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Operator Valued Weights, L^2 -von Neumann Modules
and their Relative Tensor Products

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requirements for the degree Doctor of Philosophy
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by

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DEDICATION

This dissertation is dedicated, first and foremost, to my wife Linda and our daughter Marissa, without whose support and love any accomplishment is meaningless. Over the past five years, I have learned far more from them than I did in graduate school. They demonstrate to me, each and every day, what is truly important in life.

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ABSTRACT OF THE DISSERTATION

Operator Valued Weights, L^2 -von Neumann Modules
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A result of Haagerup, generalizing a theorem of Takesaki, states the following:

If $\mathcal{N} \subset \mathcal{M}$ are von Neumann algebras, then there exists a faithful, normal and semi-finite (fns) operator valued weight $T: \mathcal{M}_+ \rightarrow \widehat{\mathcal{N}}_+$ if and only if there exist fns weights $\tilde{\varphi}$ on \mathcal{M} , and φ on \mathcal{N} satisfying $\sigma_t^\varphi(x) = \sigma_t^{\tilde{\varphi}}(x)$, $\forall x \in \mathcal{N}$, $t \in \mathbb{R}$. In fact, T can be chosen such that $\tilde{\varphi} = \varphi \circ T$; T is then uniquely determined by this condition.

This thesis presents a proof of the above which does not use any structure theory.

Additionally, we develop a theory of L^2 -von Neumann modules, which encompasses a reformulation of Connes' Spatial Derivative, and the Relative Tensor Prod-

uct of Sauvageot. We demonstrate the naturality of the relative tensor product construction in the category of L^2 -von Neumann bimodules. Finally, we give evidence for the claim that the relative tensor product is essentially the only tensor product which should be used when considering this tensor category.

CHAPTER 1

Introduction

As can be surmised from its title, this thesis comprises two parts, which are only partially related. The first task is to demonstrate that it is possible to expound a complete theory of Operator Valued Weights, and arrive at the results of Takesaki and Haagerup, without recourse to the crossed product. We then consider the subject of L^2 -modules over von Neumann algebras: their properties, their role in the development of a formulation of the Spatial Derivative (see [13]), and their Relative Tensor Products. Finally, we mention a few areas for future research.

1.1 Non-Commutative Integration

It has long been known that an abelian von Neumann algebra is essentially $L^\infty(X, \mu)$, for some measure space $\{X, \mu\}$. In the non-abelian case, then, much work has been done to try and extend some of the usual notions from Measure, Probability and Integration Theories. For example, Haagerup [3] introduced the concept of “non-commutative L^p -spaces”, extended by Kosaki [4]. Takesaki [8] described necessary and sufficient conditions for the existence of (normal) conditional expectations whenever we have one von Neumann algebra \mathcal{N} sitting inside

another, \mathcal{M} . This led Haagerup [1, 2] to the theory of Operator Valued Weights, and a generalization of Takesaki's theorem. However, Haagerup's exposition depends heavily on the existence of a crossed product decomposition for a general von Neumann algebra, i.e., on Structure Theory. It appeared possible to achieve a theory of Operator Valued Weights without resorting to structure theory. In particular, this thesis offers a proof, which does not use Structure Theory, of the following theorem:

Theorem 1.1 (Haagerup) *Let $\mathcal{N} \subset \mathcal{M}$ be von Neumann algebras. There exists a faithful, semi-finite normal operator valued weight $T: \mathcal{M}_+ \rightarrow \widehat{\mathcal{N}}_+$ if and only if there exist faithful semi-finite normal weights $\tilde{\varphi}$ on \mathcal{M} and φ on \mathcal{N} such that*

$$\sigma_t^\varphi(x) = \sigma_t^{\tilde{\varphi}}(x), \quad x \in \mathcal{N}.$$

If this condition is satisfied, then T can be chosen in such a way that $\tilde{\varphi} = \varphi \circ T$; moreover, T is uniquely determined by this identity.

The method for showing this depends critically on a result of Connes/Masuda [12] which provides a converse to the theorem positing the existence of the Radon-Nikodym cocycle derivative. (Masuda's result does not use structure theory, unlike that of Connes.) This result implies that, given an invariance condition like that in the hypothesis, a weight on \mathcal{N} induces one on \mathcal{M} . With the ability to associate weights on \mathcal{N} to weights on \mathcal{M} , it is possible to construct an operator valued weight $T: \mathcal{M}_+ \rightarrow \widehat{\mathcal{N}}_+$.

In particular, we make use of the fact that, given a fixed faithful, normal and semi-finite (fns) weight ϕ on \mathcal{N} , the existence of a one parameter family of partial isometries $\{u_t\}$ satisfying a one-cocycle condition with respect to σ_t^ϕ implies the existence of a corresponding normal weight ψ on \mathcal{M} . This ψ has the property that $u_t = (D\psi : D\phi)_t, \forall t \in \mathbf{R}$. With the ability to associate ψ to ϕ , we can construct an operator valued weight $T: \mathcal{M}_+ \rightarrow \widehat{\mathcal{N}}_+$. We then show that such a T is a faithful, normal and semi-finite operator valued weight.

The desire to recast the proof of Haagerup's Theorem in a form which does not depend on any structure theory is motivated by the belief that it is a result which is fundamental in the theory of Non-Commutative Integration. Specifically, application of an operator valued weight to a positive element of \mathcal{M} (which results in an element in the extended positive cone of \mathcal{N}) can and should be thought of as "partial integration." The "space" over which we are integrating is not always explicit; however, just as we often think of von Neumann algebras as "non-commutative L^∞ -spaces," we sense the presence of an underlying measure space indirectly, by observing the interactions of function-like objects defined on this "space." Moreover, this point of view allows us to interpret the expression $\tilde{\varphi} = \varphi \circ T$ as a form of Fubini's Theorem (or more precisely, Tonelli's Theorem). Because we believe that structure theory should, in some sense, be a consequence of the results of integration theory, and not vice-versa, a proof which does not involve the crossed product is indicated.

It should also be noted that work in this area was done by Hirakawa [7]. The work presented herein addresses the problem from a different perspective than did he, however, and therefore represents a new approach.

1.2 L^2 -von Neumann Modules and their Relative Tensor Products

It should come as no surprise (due to their origins) that von Neumann algebras play a role in current Conformal Field Theories. In particular, the tensor category of bimodules over one or several von Neumann algebras is fundamental in their exposition. Hence, it is important to understand the special nuances that arise in considering tensor products of von Neumann algebra bimodules. In general, a purely algebraic approach to their theory is insufficient. Sauvageot [5] outlined a construction for the tensor product (the Relative Tensor Product) of two bimodules which is not canonical, but depends on the choice of a faithful, normal and semi-finite (fns) weight. (In the case where the weight is actually a vector state, this choice of weight corresponds, in Field Theory, to fixing a so-called “vacuum vector.”) Some work subsequent to Sauvageot’s in this area has at times neglected the extreme care which is required when dealing with weights. However, it is possible to show that, given a bimodule \mathfrak{H} over a fixed von Neumann algebra, if the existence of another bimodule \mathfrak{K} having certain “universal, tensor product-like” properties is assumed, then \mathfrak{K} is (i.e., is isomorphic to, as \mathcal{M} - \mathcal{M} bimodules) the relative tensor product $\mathfrak{H} \boxtimes_{\tau} \mathfrak{H}$ with respect to a trace τ on \mathcal{M} . Therefore, we see

that the existence of such a bimodule implies that the von Neumann algebra must be semi-finite. Moreover, it turns out that the existence of such a \mathfrak{K} , in which the tensor product of any two arbitrary elements is defined, forces the algebra \mathcal{M} to be atomic.^f

Originally, substantial inroads into the theory were made by Sauvageot. We show herein that the relative tensor product introduced by Sauvageot is, in a sense, the only bimodule tensor product which encompasses the intricacies present when dealing with infinite von Neumann algebras. Naively, one would expect the “tensor product” of the \mathcal{M} - \mathcal{M} bimodule $L^2(\mathcal{M})$ with itself to also be an \mathcal{M} - \mathcal{M} bimodule, possessing the usual universal property of tensor products, viz., that any (continuous) “ \mathcal{M} -bilinear” map on the Cartesian product should induce an \mathcal{M} -bimodule morphism on the tensor product. If we require that an \mathcal{M} -bilinear map I include the property that $I(\xi x, \eta) = I(\xi, x\eta)$, $\forall x \in \mathcal{M}$, then we will show that the only \mathcal{M} -bimodule tensor product which exhibits the universality described above is the relative tensor product $L^2(\mathcal{M}) \otimes_{\tau} L^2(\mathcal{M})$, where τ is a trace on the *atomic* von Neumann algebra \mathcal{M} . (Once again, recall that the relative tensor product of is not canonical, but rather depends on a choice of fns weight.) This result demonstrates that no such universal object can exist when \mathcal{M} is not simply of the form

$$\mathcal{M} = \bigoplus_{\alpha} \mathcal{L}(\mathfrak{H}_{\alpha}),$$

where each \mathfrak{H}_{α} is an arbitrary Hilbert space. Since Type II and Type III algebras

can (and often do) arise in physical theories, it is obviously important to be able to decide whether one may assume the existence of a tensor product having the aforementioned characteristics. If the algebra is non-atomic, then it is impossible, in general, to define $\xi \otimes \eta$ for arbitrary ξ, η . This implies that any strictly algebraic approach to the theory will necessarily be incomplete. Hence, a satisfactory resolution of this issue is needed.

Interestingly, in formulating a theory of L^2 -von Neumann modules, a serendipitous byproduct emerges: a clear exposition of the Spatial Derivative, originally introduced by Connes [13]. Suppose we are given a right L^2 -module \mathfrak{H} over a von Neumann algebra \mathcal{N} , and we denote by \mathcal{M} the von Neumann Algebra $\mathcal{L}(\mathfrak{H}_{\mathcal{N}})$, i.e., the set of (bounded) operators on \mathfrak{H} which commute with the right action of \mathcal{N} . Then, given an fns weight ψ on \mathcal{N} (which induces an fns weight ψ' on $\mathcal{N}^\circ \cong \mathcal{L}_{\mathcal{M}}(\mathfrak{H})$, the commutant of \mathcal{M} in $\mathcal{L}(\mathfrak{H})$), and a normal, semi-finite weight ϕ on \mathcal{M} , the spatial derivative $\frac{d\phi}{d\psi'}$ arises naturally in the L^2 -module context: it appears as the relative modular operator $\Delta_{\phi, \psi}$. Hence, the L^2 -von Neumann module theory incorporates the theory of the spatial derivative.

1.3 Future Research

The Hilbert spaces on which von Neumann algebras act from both the left and right have been referred to as “ L^2 -von Neumann modules.” What should be inferred from this usage is that there exist other types of modules. Indeed,

following the work of Lance [6] on “Hilbert C^* -modules”, it is possible to define a notion of an L^∞ -von Neumann module. \mathcal{E} is an L^∞ -von Neumann module if it is the dual of a Banach space \mathcal{E}_* , and if a von Neumann algebra \mathcal{M} acts on \mathcal{E} (from either the left or the right); additionally, \mathcal{E} should be equipped with an “ \mathcal{M} -valued inner product.” Proceeding in a fashion analagous to the methods used in the L^2 theory, we can characterize L^∞ -modules as “sitting in” von Neumann algebras. Moreover, we may develop a tensor product of these modules which respects their module structure. This leads directly to a theory regarding the preduals, which may well be termed an L^1 theory. We may then proceed to a tensor product of these modules. The ultimate goal is to arrive at a satisfactory L^p theory, which would of course encompass all previous results.

In the course of developing a theory of Operator Valued Weights without the use of structure theory, an interesting byproduct arises: the need to more fully understand “generalized weights.” These extend the usual concept of weight on a von Neumann algebra in that they are not defined on (some subset of) the positive cone, nor are they restricted to take values in the extended positive reals. Intuitively, generalized weights are usual weights which have been “twisted” via a partial isometry. Such objects appear when considering 2×2 matrix algebras, $M_2(\mathcal{M})$, over a von Neumann algebra \mathcal{M} , and its corresponding weights; the generalized weights occur on the “off-diagonal.” It appears possible to formulate a type of “polar decomposition,” in which the generalized weight is written as a product of a usual weight (which plays the role of the “absolute value” of the generalized

weight), and a partial isometry. Such a generalization is necessary for the further understanding of non-commutative measure and integration theories.

In addition, it would be beneficial to understand the relationship between the existence of an fns operator valued weight $T: \mathcal{M}_+ \rightarrow \widehat{\mathcal{N}}_+$, and the presence of an “unbounded projection” $E: \mathcal{D}(E) \rightarrow L^2(\mathcal{N})$, where $\mathcal{D}(E) \subset L^2(\mathcal{M})$ is a dense subspace on which E is defined. This would clearly extend (once again) the work of Takesaki, in which the existence of a normal conditional expectation $\mathcal{E}: \mathcal{M} \rightarrow \mathcal{N}$ implied the existence of a projection from $L^2(\mathcal{M})$ onto $L^2(\mathcal{N})$. The goal is to glean as much information about the structure of the inclusion $\mathcal{N} \subset \mathcal{M}$ as one can by examining this E . Note that calling E a projection is misleading — one cannot think of $L^2(\mathcal{N})$ as “sitting in” $L^2(\mathcal{M})$.

Finally, the previous paragraph indicates how work on operator valued weights and on von Neumann bimodules can be unified. Much current research is concerned with the question: given two von Neumann algebras, $\mathcal{N} \subset \mathcal{M}$, what is the nature of this inclusion? The study of bimodules arises naturally in this setting; hence, it is not unreasonable to hope that techniques arising in the study of the unbounded projection alluded to above can be applied to research into the structure of bimodules.

CHAPTER 2

Operator Valued Weights

2.1 Notation and Preliminary Results

In order to discuss the subject of operator valued weights, we begin with a review of some of the relevant terms and concepts. When studying weights, we consider the extended positive real numbers $\mathbf{R}_+ \cup \{\infty\}$. To study “unbounded conditional expectations” (i.e., operator valued weights), we need to consider the “extended positive part” $\widehat{\mathcal{N}}_+$ of the von Neumann subalgebra \mathcal{N} of \mathcal{M} . We begin with the following:

Definition 2.1 *For a von Neumann algebra \mathcal{M} , the extended positive cone $\widehat{\mathcal{M}}_+$ of \mathcal{M} is the set of maps $m: \mathcal{M}_*^+ \rightarrow [0, \infty]$ with the following properties:*

- (i) $m(\lambda\varphi) = \lambda m(\varphi)$, $\varphi \in \mathcal{M}_*^+$, $\lambda \geq 0$,
- (ii) $m(\varphi + \psi) = m(\varphi) + m(\psi)$, $\varphi, \psi \in \mathcal{M}_*^+$,
- (iii) m is lower semi-continuous.

Clearly, the positive part \mathcal{M}_+ of \mathcal{M} is a subset of $\widehat{\mathcal{M}}_+$. It is easy to see that $\widehat{\mathcal{M}}_+$ is closed under addition, multiplication by non-negative scalars and increasing limits.

Example. Let $\{\mathcal{M}, \mathfrak{H}\}$ be a von Neumann algebra and A a positive self-adjoint operator on \mathfrak{H} affiliated with \mathcal{M} . Suppose that

$$A = \int_0^\infty \lambda de(\lambda)$$

is the spectral decomposition of A . For each $\varphi \in \mathcal{M}_*^+$, set

$$m_A(\varphi) \triangleq \int_0^\infty \lambda d\varphi(e(\lambda)).$$

Then m_A satisfies the all three conditions of Definition 2.1. The last condition, the lower semi-continuity, is a consequence of

$$m_A(\varphi) = \sup_n \varphi(A_n) \quad \text{with} \quad A_n = \int_0^n \lambda de(\lambda) \in \mathcal{M}_+.$$

It now follows that

$$m_A(\omega_\xi) = \int_0^\infty \lambda d(e(\lambda)\xi | \xi) = \begin{cases} \|A^{1/2}\xi\|^2, & \xi \in \mathcal{D}(A^{1/2}), \\ +\infty, & \xi \notin \mathcal{D}(A^{1/2}). \end{cases}$$

Hence if B is another positive self-adjoint operator on \mathfrak{H} affiliated with \mathcal{M} , then the equality $m_A = m_B$ means precisely $A = B$. Hence the map $A \mapsto m_A \in \widehat{\mathcal{M}}_+$ is injective. Thus, the set of positive self-adjoint operators affiliated with \mathcal{M} can be identified with a subset of the extended positive cone $\widehat{\mathcal{M}}_+$. \square

We shall see that the above example is in fact generic, in a sense that will be made precise in what follows.

We continue the exposition by defining several operations on elements of $\widehat{\mathcal{M}}_+$, which justify the terminology “extended positive cone.”

Definition 2.2 For $m, n \in \widehat{\mathcal{M}}_+$, $\lambda \geq 0$ and $a \in \mathcal{M}$, we define the following operations:

$$(\lambda m)(\varphi) \triangleq \lambda m(\varphi), \quad \varphi \in \mathcal{M}_*^+,$$

$$(m + n)(\varphi) \triangleq m(\varphi) + n(\varphi), \quad \varphi \in \mathcal{M}_*^+,$$

$$(a^*ma)(\varphi) \triangleq m(a\varphi a^*), \quad \varphi \in \mathcal{M}_*^+.$$

We also note here that $\sup_i m_i$ of an increasing net in $\widehat{\mathcal{M}}_+$ can be naturally defined.

Lemma 2.3 Let $\{\mathcal{M}, \mathfrak{H}\}$ be a von Neumann algebra. To each $m \in \widehat{\mathcal{M}}_+$, there corresponds uniquely a pair $\{A, \mathfrak{K}\}$ of a closed subspace \mathfrak{K} of \mathfrak{H} and a positive self-adjoint operator on \mathfrak{K} such that

(i) \mathfrak{K} is affiliated with \mathcal{M} , in the sense that the projection to \mathfrak{K} belongs to \mathcal{M} ,
and A is affiliated with \mathcal{M} ;

(ii)

$$(2.1) \quad m(\omega_\xi) = \begin{cases} \|A^{1/2}\xi\|^2, & \xi \in \mathcal{D}(A^{1/2}) \\ +\infty, & \text{otherwise} \end{cases}$$

Here ω_ξ means, of course, the functional $x \in \mathcal{M} \mapsto (x\xi | \xi)$.

The proof of the above lemma, as well as the next theorem and its corollary, is standard, and is therefore omitted. (See, for instance, [1].)

We say that an element $m \in \widehat{\mathcal{M}}_+$ is *semi-finite* if $\{\varphi \in \mathcal{M}_*^+ : m(\varphi) < +\infty\}$ is dense in \mathcal{M}_*^+ ; *faithful* if $m(\varphi) > 0$ for every non-zero $\varphi \in \mathcal{M}_*^+$.

Theorem 2.4 *Let \mathcal{M} be a von Neumann algebra. Each $m \in \widehat{\mathcal{M}}_+$ has a unique spectral decomposition of the form:*

$$(2.2) \quad m(\varphi) = \int_0^\infty \lambda d\varphi(e(\lambda)) + \infty\varphi(p), \quad \varphi \in \mathcal{M}_*^+,$$

where $\{e(\lambda) : \lambda \in \mathbf{R}_+\}$ is an increasing family of projections in \mathcal{M} which is σ -strongly continuous from the right, and $p = \mathbf{1} - \lim_{\lambda \rightarrow \infty} e(\lambda)$. Furthermore, $e(0) = 0$ if and only if m is faithful, and $p = 0$ if and only if m is semi-finite.

It is now apparent, given the above Theorem, that Example 2.1 represents a generic element in $\widehat{\mathcal{M}}_+$.

To simplify notation, we shall write

$$(2.2') \quad m = H + \infty p, \quad H = \int_0^\infty \lambda de(\lambda),$$

when m has the form of (2.2). We keep the convention $0 \cdot (+\infty) = 0$. Although we consider H as an operator affiliated with \mathcal{M} , we use the following notation:

$$\mathcal{D}(H^{1/2}) = \{\xi \in \mathfrak{H} : m(\omega_\xi) < +\infty\},$$

as long as doing so causes no confusion.

Corollary 2.5 *Any normal weight φ on \mathcal{M} has a unique extension, denoted by φ again, to $\widehat{\mathcal{M}}_+$ such that*

$$\varphi(\lambda m) = \lambda\varphi(m), \quad \lambda \geq 0, \quad m \in \widehat{\mathcal{M}}_+,$$

$$\varphi(m + n) = \varphi(m) + \varphi(n), \quad m, n \in \widehat{\mathcal{M}}_+,$$

$$\varphi(\sup_\alpha m_\alpha) = \sup_\alpha \varphi(m_\alpha),$$

for any increasing net $\{m_\alpha\}$ in $\widehat{\mathcal{M}}_+$, where, in this context, $\{m_\alpha(\omega)\}$ is increasing for every $\omega \in \mathcal{M}_+^*$.

So, we may think of the extended positive cone of \mathcal{M} as \mathcal{M}_+ , along with a myriad of “points at infinity” adjoined to it. There will be a different infinite point for each projection in \mathcal{M} . When we think about the extended positive cone in this way, it is clear why we adopt the notation in (2.2’).

Now, for each $m = H + \infty p$, we put

$$m_0 \triangleq (\mathbf{1} + H)^{-1}(\mathbf{1} - p) ,$$

$$m_\varepsilon \triangleq H(\mathbf{1} + \varepsilon H)^{-1}(\mathbf{1} - p) + \frac{1}{\varepsilon}p, \quad \varepsilon > 0 .$$

We note that both m_0 and m_ε are bounded.

Lemma 2.6

(i) For each $m, n \in \widehat{\mathcal{M}}_+$, we have the following equivalence:

$$m \leq n \Leftrightarrow m_0 \geq n_0 \Leftrightarrow m_\varepsilon \leq n_\varepsilon, \quad \varepsilon > 0.$$

(ii) Let $\{m_\alpha\} = \{H_\alpha + \infty p_\alpha\}$ be an increasing net in $\widehat{\mathcal{M}}_+$ and $m = H + \infty p$.

Then we have

$$m_\alpha \nearrow m \Leftrightarrow (m_\alpha)_0 \searrow m_0 \Leftrightarrow (m_\alpha)_\varepsilon \nearrow m_\varepsilon, \quad \varepsilon > 0.$$

Proof.

(i) Suppose $m \leq n$. Let $m = H + \infty p$ and $n = K + \infty q$. By assumption, we have

$p \leq q$. Let $\mathfrak{K} = (\mathbf{1} - p)\mathfrak{H}$ and $\mathfrak{L} = (\mathbf{1} - q)\mathfrak{H}$, where \mathfrak{H} denotes, as usual, the

underlying Hilbert space of \mathcal{M} . For any $\xi \in \mathfrak{L}$, set $\eta = (\varepsilon \mathbf{1} + H)^{-1} \xi \in \mathcal{D}(H)$ and $\zeta = (\varepsilon \mathbf{1} + K)^{-1} \xi \in \mathcal{D}(K)$. We then have

$$\begin{aligned}
((\varepsilon \mathbf{1} + K)^{-1} \xi \mid \xi)^2 &= (\xi \mid (\varepsilon \mathbf{1} + K)^{-1} \xi)^2 = ((\varepsilon \mathbf{1} + H) \eta \mid \zeta)^2 \\
&= ((\varepsilon \mathbf{1} + H)^{1/2} \eta \mid (\varepsilon \mathbf{1} + H)^{1/2} \zeta)^2 \\
&\quad (\zeta \in \mathcal{D}(K) \subset \mathcal{D}(K^{1/2}) \subset \mathcal{D}(H^{1/2}) = \mathcal{D}((\varepsilon \mathbf{1} + H)^{1/2})) \\
&\leq \|(\varepsilon \mathbf{1} + H)^{1/2} \eta\|^2 \|(\varepsilon \mathbf{1} + H)^{1/2} \zeta\|^2 \\
&= ((\varepsilon \mathbf{1} + H) \eta \mid \eta) ((\varepsilon \mathbf{1} + H) \zeta \mid \zeta) \\
&\leq ((\varepsilon \mathbf{1} + H) \eta \mid \eta) ((\varepsilon \mathbf{1} + K) \zeta \mid \zeta) \\
&= (\xi \mid (\varepsilon \mathbf{1} + H)^{-1} \xi) (\xi \mid (\varepsilon \mathbf{1} + K)^{-1} \xi),
\end{aligned}$$

so that

$$((\varepsilon \mathbf{1} + K)^{-1} \xi \mid \xi) \leq ((\varepsilon \mathbf{1} + H)^{-1} \xi \mid \xi), \quad \xi \in \mathfrak{L}.$$

Hence we get $n_0(\omega_\xi) \leq m_0(\omega_\xi)$ for any $\xi \in \mathfrak{L}$ by setting $\varepsilon = 1$. If $\xi \in \mathfrak{L}^\perp$, then $n_0(\omega_\xi) = 0$. Thus we conclude $n_0 \leq m_0$.

