I Preliminaries on bimodules

There are two notions of morphisms beetween von Neuman algebras M and N:

① $\rho : M \to N$ is a $*$ algebra homomorphism and is normal.

② $T : M \to N$ is a completely positive normal map.

The point of view ① plays an important role in the work of J. Roberts [ ] on actions of group duals on von Neumann algebras.

The point of view ② has its origin in probability theory (*) and plays a crucial role in the work of E. Effros and C. Lance [ ].

We shall introduce a third point of view and relate it to ① and ②.

Definition 1 A correspondence beetween M and N is a hilbert space $\mathcal{H}$ which is an N-M bimodule.

In other words we have commuting normal $*$ representations π_N of N and π^o_M of M^o in $\mathcal{H}$. To save notations we put $\pi_N(y)\pi_M(x^o)\xi = y\xi x \quad \forall \xi \in \mathcal{H}, \forall y \in N, \forall x \in M$.

To justify the terminology (one could simply call $\mathcal{H}$ an N-M bimodule) we consider first the special case when M and N are

(*) Where instead of a mapping from the measure space (X, μ_X) to the measure space (Y, μ_Y) one uses mappings from X to probability measures μ_x, $x \in X$, on Y such that $\int \mu_x \, d\mu_X(x) = \mu_Y$

commutative. Let then (X,ν_X), (Y,ν_Y) be standard measure spaces, $M = L^\infty(X,\nu_X)$ and $N = L^\infty(Y,\nu_Y)$. Then a correspondence $\mathcal{H}$ between M and N is given by a measure class ν on $X \times Y$ with projections $pr_X(\nu)$, $pr_Y(\nu)$ absolutely continuous with respect to ν_X, ν_Y, and an integer valued ν-measurable function $n(s,t)$ $((s,t) \in X \times Y)$. The hilbert space $\mathcal{H}$ is equal to $\int H_{(s,t)} \, d\nu(s,t)$ where $H_{(s,t)}$ is a hilbert space of dimension $n(s,t) \in \{0,1,\dots,\infty\}$ while the structure of bimodule is given by:

$$(g \xi f)(s,t) = g(t) f(s) \xi(s,t) \qquad \forall f \in M, g \in N, \xi \in \mathcal{H}.$$

In general the measure ν is not absolutely continuous with respect to $\nu_X \times \nu_Y$, this measure represents the graph of the correspondence, while the function n represents the multiplicity of the correspondence.

If in the above example we take $(X,\nu_X) = (Y,\nu_Y)$ and ν equal to the image of $\nu_X = \nu_Y$ on the diagonal $\Delta = \{(x,x), x \in X\}$ while $n(s,s) = 1$ $\forall s \in X$, we get the identity as a correspondence from M to $N = M$. The hilbert space $\mathcal{H}$ is equal to $L^2(X,\nu_X)$ and the bimodule structure corresponds to the standard representation of M.

Definition 2. Let M be a von Neumann algebra. The identity correspondence between M and M is the canonical bimodule $L^2(M)$ of the standard form of M. ([H])

To describe this bimodule, one may use an auxiliary faithful (semi finite normal) weight ν. Then the hilbert space $L^2(M,\nu)$ (completion of $\{x \in M, \nu(x^*x) < \infty\}$ with the obvious prehilbert space structure) is naturally equipped with:

A normal $*$ representation π_ν of M (by left multiplication)

An isometric antilinear involution J_ν such that:

$$J_\nu \pi_\nu(M) J_\nu = \pi_\nu(M)' \quad (\text{commutant of } \pi_\nu(M))$$

Then the equality $\pi^\circ_\nu(x^\circ) = J_\nu \pi_\nu(x)^* J_\nu$ defines a normal $*$ representation of M° in $L^2(M,\nu)$ which hence becomes an M-M bimodule.

The hilbert space $L^2(M,\nu)$ comes equipped also with a natural self dual cone $L^2(M,\nu)^+$ (cf [H]), whose elements are in bijection with the positive cone M_*^+ of the predual of M, by:

$$\xi \in L^2(M,\nu)^+ \to \omega_{\xi,\xi} \in M_*^+ \qquad ([H] \text{ Lemma } 2.10)$$

where $\omega_{\xi,\xi}(x) = \langle \pi_\nu(x)\xi, \xi \rangle$ $\forall x \in M$.

Given $\xi, \eta \in L^2(M,\nu)^+$ their scalar product $\langle \xi, \eta \rangle$ depends only on the associated elements ($\omega_{\xi,\xi}$ and $\omega_{\eta,\eta}$) of M_*^+ and not on the choice of the weight ν, which shows that the ordered bimodule is independent of the choice of ν, its positive elements can be described (without any reference to ν) as "square roots of elements of M_*^+".

Coming back to the commutative case $M = L^\infty(X,\nu_X)$, $N = L^\infty(Y,\nu_Y)$, we can take $\nu = \nu_X \times \nu_Y$, $m(s,t)=1$ $\forall (s,t)\in X\times Y$ so, to each $x\in X$ corresponds an arbitrary point of Y so that the associated correspondence is very coarse. In this case we have $\mathcal{H} = L^2(X,\nu_X)\otimes L^2(Y,\nu_Y)$ and $g(\xi\otimes\eta)f = \xi f\otimes g\eta$ $\forall \xi\in L^2(X,\nu_X)$, $f\in M$, $\eta\in L^2(Y,\nu_Y)$, $g\in N$.

<u>Definition 3</u> Let M and N be von Neumann algebras. The <u>coarse</u> correspondence between M and N is the N-M bimodule of Hilbert-Schmidt operators ρ from $L^2(M)$ to $L^2(N)$ where

$$\pi_N(y)\,\pi_M^o(x^o)\,\rho = y\cdot\rho\cdot x \quad \text{(composition of operators)}$$

$\forall y\in N$, $x\in M$.

