, $-\infty < \gamma < +\infty$ if

$$\| f \|_{p,q,\gamma} = \begin{cases} \left(\int_0^1 [t^{1/p}(1 - \log t)^\gamma f^*(t)]^q \, dt/t\right)^{1/q}, & q < \infty \\ \sup\limits_{0<t<1} [t^{1/p}(1 - \log t)^\gamma f^*(t)], & q = \infty \end{cases}$$

is finite.

All these spaces are important in classical harmonic analysis (see, e.g., [21], [18] and [2]).

The first three classes of spaces are contained as special cases in the last one [2], Theorem D.

For $1 \leqslant p < \infty$, $q = 1$ and $\gamma > 0$, the functional $\| \; \|_{p,q,\gamma}$ is a norm [14]. For $1 < p \leqslant \infty$, $1 \leqslant q \leqslant \infty$, and $-\infty < \gamma < +\infty$ $\gamma < 0$, it is possible to replace the quasinorm $\| \; \|_{p,q,\gamma}$ with an equivalent norm [2], Corollary 8.2.

Next we recall the definition of the Lorentz-Marcinkiewicz operator spaces [9].

The class of all functions φ: $(0, +\infty) \to (0, +\infty)$ continuous, with $\varphi(1) = 1$ and such that

$$\bar{\varphi}(t) = \sup_{s>0} \frac{\varphi(ts)}{\varphi(s)} < \infty \quad \text{for every} \quad t > 0$$

is represented by $\mathscr{B}$. The Boyd indices $\alpha_{\bar{\varphi}}$ and $\beta_{\bar{\varphi}}$ of the function $\bar{\varphi}$ are defined by

$$\alpha_{\bar{\varphi}} = \inf_{1<t<\infty} \frac{\log \bar{\varphi}(t)}{\log t} = \lim_{t\to+\infty} \frac{\log \bar{\varphi}(t)}{\log t}$$

$$\beta_{\bar{\varphi}} = \sup_{0<t<1} \frac{\log \bar{\varphi}(t)}{\log t} = \lim_{t\to 0} \frac{\log \bar{\varphi}(t)}{\log t}.$$

The indices $\alpha_{\bar{\varphi}}$ and $\beta_{\bar{\varphi}}$ satisfy $-\infty < \beta_{\bar{\varphi}} \leqslant \alpha_{\bar{\varphi}} < +\infty$ and indicate when $\bar{\varphi}$ belongs to $L_1((1,\infty), dt/t)$ and $L_1((0,1), dt/t)$ (see [16]).

Given two separable Hilbert spaces over the field of complex numbers H and K, given $1 \leqslant q \leqslant \infty$ and $\varphi \in \mathscr{B}$ with $0 < \beta_{\bar{\varphi}} \leqslant \alpha_{\bar{\varphi}} < 1$, $S_{\varphi,q}(H,K)$ is the collection of all compact operators T from H into K with a finite norm

$$\tau_{\varphi,q}(T) = \begin{cases} \left(\sum_{n=1}^{\infty} \left(\varphi(n)\, n^{-1} \sum_{j=1}^{n} s_j(T)\right)^q n^{-1}\right)^{1/q} & \text{for} \quad q<\infty \\ \sup_{n\geqslant 1} \left(\varphi(n)\, n^{-1} \sum_{j=1}^{n} s_j(T)\right) & \text{for} \quad q=\infty \end{cases}$$

where $(s_n(T))$ is the monotone non-increasing (non-negative) sequence converging to zero formed by the eigenvalues of the positive compact operator $[T^*T]^{1/2}$, each one repeated a number of times equal to its multiplicity.

We conclude these preliminaries by describing the real interpolation space with function parameter [19], [10], [16], [17].

Let (A_0, A_1) be a compatible couple of Banach spaces, let $1 \leqslant q \leqslant \infty$ and $\varphi \in \mathscr{B}$ with $0 < \beta_{\bar{\varphi}} \leqslant \alpha_{\bar{\varphi}} < 1$. The space $(A_0, A_1)_{\varphi,q}$ consists of all $x \in A_0 + A_1$ which have a finite norm

$$\| x \|_{\varphi,q} = \begin{cases} \left(\int_0^{\infty} (\varphi(t)^{-1} K(t,x))^q\, dt/t\right)^{1/q} & \text{if} \quad q<\infty \\ \sup_{t>0} (\varphi(t)^{-1} K(t,x)) & \text{if} \quad q=\infty \end{cases}$$

where $K(t,x)$ is the functional of J. Peetre, defined by

$$K(t,x) = \inf\{\| x_0 \|_{A_0} + t\, \| x_1 \|_{A_1} : x = x_0 + x_1,\ x_0 \in A_0,\ x_1 \in A_1\}.$$

For $\varphi(t) = t^{\vartheta}$ $(0 < \vartheta < 1)$ we get the classical real interpolation space $((A_0, A_1)_{\vartheta,q}, \|\ \|_{\vartheta,q})$ (see [3], [20]).