Conversely, suppose $n_0 \leq m_0$. It follows that $p \leq q$. Let \mathfrak{K} and \mathfrak{L} be as before. The assumption means that

$$((\mathbf{1} + H)^{-1} \xi \mid \xi) \geq ((\mathbf{1} + K)^{-1} \xi \mid \xi), \quad \xi \in \mathfrak{K}.$$

Setting $(\mathbf{1} + K)^{-1} \xi = 0$ for $\xi \in \mathfrak{L}^\perp \cap \mathfrak{K}$, we view $(\mathbf{1} + K)^{-1}$ as an operator on \mathfrak{K} and have $(\mathbf{1} + H)^{-1} \geq (\mathbf{1} + K)^{-1}$. Then we have $(\mathbf{1} + H)^{-1/2} \geq (\mathbf{1} + K)^{-1/2}$, and

$$\mathcal{D}(H^{1/2}) = (\mathbf{1} + H)^{-1/2} \mathfrak{K} \supset (\mathbf{1} + K)^{1/2} \mathfrak{K} = \mathcal{D}(K^{1/2}).$$

The argument in the first paragraph shows

$$\begin{aligned} (\mathbf{1} + H)(\mathbf{1} + \varepsilon(\mathbf{1} + H)^{-1})^{-1} &= (\varepsilon\mathbf{1} + (\mathbf{1} + H)^{-1})^{-1} \\ &\leq (\varepsilon\mathbf{1} + (\mathbf{1} + K)^{-1})^{-1} = (\mathbf{1} + K)(\mathbf{1} + \varepsilon(\mathbf{1} + K)^{-1}). \end{aligned}$$

If $\xi \in \mathcal{D}(K^{1/2})$, we have

$$\begin{aligned} \|(\mathbf{1} + K)^{1/2}\xi\|^2 &= \lim_{\varepsilon \rightarrow 0} \|(\mathbf{1} + K)^{1/2}(\mathbf{1} + \varepsilon(\mathbf{1} + K)^{-1})^{-1/2}\xi\|^2 \\ &\geq \lim_{\varepsilon \rightarrow 0} \|(\mathbf{1} + H)^{1/2}(\mathbf{1} + \varepsilon(\mathbf{1} + H)^{-1})^{-1/2}\xi\|^2 \\ &= \|(\mathbf{1} + H)^{1/2}\xi\|^2, \end{aligned}$$

so that we conclude $\mathbf{1} + n \geq \mathbf{1} + m$; equivalently $n \geq m$.

Now, we have already demonstrated $m \leq n \Leftrightarrow m_0 \geq n_0$. For a fixed $\varepsilon > 0$, we have then

$$\begin{aligned} m \leq n &\Leftrightarrow \varepsilon m \leq \varepsilon n \Leftrightarrow (\varepsilon m)_0 \geq (\varepsilon n)_0 \\ &\Leftrightarrow \mathbf{1} - (\varepsilon m)_0 \leq \mathbf{1} - (\varepsilon n)_0 \\ &\Leftrightarrow m_\varepsilon = \frac{1}{\varepsilon}(\mathbf{1} - (\varepsilon m)_0) \leq \frac{1}{\varepsilon}(\mathbf{1} - (\varepsilon n)_0) = n_\varepsilon. \end{aligned}$$

- (ii) By (i), the net $\{(m_\alpha)_0\}$ is decreasing. If $\ell = \inf_\alpha (m_\alpha)_0$, then there exists $n \in \widehat{\mathcal{M}}_+$ such that $n_0 = \ell$ because $(m_\alpha)_0 \leq \mathbf{1}$ implies $\ell \leq \mathbf{1}$. If $m = \sup_\alpha m_\alpha$, then we have $m_0 \leq (m_\alpha)_0$, so $m_0 \leq n_0$, which implies $n \leq m$, again by (i). Hence $m_0 = \inf_\alpha (m_\alpha)_0 = \lim(m_\alpha)_0$. Thus we prove the equivalence: $m_\alpha \nearrow m \Leftrightarrow (m_\alpha)_0 \searrow m_0$. Finally, the equality

$$m_\varepsilon = \frac{1}{\varepsilon}(\mathbf{1} - (\varepsilon m)_0)$$

gives the remaining equivalence.

■

We now state and prove a proposition which gives some indication of the direction in which we are headed. We are trying to see how to associate elements in the extended positive cone of a subalgebra with weights on the whole algebra.

Proposition 2.7 *Let φ be a faithful, semi-finite normal weight on \mathcal{M} , and set $\mathcal{N} = \mathcal{M}_\varphi$. (Where \mathcal{M}_φ denotes, as is customary, the centralizer of φ in \mathcal{M} .) For each $m \in \widehat{\mathcal{N}}_+$, set*

$$(2.3) \quad \varphi_m(x) = \lim_{\varepsilon \rightarrow 0} \varphi_{m_\varepsilon}(x), \quad x \in \mathcal{M}_+.$$

(Recall, as $m_\varepsilon \in \mathcal{N}_+ = (\mathcal{M}_\varphi)_+$, $\varphi_{m_\varepsilon} \triangleq \varphi(m_\varepsilon \cdot)$ gives a weight on \mathcal{M} .)

Then the map $m \in \widehat{\mathcal{N}}_+ \mapsto \varphi_m$ is an order preserving bijection from $\widehat{\mathcal{N}}_+$ onto the set of all σ_t^φ -invariant, not necessarily faithful nor semi-finite, normal weights on \mathcal{M} . Furthermore, we have

$$m_\alpha \nearrow m \text{ in } \widehat{\mathcal{N}}_+ \quad \Leftrightarrow \quad \varphi_{m_\alpha} \nearrow \varphi_m \text{ pointwise on } \mathcal{M}_+.$$

Proof. For a fixed $x \in \mathcal{M}_+$, we “define” a normal weight φ^x on \mathcal{N} by

$$\varphi^x(a) \triangleq \varphi(a^{1/2} x a^{1/2}), \quad a \in \mathcal{N}_+.$$

Of course, we must check that φ^x is, in fact, a weight, and that it is normal. Positive homogeneity is not a problem; but linearity (“additivity”) must be verified. Observe, however, if we prove the additivity of φ^x , then the normality will follow from that of φ .

Let $a, b \in \mathcal{N}_+$ and $c = a + b$. Choose $s, t \in \mathcal{N}$ as usual so that $a^{1/2} = sc^{1/2}$, $b^{1/2} = tc^{1/2}$ and $s^*s + t^*t$ is the range projection of c . If $\varphi^x(c) < +\infty$, then $y = c^{1/2}xc^{1/2} \in \mathfrak{m}_\varphi$. (Here, \mathfrak{m}_φ denotes the domain of φ , having been extended by linearity.) Hence, sys^* and tyt^* both belong to \mathfrak{m}_φ , and we get

$$\varphi(sys^*) + \varphi(tyt^*) = \varphi(ys^*s) + \varphi(yt^*t) = \varphi(y(s^*s + t^*t)) = \varphi(y) = \varphi^x(c);$$

$$\varphi(sys^*) = \varphi(a^{1/2}xa^{1/2}) = \varphi^x(a);$$

$$\varphi(tyt^*) = \varphi(b^{1/2}xb^{1/2}) = \varphi^x(b).$$

Thus, $\varphi^x(a) + \varphi^x(b) = \varphi^x(c)$. Now, we have

$$s = \lim_{\varepsilon \rightarrow 0} a^{1/2}(c + \varepsilon \mathbf{1})^{-1/2}, \quad t = \lim_{\varepsilon \rightarrow 0} b^{1/2}(c + \varepsilon \mathbf{1})^{-1/2},$$

so that

$$\begin{aligned} c^{1/2}xc^{1/2} &= \lim_{\varepsilon \rightarrow 0} (c + \varepsilon \mathbf{1})^{-1/2}cxc(c + \varepsilon \mathbf{1})^{-1/2} \\ &= \lim_{\varepsilon \rightarrow 0} (c + \varepsilon \mathbf{1})^{-1/2}(a + b)x(a + b)(c + \varepsilon \mathbf{1})^{-1/2} \\ &\leq 2 \lim_{\varepsilon \rightarrow 0} (c + \varepsilon \mathbf{1})^{-1/2}[axa + bxb](c + \varepsilon \mathbf{1})^{-1/2} \\ &= 2(s^*a^{1/2}xa^{1/2}s + t^*b^{1/2}xb^{1/2}t). \end{aligned}$$

Therefore, if $\varphi^x(a) < +\infty$ and $\varphi^x(b) < +\infty$, then $\varphi^x(c) < +\infty$; hence $\varphi^x(a + b) = \varphi^x(a) + \varphi^x(b)$ for $a, b \in \mathcal{N}_+$.

Now, for each $m \in \widehat{\mathcal{N}}_+$, we set

$$\varphi_m(x) \triangleq \varphi^x(m), \quad x \in \mathcal{M}_+.$$

This makes sense by Corollary 2.5: we have $\varphi^x(\sup_\alpha m_\alpha) = \sup_\alpha \varphi^x(m_\alpha)$ whenever $m_\alpha \nearrow m$. Hence we obtain

$$\varphi_m(x) = \varphi^x(m) = \lim_{\varepsilon \rightarrow 0} \varphi^x(m_\varepsilon) = \lim_{\varepsilon \rightarrow 0} \varphi_{m_\varepsilon}(x).$$

If $x_\alpha \nearrow x$ in \mathcal{M}_+ , then

$$\begin{aligned}\varphi^x(m) &= \sup_{\varepsilon>0} \varphi_{m_\varepsilon}(x) = \sup_{\varepsilon>0} \sup_{\alpha} \varphi_{m_\varepsilon}(x_\alpha) \\ &= \sup_{\alpha} \sup_{\varepsilon>0} \varphi_{m_\varepsilon}(x_\alpha) = \sup_{\alpha} \varphi_m(x_\alpha),\end{aligned}$$

so that φ_m is normal. The additivity of φ_m follows from the convergence in (2.3).

The invariance of φ_m under $\{\sigma_t^\varphi : t \in \mathbf{R}\}$ follows from that of φ_{m_ε} . The remainder of the proof is straightforward. \blacksquare

We now define operator valued weights. These will generalize ordinary weights, as they are allowed to assume infinite values; also, they extend the concept of conditional expectation (“projection of norm one”) in that they map from a larger von Neumann algebra onto (the extended positive cone of) a subalgebra.

Definition 2.8 *Let \mathcal{M} be a von Neumann algebra and \mathcal{N} a von Neumann subalgebra of \mathcal{M} . An operator valued weight from \mathcal{M} to \mathcal{N} is a map $T: \mathcal{M}_+ \rightarrow \widehat{\mathcal{N}}_+$ which satisfies the following conditions:*

$$(i) \quad T(\lambda x) = \lambda T(x), \quad \lambda \geq 0, \quad x \in \mathcal{M}_+.$$

$$(ii) \quad T(x + y) = T(x) + T(y), \quad x, y \in \mathcal{M}_+.$$

$$(iii) \quad T(a^*xa) = a^*T(x)a, \quad x \in \mathcal{M}_+, \quad a \in \mathcal{N}.$$

In addition, we say that T is normal if

$$(iv) \quad T(x_\alpha) \nearrow T(x) \text{ whenever } x_\alpha \nearrow x, \quad x_\alpha, x \in \mathcal{M}_+.$$

As in the case of ordinary weights, we set

$$\mathfrak{n}_T = \{x \in \mathcal{M} : \|T(x^*x)\| < +\infty\}$$

$$\mathfrak{m}_T = \mathfrak{n}_T^* \mathfrak{n}_T = \left\{ \sum_{i=1}^n y_i^* x_i : x_1, \dots, x_n, y_1, \dots, y_n \in \mathfrak{n}_T \right\}.$$

We now state the following lemma, whose proof is standard:

Lemma 2.9

(i) \mathfrak{m}_T is spanned by its positive part:

$$\mathfrak{m}_T^+ = \{x \in \mathcal{M}_+ : \|T(x)\| < +\infty\}.$$

(ii) \mathfrak{m}_T and \mathfrak{n}_T are 2-sided modules over \mathcal{N} .

(iii) T has a unique linear extension $\dot{T} : \mathfrak{m}_T \rightarrow \mathcal{N}$, which enjoys the module map property:

$$\dot{T}(axb) = a\dot{T}(x)b, \quad a, b \in \mathcal{N}, \quad x \in \mathfrak{m}_T.$$

In particular, if $T(\mathbf{1}) = \mathbf{1}$, then T is a projection of norm one from \mathcal{M} onto \mathcal{N} .

In the sequel, we shall not distinguish T and \dot{T} unless we need to.

Definition 2.10 We say that T is semi-finite if \mathfrak{n}_T is σ -weakly dense in \mathcal{M} ; faithful if $T(x^*x) \neq 0$ for $x \neq 0$. We denote by $\mathfrak{W}(\mathcal{M}, \mathcal{N})$ (resp., $\mathfrak{W}_0(\mathcal{M}, \mathcal{N})$) the set of (resp. faithful, semi-finite) normal operator valued weights from \mathcal{M} to \mathcal{N} . In the case that $\mathcal{N} = \mathbb{C}$, we write $\mathfrak{W}(\mathcal{M})$ (resp., $\mathfrak{W}_0(\mathcal{M})$) for $\mathfrak{W}(\mathcal{M}, \mathbb{C})$ (resp., for $\mathfrak{W}_0(\mathcal{M}, \mathbb{C})$).

Note that if $T: \mathcal{M}_+ \rightarrow \widehat{\mathcal{N}}_+$ is a normal operator valued weight, it can be extended to a normal “linear” map from $\widehat{\mathcal{M}}_+ \rightarrow \widehat{\mathcal{N}}_+$. Therefore, if $\mathcal{P} \subset \mathcal{N} \subset \mathcal{M}$ is an inclusion of von Neumann algebras, and if $T \in \mathfrak{W}(\mathcal{M}, \mathcal{N})$ and $S \in \mathfrak{W}(\mathcal{N}, \mathcal{P})$, then we have $S \circ T \in \mathfrak{W}(\mathcal{M}, \mathcal{P})$.

Proposition 2.11 *If $\mathcal{P} \subset \mathcal{N} \subset \mathcal{M}$ are von Neumann subalgebras and if $T \in \mathfrak{W}_0(\mathcal{M}, \mathcal{N})$ and $S \in \mathfrak{W}_0(\mathcal{N}, \mathcal{P})$, then $S \circ T \in \mathfrak{W}_0(\mathcal{M}, \mathcal{P})$.*

Proof. The only non-trivial part is the semi-finiteness of $S \circ T$. If $x \in \mathfrak{n}_T$, then $T(x^*x) \in \mathcal{N}_+$. Choose a net $\{a_\alpha\}$ in \mathfrak{n}_S such that $a_\alpha \rightarrow 1$ σ -strongly. Then we have

$$(S \circ T)(a_\alpha^* x^* x a_\alpha) = S(a_\alpha^* T(x^* x) a_\alpha) \leq \|T(x^* x)\| S(a_\alpha^* a_\alpha),$$

so that $x a_\alpha \in \mathfrak{n}_{S \circ T}$. Hence $\mathfrak{n}_{S \circ T}$ is σ -strongly dense in \mathcal{M} because \mathfrak{n}_T is. ■

Proposition 2.12 *Suppose \mathcal{N}, \mathcal{M} are von Neumann algebras, $\mathcal{N} \subset \mathcal{M}$, and $T \in \mathfrak{W}_0(\mathcal{M}, \mathcal{N})$; then*

(i) $\dot{T}(\mathfrak{m}_T)$ is a σ -weakly dense 2-sided ideal of \mathcal{N} ;

(ii) After extending T to $\dot{T}: \widehat{\mathcal{M}}_+ \rightarrow \widehat{\mathcal{N}}_+$, we have $T(\widehat{\mathcal{M}}_+) = \widehat{\mathcal{N}}_+$.

Proof.

(i) From the module map property of T (Lemma 2.9(iii))

$$\dot{T}(axb) = a\dot{T}(x)b, \quad a, b \in \mathcal{N}, \quad x \in \mathfrak{m}_T,$$

it follows that $\dot{T}(\mathfrak{m}_T)$ is a 2-sided ideal of \mathcal{N} . Let z denote the greatest projection of the σ -weak closure of $\dot{T}(\mathfrak{m}_T)$; note that z is central in \mathcal{N} . Assume $z \neq \mathbf{1}$. As \mathfrak{n}_T is σ -weakly dense in \mathcal{M} , there exists $x \in \mathfrak{n}_T$ with $x(\mathbf{1} - z) \neq 0$, so that $(\mathbf{1} - z)x^*x(\mathbf{1} - z) \in \mathfrak{m}_T \setminus \{0\}$, and as T is faithful

$$0 \neq T((\mathbf{1} - z)x^*x(\mathbf{1} - z)) = (\mathbf{1} - z)T(x^*x)(\mathbf{1} - z) = 0,$$

which is a contradiction. Hence $z = \mathbf{1}$, which means that $\dot{T}(\mathfrak{m}_T)$ is σ -weakly dense in \mathcal{N} .

- (ii) Take any $b \in \dot{T}(\mathfrak{m}_T)_+$. Then b is of the form $b = \dot{T}(h)$, $h \in \mathfrak{m}_T$. Replacing h by $\frac{1}{2}(h + h^*)$, h can be chosen to be self-adjoint. Now, by Lemma (2.9), we know there exist $a_1, a_2 \in \mathfrak{m}_T^+$ such that $b = T(a_1) - T(a_2)$. Then, we have $b \leq T(a_1)$, so we can find $s \in \mathcal{N}$ such that $b = sT(a_1)s^*$. With $a = sa_1s^*$, we have $b = T(a)$, $a \in \mathfrak{m}_T^+$. Hence $\dot{T}(\mathfrak{m}_T)_+ = \dot{T}(\mathfrak{m}_T^+)$.

Now, let $\{b_\lambda\}_{\lambda \in \Lambda}$ be a family in the positive cone of the ideal $\dot{T}(\mathfrak{m}_T)_+$, maximal with respect to the property $\sum_{\lambda \in \Lambda} b_\lambda \leq \mathbf{1}$. By this maximality, and an appeal to the Kaplansky Density Theorem, we may conclude that $\sum_{\lambda \in \Lambda} b_\lambda = \mathbf{1}$ in the σ -strong topology. Every $y \in \mathcal{N}_+$ is then of the form $y = \sum_{\lambda \in \Lambda} y^{\frac{1}{2}} b_\lambda y^{\frac{1}{2}}$, so that we have $y = \sum_{\lambda \in \Lambda} T(x_\lambda)$ with $\{x_\lambda\} \subset \mathfrak{m}_T^+$.

Finally, suppose $z \in \widehat{\mathcal{N}}_+$. From (2.2), it follows that there exists a sequence $\{y_n\} \subset \mathcal{N}_+$ such that $y_n \nearrow z$. Set $z_1 \triangleq y_1$ and $z_n \triangleq y_n - y_{n-1}$, $n \geq 2$. Then we have $z = \sum_{n=1}^{\infty} z_n$. Each z_n can be written $z_n = \sum T((x_n)_\lambda)$ with

$\{(x_n)_\lambda\} \subset \mathfrak{m}_T^+$. Hence we have

$$z = T \left(\sum_{n=1}^{\infty} \sum_{\lambda} (x_n)_\lambda \right),$$

where T has been extended to $\widehat{\mathcal{M}}_+$. Thus T maps $\widehat{\mathcal{M}}_+$ onto $\widehat{\mathcal{N}}_+$.

■

2.2 Haagerup's Theorem without Structure Theory

We now state Haagerup's theorem. It is obvious how it generalizes Takesaki's Theorem [8] which states that a necessary and sufficient condition for the existence of a normal conditional expectation $\mathcal{E}: \mathcal{M} \rightarrow \mathcal{N}$ with respect to a normal, faithful and semi-finite weight φ is the invariance of \mathcal{N} under the modular automorphism group $\{\sigma_t^\varphi\}$.

Theorem 2.13 *Let $\mathcal{N} \subset \mathcal{M}$ be von Neumann algebras. There exists a faithful semi-finite normal operator valued weight $T: \mathcal{M}_+ \rightarrow \widehat{\mathcal{N}}_+$ if and only if there exist faithful semi-finite normal weights $\tilde{\varphi}$ on \mathcal{M} and φ on \mathcal{N} such that*

$$(2.4) \quad \sigma_t^\varphi(x) = \sigma_t^{\tilde{\varphi}}(x), \quad x \in \mathcal{N}.$$

If this condition is satisfied, then T can be chosen in such a way that $\tilde{\varphi} = \varphi \circ T$; moreover, T is uniquely determined by this identity.

In order to prove this, we begin first with a Lemma which is of independent interest; note its measure-theoretic flavor. A proof similar to the one included herein can also be found in [1]; we have included it for completeness.

Lemma 2.14 *An extended positive real valued function $m: \mathcal{M}_*^+ \rightarrow [0, +\infty]$ having the properties*

(i)

$$m(\lambda\varphi) = \lambda m(\varphi), \quad \lambda \geq 0, \quad \varphi \in \mathcal{M}_*^+$$

(ii)

$$m(\varphi + \psi) = m(\varphi) + m(\psi), \quad \varphi, \psi \in \mathcal{M}_*^+,$$

is lower semi-continuous, and hence a member of the extended positive cone $\widehat{\mathcal{M}}_+$, if and only if m is countably additive in the sense that

$$m\left(\sum_{n=1}^{\infty} \varphi_n\right) = \sum_{n=1}^{\infty} m(\varphi_n)$$

for every $\{\varphi_n\} \subset \mathcal{M}_*^+$ with $\sum_{n=1}^{\infty} \|\varphi_n\| < +\infty$.

Proof. The forward implication is trivial, so we prove only the reverse direction.

Suppose m is countably additive. Represent \mathcal{M} on a Hilbert space \mathfrak{H} and consider $\bar{m}: \omega \in \mathcal{L}(\mathfrak{H})_*^+ \mapsto m(\omega|_{\mathcal{M}}) \in [0, +\infty]$. We claim first that m is lower semi-continuous if \bar{m} is. Suppose \bar{m} is lower semi-continuous. Then there exists a unique pair $\{A, \mathfrak{K}\}$ of a closed subspace \mathfrak{K} of \mathfrak{H} and a positive self-adjoint operator A on \mathfrak{K} such that (2.1) holds. As $u\omega u^*|_{\mathcal{M}} = \omega|_{\mathcal{M}}$ for every $u \in \mathcal{U}(\mathcal{M}')$, $u^*\bar{m}u = \bar{m}$, so that $\{A, \mathfrak{K}\}$ is affiliated to \mathcal{M} , and $m = m_A$. Thus, m is a member of $\widehat{\mathcal{M}}_+$. Therefore, it suffices to prove the lemma for $\mathcal{M} = \mathcal{L}(\mathfrak{H})$. Replacing m by m' defined by $m'(\varphi) = m(\varphi) + \varphi(\mathbf{1})$, we may assume $m(\varphi) \geq \|\varphi\|$, $\varphi \in \mathcal{M}_*^+$. As $\mathcal{M}_* = L^1(\mathcal{M}, \text{Tr})$ is an ideal of \mathcal{M} , we can define a map $\varphi: x \in \mathcal{M}_+ \mapsto [0, +\infty]$ by

$$\varphi(x) = \begin{cases} m(\omega_x), & x \in L^1(\mathcal{M}, \text{Tr})_+ \\ +\infty & \text{otherwise} \end{cases},$$

where $\omega_x(a) = \text{Tr}(ax)$, $a \in \mathcal{M}$, $x \in L^1(\mathcal{M}, \text{Tr})$. Then φ is a weight on \mathcal{M} , and $\varphi \geq \text{Tr}$. Let $\{x_\alpha\}_{\alpha \in A}$ be a family of positive operators with $x = \sum x_\alpha \in \mathcal{M}_+$. If $\text{Tr}(x) = +\infty$, then $\sum_\alpha \text{Tr}(x_\alpha) = \infty$ and

$$\infty = \sum \text{Tr}(x_\alpha) \leq \sum \varphi(x_\alpha) \leq \varphi(x),$$

so that $\varphi(x) = \infty = \sum \varphi(x_\alpha)$. If $\text{Tr}(x) < +\infty$, then $\sum \text{Tr}(x_\alpha) < +\infty$, so that $x_\alpha \neq 0$ for at most countably infinitely many α 's. Hence we have

$$\begin{aligned} \sum_{\alpha \in A} \varphi(x_\alpha) &= \sum_{\alpha} m(\omega_{x_\alpha}) = m\left(\sum \omega_{x_\alpha}\right) \\ &= m(\omega_x) = \varphi(x). \end{aligned}$$

Thus, φ is a completely additive weight on \mathcal{M} , and hence is normal. If $\{x_n\}$ is a sequence in $L^1(\mathcal{M}, \text{Tr})_+$ with $\lim_{n \rightarrow \infty} \|x - x_n\|_1 = 0$, then $\{x_n\}$ converges to x σ -weakly, so that

$$m(\omega_x) = \varphi(x) \leq \liminf_{n \rightarrow \infty} \varphi(x_n) = \liminf_{n \rightarrow \infty} m(\omega_{x_n}).$$

Hence m is lower semi-continuous. ■

We will also need the following Lemma, which appears without proof. It can be demonstrated by an application of one of the results in the theory of the cocycle Radon-Nikodym derivative. (See, for example, [10].)