This bimodule is isomorphic to $L^2(M)\otimes L^2(N)$ with:

$$\pi_N(y)\,\pi_M^o(x^o)(\xi\otimes\eta) = \xi x\otimes y\eta$$

(To $\xi\otimes\eta$ we associate the rank one operator from $L^2(M)$ to $L^2(N)$ which maps $\alpha\in L^2(M)$ to $\langle\alpha, J_M\xi\rangle\,\eta \in L^2(N)$.)

Let us now link our point of view (def 1) with ① - (*)

Let ρ be a normal $*$ homomorphism of M in N, we do not assume that $\rho(1)=1$, then $\rho(1)=e$ is a projection, and the Hilbert space $L^2(\rho) = \{\pi_N^o(e^o)\xi = \xi e,\ \xi\in L^2(N)\}$ is an N-M bimodule with:

$$\pi_N(y)\,\pi_M^o(x^o)\,\xi = y\,\xi\,\rho(x) \qquad \forall y\in N,\ x\in M.$$

<u>Proposition 4</u> Assume that N is properly infinite.

a) Every correspondence $\mathcal{H}$ (*) between M and N is equivalent to an $L^2(\rho)$.

b) The intertwining operators from $L^2(\rho_1)$ to $L^2(\rho_2)$ are the elements y of $\rho_2(1)\,N\,\rho_1(1)$ such that:

$$\rho_2(x)\,y = y\,\rho_1(x) \qquad \forall x\in M$$

<u>Proof</u> a) As N is properly infinite, the representation π_N of N in $\mathcal{H}$ is subequivalent to the standard representation of N in $L^2(N)$. Thus we can assume that $\mathcal{H} = L^2(N)e$, where e is a projection, $e\in N$, and that $\pi_N(y)\xi = y\xi$ $\forall y\in N$, $\xi\in L^2(N)e$. The commutant of $\pi_N(N)$ is then the algebra of right multiplications in $L^2(N)_e$ by elements of eNe, so π_M^o determines a normal

(*) To avoid useless complications we shall assume that both M and N have separable preduals and that $\mathcal{H}$ is separable.

$*$ homomorphism ρ, $\rho(1) = e$ of M in N_e, such that,

$$\pi^0_M(x^0)\xi = \xi\rho(x) \quad \forall x \in M,\ \forall \xi \in L^2(N)e -$$

b) With the obvious notations, the intertwining operators from π^1_N to π^2_N correspond to the elements of $\rho_2(1) N \rho_1(1)$, and the intertwining condition with respect to the action of M is exactly

$$y\rho_1(x) = \rho_2(x)y \quad \forall x \in M. \qquad \text{Q.E.D}$$

If N is not properly infinite, proposition 4 does not hold (in general there need not be any non zero $*$ homomorphism of M in N while there is always the coarse correspondence between M and N) this however will not create any difficulty since, letting F_∞ be the factor of type I_∞ of all bounded operators in $\ell^2(\mathbb{N})$, the von Neumann algebra $\tilde{N} = N \otimes F_\infty$ is properly infinite and replacing N by $\tilde{N}$ does not affect the correspondences from M to N (Let $\mathcal{H}$ be a correspondence from M to N, then $\mathcal{H} \otimes \ell^2$ is in an obvious way a correspondence from M to $\tilde{N}$. Conversely, let $e = 1 \otimes e_{11} \in \tilde{N}$, where $(e_{ij})_{i,j \in \mathbb{N}}$ is the canonical system of matrix units in F_∞, then if $\tilde{\mathcal{H}}$ is a correspondence from M to $\tilde{N}$ the subspace $e\tilde{\mathcal{H}}$ is a correspondence from M to $\tilde{N}_e = N$.)

Let $\mathcal{H}$ be a correspondence from M to N and M_1, N_1 be von Neumann subalgebras of M, N. It is clear that by restriction of the bimodule structure of $\mathcal{H}$ we obtain a correspondence from M_1 to N_1. This operation of restriction does not look so natural from the point of view ①, so even though ① and ③ are equivalent (prop 4) it is important to keep both of them.

<u>Example 5</u> Let Γ be a countable group acting freely by non singular transformations of the measure space (X, ν_X), then the restriction to $L^\infty(X, \nu_X) \subset M = L^\infty(X, \nu_X) \rtimes \Gamma$ (the crossed product by Γ) of the identity correspondence of M is the graph in $X \times X$, with its natural measure class, of the equivalence relation $x \sim y$ iff $\exists g \in \Gamma$, $gx = y$.

We pass now to the link of our point of view (def 1) with ②. We first assume that M and N are commutative, $M = L^\infty(X, \nu_X)$, $N = L^\infty(Y, \nu_Y)$ and we fix a, not necessarily finite, positive measure $\nu \sim \nu_Y$. Let P be a (completely) positive normal map of M in N, then we can find a measurable map μ of Y in the space of finite positive measures on X with

$$(Pf)(y) = \int_X f(x)\, d\mu_y(x)$$

Conversely any bounded measurable map μ such that $\int \mu_y\, d\nu_Y(y)$ is absolutely continuous with respect to ν_X

determines a normal (completely) positive map $P: M \to N$ -
Now assume that $P(1) \in L^1(\nu)$ and let μ be the finite measure in $X \times Y$ such that:

$$\int f\, d\mu = \int_Y \left(\int_X f(s,t)\, d\mu_t(s) \right) d\nu(t)$$

Since $pr_X(\mu)$, $pr_Y(\mu)$ are absolutely continuous with respect to μ_X, μ_Y, we see that μ defines a correspondence from M to N (we take the multiplicity $n(s,t) = 1$) and that in the bimodule $\mathfrak{H} = L^2(X \times Y, \mu)$ the vector ξ, $\xi(s,t) = 1$ $\forall s,t$, satisfies:

α) $N \xi M$ is dense in $\mathfrak{H}$ -

β) $\langle g \xi f, \xi \rangle = \nu(P(f) g) \quad \forall f \in M, g \in N$ -

We now extend this result to the general case:

Proposition 6 Let M, N be von Neumann algebras, P a completely positive map from M to N, ν a faithful weight on N such that $\nu(P(1)) < \infty$ - Then there exists a unique pair $(\mathfrak{H}, \xi)$ where $\mathfrak{H}$ is a correspondence from M to N, $\xi \in \mathfrak{H}$ and:

α) $N \xi M$ is dense in $\mathfrak{H}$

β) ξ is a ν-bounded vector (cf [ ] Def 1.) and for any $x \in M$ and $y \in N$, $\nu(y^* y) < \infty$ one has:

$$\langle y \xi x, \xi \rangle = \langle \eta_\nu(y), J_\nu \eta_\nu(P(x)) \rangle \qquad (\text{in } L^2(N,\nu))$$

Proof It is a simple extension of the results of [ ] from the case of states to the case of weights - To the faithful weight ν on N corresponds a completely positive linear map I_ν of the subspace $\mathrm{Dom}\, I_\nu$ of N_* spanned by the set $\{\varphi \in N_*^+,\ \varphi \leq \nu\}$, in N^o with:

$$\langle \eta_\nu(y), J_\nu \eta_\nu(I_\nu(\varphi)^o) \rangle = \varphi(y) \qquad \forall y \in N, \nu(y^* y) < \infty$$

The image of I_ν is exactly the linear span in N^o of $\{z^o ; z \in N^+, \nu(z) < \infty\}$ and I_ν^{-1} is also completely positive - So the equality $\varrho(x) = I_\nu^{-1}(P(x^o)^o)$ determines a completely positive normal map of M^o in the predual of N and hence by [ ] 2.2 and 2.3 there exists a binormal positive linear functional Ψ on $N \otimes_{bin} M^o$ such that:

$$\Psi(y \otimes x^o) = \varrho(x^o)(y) \qquad \forall x \in M, y \in N$$

By the Gelfand Naimark Segal construction we thus get

α) A binormal representation of $N \otimes_{bin} M^o$ in a hilbert space $\mathfrak{H}$ with cyclic vector ξ (i.e. an N-M bimodule).

β) the equality $\langle y \xi x, \xi \rangle = \Psi(y \otimes x^o) \quad \forall x \in M, y \in N$ (i.e. the equality 6 β)) Q.E.D.

Remark 7 If ν is finite the condition $\nu(P(1)) < \infty$ is automatic, moreover the equality 6 β) becomes, with $\xi_\nu = \eta(1)$

$\langle y \xi x, \xi \rangle = \langle y \xi_\nu P(x), \xi_\nu \rangle$ where we consider $L^2(N,\nu)$ as an N bimodule -

For $z \in$ Domain ν and $y \in N$ we put $S_\nu(y, z) = I_\nu^{-1}(z^\circ)(y)$, so in 6(β) we get $\langle y \xi x, \xi \rangle = S_\nu(y, P(x))$ $\forall x \in M, y \in N$.

Corollary 8 If N is properly infinite and $P : M \to N$ is a completely positive normal map, there exists a normal $*$ homomorphism $\rho : M \to N$ and a partial isometry $v \in N$, $v^* v \le \rho(1)$, $v v^* =$ Support $P(1)$, with $P(x) = P(1)^{1/2} v \rho(x) v^* P(1)^{1/2}$

Proof Let ν be a faithful state on N and $\mathcal{H}, \xi$ as in prop. 6. Then by prop. 4 we can replace $\mathcal{H}$ by $L^2(\rho)$ where ρ is a $*$ homomorphism from M to N, let $e = \rho(1) \in N$. We consider $L^2(\rho)$ as the subspace $L^2(N)e$ of $L^2(N)$, so $\xi \in L^2(N)$, $\xi e = \xi$ and:

$$\langle y \xi \rho(x), \xi \rangle = \langle y \xi_\nu P(x), \xi_\nu \rangle \quad \forall y \in N, x \in M$$

We now identify $L^2(N)$ with $L^2(N, \nu)$, so ξ and ξ_ν belong to the same hilbert space and $\langle y \xi, \xi \rangle = \langle y \xi_\nu P(1)^{1/2}, \xi_\nu P(1)^{1/2} \rangle$ $\forall y \in N$. So there exists a unique partial isometry $v \in Ne$, with final support the support of $P(1)$, such that:

$$\xi_\nu P(1)^{1/2} v = \xi$$

Then we have $\langle y_1 \xi_\nu P(1)^{1/2} v \rho(x) v^* P(1)^{1/2}, y_2 \xi_\nu \rangle$ $= \langle y_1 \xi_\nu P(x), y_2 \xi_\nu \rangle$ $\forall y_1, y_2 \in N$, and since ν is faithful we have $P(x) = P(1)^{1/2} v \rho(x) v^* P(1)^{1/2}$ $\forall x \in M$. Q.E.D.

Proposition 9 Let $\mathcal{H}$ be a correspondence from M to N, ν a faithful weight on N, and ξ a ν-bounded vector ([ ] Def 1). Then there exists a unique completely positive map P from M to N such that for any $x \in M$, $y \in N$, one has:

$$\langle y \xi x, \xi \rangle = S_\nu(y, P(x))$$

Proof Let Ψ be the binormal positive linear functional on $N \otimes_{bin} M^\circ$ such that $\Psi(y \otimes x^\circ) = \langle y \xi x, \xi \rangle$, $\forall y \in N$, $x \in M$. By [ ] 2.2. there exist a completely positive normal map of M° in N_* such that:

$$\Psi(y \otimes x^\circ) = Q(x^\circ)(y) \quad \forall x \in M, y \in N.$$

As ξ is ν-bounded the image of M° by Q is contained in the domain of I_ν (because $Q(1)$ is majorized by $c\nu$ for some $c < \infty$) so there exists a completely positive normal map P of M in N such that $I_\nu \circ Q(x^\circ) = P(x)^\circ$ $\forall x \in M$. So as in prop. 6 the equality follows. Q.E.D.