3. Results. In order to show that if $1 < q < \infty$ then the (φ, q)-method is stable for the UMD-property, we shall first prove

LEMMA 1. *Let (A_0, A_1) be a compatible couple of Banach spaces, let $1 \leqslant q < \infty$ and let $\varphi \in \mathscr{B}$ with $0 < \beta_{\bar{\varphi}} \leqslant \alpha_{\bar{\varphi}} < 1$. Then*

$$\left(L_q(A_0), L_q(A_1)\right)_{\varphi,q} = L_q\left((A_0, A_1)_{\varphi,q}\right) \quad \textit{(with equivalent norms)}.$$

Proof. It is easily checked that $A_0 \cap A_1$ is dense in $(A_0, A_1)_{\varphi,q}$ and that $L_q(A_0) \cap L_q(A_1)$ is dense in $\left(L_q(A_0), L_q(A_1)\right)_{\varphi,q}$. With this in mind and calling the collection of all functions, **S**

$$f(x) = \sum_{k=1}^{n} f^{(k)} \chi_{B_k}(x)$$

where $n \in \mathbf{N}, f^{(k)} \in A_0 \cap A_1$, χ_{B_k} is the characteristic function of the measurable set B_k, $\mu(B_k) < \infty$ and $B_j \cap B_k = \emptyset$ if $j \neq k$, one can show that **S** is dense in $\left(L_q(A_0), L_q(A_1)\right)_{\varphi,q}$ and in $L_q\left((A_0, A_1)_{\varphi,q}\right)$.

On the other hand

$$\inf_{x = x_0 + x_1} \left(\| x_0 \|_{A_0}^q + t^q \| x_1 \|_{A_1}^q\right)^{1/q} \leqslant K(t, x) \leqslant$$

$$\leqslant 2^{1-1/q} \inf_{x = x_0 + x_1} \left(\| x_0 \|_{A_0}^q + t^q \| x_1 \|_{A_1}^q\right)^{1/q}.$$

Therefore we have for every $f \in \mathbf{S}$

$$\| f \|_{(L_q(A_0),\, L_q(A_1))_{\varphi,q}}^q \sim$$

$$\sim \int_0^\infty \varphi(t)^{-q} \inf_{\substack{f = f_0 + f_1 \\ f_j \in L_q(A_j)}} \int_\Omega \left(\| f_0(x) \|_{A_0}^q + t^q \| f_1(x) \|_{A_1}^q\right) d\mu \frac{dt}{t} =$$

$$= \int_\Omega \int_0^\infty \varphi(t)^{-q} \inf_{\substack{f(x) = f_0(x) + f_1(x) \\ f_j(x) \in A_j}} \left(\| f_0(x) \|_{A_0}^q + t^q \| f_1(x) \|_{A_1}^q\right) \frac{dt}{t} d\mu \sim$$

$$\sim \int_\Omega \| f(x) \|_{(A_0,\, A_1)_{\varphi,q}}^q d\mu = \| f \|_{L_q((A_0,\, A_1)_{\varphi,q})}^q$$

where $\sim$ indicates equivalence with constants that do not depend on f. This gives the desired equality. □

Now we can establish

THEOREM 2. *Assume that* (A_0, A_1) *is a couple of UMD spaces,* $1 < q < \infty$ *and that* $\varphi \in \mathscr{B}$ *with* $0 < \beta_{\bar{\varphi}} \leqslant \alpha_{\bar{\varphi}} < 1$. *Then* $(A_0, A_1)_{\varphi,q}$ *is a UMD space.*

Proof. By Theorem A, the Hilbert transform is bounded on $L_q(A_j; \mathbf{R})$ for $j = 0, 1$. Whence it follows from the interpolation theorem and Lemma 1, that $\mathscr{H}$ is bounded on $L_q\left((A_0, A_1)_{\varphi,q}; \mathbf{R}\right)$, and this proves the result. □