Lemma 2.15 *Let $\varphi \in \mathfrak{W}(\mathcal{M})$ and $\{a_n\}$ be a sequence in $\mathcal{M}_{s(\varphi)}$. (Here, as usual, $\mathcal{M}_{s(\varphi)}$ represents the reduced algebra.) Then the following two conditions are equivalent:*

(i)

$$\varphi(x) = \sum_{n=1}^{\infty} \varphi(a_n x a_n^*), \quad x \in \mathcal{M}_+$$

(ii)

$$\{a_n\} \subset \mathcal{D}(\sigma_{-i/2}^\varphi) \quad \text{and} \quad \sum_{n=1}^{\infty} \sigma_{-i/2}^\varphi(a_n)^* \sigma_{-i/2}^\varphi(a_n) = s(\varphi)$$

Here, $\mathcal{D}(\sigma_{-i/2}^\varphi)$ is the set of $x \in \mathcal{M}$ for which the map $t \mapsto \sigma_t^\varphi$ from $\mathbf{R} \rightarrow \mathcal{M}$ extends to an analytic map $z \mapsto \sigma_z^\varphi$, taking $\{z \in \mathbf{C} : 0 \leq \Im(z) \leq -1/2\} \rightarrow \mathcal{M}$.

We can now proceed with the proof of the main theorem of this section.

Proof. (of Theorem 2.13)

Suppose that $\mathcal{N} \subset \mathcal{M}$ are von Neumann algebras, and

$$\sigma_t^{\tilde{\varphi}}(x) = \sigma_t^\varphi(x), \quad x \in \mathcal{N},$$

for some $\tilde{\varphi} \in \mathfrak{W}_0(\mathcal{M})$ and $\varphi \in \mathfrak{W}_0(\mathcal{N})$. We fix $\tilde{\varphi}$ and φ . For each $\psi \in \mathfrak{W}(\mathcal{N})$, we have the cocycle derivative $u_t = (D\psi : D\varphi)_t \in \mathcal{N}$, $t \in \mathbf{R}$. By a result of Masuda [12], there corresponds to ψ a $\tilde{\psi} \in \mathfrak{W}(\mathcal{M})$ such that $s(\tilde{\psi}) = s(\psi)$ and $(D\tilde{\psi} : D\tilde{\varphi})_t = u_t = (D\psi : D\varphi)_t$. As we have

$$\begin{aligned} (D(\lambda\psi) : D\varphi)_t &= \lambda^{it} (D\psi : D\varphi)_t = \lambda^{it} (D\tilde{\psi} : D\tilde{\varphi})_t \\ &= (D(\lambda\tilde{\psi}) : D\tilde{\varphi})_t, \end{aligned}$$

we obtain $\widetilde{\lambda\psi} = \lambda\tilde{\psi}$, $\lambda > 0$.

By the chain rule for cocycle derivatives, we know $(D\tilde{\psi}_1 : D\tilde{\psi}_2)_t = (D\psi_1 : D\psi_2)_t$ for any $\psi_1 \in \mathfrak{W}(\mathcal{N})$ and $\psi_2 \in \mathfrak{W}_0(\mathcal{N})$. Let $\{\psi_n\}$ be a sequence in \mathcal{N}_*^+ satisfying

$\psi = \sum_{n=1}^{\infty} \psi_n \in \mathcal{N}_*^+$. Let $u_i^{(n)} = (D\psi_n : D\psi)_t$ (when we restrict our consideration to $\mathcal{N}_{s(\psi)}$ and $\mathcal{M}_{s(\tilde{\psi})}$). Then we know that $u_{-i/2}^{(n)}$ is defined, and

$$\psi_n(x) = \psi((u_{-i/2}^{(n)})^* x (u_{-i/2}^{(n)})), \quad x \in \mathcal{N}.$$

Therefore, Lemma 2.15 implies

$$\begin{aligned} s(\psi) &= \sum_{n=1}^{\infty} \sigma_{-i/2}^{\psi}((u_{-i/2}^{(n)})^*)^* \sigma_{-i/2}^{\psi}((u_{-i/2}^{(n)})^*) \\ &= \sum_{n=1}^{\infty} \sigma_{i/2}^{\psi}(u_{-i/2}^{(n)}) \sigma_{i/2}^{\psi}(u_{-i/2}^{(n)})^* \\ &= \sum_{n=1}^{\infty} \sigma_{i/2}^{\tilde{\psi}}((D\tilde{\psi}_n : D\tilde{\psi})_{-i/2}) \sigma_{i/2}^{\tilde{\psi}}((D\tilde{\psi}_n : D\tilde{\psi})_{-i/2})^*. \end{aligned}$$

As $s(\psi) = s(\tilde{\psi})$, the above calculation shows that

$$\tilde{\psi}(x) = \sum_{n=1}^{\infty} \tilde{\psi}_n(x), \quad x \in \mathcal{M}_+.$$

It follows that the map $\psi \in \mathcal{N}_*^+ \mapsto \tilde{\psi} \in \mathfrak{M}(\mathcal{M})$ is homogeneous and countably additive. Hence, the map $\psi \in \mathcal{N}_*^+ \mapsto \tilde{\psi}(x) \in [0, +\infty]$, $x \in \mathcal{M}_+$, gives rise to a map $T: \mathcal{M}_+ \rightarrow \widehat{\mathcal{N}}_+$.

Note that we also have the following: for every $u \in \mathcal{U}(\mathcal{N})$,

$$\begin{aligned} (D(u\psi u^*)^\sim : D\tilde{\varphi})_t &= (D(u\psi u^*) : D\varphi)_t \\ &= u(D\psi : D\varphi)_t \sigma_t^\varphi(u^*) = u(D\tilde{\psi} : D\tilde{\varphi})_t \sigma_t^{\tilde{\varphi}}(u^*), \end{aligned}$$

so that $(u\psi u^*)^\sim = u\tilde{\psi}u^*$, $u \in \mathcal{U}(\mathcal{N})$.

We now want to show

$$(2.5) \quad T(axa^*) = aT(x)a^*, \quad a \in \mathcal{N}, \quad x \in \mathcal{M}_+.$$

First, we observe that if $a \in \mathcal{N}_\psi$ with ψ faithful, then $(Da^*\psi a : D\psi)_t = (a^*a)^{it}$.

Here $(a^*a)^{it}$ should be considered in the reduced algebra $\mathcal{N}_{s_r(a)}$, with $s_r(a)$ denoting

the right support of a , so that $(\widetilde{Da^*\psi a} : D\tilde{\psi})_t = (Da^*\psi a : D\psi)_t = (a^*a)^{it}$; hence

$\widetilde{a^*\psi a} = a^*\tilde{\psi}a$. If ψ is not faithful, then we consider $\psi'' = \psi + \psi'$ with $\psi' \in \mathfrak{W}(\mathcal{N})$

such that $s(\psi') = \mathbf{1} - s(\psi)$; we apply the above argument to ψ'' to conclude

$$\widetilde{a^*\psi a} = a^*\tilde{\psi}a, \quad a \in \mathcal{N}_\psi.$$

Now, we repeat the preceding argument using $\mathcal{N} \otimes M_2(\mathbb{C})$ and $\mathcal{M} \otimes M_2(\mathbb{C})$ in place of \mathcal{N} and \mathcal{M} , respectively. (Accordingly, we replace ψ with $\psi \otimes \text{Tr}$, and $\tilde{\psi}$ with $\tilde{\psi} \otimes \text{Tr}$.) We observe that $\sigma_t^{\psi \otimes \text{Tr}}(x) = \sigma_t^{\tilde{\psi} \otimes \text{Tr}}(x)$ for every $x \in \mathcal{N} \otimes M_2(\mathbb{C})$.

This implies that there exists a map $S: (\mathcal{M} \otimes M_2)_+ \rightarrow (\widetilde{\mathcal{N} \otimes M_2})_+$ such that

$$(\psi \otimes \text{Tr}) \circ S(x) = (\widetilde{\psi \otimes \text{Tr}})(x), \quad x \in (\mathcal{M} \otimes M_2)_+.$$

As $\mathbb{C} \otimes M_2 \subset \mathcal{N}_{\psi \otimes \text{Tr}}$ for any faithful ψ , we have $\widetilde{\psi \otimes \text{Tr}} = \tilde{\psi} \otimes \text{Tr}$. If $a \in \mathcal{N}$ such that $\|a\| \leq 1$, we set

$$u = \begin{bmatrix} a & (\mathbf{1} - aa^*)^{1/2} \\ -(\mathbf{1} - a^*a)^{1/2} & a^* \end{bmatrix} \in \mathcal{N} \otimes M_2;$$

then u is unitary. Therefore, we have

$$u^*(\widetilde{\psi \otimes \text{Tr}})u = u^*(\tilde{\psi} \otimes \text{Tr})u.$$

With $\{e_{ij}\}$ the standard matrix unit of M_2 , we have

$$[(\mathbf{1} \otimes e_{ij})^*(\psi \otimes \text{Tr})(\mathbf{1} \otimes e_{ij})]^\sim = (\mathbf{1} \otimes e_{ij})^*(\tilde{\psi} \otimes \text{Tr})(\mathbf{1} \otimes e_{ij})$$

since $(\mathbf{1} \otimes e_{ij}) \in (\mathcal{N} \otimes M_2)_{\psi \otimes \text{Tr}}$. Hence we conclude that

$$[(\mathbf{1} \otimes e_{11})u^*(\mathbf{1} \otimes e_{11})(\psi \otimes \text{Tr})(\mathbf{1} \otimes e_{11})u(\mathbf{1} \otimes e_{11})]^\sim = (a \otimes e_{11})^*(\tilde{\psi} \otimes \text{Tr})(a \otimes e_{11}),$$

i.e.,

$$[(a^* \otimes e_{11})(\psi \otimes \text{Tr})(a \otimes e_{11})]^\sim = (a^* \otimes e_{11})(\tilde{\psi} \otimes \text{Tr})(a \otimes e_{11}).$$

This means that

$$\widetilde{a^*\psi a} = a^*\tilde{\psi}a$$

for every $a \in \mathcal{N}$ with $\|a\| \leq 1$, which gives us (2.5). Therefore T is an operator valued weight of \mathcal{M}_+ onto $\widehat{\mathcal{N}}_+$ such that $\tilde{\psi} = \psi \circ T$, $\psi \in \mathfrak{W}_0(\mathcal{N})$. As $\tilde{\psi}$ is faithful for any $\psi \in \mathfrak{W}_0(\mathcal{N})$, T is faithful.

The semi-finiteness and uniqueness of T , together with the converse in Theorem 2.13, are consequences of the next Lemma. ■

Lemma 2.16 *Let $\mathcal{N} \subset \mathcal{M}$ be von Neumann algebras and $T: \mathcal{M}_+ \rightarrow \widehat{\mathcal{N}}_+$ be a normal operator valued weight.*

- (i) *If $\psi \circ T$ is semi-finite for some $\psi \in \mathfrak{W}_0(\mathcal{N})$, then T is semi-finite.*
- (ii) *If $\psi \in \mathfrak{W}_0(\mathcal{N})$ and $S \in \mathfrak{W}(\mathcal{M}, \mathcal{N})$ satisfy $\psi \circ T = \psi \circ S$, then $T = S$.*
- (iii) *If $T \in \mathfrak{W}_0(\mathcal{M}, \mathcal{N})$, then*

$$(2.6) \quad \sigma_t^{\psi \circ T}(x) = \sigma_t^\psi(x), \quad x \in \mathcal{N}.$$

Proof.

(i) Set $\tilde{\psi} = \psi \circ T$. Let $h \in \mathfrak{m}_{\tilde{\psi}}^+$ and

$$T(h) = \int_0^\infty \lambda de(\lambda) + \infty p$$

be the spectral decomposition. As $\psi \circ T(h) < +\infty$, and ψ is faithful, we have $p = 0$. Hence $e(\lambda) \nearrow \mathbf{1}$ as $\lambda \nearrow \infty$, so that $e(\lambda)he(\lambda) \rightarrow h$ strongly as $\lambda \nearrow \infty$, and $T(e(\lambda)he(\lambda)) = e(\lambda)T(h)e(\lambda) \in \mathcal{N}_+$; hence $e(\lambda)he(\lambda) \in \mathfrak{m}_T^+$. Therefore \mathfrak{m}_T^+ is σ -strongly dense in \mathcal{M}_+ .

(ii) For any $x \in \mathcal{M}_+$ and $a \in \mathcal{N}$, we have

$$\psi(a^*T(x)a) = \psi(T(a^*xa)) = \psi(S(a^*xa)) = \psi(a^*S(x)a),$$

so that $(a\psi a^*) \circ T = (a\psi a^*) \circ S$ for any $a \in \mathcal{N}$. Therefore, if $x \in \mathfrak{m}_T^+ \cap \mathfrak{m}_S^+$, then $T(x) = S(x)$, since $\{a\psi a^* : a \in \mathfrak{n}_\psi\}$ is a dense subset of \mathcal{N}_*^+ . For a general $x \in \mathcal{M}_+$, consider the spectral decompositions

$$\begin{aligned} m &= S(x) = \int_0^\infty \lambda de(\lambda) + \infty p \in \widehat{\mathcal{N}}_+; \\ n &= T(x) = \int_0^\infty \lambda df(\lambda) + \infty q \in \widehat{\mathcal{N}}_+. \end{aligned}$$

Then we have $e(\lambda)xe(\lambda) \in \mathfrak{m}_S^+ \cap \mathfrak{m}_T^+$ and

$$\begin{aligned} me(\lambda) &= e(\lambda)S(x)e(\lambda) = S(e(\lambda)xe(\lambda)) \\ &= T(e(\lambda)xe(\lambda)) = e(\lambda)T(x)e(\lambda) \\ &= e(\lambda)ne(\lambda); \end{aligned}$$

similarly

$$nf(\lambda) = f(\lambda)mf(\lambda), \quad \lambda \geq 0.$$

So we have, for every $\xi \in \bigcup_{\lambda \geq 0} e(\lambda) \mathfrak{H}_\psi$,

$$\|m^{1/2}\xi\|^2 = m(\omega_\xi) = n(\omega_\xi) = \|n^{1/2}\xi\|^2;$$

similarly

$$\|m^{1/2}\xi\|^2 = \|n^{1/2}\xi\|^2, \quad \xi \in \bigcup_{\lambda \geq 0} f(\lambda) \mathfrak{H}_\psi.$$

But $m^{1/2}$ (resp. $n^{1/2}$) is essentially self-adjoint on $\bigcup e(\lambda) \mathfrak{H}_\psi$ (resp. $\bigcup f(\lambda) \mathfrak{H}_\psi$),

so we get

$$\|m^{1/2}\xi\|^2 = \|n^{1/2}\xi\|^2, \quad \xi \in \mathcal{D}(m^{1/2}) \cap \mathcal{D}(n^{1/2}).$$

Hence there exists a partial isometry $u \in \mathcal{N}$ such that $m^{1/2} = un^{1/2}$ and $n^{1/2} = u^*m^{1/2}$; the uniqueness of the polar decomposition implies $m^{1/2} = n^{1/2}$ and $(1-p) = (1-q)$. Thus $m = n$ as elements of $\widehat{\mathcal{N}}_+$. Hence $S = T$.

(iii) We shall prove $\mathcal{G}(\sigma_{-i}^\psi) \subset \mathcal{G}(\sigma_{-i}^{\psi \circ T})$. (Here, $\mathcal{G}(\sigma_{-i}^\psi)$ represents the graph of the (densely-defined) map σ_{-i}^ψ .) Let $\tilde{\psi} = \psi \circ T$, and $(a, b) \in \mathcal{G}(\sigma_{-i}^\psi)$. As $a \in \mathcal{D}(\sigma_{-i/2}^\psi)$ and $b^* \in \mathcal{D}(\sigma_{-i/2}^\psi) = \mathcal{D}(\sigma_{i/2}^\psi)^*$, there exists $M \geq 0$ such that for every $x \in \mathcal{N}_+$

$$\psi(axa^*) \leq M^2\psi(x), \quad \psi(b^*xb) \leq M^2\psi(x).$$

Taking increasing limits, we see that the above inequalities are valid for every $x \in \widehat{\mathcal{N}}_+$, so that

$$\tilde{\psi}(axa^*) \leq M^2\tilde{\psi}(x), \quad \tilde{\psi}(b^*xb) \leq M^2\tilde{\psi}(x)$$

for every $x \in \mathcal{M}_+$; therefore $\mathfrak{n}_{\tilde{\psi}}a^* \subset \mathfrak{n}_{\tilde{\psi}}$, $\mathfrak{n}_{\tilde{\psi}}b \subset \mathfrak{n}_{\tilde{\psi}}$ and, $\forall x \in \mathfrak{n}_{\tilde{\psi}}$,

$$(2.7) \quad \begin{aligned} \|\eta_{\tilde{\psi}}(xa^*)\| &\leq M\|\eta_{\tilde{\psi}}(x)\| \quad \text{and} \\ \|\eta_{\tilde{\psi}}(xb)\| &\leq M\|\eta_{\tilde{\psi}}(x)\|. \end{aligned}$$

We will prove $(a, b) \in \mathcal{G}(\sigma_{-i}^{\tilde{\psi}})$. By invoking a result from the theory of cocycle derivatives [10], it suffices to show

$$\tilde{\psi}(ax) = \tilde{\psi}(xb), \quad x \in \mathfrak{m}_{\tilde{\psi}}.$$

Fix $x_0 = y_0^*z_0$ with $y_0, z_0 \in \mathfrak{n}_{\tilde{\psi}} \cap \mathfrak{n}_T$. Since $\mathfrak{n}_{\tilde{\psi}}a^* \subset \mathfrak{n}_{\tilde{\psi}}$, and because \mathfrak{n}_T is a right \mathcal{N} -module, we have

$$ax_0 = (y_0a^*)^*z_0 \in (\mathfrak{n}_{\tilde{\psi}} \cap \mathfrak{n}_T)^*(\mathfrak{n}_{\tilde{\psi}} \cap \mathfrak{n}_T) \subset \mathfrak{m}_{\tilde{\psi}} \cap \mathfrak{m}_T.$$

Similarly, we get

$$x_0b = y_0^*(z_0b) \in \mathfrak{m}_{\tilde{\psi}} \cap \mathfrak{m}_T.$$

Since $(a, b) \in \mathcal{G}(\sigma_{-i}^{\psi})$, we have

$$\begin{aligned} \psi \circ \dot{T}(ax_0) &= \psi(a\dot{T}(x_0)) = \psi(\dot{T}(x_0)b) \\ &= \psi \circ \dot{T}(x_0b), \end{aligned}$$

so that

$$\tilde{\psi}(ax_0) = \tilde{\psi}(x_0b).$$

Now suppose $x = y^*z$ with $y, z \in \mathfrak{n}_{\tilde{\psi}}$. Since $\psi \circ T(y^*y) < +\infty$, we have the spectral decomposition of $T(y^*y)$,

$$T(y^*y) = \int_0^\infty \lambda d\epsilon(\lambda).$$

For any $\lambda > 0$, we have $ye(\lambda) \in \mathfrak{n}_T$ and

$$\begin{aligned}\psi \circ T(e(\lambda)y^*ye(\lambda)) &= \psi(e(\lambda)T(y^*y)e(\lambda)) \\ &\leq \psi \circ T(y^*y) < +\infty,\end{aligned}$$

so that $ye(\lambda) \in \mathfrak{n}_{\tilde{\psi}}$. Furthermore,

$$\begin{aligned}\|\eta_{\tilde{\psi}}(ye(\lambda) - y)\|^2 &= \psi \circ T((ye(\lambda) - y)^*(ye(\lambda) - y)) \\ &= \psi((\mathbf{1} - e(\lambda))T(y^*y)(\mathbf{1} - e(\lambda))) \\ &= \psi\left(\int_{\lambda}^{\infty} \gamma de(\gamma)\right) \rightarrow 0 \quad \text{as } \lambda \rightarrow \infty.\end{aligned}$$

Similarly, with the spectral decomposition of $T(z^*z)$,

$$T(z^*z) = \int_0^{\infty} \lambda df(\lambda),$$

we have $zf(\lambda) \in \mathfrak{n}_{\tilde{\psi}} \cap \mathfrak{n}_T$ and

$$\lim_{\lambda \rightarrow \infty} \|\eta_{\tilde{\psi}}(zf(\lambda) - z)\| = 0.$$

By (2.7), we get

$$\begin{aligned}\lim_{\lambda \rightarrow \infty} \|\eta_{\tilde{\psi}}(ye(\lambda)a^*) - \eta_{\tilde{\psi}}(ya^*)\| &= 0 \quad \text{and} \\ \lim_{\lambda \rightarrow \infty} \|\eta_{\tilde{\psi}}(zf(\lambda)b) - \eta_{\tilde{\psi}}(zb)\| &= 0.\end{aligned}$$

Therefore, we obtain, by the previous arguments for x_0 ,

$$\begin{aligned}\tilde{\psi}(ax) &= (\eta_{\tilde{\psi}}(z) \mid \eta_{\tilde{\psi}}(ya^*)) = \lim_{\lambda \rightarrow \infty} (\eta_{\tilde{\psi}}(zf(\lambda)) \mid \eta_{\tilde{\psi}}(ye(\lambda)a^*)) \\ &= \lim_{\lambda \rightarrow \infty} \tilde{\psi}(a(ye(\lambda))^*(zf(\lambda))) \\ &= \lim_{\lambda \rightarrow \infty} \tilde{\psi}(ye(\lambda))^*(zf(\lambda))b \\ &= \lim_{\lambda \rightarrow \infty} (\eta_{\tilde{\psi}}(zf(\lambda)b) \mid \eta_{\tilde{\psi}}(ye(\lambda))) \\ &= (\eta_{\psi}(zb) \mid \eta_{\psi}(y)) = \tilde{\psi}(xb).\end{aligned}$$

Hence, we may conclude that $(a, b) \in \mathcal{G}(\sigma_{-i}^{\tilde{\psi}})$, i.e., $\mathcal{G}(\sigma_{-i}^{\psi}) \subset \mathcal{G}(\sigma_{-i}^{\tilde{\psi}})$.