Note that (see [ ] Lemma 2) the subspace $D(\mathcal{H}, \nu)$ of ν-bounded vectors is always dense in $\mathcal{H}$. For each pair ξ_1, ξ_2 of elements of $D(\mathcal{H}, \nu)$ we let $(\xi_1, \xi_2)_\nu$ be the unique normal map P of M in N such that:

$$\langle y \xi_1 x, \xi_2 \rangle = S_\nu(y, P(x)) \quad , \quad \forall x \in M, y \in N.$$

For any $a \in M$ we have $(\xi_1 a, \xi_2)_\nu (x) = (\xi_1, \xi_2)_\nu (ax)$ and $(\xi_1, \xi_2 a)_\nu (x) = (\xi_1, \xi_2)_\nu (x a^*)$ for any $x \in M$.

Lemma 10 a) Let $b \in N$ be such that $t \to \sigma_t^\nu(b) \in N$ extends analytically from $t \in \mathbb{R}$ to $\mathrm{Im}\, t \in [0, \frac{1}{2}]$. Then for any $\xi_1 \in D(\mathfrak{H}, \nu)$ one has $b\xi_1 \in D(\mathfrak{H}, \nu)$ and $(b\xi_1, \xi_2)_\nu(x) = \sigma^\nu_{i/2}(b)(\xi_1, \xi_2)_\nu(x)$ $\forall x \in M$ (and also $(\xi_1, b\xi_2)_\nu(x) = (\xi_1, \xi_2)_\nu(x)(\sigma^\nu_{i/2}(b))^*$ $\forall x \in M$)

b) Let ν' be another weight on N, with $\nu' \geq \lambda\nu$ for some $\lambda > 0$, then $D(\mathfrak{H}, \nu) \subset D(\mathfrak{H}, \nu')$, the Radon Nikodym $(D\nu' : D\nu)_t \in \mathbb{R}$ extends analytically from $t \in \mathbb{R}$ to $\mathrm{Im}\, t \in [-\frac{1}{2}, 0]$ and with $b = (D\nu' : D\nu)_{-i/2}$ one has for any $\xi_1, \xi_2 \in D(\mathfrak{H}, \nu)$, $x \in M$:

$$(\xi_1, \xi_2)_{\nu'}(x) = b^* (\xi_1, \xi_2)_\nu(x)\, b$$

Proof a) For $y \in N$, $\nu(y^*y) < \infty$, one has $\eta_\nu(yb) = J_\nu \pi_\nu(c)^* J_\nu \eta_\nu(y)$ where $c = \sigma^\nu_{i/2}(b) \in N$. So $b\xi_1 \in D(\mathfrak{H}, \nu)$ and letting $P = (\xi_1, \xi_2)_\nu$ we have for y as above, $x \in M$:

$$< y b \xi_1 x, \xi_2 > = < \eta_\nu(yb), J_\nu \eta_\nu(P(x)) > =$$

$$< \eta_\nu(y), J_\nu \eta_\nu(c P(x)) >$$

b) We have $\nu(y) = \nu'(b^* y b)$ $\forall y \in N^+$ (cf [ ]).

We then have for any $y, z \in N$, $\nu'(y^*y) < \infty$, $\nu(z^*z) < \infty$ that:

$$< \eta_\nu(y), J_\nu \eta_\nu(z) > = < \eta_{\nu'}(y), J_{\nu'} \eta_{\nu'}(b^* z b) >$$

So $I_{\nu'}(\varphi)^\circ = b^* I_\nu(\varphi)^\circ b$ for any $\varphi \in$ Domain I_ν and, as in the proof of Prop. 9, we have for $x \in M$:

$$(\xi_1, \xi_2)_{\nu'}(x) = (I_{\nu'}\, Q(x^\circ))^\circ = b^* (\xi_1, \xi_2)_\nu(x)\, b \quad \text{Q.E.D.}$$

Remark 11 We see from lemma 10 how the coefficients $(\xi_1, \xi_2)_\nu$ of the correspondence $\mathfrak{H}$ depend on the choice of ν, we could also consider the coefficients independantly of ν as completely positive maps $Q = (\xi_1, \xi_2)$ from M to the predual N°_* of N°, with:

$$< y \xi_1 x, \xi_2 > = Q(x)^\circ(y) \quad \forall x \in M, y \in N$$

However we would then loose the possibility of composing completely positive maps.

II Tensor products of bimodules (composition of correspondences)

In the previous section we have related our point of view (def 1) ③ with the two classical notions ① and ② of morphisms between von Neumann algebras. While for ① and ② the composition of morphisms is a fairly obvious notion, the definition of composition for correspondences requires some care. It will coincide with the notion of tensor product for bimodules. So we let M_1, N, M_2 be three von Neumann algebras, $\mathcal{H}_1$ a correspondence from M_1 to N and $\mathcal{H}_2$ a correspondence from N to M_2. Then $\mathcal{H}_1$ is in particular a left N-module and $\mathcal{H}_2$ a right N-module, we now want to construct the tensor product, over N, of these two modules (cf [ ] for the case when N is commutative). Exactly as for the link between ② and ③ (i.e prop 6 and 9 above) we shall fix on N an auxiliary faithful weight ν.