As a consequence we obtain

Corollary 3. *Let* $1 < p < \infty$, $1 < q < \infty$ *and* $-\infty < \gamma < +\infty$. *Then the Lorentz-Zygmund space* $L_{p,q}(\log L)^\gamma$ *has the UMD-property.*

Proof. Take $1 < p_0 < p_1 < \infty$ and $0 < \vartheta < 1$ with $1/p = (1-\vartheta)/p_0 + \vartheta/p_1$, and put

$$\varphi(t) = t^{\vartheta}(1 + |\log t|)^{-\gamma}.$$

According to [16], Theorem 1, we have that

$$(L_{p_0}, L_{p_1})_{\varphi,q} = L_{p,q}(\log L)^\gamma.$$

Therefore, Theorem 2 implies that $L_{p,q}(\log L)^\gamma$ has the UMD-property. □

In particular, since $L_p(\log L)^\gamma = L_{p,p}(\log L)^\gamma$ for $1 \leqslant p < \infty$ and $-\infty < \gamma < +\infty$ [2], Theorem D, we have

Corollary 4. *For* $1 < p < \infty$ *and* $-\infty < \gamma < +\infty$, *the Zygmund space* $L_p(\log L)^\gamma$ *has the UMD-property.*

We next consider some limit cases.

For $1 \leqslant p < \infty$ and $\gamma > 0$, the space $L_{p,1}(\log L)^\gamma$ is equal to the Lorentz space $\Lambda(W, 1)$ [14], where the weight function W is defined by

$$W(t) = t^{(1/p)-1}(1 - \log t)^\gamma.$$

So $L_{p,1}(\log L)^\gamma$ is not reflexive. Neither is the space $L_{\infty,\infty}(\log L)^{-\gamma}$ reflexive because it is the dual space of $L_{1,1}(\log L)^\gamma$ [2], Theorem 8.4. Therefore, all these spaces fail to have the UMD-property.

Consequently, taking into account that [2], Theorem D

$$K^p(\log{}^+K)^{\gamma p} = L_{p,1}(\log L)^\gamma$$

and

$$Z^\gamma = L_{\infty,\infty}(\log L)^{-\gamma}$$

we get

Theorem 5. *For* $1 \leqslant p < \infty$ *and* $\gamma > 0$, *the spaces*

$$L(\log L)^\gamma, \quad K^p(\log{}^+K)^\gamma \quad \text{and} \quad Z^\gamma$$

fail to have the UMD-property.

For the purpose of deriving a negative result complementing Theorem 2, we first establish

Lemma 6. *Let* A_0 *and* A_1 *be Banach spaces with* A_0 *continuously embedded in* A_1, *let* $\varphi, \varrho \in \mathscr{B}$ *such that*

$$0 < \beta_{\bar\varphi} \leqslant \alpha_{\bar\varphi} < \beta_{\bar\varrho} \leqslant \alpha_{\bar\varrho} < 1$$

and let $1 \leqslant q, r \leqslant \infty$. *Then* $(A_0, A_1)_{\varphi,q}$ *is continuously embedded in* $(A_0, A_1)_{\varrho,r}$.

Proof. The sub-multiplicative functions $\mu, \nu: (0, +\infty) \to (0, +\infty)$ defined by

$$\mu(t) = t\bar{\varrho}(1/t), \ \nu(t) = \bar{\varphi}(t)\bar{\varrho}(1/t)$$

satisfy

$$\beta_\mu = 1 - \alpha_{\bar{\varrho}} > 0, \ \alpha_\nu = \alpha_{\bar{\varphi}} - \beta_{\bar{\varrho}} < 0 .$$

Thus [16], Proposition 3

$$\int_0^1 \bar{\varrho}(1/t)\,dt < \infty, \ \int_1^\infty \bar{\varphi}(t)\bar{\varrho}(1/t)\,dt/t < \infty.$$

Consequently, the assertion can be proved by modifying in a natural way the method used in [20], Theorem 1.3.3/(e). □