Now, we know that $x \in \mathcal{M}$ is of exponential type relative to $\{\sigma_t^{\tilde{\psi}}\}$ if and only if

$$x \in \bigcap_{n \in \mathbb{Z}} \mathcal{D}(\sigma_{-ni}^{\tilde{\psi}}) \quad \text{and}$$

$$\sup \|\sigma_{-ni}^{\tilde{\psi}}(x)\| e^{-c|n|} < +\infty$$

for some $c > 0$. Hence $\mathcal{N}_{\text{exp}}^{\psi} \subset \mathcal{M}_{\text{exp}}^{\tilde{\psi}}$. For each $x \in \mathcal{N}_{\text{exp}}^{\psi}$, we consider

$$y(\alpha) \triangleq \sigma_{\alpha}^{\psi}(x) - \sigma_{\alpha}^{\tilde{\psi}}(x), \quad \alpha \in \mathbb{C}.$$

The preceding discussion shows that the function $f_{\omega}: \alpha \in \mathbb{C} \mapsto \omega(y(\alpha)) \in \mathbb{C}$ (where $\omega \in \mathcal{M}_{\star}$) is an entire function of exponential type, and that $f_{\omega}(-in) = 0$, $n \in \mathbb{Z}$. Hence $f_{\omega}(\alpha) = 0$ for every $\alpha \in \mathbb{C}$, and, since $\omega \in \mathcal{M}_{\star}$ was arbitrary, we must have $y(\alpha) = 0$. This means that

$$\sigma_t^{\psi}(x) = \sigma_t^{\tilde{\psi}}(x), \quad x \in \mathcal{N}_{\text{exp}}^{\psi}.$$

As $\mathcal{N}_{\text{exp}}^{\psi}$ is σ -weakly dense in \mathcal{N} , we may conclude (2.6). ■

2.3 Some Examples of Operator Valued Weights

We now wish to present a few examples in which the action of the operator valued weight can be made explicit. These examples will demonstrate the appropriateness of interpreting the application of an operator valued weight as a (partial) integration. In each of the cases to be discussed, the role of \mathcal{M} will be played by $\mathcal{L}(\mathfrak{H})$ (for an appropriate Hilbert space \mathfrak{H}), $\tilde{\varphi}$ will be Tr , and \mathcal{N} will be a subalgebra of \mathcal{M} which possesses its own trace φ . Hence, the condition

$$\sigma_t^\varphi(x) = \sigma_t^{\tilde{\varphi}}(x), \quad x \in \mathcal{N}$$

is trivially satisfied. Note also that in the following examples, no distinction will be made between T and its extension \hat{T} .

1. Let $G = \mathbf{Z}$, i.e., we are going to consider a discrete, countable abelian group.

Take any $x \in \mathcal{L}(\ell^2(\mathbf{Z}))$; we may write $x = (x_{ij})_{i,j \in \mathbf{Z}}$. Then (believe it or not), the operator valued weight

$$T: \mathcal{L}(\ell^2(\mathbf{Z}))_+ \rightarrow \widehat{\ell^\infty(\mathbf{Z})}_+$$

is given by

$$T(x) \triangleq (x_{nn})_{n \in \mathbf{Z}},$$

that is to say, we are merely taking the “diagonal” of the “infinite matrix” (x_{ij}) . For this to be an operator valued weight, one must specify how $T(x)$

acts on elements of $\ell^\infty(\mathbf{Z})_+^* \cong \ell^1(\mathbf{Z})_+$. But in this case the action is clear: given any $\omega \in \ell^\infty(\mathbf{Z})_+^*$, we may write ω as $(a_i)_{i \in \mathbf{Z}}$. Then, given any $x \in \mathcal{L}(\ell^2(\mathbf{Z}))$, we have

$$T(x)(\omega) = \sum_{n \in \mathbf{Z}} a_n x_{nn}.$$

Observe that in this very special case, $\mathfrak{m}_T = \mathcal{M}$; in fact, $T(\mathbf{1}_{\mathcal{M}}) = T(\mathbf{1}_{\mathcal{N}})$, so T is actually a conditional expectation.

2. Let $G = \mathbf{T}$, i.e., we are considering the compact, abelian group case. Then the operator valued weight is given by

$$x \in \mathcal{L}(L^2(\mathbf{T}))_+ \mapsto \sum_{k \in \mathbf{Z}} u^k x u^{-k} \in \widehat{L^\infty(\mathbf{T})}_+,$$

where $(u\xi)(\zeta) \triangleq \zeta\xi(\zeta)$. Note that if we choose $\epsilon_i(\zeta) = \zeta^i$, then $\{\epsilon_i : i \in \mathbf{Z}\}$ is a complete, orthonormal system for $L^2(\mathbf{T})$. Taking $x = t_{\epsilon_i, \epsilon_j}$ (here, we define $t_{\epsilon_i, \epsilon_j} \triangleq (\eta | \epsilon_j)\epsilon_i$, for any $\eta \in \mathcal{L}(L^2(\mathbf{T}))$), we obtain:

$$\begin{aligned} (T(x)\xi)(\zeta) &= \sum_{k \in \mathbf{Z}} \hat{\xi}(j-k)\zeta^{i-k} \\ &= \zeta^{i-j}\xi(\zeta), \end{aligned}$$

where

$$\hat{\xi}(n) = \frac{1}{2\pi} \int_{\mathbf{T}} e^{-in\zeta} \xi(\zeta) d\zeta.$$

The previous calculation shows us that $t_{\epsilon_i, \epsilon_j} \in \mathfrak{m}_T \forall i, j \in \mathbf{Z}$, so that T is indeed semi-finite. Once again, though, it is necessary to define $T(x)(\omega)$ for any $x \in L^\infty(\mathbf{T})_+$. Unlike in (1), however, T is *not* a conditional expectation, so we cannot hope to define $T(x)(\omega)$ in as elementary a fashion.

We proceed as follows: Let $\mathfrak{H} = L^2(\mathbf{T})$; then for any $x \in \mathcal{L}(L^2(\mathbf{T}))_+$, we define a map $q_x: \mathfrak{H} \rightarrow [0, +\infty]$ via

$$(2.8) \quad q_x(\xi) \triangleq \sum_{k \in \mathbf{Z}} \|x^{1/2} u_{-k} \xi\|.$$

q_x is then a lower semi-continuous quadratic form, whose domain $\mathcal{D}(q_x)$ includes $\{t_{\varepsilon_i, \varepsilon_i} : i \in \mathbf{Z}\}$ (again, by preceding calculation); hence, to it is associated a positive, self-adjoint operator H_x . Moreover, due to the form of q_x , it is clear that $H_x \eta \mathcal{N}$. Hence, $\forall \varepsilon > 0$, $H_x(\mathbf{1} + \varepsilon H_x)^{-1} \in \mathcal{N}_+ = L^\infty(\mathbf{T})_+$. Moreover, we know that $\varepsilon_1 \leq \varepsilon_2 \Rightarrow H_x(\mathbf{1} + \varepsilon_2 H_x)^{-1} \leq H_x(\mathbf{1} + \varepsilon_1 H_x)^{-1}$. We may therefore define

$$T(x)(\omega) \triangleq \lim_{\varepsilon \searrow 0} \langle \omega, H_x(\mathbf{1} + \varepsilon H_x)^{-1} \rangle.$$

Throughout the remainder of the examples, whenever the action of an operator valued weight T is exhibited as a summation (or integration), it is to be understood in the above context, viz., we should consider the quadratic form induced by our definition of T in a way entirely analogous to (2.8), and then consider the appropriate (possibly unbounded) operator.

3. Let $G = \mathbf{R}$, i.e., we are considering a separable, locally compact but non-compact, abelian group. In this case, the operator valued weight is given by

$$T: x \in \mathcal{L}(L^2(\mathbf{R}))_+ \mapsto \int_{\widehat{\mathbf{R}=\mathbf{R}}} u(\lambda) x u(-\lambda) d\lambda \in \widehat{L^\infty(\mathbf{R})}_+,$$

where

$$(u(\lambda)\xi)(s) \triangleq e^{i\lambda s}\xi(s), \quad \forall \xi \in L^2(\mathbf{R}).$$

Taking $\{\epsilon_i : \epsilon_i \in L^2(\mathbf{R}) \cap L^\infty(\mathbf{R}), i \in \mathbf{N}\}$, a complete orthonormal system for $L^2(\mathbf{R})$, and again taking x to be $t_{\epsilon_i, \epsilon_j}$, we obtain:

$$(T(t_{\epsilon_i, \epsilon_j})\xi)(s) = \epsilon_i(s)\overline{\epsilon_j(s)}\xi(s).$$

In the general setting, where G is any (separable) locally compact, abelian group, with dual group \widehat{G} , we will have $T: \mathcal{L}(L^2(G))_+ \rightarrow \widehat{L^\infty(G)}_+$ given by

$$T(x) \triangleq \int_{\widehat{G}} u(p)xu(-p) dp,$$

where

$$(u(p)\xi)(g) \triangleq \langle p, g \rangle \xi(g), \quad \forall \xi \in L^2(G).$$

Here, of course, $\langle \cdot, \cdot \rangle$ denotes the duality between \widehat{G} and G .

4. Suppose G is a discrete, separable group, but not necessarily abelian. If G is ICC (i.e., infinite conjugacy class), then $\mathcal{R}_\ell(G) \subset \mathcal{L}(\ell^2(G))$ is a II_1 -factor, and therefore possesses a tracial state τ . So again, we are motivated to look for an operator valued weight $T: \mathcal{L}(\ell^2(G))_+ \rightarrow \widehat{\mathcal{R}_\ell(G)}_+$.

In this setting, the operator valued weight is given by

$$T(x) \triangleq \sum_{g \in G} \rho(g)x\rho(g)^*,$$

where

$$\rho(g)\xi(h) \triangleq \xi(hg), \quad \forall \xi \in \ell^2(G).$$

To see that this is so, we note immediately that T is clearly positive homogeneous and linear on all of $\mathcal{L}(\ell^2(G))_+$. If we define $\epsilon_g: G \rightarrow \mathbf{C}$, $g \in G$ via $\epsilon_g(s) \triangleq \delta_{s,g}$, then clearly $\{\epsilon_g : g \in G\}$ is a complete, orthonormal system for $\ell^2(G)$. Take $x = t_{\epsilon_g, \epsilon_h}$; then $x \in \mathcal{LF}(\ell^2(G))$ (the set of finite rank operators); we calculate:

$$\begin{aligned} T(x) &= T(t_{\epsilon_g, \epsilon_h}) = \sum_{s \in G} \rho(s) t_{\epsilon_g, \epsilon_h} \rho(s)^*, \\ &= \sum_{s \in G} \rho(s) t_{\epsilon_g, \epsilon_h} \rho(s^{-1}) = \sum_{s \in G} \rho(s) t_{\epsilon_g, \epsilon_{hs^{-1}}} \\ &= \sum_{s \in G} t_{\epsilon_{gs^{-1}}, \epsilon_{hs^{-1}}}. \end{aligned}$$

Now, we will demonstrate that $T(t_{\epsilon_g, \epsilon_h})$ is actually equal to $\lambda(gh^{-1})$, where (as usual)

$$\lambda(g)\xi(h) \triangleq \xi(g^{-1}h), \quad \forall \xi \in \ell^2(G).$$

For any $k \in G$, we compute:

$$\begin{aligned} T(t_{\epsilon_g, \epsilon_h})\epsilon_k &= \sum_{s \in G} t_{\epsilon_{gs^{-1}}, \epsilon_{hs^{-1}}} \epsilon_k \\ &= \sum_{s \in G} (\epsilon_k | \epsilon_{hs^{-1}}) \epsilon_{gs^{-1}} \\ &= \sum_{s \in G} \delta_{k, hs^{-1}} \epsilon_{gs^{-1}} \\ &= \sum_{s \in G} \delta_{s, k^{-1}h} \epsilon_{gs^{-1}} \\ &= \epsilon_{gh^{-1}k} = \lambda(gh^{-1})\epsilon_k. \end{aligned}$$

Hence, we see $T(\{t_{\epsilon_g, \epsilon_h} : g, h \in G\}) \subset \mathcal{R}_\ell(G)$; thus T is semi-finite. Also, T is faithful: to say $T(x^*x) = 0$ implies $\sum_{s \in G} \rho(s)x^*x\rho(s)^* = 0$. But

$\rho(s)x^*x\rho(s)^* = (x\rho(s)^*)^*(x\rho(s)^*)$, so $T(x^*x)$ is a sum of positive elements, which can only be 0 if each $(x\rho(s)^*)^*(x\rho(s)^*) = 0$. But since each $\rho(s)$ is unitary, this will only be so if $x^*x = x = 0$. Finally, the definition of T (i.e., as a “sum of positive elements”) ensures its normality.

It is also interesting to note that the form of T is that of “integration over the commutant;” specifically, we are summing over $\{\rho(s) : s \in G\}$, and we recognize that $\{\rho(s) : s \in G\}'' = \mathcal{R}_r(G) \cong \mathcal{R}_l(G)'$. This lends credence to our earlier claim that the application of an operator valued weight should be interpreted as integration.

As an aside, we mention that, on \mathfrak{m}_T , T has the form

$$T(x) = \sum_{g \in G} \text{Tr}(\lambda(g)^*x)\lambda(g).$$

Expressing T via the above allows us to view the application of T as a kind of “Fourier transform.”

5. Let $A_n = M_2(\mathbb{C}) \otimes \cdots \otimes M_2(\mathbb{C}) \cong M_{2^n}(\mathbb{C})$, $\mathcal{A}_0 = \varinjlim A_n$, the algebraic direct limit of the A_n , and $\mathcal{A} = \widehat{\mathcal{A}}_0$, the C^* -algebraic direct limit of the A_n . Note that in this context we may (and shall) consider \mathcal{A} to be the “ C^* -algebraic infinite tensor product” $M_2(\mathbb{C}) \otimes M_2(\mathbb{C}) \otimes \cdots$. Now, each A_n possesses a tracial state $\tau_n \triangleq 2^{-n}\text{Tr}_n$, where Tr_n denotes the usual trace on $M_{2^n}(\mathbb{C})$. These, in turn, induce a tracial state on \mathcal{A} , which we shall call τ .

In the usual way (i.e., via the GNS construction), we obtain a representation of \mathcal{A} , viz., $\{\pi_\tau, \mathfrak{H}_\tau, \xi_\tau\}$. Then, we know that $\pi_\tau(\mathcal{A})'' \subset \mathcal{L}(\mathfrak{H}_\tau)$ is a II_1 -factor

– in fact, it is *the* AFD II₁-factor \mathcal{R}_0 . The trace on \mathcal{R}_0 , which will also be denoted by τ , is given by ω_{ξ_τ} . Hence, we are now looking for the operator valued weight $T: \mathcal{L}(\mathfrak{H}_\tau)_+ \rightarrow \widehat{\mathcal{R}_{0+}}$ which satisfies $\text{Tr} = \tau \circ T$.

To find T , it is helpful to change perspective. Let's start by defining $\tilde{\mathfrak{H}}$ to be $\{M_2(\mathbb{C}), \frac{1}{2}\text{Tr}_1\}$, i.e., $M_2(\mathbb{C})$ considered as a Hilbert space, via the inner product $(X | Y) \triangleq \text{Tr}_1(Y^*X)$. We next define \mathfrak{H}_n via $\mathfrak{H}_n \triangleq \tilde{\mathfrak{H}} \otimes \cdots \otimes \tilde{\mathfrak{H}}$ (n times). We embed $\mathfrak{H}_n \hookrightarrow \mathfrak{H}_{n+1}$ via $\xi \mapsto \xi \otimes \mathbf{1}_1$, where $\mathbf{1}_1$ is the identity matrix in $M_2(\mathbb{C})$. Since this map is clearly an isometry, we are free to consider the “Hilbert space infinite tensor product;” we shall call this \mathfrak{H} .

Now, it is easy to see that \mathcal{A}_0 acts on \mathfrak{H} in a natural way via both left and right multiplication; moreover, this action is bounded. Thus, we may consider $\lambda_\ell(\mathcal{A}_0), \lambda_r(\mathcal{A}_0) \subset \mathcal{L}(\mathfrak{H})$. It is also not difficult to see that we actually have

$$\mathfrak{H} \cong \mathfrak{H}_\tau, \quad \text{and} \quad \lambda_\ell(\mathcal{A}_0)'' \cong \mathcal{R}_0.$$

Let's contemplate what occurs at the n^{th} “stage” of the direct limiting process. We have A_n acting on \mathfrak{H}_n from both the left and right; in fact, $\mathcal{L}(\mathfrak{H}_n) \cong A_{2n} = A_n \otimes A_n$. Note also that we have $\text{Tr}_{2n} = \text{Tr}_n \otimes \text{Tr}_n$, and $\tau_{2n} = \tau_n \otimes \tau_n$. We can obtain a conditional expectation $\mathcal{E}_n: \mathcal{L}(\mathfrak{H}_n) \rightarrow A_n$ via

$$\mathcal{E}_n(x) \triangleq \int_{\mathcal{U}(A_n)} u x u^* du, \quad \forall x \in \mathcal{L}(\mathfrak{H}_n).$$

Here, we are integrating with respect to the normalized Haar measure over the compact group $\mathcal{U}(A_n) \cong \text{U}(2^n)$. It is clear that we have $\tau_{2n} = \tau_n \circ \mathcal{E}_n$.

However, note that if we want to have $T_n: \mathcal{L}(\mathfrak{H}_n) \rightarrow A_n$ such that $\text{Tr}_{2^n} = \tau_n \circ T_n$, we must define $T_n \triangleq 2^{2^n} \mathcal{E}_n$.

These observations guide us to the proper formulation of T . For any $x \in \mathcal{L}(\mathfrak{H}_\tau)_+$, we define:

$$T(x) \triangleq \lim_{n \rightarrow \infty} 2^{2^n} \int_{\mathcal{U}(A_n)} \text{Ad}(J_\tau u J_\tau) x \, du,$$

where J_τ is the modular conjugation. (As τ is a trace, we have $J_\tau \eta_\tau(y) = \eta_\tau(y^*)$, with $\eta_\tau: \mathcal{A} \rightarrow \mathfrak{H}_\tau$.) For any $x \in \mathcal{A}_0$, our previous considerations show that this definition yields the desired results; as \mathcal{A}_0 is σ -weakly dense in $\mathcal{L}(\mathfrak{H}_\tau)$, we observe (once again by an appeal to normality) that T is the correct operator valued weight.

It is also interesting to note that it is possible to consider a different limiting process applied to the previously defined \mathcal{E}_n 's. As each \mathcal{E}_n is a projection of norm one, for any $x \in \mathcal{L}(\mathfrak{H})$, $\{(\tau \circ \mathcal{E}_n)(x) : n \in \mathbf{N}\} \in \ell^\infty(\mathbf{N})$; if we choose ω a free ultrafilter (i.e., $\omega \in \beta\mathbf{N} \setminus \mathbf{N}$, where, as usual, $\beta\mathbf{N}$ is the Stone-Ćech compactification of \mathbf{N}), we may define

$$\rho_\omega(x) \triangleq \lim_{n \rightarrow \omega} (\tau \circ \mathcal{E}_n)(x).$$

Then ρ_ω is a hypertrace on $\mathcal{L}(\mathfrak{H})$. It is, however, far from normal; it is *singular* (see [9]). Hence ρ_ω is a transcendental object. Similarly, if we consider, for any $x \in \mathcal{L}(\mathfrak{H})$, $\{\mathcal{E}_n(x) : n \in \mathbf{N}\} \in \ell^\infty(\mathbf{N}, \mathcal{L}(\mathfrak{H}))$, we may define

$$\Phi_\omega(x) \triangleq \lim_{n \rightarrow \omega} \mathcal{E}_n(x).$$

$\Phi_\omega: \mathcal{L}(\mathfrak{H}) \rightarrow \mathcal{R}_0$ is a conditional expectation, but it too is highly singular. So, it seems, we are faced with a choice: to forego boundedness for the sake of normality (i.e., work with T), or retain boundedness and give up σ -weak continuity. Both techniques have utility; circumstances will dictate which option is more useful.

CHAPTER 3

*L*²-von Neumann Modules and the Spatial Derivative

3.1 Left and Right Modules over a von Neumann Algebra

It is commonplace to think about von Neumann algebras presented spatially, i.e., we consider the pair $\{\mathcal{M}, \mathfrak{H}\}$. Then, \mathfrak{H} has a natural structure as a *left* \mathcal{M} -module. We now want to consider a *right* action of a von Neumann algebra on a Hilbert space. Hence, we are motivated to the following:

Definition 3.1

(i) Given a von Neumann algebra \mathcal{N} , the opposite von Neumann algebra \mathcal{N}° means the von Neumann algebra obtained by reversing the product in \mathcal{N} , i.e., as a linear space equipped with $*$ -operation we take \mathcal{N}° to be \mathcal{N} , denote by x° the element in \mathcal{N}° corresponding to $x \in \mathcal{N}$, and then define the product in \mathcal{N}° via

$$(3.1) \quad x^\circ y^\circ \triangleq (yx)^\circ, \quad \forall x, y \in \mathcal{N}.$$

(ii) A right \mathcal{N} -module is a Hilbert space \mathfrak{H} on which \mathcal{N} acts from the right, i.e., \mathfrak{H} equipped with a normal anti-representation, $\pi'_\mathfrak{H}$, of \mathcal{N} on \mathfrak{H} ; equivalently a Hilbert space equipped with a normal representation of \mathcal{N}° . To avoid uninteresting notational complexity, we consider only faithful right \mathcal{N} -modules \mathfrak{H} , in the sense that $\pi'_\mathfrak{H}(x) \neq 0$ for every non-zero $x \in \mathcal{N}$. We denote the right \mathcal{N} -module \mathfrak{H} by $\mathfrak{H}_\mathcal{N}$ to emphasize that \mathfrak{H} is being viewed as a right \mathcal{N} -module.

(iii) For a pair \mathcal{M}, \mathcal{N} of von Neumann algebras, an \mathcal{M} - \mathcal{N} bimodule means a Hilbert space \mathfrak{H} , (often denoted ${}_\mathcal{M}\mathfrak{H}_\mathcal{N}$ to emphasize its bimodule structure), equipped with a normal representation π of \mathcal{M} on \mathfrak{H} and a normal anti-representation π' of \mathcal{N} on \mathfrak{H} such that $\pi(\mathcal{M})$ and $\pi'(\mathcal{N})$ commute. We write:

$$(3.2) \quad x\xi y = \pi(x)\pi'(y)\xi, \quad \forall x \in \mathcal{M}, y \in \mathcal{N}.$$

The commutativity of $\pi(\mathcal{M})$ and $\pi'(\mathcal{N})$ is equivalent to associativity: $x(\xi y) = (x\xi)y$, $x \in \mathcal{M}$ and $y \in \mathcal{N}$. Once again, we will consider only faithful bimodules.

Now, let's fix von Neumann algebras \mathcal{M} and \mathcal{N} . If \mathfrak{H} is an \mathcal{M} - \mathcal{N} bimodule, then its Banach space dual $\overline{\mathfrak{H}}$ is canonically an \mathcal{N} - \mathcal{M} bimodule by the action:

$$(3.3) \quad x\overline{\xi}y \triangleq \overline{y^*\xi x^*}, \quad x \in \mathcal{M}, y \in \mathcal{N}$$

where $\overline{\xi}$ denotes the vector in $\overline{\mathfrak{H}}$ corresponding to $\xi \in \mathfrak{H}$ by the pairing: $\langle \eta, \overline{\xi} \rangle = (\eta | \xi)$, with $\eta \in \mathfrak{H}$ and $\overline{\xi} \in \overline{\mathfrak{H}}$. This left \mathcal{N} -module $\overline{\mathfrak{H}}$ will be called the *conjugate* bimodule or the *bimodule dual* to the original bimodule \mathfrak{H} .

Of special interest is a von Neumann algebra in standard form. Let us fix an fns weight ψ on \mathcal{N} (so we can and will write $\psi \in \mathfrak{W}_0(\mathcal{N})$), and consider the standard form, which we will denote by $\{L^2(\mathcal{N}), L^2(\mathcal{N})_+, \mathbf{J}\}$. The right action of \mathcal{N} is given by

$$(3.4) \quad \xi x \triangleq \mathbf{J}x^*\mathbf{J}\xi, \quad x \in \mathcal{N}.$$

Thus we obtain an \mathcal{N} - \mathcal{N} bimodule $L^2(\mathcal{N})$, which will be called the *standard bimodule*. Sometimes, we write ξ^* for $\mathbf{J}\xi$, $\xi \in L^2(\mathcal{N})$. We state here the following easy but important proposition:

Proposition 3.2 *For a von Neumann algebra \mathcal{N} , the standard bimodule $L^2(\mathcal{N})$ is self-dual under the correspondence: $\xi^* \leftrightarrow \bar{\xi}$, $\xi \in L^2(\mathcal{N})$.*

The proof is straightforward, so will be omitted.