On the algebraic tensor product $\mathcal{H}_2 \odot D(\mathcal{H}_1,\nu)$ of $\mathcal{H}_2$ by the dense subspace of ν-bounded vectors in $\mathcal{H}_1$ we define a sesquilinear form by the equality:

$$\langle \xi_2\otimes\xi_1, \eta_2\otimes\eta_1\rangle = \varphi_2(I_\nu(\varphi_1))$$

where $\varphi_1(y) = \langle y\xi_1,\eta_1\rangle$ $\forall y\in N$ (note that $\varphi_1 \in \text{Domain } I_\nu$)

and $\varphi_2(y) = \langle \xi_2 y, \eta_2\rangle$ $\forall y\in N$

Proposition 12 a) The above sesquilinear form is positive, we let $\mathcal{H}_2\otimes_\nu\mathcal{H}_1$ be the corresponding Hilbert space.

b) One obtains the same result if one completes the tensor product $D(\mathcal{H}_2,\nu^o)\odot\mathcal{H}_1$ with $\langle\xi_2\otimes\xi_1,\eta_2\otimes\eta_1\rangle = \varphi_1(I_{\nu^o}(\varphi_2))$

c) For every $A\in\mathcal{L}_N(\mathcal{H}_2)$ there is a unique bounded operator $A\otimes_\nu 1$ in $\mathcal{H}_2\otimes_\nu\mathcal{H}_1$ such that:

$$(A\otimes_\nu 1)(\xi_2\otimes_\nu\xi_1) = A\xi_2\otimes_\nu\xi_1 \qquad \forall \xi_2\in\mathcal{H}_2,\ \xi_1\in D(\mathcal{H}_1,\nu)$$

We let $\xi_2\otimes_\nu\xi_1$ be the image in $\mathcal{H}_2\otimes_\nu\mathcal{H}_1$ of the element $\xi_2\otimes\xi_1$ of $\mathcal{H}_2\odot D(\mathcal{H}_1,\nu)$.

Proof a) the positivity follows from the complete positivity of I_ν. One could also check it by taking the N-modules $\mathcal{H}_2$ and $\mathcal{H}_1$ equal to $L^2(N)$ and then passing to the general case.

b) If $\xi_j,\eta_j\in D(\mathcal{H}_j,\nu)$ for $j=1,2$ then one has:

$$\varphi_2(I_\nu(\varphi_1)) = \langle \eta_\nu(I_\nu(\varphi_1)^o), J_\nu\,\eta_\nu(I_{\nu^o}(\varphi_2))\rangle = \varphi_1(I_{\nu^o}(\varphi_2));$$

It remains to check that for $\xi_2\in\mathcal{H}_2$, $\xi_1\in D(\mathcal{H}_1,\nu)$ the vector $\xi_2\otimes_\nu\xi_1$ is a limit of vectors $\xi_2'\otimes_\nu\xi_1$, $\xi_2'\in D(\mathcal{H}_2,\nu^o)$. But we know that $D(\mathcal{H}_2,\nu^o)$ is dense in $\mathcal{H}_2$ and we have:

$$\|\xi'\otimes_\nu\xi_1\|^2 \le \|\xi'\|^2\,\|I_\nu(\varphi_1)\|^2 \qquad \forall\xi'\in\mathcal{H}_2.$$

c) The uniqueness is clear. The existence also, since it is enough to treat the case of unitaries $A\in\mathcal{L}_N(\mathcal{H})$. Q.E.D.

Corollary 13 a) Let H_1 (resp H_2) be a correspondence from M_1 to N (resp N to M_2). Then $H_2 \otimes_\nu H_1$ defines a correspondence from M_1 to M_2.

b) For $\xi_1, \eta_1 \in D(H_1, \nu)$, $\xi_2, \eta_2 \in H_2$ one has:

$$(\xi_2 \otimes_\nu \xi_1, \eta_2 \otimes_\nu \eta_1) = (\xi_2, \eta_2) \circ (\xi_1, \eta_1)_\nu$$

Proof a) follows from 12 c).

b) Let $P_1 = (\xi_1, \eta_1)_\nu$, $Q_2 = (\xi_2, \eta_2)$, $Q = (\xi_2 \otimes_\nu \xi_1, \eta_2 \otimes_\nu \eta_1)$, we have:

$$\langle x_2 (\xi_2 \otimes_\nu \xi_1) x_1, \eta_2 \otimes_\nu \eta_1 \rangle = Q(x_1)^\circ (x_2) \quad \forall x_i \in M_i \text{. But}$$

$$\langle (x_2 \xi_2) \otimes_\nu (\xi_1 x_1), \eta_2 \otimes_\nu \eta_1 \rangle = \varphi_2'(I_\nu(\varphi_1')^\circ) \quad \text{where:}$$

$$\varphi_1'(y) = \langle y(\xi_1 x_1), \eta_1 \rangle, \quad \varphi_2'(y) = \langle (x_2 \xi_2) y, \eta_2 \rangle \quad \forall y \in N$$

So $\varphi_1'(y) = \langle \eta_\nu(y), J_\nu \eta_\nu(P_1(x_1)) \rangle$ $\forall y \in N$, $\nu(y^* y) < \infty$ while $\varphi_2'(y) = Q_2(y)^\circ (x_2)$ $\forall y \in N$. So we have $I_\nu(\varphi_1')^\circ = P_1(x_1)$ and making $y = P_1(x_1)$ we get:

$$\varphi_2'(I_\nu(\varphi_1')^\circ) = Q_2(P_1(x_1))^\circ (x_2) \quad \text{i.e.}$$

$$Q(x_1) = Q_2(P_1(x_1)) \quad \forall x_1 \in M_1 \text{ . Q.E.D.}$$

We note that in particular, if ν_2 is a faithful weight on M_2 we deduce from b) that for $\xi_2, \eta_2 \in D(H_2, \nu_2)$ we have:

$$(\xi_2 \otimes_\nu \xi_1, \eta_2 \otimes_\nu \eta_1)_{\nu_2} = (\xi_2, \eta_2)_{\nu_2} \circ (\xi_1, \eta_1)_\nu$$

So if we deal with correspondences from N to N we get that the coefficients of the composition $H_2 \otimes_\nu H_1$ are the composition of the coefficients.