LEMMA 7. *Assume that A_0 and A_1 are Banach spaces with A_0 continuously embedded in A_1, that $\varphi \in \mathscr{B}$ with $0 < \beta_{\bar{\varphi}} \leqslant \alpha_{\bar{\varphi}} < 1$ and that $1 \leqslant q \leqslant \infty$. If A_0 is not closed in A_1, then $(A_0, A_1)_{\varphi,q}$ contains a subspace isomorphic to l_q.*

Proof. Take $1 < r, \eta < \infty$ such that

$$0 < \frac{1}{r} < \beta_{\bar{\varphi}} \leqslant \alpha_{\bar{\varphi}} < \frac{(\eta - 1)r + 1}{\eta r}$$

and put

$$\varrho_0(t) = t^{1/r} \quad \text{and} \quad \varrho_1(t) = \left(\varphi(t)/t^{1/\eta r}\right)^{\eta/(\eta-1)}.$$

Both functions belong to $\mathscr{B}$ and their indices are

$$0 < \beta_{\bar{\varrho}_0} = \alpha_{\bar{\varrho}_0} = \frac{1}{r} < 1$$

$$\beta_{\bar{\varrho}_1} = \frac{\eta}{\eta - 1}\left[\beta_{\bar{\varphi}} - \frac{1}{\eta r}\right] > 0, \quad \alpha_{\bar{\varrho}_1} = \frac{\eta}{\eta - 1}\left[\alpha_{\bar{\varphi}} - \frac{1}{\eta r}\right] < 1.$$

Therefore, [17], Thm. 2 gives

$$(A_0, A_1)_{\varphi,q} = \left((A_0, A_1)_{\varrho_0,2}, \ (A_0, A_1)_{\varrho_1,2}\right)_{(\eta-1)/\eta,q}.$$

In addition, we have from Lemma 6

$$B_0 = (A_0, A_1)_{\varrho_0,2} \subsetneq B_1 = (A_0, A_1)_{\varrho_1,2}.$$

Let us now see that B_0 is not closed in B_1:

It is not hard to verify that B_0 is dense in B_1. So, if we suppose that B_0 is closed in B_1, we would have $B_0 = B_1$. Then, taking $1/r < \vartheta < \beta_{\bar{\varphi}_1}$ and again applying Lemma 6, we would obtain

$$(A_0, A_1)_{\vartheta,2} = B_0 = (A_0, A_1)_{1/r,2}.$$

But this contradicts [11], Theorem 3.1.

Hence $B_0 = B_0 \cap B_1$ is not closed in $B_1 = B_0 + B_1$ and $(A_0, A_1)_{\varphi,q} = (B_0, B_1)_{(\eta-1)/\eta,q}$. Whence the result follows using [13], Theorem. □

As an immediate consequence of this lemma we get

THEOREM 8. *Assume that A_0 and A_1 are Banach spaces with A_0 continuously embedded in A_1, and that $\varphi \in \mathscr{B}$ with $0 < \beta_{\bar{\varphi}} \leqslant \alpha_{\bar{\varphi}} < 1$. If A_0 is not closed in A_1, then the spaces $(A_0, A_1)_{\varphi,1}$ and $(A_0, A_1)_{\varphi,\infty}$ fail to have the UMD-property.*

Finally, we apply these results to the Lorentz-Marcinkiewicz operator spaces.

COROLLARY 9. *Let $\varphi \in \mathscr{B}$ with $0 < \beta_{\bar{\varphi}} \leqslant \alpha_{\bar{\varphi}} < 1$. Then the following holds.*
(i) *For $1 < q < \infty$, $S_{\varphi,q}(H, K)$ has the UMD-property.*
(ii) *The spaces $S_{\varphi,1}(H, K)$ and $S_{\varphi,\infty}(H, K)$ fail to have the UMD-property.*

Proof. Choose $1 < p_0 < p_1 < \infty$ such that $1/p_1 < \beta_{\bar{\varphi}} \leqslant \alpha_{\bar{\varphi}} < 1/p_0$ and consider the function

$$\varrho(t) = t^{p_1/(p_1-p_0)} \left(\varphi\left(t^{p_0 p_1/(p_1-p_0)}\right)\right)^{-1}.$$

According to [9], Theorem 5.1, we have

$$S_{\varphi,q}(H, K) = \left(S_{p_0}(H, K), S_{p_1}(H, K)\right)_{\varrho,q}.$$

Moreover, the Schatten classes $S_{p_j}(H, K)$ are UMD spaces, by [5]. Therefore Theorems 2 and 8 give the result. □

The problems treated in this note have been suggested to us during a conversation with Professor José L. Rubio de Francia. We should like to thank him for his helpful comments. Our thanks also to Professor José L. Torrea for some valuable information on the UMD-property.