With $\psi \in \mathfrak{W}_0(\mathcal{N})$, the left action on $L^2(\mathcal{N})$ is nothing but the semi-cyclic representation π_ψ on \mathfrak{H}_ψ . The right action π'_ψ of \mathcal{N} is then given by:

$$(3.5) \quad \pi'_\psi(x) = \mathbf{J}_\psi \pi_\psi(x^*) \mathbf{J}_\psi, \quad x \in \mathcal{N}.$$

It then follows from the theory of the cocycle Radon-Nikodym derivative (see [10]) that the right action of \mathcal{N} is also given by the following:

$$(3.6) \quad \eta_\psi(x)b = \eta_\psi(x\sigma_{-i/2}^\psi(b)), \quad x \in \mathfrak{n}_\psi, \quad b \in \mathcal{D}(\sigma_{-i/2}^\psi).$$

This twist on the right action suggests that we write $x\psi^{1/2}$ for $\eta_\psi(x)$, $x \in \mathfrak{n}_\psi$, viewing $\psi^{1/2}$ as a vector of infinite magnitude "in" $L^2(\mathcal{N})$. Then (3.6) can be

written more suggestively as

$$(3.6') \quad x\psi^{1/2}b = (x\psi^{1/2}b\psi^{-1/2})\psi^{1/2} = (x\sigma_{-i/2}^\psi(b))\psi^{1/2}, \quad x \in \mathfrak{n}_\psi, \quad b \in \mathcal{D}(\sigma_{-i/2}^\psi).$$

We now introduce a new notation:

$$(3.7) \quad \eta'_\psi(x) \triangleq J_\psi \eta_\psi(x^*), \quad x \in \mathfrak{n}_\psi^*,$$

which can be written as $\psi^{1/2}x$, $x \in \mathfrak{n}_\psi^*$. This new map $\eta'_\psi : x \in \mathfrak{n}_\psi^* \mapsto \eta'_\psi(x) \in L^2(\mathcal{N})$

allows us to write (3.5) as simply

$$(3.8) \quad \pi'_\psi(b)\eta'_\psi(x) = \eta'_\psi(xb) = \eta'_\psi(x)b, \quad x \in \mathfrak{n}_\psi^*, \quad b \in \mathcal{N}.$$

We now consider a general right \mathcal{N} -module \mathfrak{H} . First, we define, given a pair $\{\mathfrak{H}_1, \mathfrak{H}_2\}$ of right \mathcal{N} -modules,

$$(3.9) \quad \mathcal{L}((\mathfrak{H}_1)_\mathcal{N}, (\mathfrak{H}_2)_\mathcal{N}) \triangleq \{t \in \mathcal{L}(\mathfrak{H}_1, \mathfrak{H}_2) : t(\xi y) = (t\xi)y, \quad y \in \mathcal{N}\},$$

and for $\mathcal{L}(\mathfrak{H}_\mathcal{N}, \mathfrak{H}_\mathcal{N})$, we shall write $\mathcal{L}(\mathfrak{H}_\mathcal{N})$. With this notation, the right \mathcal{N} -module \mathfrak{H} becomes canonically an $\mathcal{L}(\mathfrak{H}_\mathcal{N})$ - \mathcal{N} bimodule. Also, we note that $\mathcal{L}(L^2(\mathcal{N})_\mathcal{N}) = \mathcal{N}$ (a direct consequence of Tomita-Takesaki theory) — a fact that will be used throughout. For the pair $\{\mathfrak{H}_1, \mathfrak{H}_2\}$, we shall also consider the direct sum right \mathcal{N} -module $\mathfrak{H}_\mathcal{N} = (\mathfrak{H}_1)_\mathcal{N} \oplus (\mathfrak{H}_2)_\mathcal{N}$; if we let e_1 and e_2 denote the projections of \mathfrak{H} down to \mathfrak{H}_1 and \mathfrak{H}_2 respectively, then we have $\mathcal{L}((\mathfrak{H}_1)_\mathcal{N}, (\mathfrak{H}_2)_\mathcal{N}) = e_2\mathcal{L}(\mathfrak{H}_\mathcal{N})e_1$.

Now let $\{\mathcal{M}, \mathfrak{H}\}$ be a von Neumann algebra. We want to study the relation between a semi-finite, normal weight φ on \mathcal{M} and an fns weight ψ' on \mathcal{M}' . Set $\mathcal{N} = (\mathcal{M}')^\circ$, which allows us to view \mathfrak{H} as an \mathcal{M} - \mathcal{N} bimodule. Let ψ be the weight

on \mathcal{N} defined by

$$\psi(y) \triangleq \psi'(y^\circ), \quad y \in \mathcal{N}_+.$$

We first pair the von Neumann algebra $\{\mathcal{M}, \mathfrak{H}\}$ with one in standard form, in the following manner: let $\tilde{\mathfrak{H}} = L^2(\mathcal{N}) \oplus \mathfrak{H}$ as a right \mathcal{N} -module. Then, set $\mathcal{R} = \mathcal{L}(\tilde{\mathfrak{H}}_{\mathcal{N}})$. It is easy to verify that $\mathcal{L}(L^2(\mathcal{N})_{\mathcal{N}}, \mathfrak{H}_{\mathcal{N}}) = f\mathcal{R}e$, where e and f are the projections of $\tilde{\mathfrak{H}}$ onto $L^2(\mathcal{N})$ and \mathfrak{H} , respectively. The semi-finite, normal weights ψ on \mathcal{N} and φ on \mathcal{M} give rise to a semi-finite, normal weight ρ on \mathcal{R} given by

$$\rho(x) \triangleq \psi(exe) + \varphi(fxf), \quad x \in \mathcal{R}.$$

We set

$$(3.10) \quad \begin{aligned} \mathfrak{n}_\psi(\mathfrak{H}) &= f\mathfrak{n}_\rho e = \{t \in \mathcal{L}(L^2(\mathcal{N})_{\mathcal{N}}, \mathfrak{H}_{\mathcal{N}}) : \psi(t^*t) < +\infty\}; \\ \mathfrak{D}(\mathfrak{H}, \psi) &= \{\xi \in \mathfrak{H} : \|\xi x\| \leq C_\xi \|\eta'_\psi(x)\|, \quad x \in \mathfrak{n}_\psi^* \text{ for some } C_\xi \geq 0\}. \end{aligned}$$

Observe that each $\xi \in \mathfrak{D}(\mathfrak{H}, \psi)$ gives rise to an operator, denoted $L_\psi(\xi)$, which belongs to $\mathcal{L}(L^2(\mathcal{N})_{\mathcal{N}}, \mathfrak{H}_{\mathcal{N}})$; it is defined by the equation

$$(3.11) \quad L_\psi(\xi)\eta'_\psi(x) \triangleq \xi x, \quad x \in \mathfrak{n}_\psi^*, \quad \xi \in \mathfrak{D}(\mathfrak{H}, \psi).$$

Lemma 3.3

(i)

$$\mathfrak{n}_\psi(\mathfrak{H}) = \mathcal{L}(L^2(\mathcal{N})_{\mathcal{N}}, \mathfrak{H}_{\mathcal{N}})\mathfrak{n}_\psi$$

and

$$\mathfrak{D}(\mathfrak{H}, \psi) = \mathcal{L}(L^2(\mathcal{N})_{\mathcal{N}}, \mathfrak{H}_{\mathcal{N}})\mathfrak{B}_\psi,$$

where $\mathfrak{B}_\psi = \eta_\psi(\mathfrak{n}_\psi) \subset L^2(\mathcal{N})$.

(ii) The map $t \otimes_{\mathcal{N}} y \in \mathcal{L}(L^2(\mathcal{N})_{\mathcal{N}}, \mathfrak{H}_{\mathcal{N}}) \odot_{\mathcal{N}} \mathfrak{n}_\psi \mapsto t\eta_\psi(y) \in \mathfrak{D}(\mathfrak{H}, \psi)$ gives rise to a map, denoted $\tilde{\eta}_\psi$, from $\mathfrak{n}_\psi(\mathfrak{H})$ onto $\mathfrak{D}(\mathfrak{H}, \psi)$ such that

$$(3.12) \quad \begin{aligned} \tilde{\eta}_\psi(at) &= a\tilde{\eta}_\psi(t), \quad a \in \mathcal{M}, t \in \mathfrak{n}_\psi(\mathfrak{H}); \\ \tilde{\eta}_\psi(t\sigma_{-i/2}^\psi(b)) &= \tilde{\eta}_\psi(t)b, \quad t \in \mathfrak{n}_\psi(\mathfrak{H}), \quad b \in \mathcal{D}(\sigma_{-i/2}^\psi). \end{aligned}$$

Here, $\mathcal{L}(L^2(\mathcal{N})_{\mathcal{N}}, \mathfrak{H}_{\mathcal{N}}) \odot_{\mathcal{N}} \mathfrak{n}_\psi$ represents the algebraic tensor product of the (algebraic) right \mathcal{N} -module $\mathcal{L}(L^2(\mathcal{N})_{\mathcal{N}}, \mathfrak{H}_{\mathcal{N}})$ and the (algebraic) left \mathcal{N} -module \mathfrak{n}_ψ .

(iii) $\mathfrak{D}(\mathfrak{H}, \psi)$ is dense in \mathfrak{H} .

(iv) The maps $L_\psi: \xi \in \mathfrak{D}(\mathfrak{H}, \psi) \mapsto L_\psi(\xi) \in \mathfrak{n}_\psi(\mathfrak{H})$ and $\tilde{\eta}_\psi: t \in \mathfrak{n}_\psi(\mathfrak{H}) \mapsto \tilde{\eta}_\psi(t) \in \mathfrak{D}(\mathfrak{H}, \psi)$ are the inverse of each other.

(v)

$$(3.12') \quad L_\psi(\xi\sigma_{i/2}^\psi(b)) = L_\psi(\xi)b, \quad \xi \in \mathfrak{D}(\mathfrak{H}, \psi), \quad b \in \mathcal{D}(\sigma_{i/2}^\psi).$$

(vi) With the semi-finite, normal weight $\bar{\psi}$ on \mathcal{R} defined by $\bar{\psi}(x) \triangleq \psi(exe) = \rho(exe)$, $x \in \mathcal{R}_+$, we have $\mathfrak{n}_{\bar{\psi}} = \mathfrak{n}_\psi \oplus \mathfrak{n}_\psi(\mathfrak{H}) \oplus \mathcal{R}f$, with $\mathcal{R}f \subset N_{\bar{\psi}}$, where $N_{\bar{\psi}}$ means the left kernel of $\bar{\psi}$ (i.e., $\{y \in \mathcal{N} : \bar{\psi}(y^*y) = 0\}$). Moreover, the action of \mathcal{R} on $\tilde{\mathfrak{H}}$ is semi-cyclic relative to the semi-finite normal weight $\bar{\psi}$ under the identification $\eta_{\bar{\psi}}((x, t, 0)) \in \mathfrak{H}_{\bar{\psi}} \leftrightarrow (\eta_\psi(x), \tilde{\eta}_\psi(t)) \in L^2(\mathcal{N}) \oplus \mathfrak{H}$, $x \in \mathfrak{n}_\psi$, $t \in \mathfrak{n}_\psi(\mathfrak{H})$.

Proof.

- (i) If t is in $\mathfrak{n}_\psi(\mathfrak{H})$, then the absolute value $|t|$ belongs to \mathfrak{n}_ψ by definition, so that the polar decomposition of t shows that $t = u|t| \in \mathcal{L}(L^2(\mathcal{N})_{\mathcal{N}}, \mathfrak{H}_{\mathcal{N}})\mathfrak{n}_\psi$. Conversely, if $a \in \mathcal{L}(L^2(\mathcal{N})_{\mathcal{N}}, \mathfrak{H}_{\mathcal{N}})$ and $x \in \mathfrak{n}_\psi$, then the inequality $x^*a^*ax \leq \|a\|^2x^*x$ implies that $ax \in \mathfrak{n}_\psi(\mathfrak{H})$. If $\xi \in \mathfrak{D}(\mathfrak{H}, \psi)$, then $a = L_\psi(\xi)$ belongs to $\mathcal{L}(L^2(\mathcal{N})_{\mathcal{N}}, \mathfrak{H}_{\mathcal{N}})$, and with the polar decomposition $a = u|a|$ we conclude first that $|a|$ belongs to \mathfrak{n}_ψ , and also that $\xi = u\eta_\psi(|a|) \in \mathcal{L}(L^2(\mathcal{N})_{\mathcal{N}}, \mathfrak{H}_{\mathcal{N}})\mathfrak{B}_\psi$.
- (ii) If $ty = 0$ with $t \in \mathcal{L}(L^2(\mathcal{N})_{\mathcal{N}}, \mathfrak{H}_{\mathcal{N}})$ and $y \in \mathfrak{n}_\psi$, then again, writing $t = u|t|$, we have $|t|y = 0$ and $t\eta_\psi(y) = u|t|\eta_\psi(y) = u\eta_\psi(|t|y) = 0$. This means that if $t_1y_1 = t_2y_2$ with $t_1, t_2 \in \mathcal{L}(L^2(\mathcal{N})_{\mathcal{N}}, \mathfrak{H}_{\mathcal{N}})$ and $y_1, y_2 \in \mathfrak{n}_\psi$, then we have $t_1\eta_\psi(y_1) = t_2\eta_\psi(y_2)$, so that the map $\tilde{\eta}_\psi$ is well-defined. The rest follows easily by calculation.
- (iii) From (i) it follows that

$$[\mathfrak{D}(\mathfrak{H}, \psi)] = [\mathcal{L}(L^2(\mathcal{N})_{\mathcal{N}}, \mathfrak{H}_{\mathcal{N}})\mathfrak{B}_\psi] = [\mathcal{L}(L^2(\mathcal{N})_{\mathcal{N}}, \mathfrak{H}_{\mathcal{N}})L^2(\mathcal{N})].$$

Let $\xi \in \mathfrak{H}$. Consider $\omega = \omega_\xi$ as a functional over \mathcal{N} , and let $\xi(\omega)$ be the representing vector in $L^2(\mathcal{N})_+$ of ω for the *right* action of \mathcal{N} on $L^2(\mathcal{N})$, i.e., $\langle \omega, x \rangle = (\xi(\omega)x \mid \xi(\omega))$, for $x \in \mathcal{N}$. Then we have a partial isometry u in $\mathcal{L}(L^2(\mathcal{N})_{\mathcal{N}}, \mathfrak{H}_{\mathcal{N}})$ such that $u\xi(\omega) = \xi$. Hence, $\mathcal{L}(L^2(\mathcal{N})_{\mathcal{N}}, \mathfrak{H}_{\mathcal{N}})L^2(\mathcal{N}) = \mathfrak{H}$; this implies, then, that $\mathfrak{D}(\mathfrak{H}, \psi)$ is dense in \mathfrak{H} .

(iv) Let $\xi = \tilde{\eta}_\psi(t)$ with $t \in \mathfrak{n}_\psi(\mathfrak{H})$, and take $t = u|t|$. Then $|t| \in \mathfrak{n}_\psi$, $u \in \mathcal{L}(L^2(\mathcal{N})_{\mathcal{N}}, \mathfrak{H}_{\mathcal{N}})$ and $\xi = u\eta_\psi(|t|)$ by (i). Now for each $y \in \mathfrak{n}_\psi \cap \mathfrak{n}_\psi^* \cap \mathcal{D}(\sigma_{-i/2}^\psi)$ such that $\sigma_{-i/2}^\psi(y) \in \mathfrak{n}_\psi$, we have

$$\begin{aligned} L_\psi(\xi)J_\psi\eta_\psi(y^*) &= \xi y = u\eta_\psi(|t|)y = u\eta_\psi(|t|\sigma_{-i/2}^\psi(y)) \\ &= u|t|\eta_\psi(\sigma_{-i/2}^\psi(y)) = t\Delta_\psi^{1/2}\eta_\psi(y) = tJ_\psi\eta_\psi(y^*), \end{aligned}$$

where we have used (3.8). Therefore, we have $t = L_\psi(\xi)$. Conversely, suppose $t = L_\psi(\xi)$ with $\xi \in \mathcal{D}(\mathfrak{H}, \psi)$. With $t = u|t|$ the polar decomposition, we have, for each $y \in \mathfrak{n}_\psi \cap \mathfrak{n}_\psi^* \cap \mathcal{D}(\sigma_{-i/2}^\psi)$ such that $\sigma_{-i/2}^\psi(y) \in \mathfrak{n}_\psi$,

$$|t|J_\psi\eta_\psi(y^*) = u^*tJ_\psi\eta_\psi(y^*) = u^*(\xi y) = (u^*\xi)y,$$

so that the vector $u^*\xi \in L^2(\mathcal{N})$ is left bounded relative to the left Hilbert algebra $\mathfrak{A}_\psi = \eta_\psi(\mathfrak{n}_\psi \cap \mathfrak{n}_\psi^*)$, and $|t| = \pi_\ell(u^*\xi)$. This means that $|t| \in \mathfrak{n}_\psi$, and so $t \in \mathfrak{n}_\psi(\mathfrak{H})$. It is easy to see now that $\xi = \tilde{\eta}_\psi(t)$.

(v) This follows from (ii), (iv) and results from the theory of cocycle derivatives.

(See [10]).

(vi) This assertion follows from a routine calculation of the actions of \mathcal{R} on $\tilde{\mathfrak{H}}$ and $\mathfrak{H}_{\overline{\psi}}$. ■

Now, it is a fundamental fact of Tomita-Takesaki theory that $\mathfrak{a}_\psi = \mathfrak{n}_\psi \cap \mathfrak{n}_\psi^*$, or more precisely its image $\eta_\psi(\mathfrak{a}_\psi)$, forms a left Hilbert algebra. Likewise, $\mathcal{A} = \mathcal{D}(\sigma_{-i/2}^\psi) \cap \mathcal{D}(\sigma_{i/2}^\psi) = \mathcal{D}(\sigma_{i/2}^\psi) \cap \mathcal{D}(\sigma_{i/2}^\psi)^*$ is a self-adjoint subalgebra of \mathcal{N} which

multiplies \mathfrak{a}_ψ and \mathfrak{n}_ψ^* from both sides. We then have the following tautological statement:

Lemma 3.4 *The anti-representation π'_ψ of \mathcal{A} defined by*

$$(3.12'') \quad \pi'_\psi(b)\tilde{\eta}_\psi(t) \triangleq \tilde{\eta}_\psi(t\sigma_{-i/2}^\psi(b)), \quad t \in \mathfrak{n}_\psi(\mathfrak{H}), b \in \mathcal{A},$$

extends to the original right action of \mathcal{N} on \mathfrak{H} .

Proof. This assertion follows directly from (3.6), (3.12) and (3.12'). ■

We now continue our investigation of the action of \mathcal{R} on $\tilde{\mathfrak{H}}$. The direct sum decomposition, $\tilde{\mathfrak{H}} = L^2(\mathcal{N}) \oplus \mathfrak{H}$, yields the following matrix representation of \mathcal{R} :

$$x = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}, \quad \text{with} \quad \begin{aligned} x_{11} &\in \mathcal{N}, \quad x_{12} \in \mathcal{L}(\mathfrak{H}_\mathcal{N}, L^2(\mathcal{N})_\mathcal{N}), \\ x_{21} &\in \mathcal{L}(L^2(\mathcal{N})_\mathcal{N}, \mathfrak{H}_\mathcal{N}), \quad x_{22} \in \mathcal{M} = \mathcal{L}(\mathfrak{H}_\mathcal{N}) \end{aligned}$$

for each $x \in \mathcal{R}$.

Notice that we have not yet made use of the semi-finite, normal weight $\varphi \in \mathfrak{W}(\mathcal{M})$; all our considerations thus far have involved only $\psi \in \mathfrak{W}_0(\mathcal{N})$. We recall that the "balanced" weight $\rho = \psi \oplus \varphi$ on \mathcal{R} gives a semi-cyclic representation $\{\pi_\rho, \mathfrak{H}_\rho\}$ of \mathcal{R} . We wish to characterize the representation π_ρ in terms of \mathfrak{H} and π_ψ . To do this, we consider the weights $\bar{\psi}$ and $\bar{\varphi}$ on \mathcal{R} given by $\bar{\psi}(x) \triangleq \psi(exe)$ and $\bar{\varphi}(x) \triangleq \varphi(fxf)$, $x \in \mathcal{R}_+$. We then obtain the decomposition

$$(3.13) \quad \mathfrak{H}_\rho = \left[\begin{pmatrix} \eta_\rho(en_\rho e) & \eta_\rho(en_\rho f) \\ \eta_\rho(fn_\rho e) & \eta_\rho(fn_\rho f) \end{pmatrix} \right] \cong \begin{pmatrix} L^2(\mathcal{N}) \oplus [\eta_\rho(en_\rho f)] \\ \mathfrak{H} \oplus \mathfrak{H}_\varphi \end{pmatrix},$$

where $[\dots]$ stands for the closure in the Hilbert space of the linear span, as usual. We have already seen that $\eta_\rho(f\mathfrak{n}_\rho e) = \tilde{\eta}_\psi(\mathfrak{n}_\psi(\mathfrak{H})) = \mathfrak{D}(\mathfrak{H}, \psi)$. In addition, we know $f\mathcal{R}e = \mathcal{L}(L^2(\mathcal{N})_{\mathcal{N}}, \mathfrak{H}_{\mathcal{N}})$ and $e\mathcal{R}f = \mathcal{L}(\mathfrak{H}_{\mathcal{N}}, L^2(\mathcal{N})_{\mathcal{N}})$. We now want to investigate $e\mathfrak{n}_\rho f$ and its image under the map η_ρ . As we did not assume the faithfulness of φ , we don't have complete symmetry between φ and ψ . At any rate, we do have

$$(3.10') \quad \begin{aligned} e\mathfrak{n}_\rho f &= \{s \in \mathcal{L}(\mathfrak{H}_{\mathcal{N}}, L^2(\mathcal{N})_{\mathcal{N}}) : \varphi(s^*s) < +\infty\} \\ &= \{t^* : t \in \mathcal{L}(L^2(\mathcal{N})_{\mathcal{N}}, \mathfrak{H}_{\mathcal{N}}), \varphi(tt^*) < +\infty\}. \end{aligned}$$

Given the decomposition described by (3.13), it is then natural to define

$$\begin{aligned} \mathfrak{H}_{11} &= L^2(\mathcal{N}), & \mathfrak{H}_{12} &= [\eta_\rho(e\mathfrak{n}_\rho f)] \\ \mathfrak{H}_{21} &= \mathfrak{H}, & \mathfrak{H}_{22} &= \mathfrak{H}_\varphi \end{aligned}$$

We conclude this section with a Lemma which indicates the relationship between the weights φ and ψ , at least on the level of their semi-cyclic representation spaces.

Lemma 3.5

- (i) *The restriction of π_ρ to the second column space of (3.13), $\mathfrak{H}_{12} \oplus \mathfrak{H}_{22}$, is semi-cyclic relative to the weight $\bar{\varphi}$.*
- (ii) *The Hilbert space \mathfrak{H}_{12} is isomorphic to $\overline{s(\varphi)\mathfrak{H}_{21}}$ (and hence $\cong \overline{s(\varphi)\mathfrak{H}}$) as an \mathcal{N} - $\mathcal{M}_{s(\varphi)}$ bimodule under the natural map.*

Proof. First consider the case when φ is faithful. Then with $\mathfrak{a}_\rho = \mathfrak{n}_\rho \cap \mathfrak{n}_\rho^*$, $\mathfrak{A}_\rho = \eta_\rho(\mathfrak{a}_\rho)$ is a left Hilbert algebra. Furthermore, the \mathcal{R} - \mathcal{R} bimodule $L^2(\mathcal{R})$ can

be naturally identified with \mathfrak{H}_ρ . Under this identification, the components of \mathfrak{H}_ρ defined in (3.13) allow us to write

$$\begin{aligned}\mathfrak{H}_{11} &= eL^2(\mathcal{R})e, & \mathfrak{H}_{12} &= eL^2(\mathcal{R})f, \\ \mathfrak{H}_{21} &= fL^2(\mathcal{R})e, & \mathfrak{H}_{22} &= fL^2(\mathcal{R})f.\end{aligned}$$

The modular conjugation J implements the desired isomorphism between $\overline{\mathfrak{H}}_{21}$ and \mathfrak{H}_{12} . This gives assertion (ii). Assertion (i) follows from the symmetry between ψ on \mathcal{N} and φ° on \mathcal{M}° .