We now relate the composition of correspondences with the composition of $*$ homomorphisms. We let ρ_1 be a homomorphism of M_1 in N and ρ_2 of N in M_2.

Proposition 14 The correspondence $L^2(\rho_2) \otimes_\nu L^2(\rho_1)$ is canonically equivalent with $L^2(\rho_2 \circ \rho_1)$.

It follows from a slightly more general statement: let H_2 be any correspondence from N to M_2, and consider on the subspace $H_2 \rho_1(1)$ the structure of bimodule given by:

$$\pi_{M_2}(x_2)\, \pi^\circ_{M_1}(x_1^\circ)\, \xi_2 = x_2 \xi_2 \rho_1(x_1) \quad \forall x_i \in M_i \text{ .}$$

Lemma 15 The above bimodule $H_2 \rho_1(1)$ is canonically equivalent with $H_2 \otimes_\nu L^2(\rho_1)$

Proof To each ν-bounded vector ξ_1 in $L^2(\rho_1) = L^2(N) \rho_1(1)$ we want to associate a bounded linear map $A(\xi_1)$ of H_2 in $H_2 \rho_2(1)$ so that $\xi_2 \otimes_\nu \xi_1 \to A(\xi_1) \xi_2$ defines the required equivalence. First we identify $L^2(N)$ with $L^2(N, \nu)$, hence, as ξ_1 is ν-bounded, there exists a unique $y_1 \in N$ such that $\xi_1 = J_\nu \eta_\nu(y_1^*)$. As $\xi_1 \rho_1(1) = \xi_1$ we have $J_\nu \pi_\nu(\rho_1(1))^* J_\nu \xi_1 = \xi_1$ and hence $y_1 \rho_1(1) = y_1$. We put $A(\xi_1) \xi_2 = \xi_2 y_1$ $\forall \xi_2 \in H_2$.

We first have to check the equality:

$$\langle A(\xi_1) \xi_2, A(\eta_1) \eta_2 \rangle = \langle \xi_2 \otimes_\nu \xi_1, \eta_2 \otimes_\nu \eta_1 \rangle$$

With the obvious notations, we have, for any $y \in N$:

$$\varphi_1(y) = \langle y \xi_1, \eta_1 \rangle = \langle \pi_\nu(y)\, J_\nu \eta_\nu(y_1^*), J_\nu \eta_\nu(z_1^*) \rangle$$

If $\nu(y^*y)<\infty$ we have $\pi_\nu(y)\, J_\nu\, \eta_\nu(y_1^*) = J_\nu\, \pi_\nu(y_1^*)\, J_\nu\, \eta_\nu(y)$

Thus we get $I_\nu(\varphi_1)^0 = y_1\xi_1^*$, $\varphi_2(I_\nu(\varphi_1)^0) = \varphi_2(y_1\xi_1^*)$

$= \langle \xi_2 y_1, \eta_2\xi_1\rangle = \langle A(\xi_1)\xi_2, A(\eta_1)\eta_2\rangle$ -

Let U be the isometry of $\mathfrak{H}_2\otimes_\nu L^2(\rho_1)$ in $\mathfrak{H}_2\,\rho_1(1)$ such that $U(\xi_2\otimes_\nu\xi_1) = A(\xi_1)\xi_2$ - For any $y\in N$, $\nu(yy^*)<\infty$ the vector $J_\nu\, \eta_\nu(\rho_1(1)^*y^*) = \xi_1$ belongs to $L^2(\rho_1)$, is ν-bounded, and $A(\xi_1)$ is the right multiplication by $y\rho_1(1)$ - This shows that U is onto -

Finally the equality $(J_\nu\,\pi_\nu(\rho_1(x_1)^*)\,J_\nu)\,J_\nu\,\eta_\nu(y_1^*) = J_\nu\,\eta_\nu((y_1\rho_1(x_1))^*)$ shows that $U(\xi_2\otimes_\nu\xi_1\rho_1(x_1)) = U(\xi_2\otimes_\nu\xi_1)\rho_1(x_1)$ $\forall x_1\in M_1$ so that one checks that U is an intertwining operator. Q.E.D.

Remark 16 a) Unlike the usual tensor product of hilbert spaces the tensor product $\mathfrak{H}_2\otimes_\nu\mathfrak{H}_1$ is no longer commutative - (cf. Prop 14)

b) Unless the weight ν is a trace we do not have the equality $\xi_1 y\otimes_\nu\xi_2 = \xi_1\otimes_\nu y\xi_2$; it has to be replaced by:
$\xi_1 y_1\otimes_\nu\xi_2 = \xi_1\otimes_\nu y_2\xi_2$ where $t\to\sigma_t^\nu(y_1)$ extends analytically for $\mathrm{Im}\, t\in[-\frac{1}{2},0]$ and $y_2=\sigma^\nu_{-i/2}(y_1)$ -

c) From lemma 15 we see that $\mathfrak{H}_2\otimes_\nu L^2(\mathrm{id}_N)$ is canonically equivalent to $\mathfrak{H}_2$ for any correspondence $\mathfrak{H}_2$ from N to M_2 - Using the above proposition 17 one sees that the same fact is true for $L^2(\mathrm{id}_N)\otimes_\nu\mathfrak{H}_1$ -