DEPARTMENT OF MATHEMATICS, UNIVERSIDAD AUTONOMA DE MADRID, 28049 MADRID (SPAIN)

REFERENCES

[1] D. J. Aldous, *Unconditional bases and martingales in $L_p(F)$*, Math. Proc. Cambridge Philos. Soc., **85** (1979), 117–123.

[2] C. Bennett, K. Rudnicki, *On Lorentz-Zygmund spaces*, Dissertationes Math., **175** (1980), 1–67.

[3] J. Bergh, J. Löfström, *Interpolation spaces, an introduction*, Springer, Berlin-Heidelberg-New York, 1976.

[4] J. Bourgain, *Some remarks on Banach spaces in which martingale difference sequences are unconditional*, Ark. Mat., **22** (1983), 163–168.

[5] J. Bourgain, *Vector values singular integrals and the H^1-BMO duality*, in: *Probability Theory and Harmonic Analysis*, Marcel Dekker, New York-Basel, 1986.

[6] D. L. Burkholder, *Martingale transforms*, Ann. Math. Statist., **37** (1966), 1494–1504.

[7] D. L. Burkholder, *A geometrical characterization of Banach spaces in which martingale difference sequences are unconditional*, Ann. Probability, **9** (1981), 997–1011.

[8] D. L. Burkholder, *A geometrical condition that implies the existence of certain singular integrals of Banach-space-valued functions*, Proc. Conf. Harmonic Analysis in Honour of Antony Zygmund, Chicago, 1981. Wadsworth, Belmont, 1983.

[9] F. Cobos, *On the Lorentz-Marcinkiewicz operator ideal*, Math. Nachr., **126** (1986), 281–300.

[10] J. Gustavsson, *A function parameter in connection with interpolation of Banach spaces*, Math. Scand., **42** (1978), 289–305.

[11] S. Janson, P. Nilsson, J. Peetre, *Notes on Wolff's note on interpolation spaces*, Proc. London Math. Soc., **48** (1984), 283–299.

[12] T. F. Kalugina, *Interpolation of Banach spaces with a functional parameter, The reiteration theorem*, Moscow Univ. Math. Bull. **30** (6) (1975), 108–116.

[13] M. Levy, *L'espace d'interpolation réel* $(A_0, A_1)_{\vartheta,p}$ *contient* l^p, Compt. Rend. Acad. Sci. Paris, **289** (1979), A 675–A 677.

[14] G. G. Lorentz, *On the theory of spaces* Λ, Pacific J. Math., **1** (1951), 411–429.

[15] B. Maurey, *Systeme de Haar*, Séminaire Maurey-Schwartz (1974–1975, Ecole Polytechnique, Paris.

[16] C. Merucci, *Interpolation réelle avec paramètre fonctionnel des espaces* $L^{p,q}$, Compt. Rend. Acad. Sci. Paris, Sér. I, **294** (1982), 653–656.

[17] C. Merucci, *Interpolation réelle avec fonction paramètre: réiteration et applications aux espaces* $\Lambda^p(\varphi)$ $(0 < p \leqslant \infty)$, Compt. Rend. Acad. Sci. Paris, Sér. I, **295** (1982), 427–430.

[18] R. O'Neil, *Les fonctions conjuguées et les intégrales fractionnaires de la classe* $L(\log^+ L)^s$, Compt. Rend. Acad. Sci. Paris, **263** (1966), A 463–A 466.

[19] J. Peetre, *A theory of interpolation of normed spaces*, Lecture notes, Brasilia 1963 [Notes Mat., **39** (1968), 1–86].

[20] H. Triebel, *Interpolation Theory, Function Spaces, Differential Operators*, North--Holland, Amsterdam-New York-Oxford, 1978.

[21] A. Zygmund, *Trigonometric Series*, Vol. I and II. Cambridge Univ. Press, Cambridge, 1977.

Ф. Кобос, **О пространствах, в которых последовательности мартингальных разностей безусловны**

В статье обсуждается случай, когда пространства, рассматриваемые в гармоническом анализе, имеют свойство безусловности для мартингальных расностей. Приводятся также примеры операторных пространств, обладающих этим свойством.