In the general case, (i.e., if φ is not faithful), we consider an auxiliary semi-finite, normal weight φ' on \mathcal{M} with $s(\varphi') = \mathbf{1}_{\mathcal{M}} - s(\varphi)$. We then define

$$q \triangleq \begin{pmatrix} 0 & 0 \\ 0 & s(\varphi) \end{pmatrix}, \quad p \triangleq \begin{pmatrix} 0 & 0 \\ 0 & s(\varphi') \end{pmatrix},$$

i.e., $p = f - q$. We can now form an fns weight $\rho' \triangleq \rho + \overline{\varphi}'$ on \mathcal{R} , where $\overline{\varphi}'$ is defined via $\overline{\varphi}'(x) \triangleq \varphi'(pxp)$, $x \in \mathcal{R}_+$. Observe that η_ρ and $\eta_{\rho'}$ agree on $e\mathcal{R}e$ and $f\mathcal{R}e$ and that $\eta_\rho(x) = \eta_{\rho'}(xq)$, $x \in \mathfrak{n}_\rho q$. Hence we get $\mathfrak{H}_{11} = eL^2(\mathcal{R})e$, $\mathfrak{H}_{21} = fL^2(\mathcal{R})e$, $\mathfrak{H}_{12} = eL^2(\mathcal{R})q$ and $\mathfrak{H}_{22} = fL^2(\mathcal{R})q$. So, we may conclude $Jq\mathfrak{H}_{21} = JqL^2(\mathcal{R})e = eL^2(\mathcal{R})q = \mathfrak{H}_{12}$. This completes the proof of (i). For (ii), we have $\mathfrak{H}_{12} \oplus \mathfrak{H}_{22} = L^2(\mathcal{R})q$ as an \mathcal{N} - \mathcal{M}_q bimodule. Therefore, the representation $\{\pi_\rho, \mathfrak{H}_{12} \oplus \mathfrak{H}_{22}\}$ is precisely the semi-cyclic representation $\{\pi_{\overline{\varphi}'}, \mathfrak{H}_{\overline{\varphi}'}, \eta_{\overline{\varphi}'}\}$. ■

3.2 The Spatial Derivative

If we examine the details of the preceding proof, we recognize that the conjugation operator $J: L^2(\mathcal{R}) \rightarrow L^2(\mathcal{R})$ restricts to the conjugate linear operator $S_{\varphi,\psi}: \tilde{\eta}_\psi(t) \in f(\mathfrak{n}_\rho \cap \mathfrak{n}_\rho^*)e \mapsto \eta_\rho(t^*) \in \mathfrak{H}_{12}$, which can also be viewed as the restriction of the map $S_{\varphi+\varphi',\psi}$ for ρ' to the smaller domain $gL^2(\mathcal{R})e$. Hence, $S_{\varphi,\psi}$ can be defined directly as the closure of the operator given by $\tilde{\eta}_\psi(t) \mapsto \eta_\rho(t^*)$ for $t \in \mathfrak{n}_\psi(\mathfrak{H}) \cap \mathfrak{n}_\varphi(\mathfrak{H})^*$, where $\mathfrak{n}_\varphi(\mathfrak{H}) \triangleq \{s \in \mathcal{L}(\mathfrak{H}_{\mathcal{N}}, L^2(\mathcal{N})_{\mathcal{N}}) : \varphi(s^*s) < +\infty\}$. Thus we make the following definition:

Definition 3.6 *The absolute value $\Delta_{\varphi,\psi}$ of $S_{\varphi,\psi}$ is called the spatial derivative of the semi-finite, normal weight φ on \mathcal{M} relative to the fns weight ψ' on the commutant \mathcal{M}' , and is denoted $\frac{d\varphi}{d\psi'}$, since it is determined by φ on \mathcal{M} and ψ' on \mathcal{M}' .*

Dualizing (3.10), we set

$$(3.10'') \quad \mathfrak{D}'(\mathfrak{H}, \varphi) \triangleq \{\xi \in \mathfrak{H} : \|x\xi\|^2 \leq C_\xi \varphi(x^*x), \quad x \in \mathfrak{n}_\varphi, \text{ for some } C_\xi \geq 0\}.$$

To each $\xi \in \mathfrak{D}'(\mathfrak{H}, \varphi)$ there corresponds an operator $R_\varphi(\xi)$ defined by

$$R_\varphi(\xi)\eta_\varphi(x) \triangleq x\xi, \quad x \in \mathfrak{n}_\varphi,$$

which belongs to $\mathcal{L}({}_{\mathcal{M}}L^2(\mathcal{M}), {}_{\mathcal{M}}\mathfrak{H})$. As φ is not assumed to be faithful, ψ and φ are not symmetric. In fact, we have the following:

Lemma 3.7 *The closure of $\mathfrak{D}'(\mathfrak{H}, \varphi)$ is the range of the projection $s(\varphi)$, i.e.,*

$$[\mathfrak{D}'(\mathfrak{H}, \varphi)] = s(\varphi)\mathfrak{H}.$$

Proof. If $\xi \in \mathfrak{D}'(\mathfrak{H}, \varphi)$, then we have $\|(1 - s(\varphi))\xi\|^2 \leq C_\xi \varphi((1 - s(\varphi))) = 0$.

Hence $\mathfrak{D}'(\mathfrak{H}, \varphi) \subset s(\varphi)\mathfrak{H}$. Conversely, suppose $\xi \perp \mathfrak{D}'(\mathfrak{H}, \varphi)$. With $\Phi = \{\omega \in \mathcal{M}_*^+ : \omega \leq \varphi\}$, we know $\varphi(x) = \sup_{\omega \in \Phi} \omega(x)$, $x \in \mathcal{M}_+$, and that

$$\eta \in \mathfrak{D}'(\mathfrak{H}, \varphi) \iff \omega_\eta \in \bigcup_{C>0} C\Phi.$$

Also every $\omega \in \Phi$ can be written as a countable sum of ω_{η_n} with $\eta_n \in \mathfrak{D}'(\mathfrak{H}, \varphi)$, so that $s(\varphi) = \sup\{s(\omega_\eta) : \eta \in \mathfrak{D}'(\mathfrak{H}, \varphi)\}$; thus we may conclude $s(\varphi)\xi = 0$. ■

We now state the main result of this section. Note that the spatial derivative was originally defined by Connes [13]; however, his approach did not use (explicitly) the notions of von Neumann bimodules.

Theorem 3.8 *Let $\{\mathcal{M}, \mathfrak{H}\}$ be a von Neumann algebra, φ a semi-finite, normal weight on \mathcal{M} and ψ' an fns weight on the commutant \mathcal{M}' . Then the spatial derivative $\frac{d\varphi}{d\psi'}$ has the following properties:*

- (i) *The support $s(\frac{d\varphi}{d\psi'})$ of the spatial derivative $\frac{d\varphi}{d\psi'}$ is equal to $s(\varphi)$. Note that here what is meant by the support of a self-adjoint operator is the projection to the closure of its range.*

(ii) On the reduced von Neumann algebra $\{\mathcal{M}_{s(\varphi)}, s(\varphi)\mathfrak{H}\}$ and its commutant

$\mathcal{M}'_{s(\varphi)}$, we have

$$(3.14) \quad \left. \begin{aligned} \left(\frac{d\varphi}{d\psi'}\right)^{it} x \left(\frac{d\varphi}{d\psi'}\right)^{-it} &= \sigma_t^\varphi(x), & x \in \mathcal{M}_{s(\varphi)}, \\ \left(\frac{d\varphi}{d\psi'}\right)^{it} y \left(\frac{d\varphi}{d\psi'}\right)^{-it} &= \sigma_{-t}^{\psi'}(y), & y \in \mathcal{M}'_{s(\varphi)}. \end{aligned} \right\}$$

(iii) If φ_1 and φ_2 are fns weights on \mathcal{M} , then

$$(3.15) \quad \left(\frac{d\varphi_2}{d\psi'}\right)^{it} = (D\varphi_2 : D\varphi_1)_t \left(\frac{d\varphi_1}{d\psi'}\right)^{it}.$$

(iv) If φ is faithful, then

$$(3.16) \quad \frac{d\psi'}{d\varphi} = \left(\frac{d\varphi}{d\psi'}\right)^{-1}.$$

(v) With $\mathcal{N} = (\mathcal{M}')^\circ$ and $\psi = (\psi')^\circ$, the square root of the spatial derivative,

$\left(\frac{d\varphi}{d\psi'}\right)^{\frac{1}{2}}$, is essentially self-adjoint on

$$\mathfrak{D}_{\varphi, \psi}(\mathfrak{H}) \triangleq \{\xi \in \mathfrak{D}(\mathfrak{H}, \psi) : L_\psi(\xi)^* \in \mathfrak{n}_\varphi(\mathfrak{H})\}$$

and is determined by

$$(3.17) \quad \left(\left(\frac{d\varphi}{d\psi'}\right)^{\frac{1}{2}}\xi \mid \left(\frac{d\varphi}{d\psi'}\right)^{\frac{1}{2}}\eta\right) = \varphi(L_\psi(\xi)L_\psi(\eta)^*), \quad \xi, \eta \in \mathfrak{D}_{\varphi, \psi}(\mathfrak{H}).$$

Therefore, the spatial derivative $\frac{d\varphi}{d\psi'}$ of φ relative to ψ' is directly computable from φ and ψ' . (Again, see [13])

Proof. From the previous arguments involving \mathcal{M} , \mathcal{N} , φ and ψ , we know that the spatial derivative $\frac{d\varphi}{d\psi'}$ is precisely the relative modular operator $\Delta_{\varphi, \psi}$ on the subspace $s(\varphi)\mathfrak{H}$, when we replace \mathfrak{H} by $s(\varphi)\mathfrak{H}$ and assume that φ is faithful. Then

the assertions (i) through (v) are really statements about the relative modular operator; all of these are standard results in the theory of the cocycle derivative. (See, for example, [10].) ■

Again, we wish to emphasize that, via (3.17), the spatial derivative $\frac{d\varphi}{d\psi}$ is completely determined by the weights φ and ψ , without making use of the auxiliary von Neumann algebra \mathcal{R} . We will now investigate additional properties of the spatial derivative; we begin with a Lemma.

Lemma 3.9 *The linear span \mathcal{J}_ψ of $\{L_\psi(\xi)L_\psi(\eta)^* : \xi, \eta \in \mathfrak{D}(\mathfrak{H}, \psi)\}$ is a σ -weakly dense ideal of \mathcal{M} ; moreover, we have*

$$\mathcal{J}_\psi^+ = \left\{ \sum_{i=1}^n L_\psi(\xi_i)L_\psi(\xi_i)^* : \xi_i \in \mathfrak{D}(\mathfrak{H}, \psi), i = 1, \dots, n \right\}$$

Proof. It is easy to see that $L_\psi(a\xi) = aL_\psi(\xi)$ for any $a \in \mathcal{M}$ and $\xi \in \mathfrak{D}(\mathfrak{H}, \psi)$. Hence \mathcal{J}_ψ is an ideal of \mathcal{M} . The characterization of the positive cone is accomplished by using polarization, which is a standard technique, so we omit that portion of the argument.

To demonstrate the σ -weak density of \mathcal{J}_ψ , it is sufficient to prove that, if $a\mathfrak{n}_\psi(\mathfrak{H}) = \{0\}$, $a \in \mathcal{M}$, then $a = 0$, since $L_\psi(\mathfrak{D}(\mathfrak{H}, \psi)) = \mathfrak{n}_\psi(\mathfrak{H})$. So, suppose $a\mathfrak{n}_\psi(\mathfrak{H}) = \{0\}$ for some $a \in \mathcal{M}$. This implies $aL_\psi(\xi) = 0$ for every $\xi \in \mathfrak{D}(\mathfrak{H}, \psi)$. Thus for every $x \in \mathfrak{n}_\psi^*$ we have

$$0 = aL_\psi(\xi)\eta'_\psi(x) = a(\xi x) = (a\xi)x.$$

Since \mathfrak{n}_ψ^* is σ -weakly dense in \mathcal{N} , we have $a\xi = 0$. The density of $\mathfrak{D}(\mathfrak{H}, \psi)$ in \mathfrak{H} then gives $a = 0$. ■

Proposition 3.10 *The spatial derivative $\frac{d\varphi}{d\psi'}$ has the following additional properties:*

(i)

$$\varphi_1 \leq \varphi_2 \iff \frac{d\varphi_1}{d\psi'} \leq \frac{d\varphi_2}{d\psi'}$$

(ii) *If φ_1 and φ_2 are both finite, then*

$$(3.18) \quad \frac{d(\varphi_1 + \varphi_2)}{d\psi'} = \frac{d\varphi_1}{d\psi'} + \frac{d\varphi_2}{d\psi'}$$

where the above sum should be interpreted as a form sum.

(iii) *If $a \in \mathcal{M}$ is invertible, then*

$$(3.19) \quad \frac{d(a\varphi a^*)}{d\psi'} = a \left(\frac{d\varphi}{d\psi'} \right) a^*$$

(iv) *The support of $\frac{d\varphi}{d\psi'}$, $s\left(\frac{d\varphi}{d\psi'}\right)$, is equal to the support, $s(\varphi)$, of φ .*

Proof.

(i) Suppose $\varphi_1 \leq \varphi_2$. Then we have

$$\left\| \left(\frac{d\varphi_1}{d\psi'} \right)^{1/2} \xi \right\|^2 = \varphi_1(L_\psi(\xi)L_\psi(\xi)^*) \leq \varphi_2(L_\psi(\xi)L_\psi(\xi)^*) = \left\| \left(\frac{d\varphi_2}{d\psi'} \right)^{1/2} \xi \right\|^2$$

for every $\xi \in \mathfrak{D}(\mathfrak{H}, \psi)$. Hence $\frac{d\varphi_1}{d\psi'} \leq \frac{d\varphi_2}{d\psi'}$.

Conversely, suppose $\frac{d\varphi_1}{d\psi'} \leq \frac{d\varphi_2}{d\psi'}$. This means that we must have $\varphi_1(a) \leq \varphi_2(a)$

for every $a \in \mathcal{M}_+$ of the form $a = \sum_{i=1}^n L_\psi(\xi_i)L_\psi(\xi_i)^*$. Our assertion then

follows from Lemma 3.9.

(ii) Suppose that $\varphi_1, \varphi_2 \in \mathcal{M}_*^+$, and set $\varphi \triangleq \varphi_1 + \varphi_2$. The boundedness of φ_1 and φ_2 of course imply that φ is bounded; we also have seen that the square roots of all the spatial derivatives $\frac{d\varphi_1}{d\psi'}$, $\frac{d\varphi_2}{d\psi'}$ and $\frac{d\varphi}{d\psi'}$ are essentially self-adjoint on $\mathfrak{D}(\mathfrak{H}, \psi)$. Let $H_1 \triangleq \frac{d\varphi_1}{d\psi'}$, $H_2 \triangleq \frac{d\varphi_2}{d\psi'}$ and $H \triangleq \frac{d\varphi}{d\psi'}$. Then we have $\|H^{\frac{1}{2}}\xi\|^2 = \|H_1^{\frac{1}{2}}\xi\|^2 + \|H_2^{\frac{1}{2}}\xi\|^2$, $\xi \in \mathfrak{D}(\mathfrak{H}, \psi)$. Hence our assertion follows.

(iii) Again, take $H \triangleq \frac{d\varphi}{d\psi'}$. It follows that aHa^* is a positive, self-adjoint operator with domain $(a^*)^{-1}\mathfrak{D}(H)$, and that for each $\xi \in \mathfrak{D}(\mathfrak{H}, \psi)$

$$\begin{aligned} \|H^{\frac{1}{2}}a^*\xi\|^2 &= \varphi(L_\psi(a^*\xi)L_\psi(a^*\xi)^*) = \varphi(a^*L_\psi(\xi)L_\psi(\xi)^*a) \\ &= (a\varphi a^*)(L_\psi(\xi)L_\psi(\xi)^*). \end{aligned}$$

(Note that $\|H^{\frac{1}{2}}a^*\xi\|^2$ can be $+\infty$ if φ is not finite. In fact, $\|H^{\frac{1}{2}}a^*\xi\|^2 < +\infty \iff a^*\xi \in \mathfrak{D}(H^{\frac{1}{2}})$.)

Since a is invertible, $\mathfrak{D}((aHa^*)^{\frac{1}{2}}) = \mathfrak{D}(H^{\frac{1}{2}}a^*)$ and the absolute value of $H^{\frac{1}{2}}a^*$ is precisely $(aHa^*)^{\frac{1}{2}}$. Hence we may conclude (3.19).

(iv) Let p be the support of $H = \frac{d\varphi}{d\psi'}$ and $q = s(\varphi)$. Then p is characterized by the fact that $\mathbf{1} - p$ is the projection of \mathfrak{H} onto the null space of H , i.e. onto the subspace $\mathfrak{K} = \{\xi \in \mathfrak{H} : H\xi = 0\}$. Let $(\mathfrak{A}_{\rho'})_0$ be the maximal Tomita algebra associated with the left Hilbert algebra $\mathfrak{A}_{\rho'} = \eta_{\rho'}(\mathfrak{n}_{\rho'} \cap \mathfrak{n}_{\rho'}^*)$. (Recall that we defined ρ' in the proof of Lemma 3.5 by adding an auxiliary weight to our original weight ρ in order to make ρ' faithful.) Because $f \in \mathcal{R}$, we have, with $f' = JfJ$, $f'(\mathfrak{A}_{\rho'})_0 = J(\mathfrak{A}_{\rho'})_0 \subset (\mathfrak{A}_{\rho'})_0$. If $\xi \in \mathfrak{K}$, then there exists

a sequence $\{\xi_n\} \subset \mathfrak{D}(\mathfrak{H}, \psi)$ such that $\xi_n \rightarrow \xi$ and $H\xi_n \rightarrow 0$, as $\mathfrak{D}(\mathfrak{H}, \psi)$ is a core for H . For each $\eta \in f'(\mathfrak{A}_{\rho'})_0$, we have $\pi_r(\eta)\xi_n \rightarrow \pi_r(\eta)\xi$ and

$$\begin{aligned} \|H^{\frac{1}{2}}\pi_r(\eta)\xi\|^2 &= \lim_n \|H^{\frac{1}{2}}\pi_r(\eta)\xi_n\|^2 = \lim_n \|H^{\frac{1}{2}}\pi_\ell(\xi_n)\eta\|^2 \\ &= \lim_n \varphi(\pi_\ell(\xi_n)\eta)\pi_\ell(\xi_n\eta)^* \\ &\leq \|\pi_\ell(\eta)\|^2 \lim_n \varphi(\pi_\ell(\xi_n)\eta)\pi_\ell(\xi_n\eta)^* = 0, \end{aligned}$$

which gives $\pi_r(\eta)\xi \in \mathfrak{K}$. Since $\{\pi_r(\eta)_e : \eta \in f'(\mathfrak{A}_{\rho'})_0\}$ is σ -weakly dense in \mathcal{N} , the projection p belongs to $\mathcal{M} \cong \mathcal{N}'$ (in \mathcal{R}).

Now, if $\xi \in (\mathbf{1} - q)\mathfrak{D}(\mathfrak{H}, \psi)$, then $\varphi(L_\psi(\xi)L_\psi(\xi)^*) = \varphi(qL_\psi(\xi)L_\psi(\xi)^*q) = \varphi(L_\psi(q\xi)L_\psi(q\xi)^*) = 0$, so $H^{\frac{1}{2}}\xi = 0$. If $\xi \in (\mathbf{1} - q)\mathfrak{H}$, then we choose a sequence $\xi_n \in \mathfrak{D}(\mathfrak{H}, \psi)$ with $\xi_n \rightarrow \xi$. It follows that $(\mathbf{1} - q)\xi_n \rightarrow \xi$, and since $H^{\frac{1}{2}}(\mathbf{1} - q)\xi_n = 0 \quad \forall n$, we see that $\xi \in \mathfrak{D}(H^{\frac{1}{2}})$, and $H^{\frac{1}{2}}\xi = 0$. Thus $\mathbf{1} - q \leq \mathbf{1} - p$, i.e., $p \leq q$. On the other hand, φ is a faithful weight on \mathcal{M}_q . It is then easy to check that we can view the weight ψ' as one on $\mathcal{M}'_{s(\varphi)}$ without changing $H = \frac{d\varphi}{d\psi'}$ other than to change the underlying Hilbert space from \mathfrak{H} to $s(\varphi)\mathfrak{H}$. By Theorem 3.8, $\frac{d\varphi}{d\psi'}$ is non-singular on $s(\varphi)\mathfrak{H}$, which means $p = s(\varphi)$. ■

We continue our investigation of the properties of the spatial derivative. In particular, we are interested in answering the question: given a positive, self-adjoint operator H in \mathfrak{H} , and an fns weight ψ' on \mathcal{M}' (or equivalently an fns weight ψ on \mathcal{N}), when is there a weight $\varphi \in \mathfrak{W}(\mathcal{M})$ such that $\frac{d\varphi}{d\psi'} = H$?

Theorem 3.11 *Let $\{\mathcal{M}, \mathfrak{H}\}$ and $\mathcal{N} = (\mathcal{M}')^\circ$ be as before, and fix an fns weight ψ on \mathcal{N} . For a positive self-adjoint operator H in \mathfrak{H} , the following three conditions are equivalent:*

- (i) *There exists a semi-finite, normal weight φ on \mathcal{M} with $\frac{d\varphi}{d\psi} = H$.*
- (ii) *For every $y \in \mathcal{N}$, $H^{it}\sigma_t^\psi(y) = yH^{it}$, $t \in \mathbf{R}$, where H^{it} is considered only on the closure of the range of H . (Note that we are not assuming that H is non-singular.)*
- (iii) *$\mathfrak{D}(\mathfrak{H}, \psi) \cap \mathfrak{D}(H^{\frac{1}{2}})$ is a core for $H^{\frac{1}{2}}$, and the scalar $\sum_{i=1}^n \|H^{\frac{1}{2}}\xi_i\|^2$ depends only on the operator $\sum_{i=1}^n L_\psi(\xi_i)L_\psi(\xi_i)^*$ for $\{\xi_1, \dots, \xi_n\} \subset \mathfrak{D}(\mathfrak{H}, \psi) \cap \mathfrak{D}(H^{\frac{1}{2}})$.*

Proof. (i) \iff (ii): Let p be the support of H . Each of conditions (i) and (ii) implies $p \in \mathcal{M}$. Hence we may and do assume the non-singularity of H . (We need only consider the reduced algebra \mathcal{M}_p .) The implication (i) \Rightarrow (ii) follows from Theorem 3.8. Conversely, assume (ii); take any faithful weight φ' on \mathcal{M} . Put $K = \frac{d\varphi'}{d\psi}$, and define $u_t \triangleq H^{it}K^{-it}$, $t \in \mathbf{R}$. It then follows that u_t is a $\sigma_t^{\varphi'}$ -cocycle in \mathcal{M} . Again, the result of Connes-Masuda [12] guarantees the existence of a faithful weight φ on \mathcal{M} with $(D\varphi : D\varphi')_t = u_t$, $t \in \mathbf{R}$, and hence $H = \frac{d\varphi}{d\psi'}$.

(i) \Rightarrow (iii): By construction, we have

$$\sum_{i=1}^n \|H^{\frac{1}{2}}\xi_i\|^2 = \varphi \left(\sum_{i=1}^n L_\psi(\xi_i)L_\psi(\xi_i)^* \right), \quad \{\xi_1, \dots, \xi_n\} \subset \mathfrak{D}(\mathfrak{H}, \psi),$$

so that the assertion follows.