As for unitary representations of groups, to each correspondence $\mathfrak{H}$ from M to N is associated canonically its <u>contragredient</u> $\bar{\mathfrak{H}}$, which is a correspondence from N to M with underlying hilbert space the conjugate of $\mathfrak{H}$ (we let $\xi\to\bar\xi$ be the canonical antilinear isometry of $\mathfrak{H}$ onto $\bar{\mathfrak{H}}$) and bimodule structure given by: $x\bar\xi y = (y^*\xi x^*)^-$ $\forall \xi\in\mathfrak{H}$, $x\in M$, $y\in N$ -

Proposition 17 a) Let $\rho: M\to N$ be a $*$ isomorphism, then $\overline{L^2(\rho)}$ is canonically equivalent to $L^2(\rho^{-1})$

b) With the notations of proposition 12, $(\mathfrak{H}_2\otimes_\nu\mathfrak{H}_1)^-$ is canonically equivalent to $\bar{\mathfrak{H}}_1\otimes_\nu\bar{\mathfrak{H}}_2$ -

Proof a) Let U be the unique isometry of $L^2(N)^+$ on $L^2(M)^+$ such that for any $\xi\in L^2(N)^+$ one has for $x\in M$:
$\langle\rho(x)\xi,\xi\rangle = \langle xU\xi,U\xi\rangle$

Let J_N, J_M be the canonical isometric involutions of $L^2(N)$, $L^2(M)$. Then $L^2(\rho)$ is equal to $L^2(N)$ with bimodule structure given by
$y\xi x = y\,J_N\,\rho(x)^*\,J_N\,\xi$ $\forall y\in N$, $x\in M$, $\xi\in L^2(N)$.

While $L^2(\rho^{-1})$ is equal to L^2M with:
$x\xi y = x\,J_M\,\rho^{-1}(y)^*\,J_M\,\xi$ $\forall x\in M$, $y\in N$, $\xi\in L^2M$,

thus with $V\bar\xi = UJ_N\xi$ $\forall\xi\in L^2(N)$ one gets the required equivalence since $\rho(x) = U^{-1}xU$ $\forall x\in M$ and $UJ_N = J_M U$.

b) Put $V(\overline{\xi_2\otimes_\nu\xi_1}) = \bar\xi_1\otimes_\nu\bar\xi_2$ for $\xi_j\in D(\mathfrak{H}_j,\nu)$ $j=1,2$ and check that V is an isometry. Q.E.D.

III Completely positive maps and operators in L^2

Let P be a completely positive map from M to N, ν_N a faithful weight on N such that $\nu_N(P(1))<\infty$. Then let $(\mathcal{H},\xi)$ be as in proposition 6, the correspondence $\overline{\mathcal{H}}$ from N to M contains the vector $\overline{\xi}$, the first question is when is $\overline{\xi}$ a ν_M-bounded vector where ν_M is a faithful weight on M.

Lemma 18 Let ν_M be a faithful weight on M, then $\overline{\xi}$ is ν_M bounded iff there exists $c<\infty$ such that $\nu_N\circ P\le c\,\nu_M$.

Proof By definition ([ ] Def 1) $\overline{\xi}$ is ν_M bounded iff there exists $c<\infty$ such that $\|\xi x^*\|^2\le c\,\nu_M(x^*x)$ $\forall x\in M$, $\nu_M(x^*x)<\infty$.
For any $y\in N$, and $x\in M$, one has $S_\nu(y,P(x^*x))=\langle y\xi x^*x,\xi\rangle = \langle y\xi x^*,\xi x^*\rangle$ so that one gets:

$$\|\xi x^*\|^2=\nu_N(P(x^*x)) \qquad \forall x\in M \qquad \text{Q.E.D.}$$

So let ν_M be a faithful weight on M such that $\nu_N\circ P\le c\,\nu_M$, then by proposition 9, there exists a unique completely positive map P^* of N in M satisfying the following equality:

$$\langle x\overline{\xi}y,\overline{\xi}\rangle = S_\nu(x,P^*(y)) \qquad \forall x\in M,\ y\in N$$

But we have: $\langle x\overline{\xi}y,\overline{\xi}\rangle=\langle\overline{y^*\xi x^*},\overline{\xi}\rangle=\langle\xi,y^*\xi x^*\rangle$ $=\langle y\xi x,\xi\rangle = S_\nu(y,P(x))$. Thus we get:

$$S_\nu(x,P^*(y))=S_\nu(y,P(x)) \qquad \forall x\in M,\ y\in N$$

We now interpret this adjoint P^* of P. Given a faithful weight ν on the von Neumann algebra N, there is a canonical map $\eta_\nu^\natural$ of $\{y\in N,\ \nu(y^*y)<\infty,\ \nu(yy^*)<\infty\}$ in $L^2(N,\nu)$ (the canonical identity correspondence) such that:

$$\eta_\nu^\natural(y)=\Delta_\nu^{1/4}\,\eta_\nu(y)$$

One has $\eta_\nu^\natural(y^*)=J_\nu\,\eta_\nu^\natural(y)$ and the image by $\eta_\nu^\natural$ of the intersection of its domain with N^+ is dense in the self dual cone $L^2(N,\nu)^+$.

Proposition 19 a) Let ν_M,ν_N be faithful weights on M and N, P a completely positive map from M to N with $\nu_N\circ P\le c\,\nu_M$, $\nu_N(P(1))<\infty$ and let $\lambda\in[0,\frac12]$. There exists a unique bounded operator $\pi_\lambda(P)$ of $L^2(M,\nu_M)$ in $L^2(N,\nu_N)$ such that:

$$\pi_\lambda(P)\,\Delta_M^\lambda\,\eta_\nu(x)=\Delta_N^\lambda\,\eta_\nu(P(x)) \qquad \forall x\in M,\ \nu_M(x^*x+xx^*)<\infty$$

b) With the above notations $\pi_{1/4}(P^*)$ is the adjoint of $\pi_{1/4}(P)$.

c) $\pi_\lambda(P_2\circ P_1)=\pi_\lambda(P_2)\circ\pi_\lambda(P_1)$

d) $\pi_{1/4}(P)(L^2(M,\nu_M)^+)\subset L^2(N,\nu_N)^+$.

Proof a) First, as $\nu_N(P(1))<\infty$ one has $P(x)\in$ Domain η_N for any $x\in M$ so that $\Delta_N^\lambda\,\eta_\nu(P(x))$ makes sense. The uniqueness of $\pi_\lambda(P)$ follows from the density in $L^2(M,\nu_M)$ of $\{\Delta_M^\lambda\,\eta_\nu(x),\ \nu(x^*x+xx^*)<\infty\}$.

Let $(\mathfrak{h}, \xi)$ be associated to P as in proposition 6 – For $x \in M$, $y \in N$ we have $|\langle y\xi x, \xi\rangle| \leq \|y\xi\| \, \|\xi x^*\| \leq$ $c_1 \nu_N(y^*y)\; c_2 \nu_M(x^*x)$ – Thus there exists a bounded operator T_1 of $L^2(M, \nu_M)$ in $L^2(N, \nu_N)$ such that:

$$\langle T_1 \eta_M(x), J_N \eta_N(y)\rangle = \langle y\xi x, \xi\rangle, \quad \forall x \in M, y \in N$$

$\nu_M(x^*x) < \infty$, $\nu_N(y^*y) < \infty$ –

But we also have $|\langle y\xi x, \xi\rangle| \leq \|y^*\xi\| \, \|\xi x\| \leq$ $c_1 c_2\, \nu_N(yy^*)\, \nu_M(xx^*)$, hence the existence of T_2 with:

$$\langle T_2 J_M \eta_M(x^*), \eta_N(y^*)\rangle = \langle y\xi x, \xi\rangle \quad \forall x \in M, y \in N$$

$\nu_M(xx^*) < \infty$, $\nu_N(yy^*) < \infty$ –

Let $x \in M$, $y \in N$, $\nu_M(x^*x + xx^*) < \infty$, $\nu_N(y^*y + yy^*) < \infty$; then $\eta_M(x)$ (resp $\eta_N(y)$) is in the domain of $\Delta_M^{1/2}$ (resp $\Delta_N^{1/2}$) so:

$$\langle T_2 \Delta_M^{1/2} \eta_M(x), \Delta_N^{-1/2} J_N \eta_N(y)\rangle = \langle T_1 \eta_M(x), J_N \eta_N(y)\rangle$$

Thus for any $\alpha \in$ Domain $\Delta_M^{1/2}$, $\beta \in$ Domain $\Delta_N^{-1/2}$ one has:

$$\langle T_2 \Delta_M^{1/2} \alpha, \Delta_N^{-1/2} \beta\rangle = \langle T_1 \alpha, \beta\rangle$$

By interpolation (cf [ ]) one gets that for $\alpha \in \mathrm{Dom}\, \Delta_M^{\lambda}$ $(\lambda \in [0, \frac{1}{2}])$ $\beta \in \mathrm{Dom}\, \Delta_N^{-\lambda}$ one has: $|\langle T_1 \alpha, \beta\rangle| \leq c' \|\Delta_M^{\lambda} \alpha\| \, \|\Delta_N^{-\lambda} \beta\|$ –

So with x and y as above we get:

$$|\langle y\xi x, \xi\rangle| \leq c' \|\Delta_M^{\lambda} \eta_\nu(x)\| \, \|\Delta_N^{\lambda} \eta_\nu(y)\|$$

But $\langle \Delta_N^{\lambda} \eta_\nu(P(x)), J_\nu \Delta_N^{\lambda} \eta_\nu(y)\rangle = \langle \eta_\nu(P(x)), J_\nu \eta_\nu(y)\rangle$ $= \langle \eta_\nu(y), J_\nu \eta_\nu(P(x))\rangle = \langle y\xi x, \xi\rangle$ –

This shows that $\pi_\lambda(P)$ is bounded –

b) For $x \in M$, $y \in N$, $\nu_M(x^*x + xx^*) < \infty$, $\nu_N(y^*y + yy^*) < \infty$ one has $\langle \pi_{\frac14}(P)\eta^{\natural}(x), J_N \eta^{\natural}(y)\rangle = \langle y\xi x, \xi\rangle =$

$$\langle \pi_{\frac14}(P^*)\eta^{\natural}(y), J_M \eta^{\natural}(x)\rangle = \langle \eta^{\natural}(x), \pi_{\frac14}(P^*) J_N \eta^{\natural}(y)\rangle$$

because the equality $P^*(x^*) = P^*(x)^*$ $\forall x \in M$ shows that $\pi_{\frac14}(P^*) J_N = J_M \pi_{\frac14}(P^*)$ –

c) For $x \in M_1$, $\nu_{M_1}(x^*x + xx^*) < \infty$ one has:

$\pi_\lambda(P_2 \circ P_1)\Delta_M^{\lambda}\eta(x) = \Delta_M^{\lambda}\eta\, P_2(P_1(x)) = \pi_\lambda(P_2)\pi_\lambda(P_1)\Delta_M^{\lambda}\eta(x)$

d) For $x \in M^+$, $\nu_M(x^2) < \infty$, one has $\eta^{\natural}(x) \in L^2(M, \nu)^+$ and $\pi_{\frac14}(P)\eta^{\natural}(x) = \eta^{\natural}(P(x)) \in L^2(N, \nu)^+$ – As $L^2(M, \nu)^+$ is the closure of the set of such $\eta^{\natural}(x)$ one gets the conclusion – Q.E.D.

<u>Remark 20</u>