(iii) \Rightarrow (i): We need only construct a semi-finite normal weight φ on \mathcal{M} such that

$$\|H^{\frac{1}{2}}\xi\|^2 = \varphi(L_\psi(\xi)L_\psi(\xi)^*), \quad \xi \in \mathfrak{D}(\mathfrak{H}, \psi) \cap \mathfrak{D}(H^{\frac{1}{2}}).$$

Any weight with this property is automatically semi-finite because $\mathfrak{D}(H^{\frac{1}{2}}) \cap \mathfrak{D}(\mathfrak{H}, \psi)$ is dense in \mathfrak{H} , which means that $\{L_\psi(\xi)L_\psi(\xi)^* : \xi \in \mathfrak{D}(H^{\frac{1}{2}}) \cap \mathfrak{D}(\mathfrak{H}, \psi)\}$ is non-degenerate. The rest of the proof then follows from the next Lemma. \blacksquare

Lemma 3.12

(i) *Let H be as in Theorem 3.11(iii). Then there exists a preweight φ_1 on \mathcal{J}_ψ such that*

$$(3.20) \quad \varphi_1(L_\psi(\xi)L_\psi(\xi)^*) = \|H^{\frac{1}{2}}\xi\|^2, \quad \xi \in \mathfrak{D}(\mathfrak{H}, \psi),$$

where $\|H^{\frac{1}{2}}\xi\|^2 = +\infty$ if $\xi \notin \mathfrak{D}(H^{\frac{1}{2}})$. The preweight φ_1 has the property that for any net $\{x_\alpha\} \subset \mathcal{M}$ converging strongly to $\mathbf{1}$

$$(3.21) \quad \liminf_\alpha \varphi_1(x_\alpha y x_\alpha^*) \geq \varphi_1(y), \quad y \in \mathcal{J}_\psi^+.$$

Here, when we say that φ_1 is a preweight on \mathcal{J}_ψ , we mean that it is an extended real valued map on \mathcal{J}_ψ , satisfying the usual requirements of positive homogeneity and (finite) additivity. In this case, however, the domain is not the positive cone of a von Neumann algebra (recall that \mathcal{J}_ψ is merely a σ -weakly dense ideal in \mathcal{M}), and so we refrain from calling φ_1 a weight.

However,

(ii) Any preweight φ_1 on \mathcal{J}_ψ with the property given by (3.21) extends to a normal weight φ on \mathcal{M} .

Proof.

(i) By assumption, φ_1 defined by (3.17) on $L_\psi(\xi)L_\psi(\xi)^*$ extends to a preweight on \mathcal{J}_ψ^+ by Theorem 3.9, which we will continue to denote by φ_1 . Suppose $y = \sum_{k=1}^n L_\psi(\xi_k)L_\psi(\xi_k)^* \in \mathcal{J}_\psi^+$. We then have

$$x_\alpha y x_\alpha^* = \sum_{k=1}^n L_\psi(x_\alpha \xi_k)L_\psi(x_\alpha \xi_k)^*,$$

so that

$$\varphi(x_\alpha y x_\alpha^*) = \sum_{k=1}^n \|H^{\frac{1}{2}} x_\alpha \xi_k\|^2.$$

Hence inequality (3.21) follows from the lower semi-continuity of the positive quadratic form associated with H .

(ii) Since \mathcal{J}_ψ is a σ -weakly dense ideal of \mathcal{M} , every element of \mathcal{M}_+ can be approximated by \mathcal{J}_ψ^+ from below. So we put

$$\varphi(x) \triangleq \sup\{\varphi_1(y) : y \in \mathcal{J}_\psi^+, y \leq x\}, \quad x \in \mathcal{M}_+.$$

It follows that φ agrees with φ_1 on \mathcal{J}_ψ^+ . Since $x_\alpha \nearrow x$ and $y_\alpha \nearrow y \Rightarrow (x_\alpha + y_\alpha) \nearrow (x + y)$, the additivity of φ follows from that of φ_1 . We need only check the normality of φ . Suppose that $x_\alpha \nearrow x$ in \mathcal{M}_+ . Then $x_\alpha^{1/2} = a_\alpha x^{1/2}$ for a unique $a_\alpha \in \mathcal{M}$, with $s_r(a_\alpha) \leq s(x)$. Put $b_\alpha = a_\alpha + (\mathbf{1} - s(x))$. Then $x_\alpha = b_\alpha x b_\alpha^*$ and $\{b_\alpha\}$ converges strongly to $\mathbf{1}$. For any $y \in \mathcal{J}_\psi^+$ with $y \leq x$

we must show that $\sup_{\alpha} \varphi(x_{\alpha}) \geq \varphi_1(y)$. But we have, by (3.21),

$$\varphi_1(y) \leq \liminf_{\alpha} \varphi_1(b_{\alpha} y b_{\alpha}^*) \leq \liminf_{\alpha} \varphi(b_{\alpha} x b_{\alpha}^*),$$

and we have seen $b_{\alpha} x b_{\alpha}^* = x_{\alpha}$. ■

We conclude this chapter with a Corollary which will relate convergence in $\mathfrak{M}(\mathcal{M})$, convergence (in the strongly resolvent sense) amongst positive, self-adjoint and non-singular operators in \mathfrak{H} , and convergence in $\text{Aut}(\mathcal{M})$.

Corollary 3.13 *Let \mathcal{M} , \mathcal{N} and \mathfrak{H} be as before, and fix ψ' an fns weight on \mathcal{M}' . If $\{\varphi_n\}$ is an increasing sequence of fns weights \mathcal{M} and if $\varphi = \sup_n \varphi_n$ is semi-finite, then $\{\frac{d\varphi_n}{d\psi'}\}$ is increasing, and converges to $\frac{d\varphi}{d\psi'}$ in the strongly resolvent sense; hence, $\{\sigma_t^{\varphi_n}\}$ converges to σ_t^{φ} in $\text{Aut}(\mathcal{M})$ uniformly on any finite interval (of \mathbf{R}).*

Proof. Let $H_n \triangleq \frac{d\varphi_n}{d\psi'}$. By Proposition 3.10, $\{H_n\}$ is increasing and bounded by $H \triangleq \frac{d\varphi}{d\psi'}$ from above. Hence $\{H_n\}$ converges to a positive self-adjoint operator K in the strongly resolvent sense. Since $H_n^{it} y H_n^{-it} = \sigma_{-t}^{\psi}(y)$ for every $y \in \mathcal{N}$, $K^{it} y K^{it} = \sigma_{-t}^{\psi}(y)$, $y \in \mathcal{N}$. By Theorem 3.11, there exists a unique weight μ on \mathcal{M} with $K = \frac{d\mu}{d\psi'}$. The inequalities

$$H_n \leq K \leq H$$

show that $\varphi_n \leq \mu \leq \varphi$. Hence $\mu = \varphi$ and $K = H$. The rest follows from general facts about monotone convergence. ■

CHAPTER 4

The Relative Tensor Product of L^2 -von Neumann Modules

4.1 Definition of the Relative Tensor Product

We now proceed to define the *relative tensor product* of a right module and a left module over the same von Neumann algebra \mathcal{N} . Unlike the ordinary (i.e., spatial) tensor product, the construction of the relative tensor product depends on the choice of an fns weight on \mathcal{N} — hence, the use of the adjective *relative*. Furthermore, given a right \mathcal{N} -module \mathfrak{H} and a left \mathcal{N} -module \mathfrak{K} , the tensor product of an arbitrary pair of vectors from \mathfrak{H} and \mathfrak{K} cannot, in general, be defined. (In fact, as we shall see in the next Section, the existence of the tensor product for all possible pairs of vectors severely limits the possible type of the von Neumann algebra \mathcal{N} .) The formation of the relative tensor product is restricted to a subset of vectors from \mathfrak{H} and \mathfrak{K} which depends on the choice of weight. It is interesting to note that the tensor product actually behaves like the product of closed, unbounded operators. We shall begin our discussion by introducing some notation and terminology to be used throughout the Chapter.

As in the case of right modules, for two left \mathcal{N} -modules ${}_{\mathcal{N}}\mathfrak{K}_1$ and ${}_{\mathcal{N}}\mathfrak{K}_2$ we

consider $\mathcal{L}({}_{\mathcal{N}}\mathfrak{K}_1, {}_{\mathcal{N}}\mathfrak{K}_2) = \{t \in \mathcal{L}(\mathfrak{K}_1, \mathfrak{K}_2) : ta\eta = at\eta, \eta \in \mathfrak{K}_1, a \in \mathcal{N}\}$. For $\mathcal{L}({}_{\mathcal{N}}\mathfrak{K}, {}_{\mathcal{N}}\mathfrak{K})$ we write $\mathcal{L}({}_{\mathcal{N}}\mathfrak{K})$. Throughout the remainder of this Chapter, \mathfrak{H} will denote a right \mathcal{N} -module, \mathfrak{K} a left \mathcal{N} -module. Observe that a right \mathcal{N} -module \mathfrak{H} is also canonically an $\mathcal{L}(\mathfrak{H}_{\mathcal{N}})\text{-}\mathcal{N}$ bimodule, while a left \mathcal{N} -module \mathfrak{K} can always be considered an $\mathcal{N}\text{-}\mathcal{L}({}_{\mathcal{N}}\mathfrak{K})^\circ$ bimodule in a canonical way. We are now going to construct the relative tensor product $\mathfrak{H} \boxtimes_\psi \mathfrak{K}$ of a right \mathcal{N} -module \mathfrak{H} and a left \mathcal{N} -module \mathfrak{K} , which will depend on the choice of a fns weight ψ on \mathcal{N} .

So, we fix a von Neumann algebra \mathcal{N} , a right \mathcal{N} -module \mathfrak{H} and a left \mathcal{N} -module \mathfrak{K} . We also fix a faithful, normal and semi-finite weight ψ on \mathcal{N} . We have seen (Lemma 3.3) that the right module \mathfrak{H} can be recovered from $\mathfrak{D}(\mathfrak{H}, \psi)$, and that the left module \mathfrak{K} is also recoverable from $\mathfrak{D}'(\mathfrak{K}, \psi)$. (Observe that in this case, the roles of ψ and ψ° are symmetric, as they are both faithful). We state here a few facts about $\mathfrak{D}(\mathfrak{H}, \psi)$ (resp., $\mathfrak{D}'(\mathfrak{K}, \psi)$) and L_ψ (resp., R_ψ) which have been implicit in our previous results.

$$\begin{aligned}
(\xi_1 \mid \xi_2) &= \psi(L_\psi(\xi_2)^* L_\psi(\xi_1)), \quad \xi_1, \xi_2 \in \mathfrak{D}(\mathfrak{H}, \psi); \\
(\eta_1 \mid \eta_2) &= \psi(J_\psi R_\psi(\eta_1)^* R_\psi(\eta_2) J_\psi), \quad \eta_1, \eta_2 \in \mathfrak{D}'(\mathfrak{K}, \psi); \\
\eta_\psi(L_\psi(\xi_2)^* L_\psi(\xi_1)) &= L_\psi(\xi_2)^* \xi_1, \quad \xi_1, \xi_2 \in \mathfrak{D}(\mathfrak{H}, \psi); \\
\eta_\psi(J_\psi R_\psi(\eta_1)^* R_\psi(\eta_2) J_\psi) &= R_\psi(\eta_2)^* \eta_1, \quad \eta_1, \eta_2 \in \mathfrak{D}'(\mathfrak{K}, \psi).
\end{aligned}
\tag{4.1}$$

It is also easy to see that if we consider $\mathfrak{H} = L^2(\mathcal{N}, \psi)$, with the standard right action of \mathcal{N} (resp., $\mathfrak{K} = L^2(\mathcal{N}, \psi)$, with the usual left action of \mathcal{N}), as a right

(resp., left) module, then

$$\mathfrak{D}(\mathfrak{H}, \psi) = \eta_\psi(\mathfrak{n}_\psi) = \mathfrak{B}_\psi, \quad L_\psi(\xi) = \pi_\ell(\xi), \quad \xi \in \mathfrak{B}_\psi,$$

$$\text{(resp., } \mathfrak{D}'(\mathfrak{K}, \psi) = \mathfrak{B}'_\psi, \quad R_\psi(\eta) = \pi_r(\eta), \quad \eta \in \mathfrak{B}'_\psi),$$

where \mathfrak{B}_ψ (resp., \mathfrak{B}'_ψ) means the algebra of all left (resp., right) bounded vectors in $L^2(\mathcal{N}, \psi)$.

Proposition 4.1

(i) The sesquilinear form $B: \mathfrak{D}(\mathfrak{H}, \psi) \odot \mathfrak{K} \rightarrow \mathbb{C}$ determined by

$$(4.2) \quad B(\xi_1 \otimes \eta_1, \xi_2 \otimes \eta_2) \triangleq (\pi_{\mathfrak{K}}(L_\psi(\xi_2)^* L_\psi(\xi_1))\eta_1 \mid \eta_2)$$

is positive semi-definite, and so defines an inner product on $\mathfrak{D}(\mathfrak{H}, \psi) \odot \mathfrak{K}$, which is, in many cases, degenerate.

(ii) If $\xi_1, \xi_2 \in \mathfrak{D}(\mathfrak{H}, \psi)$ and $\eta_1, \eta_2 \in \mathfrak{D}'(\mathfrak{K}, \psi)$, then

$$(4.3) \quad (\pi_{\mathfrak{K}}(L_\psi(\xi_2)^* L_\psi(\xi_1))\eta_1 \mid \eta_2) = (\pi'_{\mathfrak{H}}(J_\psi R_\psi(\eta_1)^* R_\psi(\eta_2) J_\psi)\xi_1 \mid \xi_2).$$

(i') Dual to (i), the sesquilinear form B' defined on $\mathfrak{H} \odot \mathfrak{D}'(\mathfrak{K}, \psi)$ and determined

by

$$(4.2') \quad B'(\xi_1 \otimes \eta_1, \xi_2 \otimes \eta_2) \triangleq (\pi'_{\mathfrak{H}}(J_\psi R_\psi(\eta_1)^* R_\psi(\eta_2) J_\psi)\xi_1 \mid \xi_2),$$

is also positive semi-definite, and agrees with B on $\mathfrak{D}(\mathfrak{H}, \psi) \odot \mathfrak{D}'(\mathfrak{K}, \psi)$

Proof.

(i) Suppose that $\{\xi_1, \dots, \xi_n\} \subset \mathfrak{D}(\mathfrak{H}, \psi)$. Let $a_{kj} \triangleq L_\psi(\xi_k)^* L_\psi(\xi_j)$, with $j, k \in \{1, \dots, n\}$; then $a = (a_{kj})$ is an $n \times n$ matrix over \mathcal{N} . If $\{x_1, \dots, x_n\} \subset \mathcal{A}$, where, as before, $\mathcal{A} = \mathcal{D}(\sigma_{i/2}^\psi) \cap \mathcal{D}(\sigma_{-i/2}^\psi)$, then we have, by (3.6'),

$$\begin{aligned} \sum_{j,k=1}^n x_k^* a_{kj} x_j &= \sum_{j,k=1}^n x_k^* L_\psi(\xi_k)^* L_\psi(\xi_j) x_j \\ &= \sum_{j,k=1}^n L_\psi(\xi_k \sigma_{i/2}^\psi(x_k))^* L_\psi(\xi_j \sigma_{i/2}^\psi(x_j)) \\ &= \left(\sum_{k=1}^n L_\psi(\xi_k \sigma_{i/2}^\psi(x_k)) \right)^* \left(\sum_{j=1}^n L_\psi(\xi_j \sigma_{i/2}^\psi(x_j)) \right) \geq 0. \end{aligned}$$

Because \mathcal{A} is σ -weakly dense in \mathcal{N} , the matrix a is positive in $M_n(\mathcal{N}) = \mathcal{N} \otimes M_n(\mathbb{C})$; hence there exists a $b = (b_{\ell k}) \in M_n(\mathcal{N})$ such that $a = b^* b$, i.e., $a_{kj} = \sum_{\ell=1}^n b_{\ell k}^* b_{\ell j}$, $j, k \in \{1, \dots, n\}$. We then have, for any $\{\eta_1, \dots, \eta_n\} \subset \mathfrak{K}$,

$$B \left(\sum_{j=1}^n \xi_j \otimes \eta_j, \sum_{k=1}^n \xi_k \otimes \eta_k \right) = \sum_{j,k=1}^n (a_{kj} \eta_j \mid \eta_k) = \sum_{k=1}^n \left\| \sum_{j=1}^n b_{kj} \eta_j \right\|^2 \geq 0.$$

Hence the sesquilinear form B is positive.

(ii) Suppose $\xi_1, \xi_2 \in \mathfrak{D}(\mathfrak{H}, \psi)$ and $\eta_1, \eta_2 \in \mathfrak{D}'(\mathfrak{K}, \psi)$. Then, as both $L_\psi(\xi_2)^* L_\psi(\xi_1)$ and $J_\psi R_\psi(\eta_1)^* R_\psi(\eta_2) J_\psi$ are elements of $\mathfrak{m}_\psi \subset \mathfrak{n}_\psi \cap \mathfrak{n}_\psi^*$,

$$\begin{aligned} (\pi_{\mathfrak{K}}(L_\psi(\xi_2)^* L_\psi(\xi_1)) \eta_1 \mid \eta_2) &= (R_\psi(\eta_1) \eta_\psi (L_\psi(\xi_2)^* L_\psi(\xi_1)) \mid \eta_2) \\ &= (L_\psi(\xi_2)^* \xi_1 \mid R_\psi(\eta_1)^* \eta_2) = (L_\psi(\xi_2)^* \xi_1 \mid \eta'_\psi (J_\psi R_\psi(\eta_2)^* R_\psi(\eta_1) J_\psi)) \\ &= (\xi_1 \mid L_\psi(\xi_2) \eta'_\psi (J_\psi R_\psi(\eta_2)^* R_\psi(\eta_1) J_\psi)) = (\xi_1 \mid \pi'_{\mathfrak{H}} (J_\psi R_\psi(\eta_2)^* R_\psi(\eta_1) J_\psi) \xi_2) \\ &= (\pi'_{\mathfrak{H}} (J_\psi R_\psi(\eta_1)^* R_\psi(\eta_2) J_\psi) \xi_1 \mid \xi_2). \end{aligned}$$

(i') The positive semi-definiteness follows from (i) by symmetry. The second assertion follows from (ii).



Definition 4.2 Let \mathfrak{N} be the subspace of $\mathfrak{D}(\mathfrak{H}, \psi) \odot \mathfrak{K}$ comprising those vectors ζ with $B(\zeta, \zeta) = 0$. The Hilbert space obtained as the completion of the quotient space $\mathfrak{D}(\mathfrak{H}, \psi) \odot \mathfrak{K} / \mathfrak{N}$ relative to the inner product induced by the positive-definite sesquilinear form B will be called the relative tensor product of the right \mathcal{N} -module \mathfrak{H} and the left \mathcal{N} -module \mathfrak{K} with respect to the fns weight ψ and will be written $\mathfrak{H} \boxtimes_{\psi} \mathfrak{K}$. The image of $\xi \otimes \eta$ will similarly be denoted $\xi \boxtimes_{\psi} \eta$ for $\xi \in \mathfrak{D}(\mathfrak{H}, \psi)$, $\eta \in \mathfrak{K}$. By Proposition 4.1, the relative tensor product $\mathfrak{H} \boxtimes_{\psi} \mathfrak{K}$ can also be obtained as the completion of the quotient space of the algebraic tensor product $\mathfrak{H} \odot \mathfrak{D}'(\mathfrak{K}, \psi)$ by the subspace \mathfrak{N}' , where \mathfrak{N}' consists of null vectors with respect to the positive-definite sesquilinear form B' . In this way, we can consider the tensor product $\xi \boxtimes_{\psi} \eta$ for a pair $\xi \in \mathfrak{H}$, $\eta \in \mathfrak{D}'(\mathfrak{K}, \psi)$.

Theorem 4.3 Let \mathcal{N} be a von Neumann algebra equipped with an fns weight ψ , \mathfrak{H} a right \mathcal{N} -module and \mathfrak{K} a left \mathcal{N} -module. Set $\mathcal{P} \triangleq \mathcal{L}(\mathfrak{H}_{\mathcal{N}})$ and $\mathcal{Q} \triangleq \mathcal{L}({}_{\mathcal{N}}\mathfrak{K})$. We construct the direct sum $\tilde{\mathfrak{H}} \triangleq L^2(\mathcal{N}, \psi) \oplus \mathfrak{H} \oplus \overline{\mathfrak{K}}$ as a right \mathcal{N} -module and then consider $\mathcal{R} \triangleq \mathcal{L}(\tilde{\mathfrak{H}}_{\mathcal{N}})$, together with the "balanced" fns weight $\rho = \psi \oplus \varphi \oplus \nu$, where φ is a faithful, normal and semi-finite weight on \mathcal{P} and ν is an fns weight on \mathcal{Q} . Let e, f and g be, respectively, the projections of $\tilde{\mathfrak{H}}$ onto $L^2(\mathcal{N}, \psi)$, \mathfrak{H} and $\overline{\mathfrak{K}}$, which, we note, belong to \mathcal{R} . Represent the standard Hilbert space \mathfrak{H}_{ρ} as the space of 3×3

matrices

$$(3.13') \quad \mathfrak{H}_\rho \cong \begin{pmatrix} L^2(\mathcal{N}, \psi) & \bar{\mathfrak{H}} & \mathfrak{K} \\ \mathfrak{H} & L^2(\mathcal{P}, \varphi) & \mathfrak{H}_{23} \\ \bar{\mathfrak{K}} & \mathfrak{H}_{32} & L^2(\mathcal{Q}, \nu) \end{pmatrix}.$$

Then there exists a natural isomorphism between $\mathfrak{H} \boxtimes_\psi \mathfrak{K}$ and \mathfrak{H}_{23} .

Proof. Let \mathfrak{A} ($= \mathfrak{A}_\rho$) be the left Hilbert algebra associated with ρ , \mathfrak{B} ($= \mathfrak{B}_\rho$) the algebra of left bounded elements in $L^2(\mathcal{R}, \rho)$ and, as usual, $\mathfrak{n}_\rho = \{x \in \mathcal{R} : \rho(x^*x) < +\infty\}$. Since e, f and g are all in \mathcal{R} , \mathfrak{A} and \mathfrak{B} can each be decomposed into the matrix direct sum relative to (3.13'), i.e.,

$$(3.13'') \quad \mathfrak{A} = \begin{pmatrix} \mathfrak{A}_{11} & \mathfrak{A}_{12} & \mathfrak{A}_{13} \\ \mathfrak{A}_{21} & \mathfrak{A}_{22} & \mathfrak{A}_{23} \\ \mathfrak{A}_{31} & \mathfrak{A}_{32} & \mathfrak{A}_{33} \end{pmatrix}, \quad \mathfrak{B} = \begin{pmatrix} \mathfrak{B}_{11} & \mathfrak{B}_{12} & \mathfrak{B}_{13} \\ \mathfrak{B}_{21} & \mathfrak{B}_{22} & \mathfrak{B}_{23} \\ \mathfrak{B}_{31} & \mathfrak{B}_{32} & \mathfrak{B}_{33} \end{pmatrix}.$$

It follows from Lemma 3.3 that $\mathfrak{B}_{21} = \mathfrak{D}(\mathfrak{H}, \psi)$ and $\mathfrak{B}_{31} = \mathfrak{D}(\bar{\mathfrak{K}}, \psi)$. Also, we note that $L_\psi(\xi) = \pi_\ell(\xi)|_{\mathfrak{H}_{11}}$, $\xi \in \mathfrak{D}(\mathfrak{H}, \psi) = \mathfrak{B}_{21}$, and $L_\psi(\bar{\eta}) = \pi_\ell(\bar{\eta})|_{\mathfrak{H}_{12}}$, $\bar{\eta} \in \mathfrak{D}(\bar{\mathfrak{K}}, \psi) = \mathfrak{B}_{31}$, where π_ℓ means the left multiplication representation of \mathfrak{B} on \mathfrak{H}_ρ . At this point, one can see (through symmetry) that the right Hilbert algebra \mathfrak{A}' , and the algebra \mathfrak{B}' of right bounded vectors, admit similar matrix decompositions; we can use these to obtain $\mathfrak{B}'_{21} = \mathfrak{D}'(\bar{\mathfrak{H}}, \psi)$, $\mathfrak{B}'_{31} = \mathfrak{D}'(\mathfrak{K}, \psi)$ and $R_\psi(\eta) = \pi_r(\eta)|_{\mathfrak{H}_{11}}$, $\eta \in \mathfrak{D}'(\mathfrak{K}, \psi)$.

We claim that $\xi \boxtimes_\psi \eta$, with $\xi \in \mathfrak{D}(\mathfrak{H}, \psi)$ and $\eta \in \mathfrak{K}$, can be naturally identified with $\pi_\ell(\xi)\eta \in \mathfrak{H}_{23}$. Let U_0 be the map from $\mathfrak{D}(\mathfrak{H}, \psi) \odot \mathfrak{K}$ into \mathfrak{H}_ρ determined by

$U_0(\xi \otimes \eta) \triangleq \pi_\ell(\xi)\eta$ for $\xi \in \mathfrak{D}(\mathfrak{H}, \psi)$, $\eta \in \mathfrak{K}$. Since $\xi \in \mathfrak{B}_{21}$ and $\eta \in \mathfrak{K} \cong \mathfrak{H}_{13}$, $\pi_\ell(\xi)\eta$ belongs to \mathfrak{H}_{23} . Now we have, for $\xi_1, \xi_2 \in \mathfrak{D}(\mathfrak{H}, \psi)$ and $\eta_1, \eta_2 \in \mathfrak{K}$,

$$\begin{aligned} (U_0(\xi_1 \otimes \eta_1) \mid U_0(\xi_2 \otimes \eta_2)) &= (\pi_\ell(\xi_1)\eta_1 \mid \pi_\ell(\xi_2)\eta_2) = (\pi_\ell(\xi_2)^* \pi_\ell(\xi_1)\eta_1 \mid \eta_2) \\ &= (\pi_{\mathfrak{K}}(L_\psi(\xi_2)^* L_\psi(\xi_1))\eta_1 \mid \eta_2) = (\xi_1 \boxtimes_\psi \eta_1 \mid \xi_2 \boxtimes_\psi \eta_2). \end{aligned}$$

Therefore, the map U_0 gives rise to an isometry U of $\mathfrak{H} \boxtimes_\psi \mathfrak{K}$ into \mathfrak{H}_{23} . Let $\mathfrak{M} = U(\mathfrak{H} \boxtimes_\psi \mathfrak{K}) = [\pi_\ell(\mathfrak{B}_{21})\mathfrak{K}]$. First, we observe that $\mathfrak{H}_{23} = fL^2(\mathcal{R}, \psi)g$, $\mathcal{P} = \mathcal{R}_f$ and $\mathcal{Q} = \mathcal{R}_g$. Hence $\pi_{\mathfrak{H}_{23}}(\mathcal{P})' = \pi_{\mathfrak{H}_{23}}'(\mathcal{Q})$. We know that \mathfrak{M} is invariant under the right action of \mathcal{Q} . Thus, the projection \bar{p} of \mathfrak{H}_{23} onto \mathfrak{M} belongs to $\pi_{\mathfrak{H}_{23}}(\mathcal{P})$, i.e., \bar{p} can be identified with left multiplication by a projection in \mathcal{P} , which we shall call p . This implies that $\mathfrak{M} = p\mathfrak{H}_{23}$, with $p \in \text{Proj}(\mathcal{P})$. But as $\pi_\ell(a\xi) = a\pi_\ell(\xi)$ for $a \in \mathcal{P}$ and $\xi \in \mathfrak{D}(\mathfrak{H}, \psi) = \mathfrak{B}_{21}$, \mathfrak{B}_{23} is invariant under left multiplication by elements of \mathcal{P} , which in turn implies the invariance of \mathfrak{M} under the left action of \mathcal{P} . We may therefore conclude that the projection p belongs to the center $\mathcal{Z}(\mathcal{P})$ of \mathcal{P} , which is of the form $\mathcal{Z}(\mathcal{P}) = \mathcal{Z}(\mathcal{R})_f$. So p may be viewed as a projection in $\mathcal{Z}(\mathcal{R})$. When we view p as an element in $\text{Proj}(\mathcal{Z}(\mathcal{R}))$, we see that we may write $(f-p)\mathfrak{M} = \{0\}$, so that $0 = (f-p)\pi_\ell(\xi)\eta = \pi_\ell((f-p)\xi)\eta$ for every $\xi \in \mathfrak{B}_{21}$ and $\eta \in \mathfrak{K}$. Thus, $\pi_{\mathfrak{K}}(\pi_\ell((f-p)\xi)^* \pi_\ell((f-p)\xi)) = 0$. Because \mathfrak{K} is a faithful left \mathcal{N} -module, we must have $\pi_\ell((f-p)\xi) = 0$, $\xi \in \mathfrak{D}(\mathfrak{H}, \psi)$, which means that $f-p=0$. Therefore, we see that $f=p$, i.e., as an element of $\text{Proj}(\mathcal{M})$, $p = \mathbf{1}_{\mathcal{M}}$, which in turn implies $\mathfrak{M} = \mathfrak{H}_{23}$

Thus, we may conclude that, via the isometry U , $\mathfrak{H} \boxtimes_\psi \mathfrak{K}$ can be identified with



Using the preceding theorem, it is not difficult to arrive at the following Corollary, which is presented without proof.

Corollary 4.4

(i) If \mathfrak{H} and \mathfrak{K} are, respectively, right and left \mathcal{N} -modules, with \mathcal{N} a von Neumann algebra equipped with a fns weight ψ , then the relative tensor product $\mathfrak{H} \boxtimes_{\psi} \mathfrak{K}$ is naturally an $\mathcal{L}(\mathfrak{H}_{\mathcal{N}})$ - $\mathcal{L}(\mathcal{N}\mathfrak{K})^{\circ}$ bimodule, whose bimodule structure is given by

(4.4)

$$a(\xi \boxtimes_{\psi} \eta)b \triangleq (a\xi) \boxtimes_{\psi} (\eta b), \quad a \in \mathcal{L}(\mathfrak{H}_{\mathcal{N}}), \quad b \in \mathcal{L}(\mathcal{N}\mathfrak{K})^{\circ}, \quad \xi \in \mathcal{D}(\mathfrak{H}, \psi), \quad \eta \in \mathfrak{K}.$$

(ii) In terms of operators acting from the left (as usual), if $x \in \mathcal{L}(\mathfrak{H}_{\mathcal{N}})$ and $y \in \mathcal{L}(\mathcal{N}\mathfrak{K})$, then there exists a unique operator $x \boxtimes_{\psi} y \in \mathcal{L}(\mathfrak{H} \boxtimes_{\psi} \mathfrak{K})$ defined by

$$(4.5) \quad (x \boxtimes_{\psi} y)(\xi \boxtimes_{\psi} \eta) \triangleq (x\xi) \boxtimes_{\psi} (y\eta), \quad \xi \in \mathcal{D}(\mathfrak{H}, \psi), \quad \eta \in \mathfrak{K}.$$

The map $(x, y) \in \mathcal{L}(\mathfrak{H}_{\mathcal{N}}) \times \mathcal{L}(\mathcal{N}\mathfrak{K}) \mapsto x \boxtimes_{\psi} y \in \mathcal{L}(\mathfrak{H} \boxtimes_{\psi} \mathfrak{K})$ extends canonically to an injective $*$ -homomorphism from the algebraic tensor product, $\mathcal{L}(\mathfrak{H}_{\mathcal{N}}) \odot \mathcal{L}(\mathcal{N}\mathfrak{K})$, into $\mathcal{L}(\mathfrak{H} \boxtimes_{\psi} \mathfrak{K})$.

(iii) Although \mathcal{N} does not act on the relative tensor product $\mathfrak{H} \boxtimes_{\psi} \mathfrak{K}$, we have

$$(4.6) \quad (\xi b) \boxtimes_{\psi} \eta = \xi \boxtimes_{\psi} (\sigma_{-i/2}^{\psi}(b)\eta), \quad b \in \mathcal{D}(\sigma_{-i/2}^{\psi}), \quad \xi \in \mathcal{D}(\mathfrak{H}, \psi), \quad \eta \in \mathfrak{K}.$$

In order to summarize the preceding arguments, we restate the matrix decomposition of \mathfrak{H}_ρ in the following form, making explicit use of our results up to this point:

$$(4.7) \quad L^2(\mathcal{R}, \rho) = \begin{pmatrix} L^2(\mathcal{N}, \psi) & \bar{\mathfrak{H}} & \mathfrak{K} \\ \mathfrak{H} & L^2(\mathcal{P}, \varphi) & \mathfrak{H} \boxtimes_\psi \mathfrak{K} \\ \bar{\mathfrak{K}} & \bar{\mathfrak{K}} \boxtimes_\psi \bar{\mathfrak{H}} & L^2(\mathcal{Q}, \nu) \end{pmatrix}.$$

Proposition 4.5

(i) Viewing $L^2(\mathcal{N}, \psi)$ as a right \mathcal{N} -module, the map

$$V_{\mathfrak{K}}^\psi : \eta_\psi(y) \boxtimes_\psi \eta \in L^2(\mathcal{N}, \psi) \boxtimes_\psi \mathfrak{K} \mapsto y\eta \in \mathfrak{K}, \quad y \in \mathfrak{n}_\psi, \eta \in \mathfrak{K}$$

gives rise to an isomorphism of $L^2(\mathcal{N}, \psi) \boxtimes_\psi \mathfrak{K}$ onto \mathfrak{K} as \mathcal{N} - $\mathcal{L}(\mathcal{N}\text{-}\mathfrak{K})^\circ$ bimodules.

(ii) Similarly, if we regard $L^2(\mathcal{N}, \psi)$ as a left \mathcal{N} -module, then the map

$$U_{\mathfrak{H}}^\psi : \xi \boxtimes_\psi \eta'_\psi(y) \in \mathfrak{H} \boxtimes_\psi L^2(\mathcal{N}, \psi) \mapsto \xi y \in \mathfrak{H}, \quad \xi \in \mathfrak{H}, y \in \mathfrak{n}_\psi^*$$

extends to an isomorphism of $\mathfrak{H} \boxtimes_\psi L^2(\mathcal{N}, \psi)$ onto \mathfrak{H} as $\mathcal{L}(\mathfrak{H}\mathcal{N})$ - \mathcal{N} bimodules.

The proof of the preceding Proposition is entirely routine, and is omitted. Note that in light of this Proposition, it is reasonable to refer to $L^2(\mathcal{N}, \psi)$ as a sort of “identity” (both right and left) amongst \mathcal{N} modules, relative to the weight ψ .

It is also easy to verify, using our previous technique, viz., the 3×3 matrix decomposition, that we have the following identities, after making the necessary

(implicit) identifications:

$$(4.8) \quad \left. \begin{aligned} (\mathfrak{H}_1 \oplus \mathfrak{H}_2) \boxtimes_{\psi} \mathfrak{K} &\simeq (\mathfrak{H}_1 \boxtimes_{\psi} \mathfrak{K}) \oplus (\mathfrak{H}_2 \boxtimes_{\psi} \mathfrak{K}), \\ \mathfrak{H} \boxtimes_{\psi} (\mathfrak{K}_1 \oplus \mathfrak{K}_2) &\simeq (\mathfrak{H} \boxtimes_{\psi} \mathfrak{K}_1) \oplus (\mathfrak{H} \boxtimes_{\psi} \mathfrak{K}_2), \end{aligned} \right\}$$

where $\mathfrak{H}, \mathfrak{H}_1$ and \mathfrak{H}_2 are all right \mathcal{N} -modules, while $\mathfrak{K}, \mathfrak{K}_1$ and \mathfrak{K}_2 are left \mathcal{N} -modules.

Given the distributivity evidenced above, it is natural to inquire about the associativity of the relative tensor product. This issue is dealt with in the next Theorem.

Theorem 4.6 *Let \mathcal{M} and \mathcal{N} be two von Neumann algebras equipped with fns weights φ and ψ , respectively. If \mathfrak{H} is a right \mathcal{M} -module, \mathfrak{K} an \mathcal{M} - \mathcal{N} bimodule and \mathfrak{M} a right \mathcal{N} -module, then after natural identifications we have*

$$(4.9) \quad (\mathfrak{H} \boxtimes_{\varphi} \mathfrak{K}) \boxtimes_{\psi} \mathfrak{M} \simeq \mathfrak{H} \boxtimes_{\varphi} (\mathfrak{K} \boxtimes_{\psi} \mathfrak{M}),$$

as $\mathcal{L}(\mathfrak{H}_{\mathcal{M}})$ - $\mathcal{L}(\mathcal{N}_{\mathfrak{M}})^{\circ}$ bimodules.

Proof. For each $\xi \in \mathfrak{D}(\mathfrak{H}, \varphi)$, $\eta \in \mathfrak{K}$ and $\zeta \in \mathfrak{D}'(\mathfrak{M}, \psi)$, set

$$U((\xi \boxtimes_{\varphi} \eta) \boxtimes_{\psi} \zeta) \stackrel{\Delta}{=} \xi \boxtimes_{\varphi} (\eta \boxtimes_{\psi} \zeta).$$

Let ξ_i, η_i and ζ_i , $i = 1, 2$, denote elements in $\mathfrak{D}(\mathfrak{H}, \varphi)$, \mathfrak{K} and $\mathfrak{D}'(\mathfrak{M}, \psi)$. We want

to show

$$(U((\xi_1 \boxtimes_{\varphi} \eta_1) \boxtimes_{\psi} \zeta_1) \mid U((\xi_2 \boxtimes_{\varphi} \eta_2) \boxtimes_{\psi} \zeta_2)) = ((\xi_1 \boxtimes_{\varphi} \eta_1) \boxtimes_{\psi} \zeta_1 \mid (\xi_2 \boxtimes_{\varphi} \eta_2) \boxtimes_{\psi} \zeta_2),$$

as this will demonstrate that U is well-defined and a unitary. We compute

$$\begin{aligned}
(U((\xi_1 \boxtimes_{\varphi} \eta_1) \boxtimes_{\psi} \zeta_1) \mid U((\xi_2 \boxtimes_{\varphi} \eta_2) \boxtimes_{\psi} \zeta_2)) &= (\xi_1 \boxtimes_{\varphi} (\eta_1 \boxtimes_{\psi} \zeta_1) \mid \xi_2 \boxtimes_{\varphi} (\eta_2 \boxtimes_{\psi} \zeta_2)) \\
&= (\pi_{\mathfrak{K} \boxtimes_{\psi} \mathfrak{M}}(L_{\varphi}(\xi_2)^* L_{\varphi}(\xi_1))(\eta_1 \boxtimes_{\psi} \zeta_1) \mid \eta_2 \boxtimes_{\psi} \zeta_2) \\
&= ((\pi_{\mathfrak{K}}(L_{\varphi}(\xi_2)^* L_{\varphi}(\xi_1))\eta_1) \boxtimes_{\psi} \zeta_1 \mid \eta_2 \boxtimes_{\psi} \zeta_2) \\
&= (\pi'_{\mathfrak{K}}(J_{\psi} R_{\psi}(\zeta_1)^* R_{\psi}(\zeta_2) J_{\psi}) \pi_{\mathfrak{K}}(L_{\varphi}(\xi_2)^* L_{\varphi}(\xi_1))\eta_1 \mid \eta_2) \quad (\text{by (4.3)}) \\
&= (\pi_{\mathfrak{K}}(L_{\varphi}(\xi_2)^* L_{\varphi}(\xi_1)) \pi'_{\mathfrak{K}}(J_{\psi} R_{\psi}(\zeta_1)^* R_{\psi}(\zeta_2) J_{\psi})\eta_1 \mid \eta_2) \\
&= (\xi_1 \boxtimes_{\varphi} (\pi'_{\mathfrak{K}}(J_{\psi} R_{\psi}(\zeta_1)^* R_{\psi}(\zeta_2) J_{\psi})\eta_1 \mid \xi_2 \boxtimes_{\varphi} \eta_2) \\
&= (\pi'_{\mathfrak{H} \boxtimes_{\varphi} \mathfrak{K}}(J_{\psi} R_{\psi}(\zeta_1)^* R_{\psi}(\zeta_2) J_{\psi})(\xi_1 \boxtimes_{\varphi} \eta_1) \mid \xi_2 \boxtimes_{\varphi} \eta_2) \\
&= ((\xi_1 \boxtimes_{\varphi} \eta_1) \boxtimes_{\psi} \zeta_1 \mid (\xi_2 \boxtimes_{\varphi} \eta_2) \boxtimes_{\psi} \zeta_2).
\end{aligned}$$

Now we also have, for each $a \in \mathcal{L}(\mathfrak{H}_{\mathcal{M}})$ and $b \in \mathcal{L}(\mathcal{N}\mathfrak{M})^{\circ}$,

$$\begin{aligned}
U(a((\xi \boxtimes_{\varphi} \eta) \boxtimes_{\psi} \zeta)b) &= U(((a\xi) \boxtimes_{\varphi} \eta) \boxtimes_{\psi} (\zeta b)) \\
&= (a\xi) \boxtimes_{\varphi} (\eta \boxtimes_{\psi} (\zeta b)) = a(\xi \boxtimes_{\varphi} (\eta \boxtimes_{\psi} \zeta))b \\
&= a(U((\xi \boxtimes_{\varphi} \eta) \boxtimes_{\psi} \zeta))b.
\end{aligned}$$

Hence, we see that U is indeed an isomorphism of $(\mathfrak{H} \boxtimes_{\varphi} \mathfrak{K}) \boxtimes_{\psi} \mathfrak{M}$ onto $\mathfrak{H} \boxtimes_{\varphi} (\mathfrak{K} \boxtimes_{\psi} \mathfrak{M})$ as $\mathcal{L}(\mathfrak{H}_{\mathcal{M}})$ - $\mathcal{L}(\mathcal{N}\mathfrak{M})^{\circ}$ bimodules. ■

It is natural at this point to ask what happens to the relative tensor product $\mathfrak{H} \boxtimes_{\psi} \mathfrak{K}$ when we change the fns reference weight ψ . In order to investigate this issue, let us first recall some notation: $\mathfrak{W}(\mathcal{N})$ is the set of semi-finite, normal weights on the von Neumann algebra \mathcal{N} , while $\mathfrak{W}_0(\mathcal{N})$ represents the set of all faithful such. Once again, we fix \mathcal{N} , the right \mathcal{N} -module \mathfrak{H} , and the left \mathcal{N} -module \mathfrak{K} .

Theorem 4.7 Let \mathcal{N} be a von Neumann algebra, and let \mathfrak{H} and \mathfrak{K} be, respectively, right and left \mathcal{N} -modules. To each pair $(\psi_1, \psi_2) \in \mathfrak{M}_0(\mathcal{N}) \times \mathfrak{M}_0(\mathcal{N})$, there corresponds a unique $\mathcal{L}(\mathfrak{H}_{\mathcal{N}})\text{-}\mathcal{L}(\mathcal{N}\mathfrak{K})^\circ$ bimodule isomorphism, $U_{\mathfrak{H}, \mathfrak{K}}^{\psi_2, \psi_1}$, from $\mathfrak{H} \boxtimes_{\psi_1} \mathfrak{K}$ onto $\mathfrak{H} \boxtimes_{\psi_2} \mathfrak{K}$, which makes the following diagram commute:

$$(4.10) \quad \begin{array}{ccc} \mathfrak{H} \boxtimes_{\psi_1} \mathfrak{K} & \xrightarrow{U_{\mathfrak{H}, \mathfrak{K}}^{\psi_2, \psi_1}} & \mathfrak{H} \boxtimes_{\psi_2} \mathfrak{K} \\ \downarrow a_1 \boxtimes_{\psi_1} b_1^* & & \downarrow a_2 \boxtimes_{\psi_2} b_2^* \\ L^2(\mathcal{N}, \psi_1) \boxtimes_{\psi_1} L^2(\mathcal{N}, \psi_1) & & L^2(\mathcal{N}, \psi_2) \boxtimes_{\psi_2} L^2(\mathcal{N}, \psi_2) \\ \downarrow U_{L^2(\mathcal{N}, \psi_1)}^{\psi_1} & & \downarrow U_{L^2(\mathcal{N}, \psi_2)}^{\psi_2} \\ L^2(\mathcal{N}, \psi_1) & \xrightarrow{U_{\psi_2, \psi_1}} & L^2(\mathcal{N}, \psi_2) \end{array}$$

Here, we take the pair $(a_i, b_i) \in \mathcal{L}(\mathfrak{H}_{\mathcal{N}}, L^2(\mathcal{N}, \psi_i)_{\mathcal{N}}) \times \mathcal{L}(\mathcal{N}L^2(\mathcal{N}, \psi_i)_{\mathcal{N}}, \mathfrak{K})$, for $i = 1, 2$, such that $a_2 = U_{\psi_2, \psi_1} a_1$, $b_2 = b_1 U_{\psi_2, \psi_1}$, with U_{ψ_2, ψ_1} representing the canonical unitary which implements the equivalence of the standard forms, i.e.,

$$U_{\psi_2, \psi_1} : \{\mathcal{N}, L^2(\mathcal{N}, \psi_1), \mathfrak{P}_{\psi_1}, J_{\psi_1}\} \rightarrow \{\mathcal{N}, L^2(\mathcal{N}, \psi_2), \mathfrak{P}_{\psi_2}, J_{\psi_2}\}.$$

Moreover, the correspondence

$$(\psi_1, \psi_2) \in \mathfrak{M}_0(\mathcal{N}) \times \mathfrak{M}_0(\mathcal{N}) \mapsto U_{\mathfrak{H}, \mathfrak{K}}^{\psi_2, \psi_1}$$

satisfies the chain rule, viz.,

$$(4.11) \quad U_{\mathfrak{H}, \mathfrak{K}}^{\psi_3, \psi_2} U_{\mathfrak{H}, \mathfrak{K}}^{\psi_2, \psi_1} = U_{\mathfrak{H}, \mathfrak{K}}^{\psi_3, \psi_1}, \quad \psi_1, \psi_2, \psi_3 \in \mathfrak{M}_0(\mathcal{N}).$$

Proof. We start with existence: we will use the notation established in Theorem 4.3. Choose fns weights $\varphi \in \mathfrak{M}_0(\mathcal{P})$ and $\nu \in \mathfrak{M}_0(\mathcal{Q})$ and set $\rho_i = \psi_i \oplus \varphi \oplus \nu$, for

$i = 1, 2$. We observe that the construction of \mathcal{R} does not depend on the choice of the ψ 's: there is a canonical isometry U_{ρ_2, ρ_1} from $L^2(\mathcal{R}, \rho_1)$ onto $L^2(\mathcal{R}, \rho_2)$. Moreover, this isometry implements an \mathcal{R} - \mathcal{R} bimodule isomorphism. As the projections e, f and g commute with the fns weights ρ_1 and ρ_2 (by their definitions), it is easy to check that U_{ρ_2, ρ_1} preserves the matrix decompositions in of $L^2(\mathcal{R}, \rho_i)$, $i = 1, 2$, which were given by (3.13'). With J the conjugation operator $\bar{\eta} \in \bar{\mathfrak{K}} \mapsto \eta \in \mathfrak{K}$, set $b_i^\circ \triangleq Jb_i^*J \in \mathcal{L}(\bar{\mathfrak{K}}_{\mathcal{N}}, L^2(\mathcal{N}, \psi_i))$, $i = 1, 2$. We then have

$$\bar{a}_i = \begin{pmatrix} 0 & a_i & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \bar{b}_i = \begin{pmatrix} 0 & 0 & b_i^\circ \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \mathcal{R}, \quad i = 1, 2.$$

Hence, we see that the restriction of the operators $\pi_{\rho_i}(\bar{a}_i)\pi'_{\rho_i}(\bar{b}_i)^*$ to the (2,3)-component of $L^2(\mathcal{R}, \rho_i)$ is equal to $U_{L^2(\mathcal{N}, \psi_i)}^{\psi_i}(a_{\mathfrak{E}_{\psi_i}} b_i^*)$, with π'_{ρ_i} the semi-cyclic anti-representation of \mathcal{R} defined by $\pi'_{\rho_i}(x) \triangleq J\pi_{\rho_i}(x)^*J$, $x \in \mathcal{R}$. Since U_{ρ_2, ρ_1} is an \mathcal{R} - \mathcal{R} bimodule isomorphism of $L^2(\mathcal{R}, \rho_1)$ onto $L^2(\mathcal{R}, \rho_2)$, and carries the matrix decomposition (3.13') of $L^2(\mathcal{R}, \rho_1)$ onto that of $L^2(\mathcal{R}, \rho_2)$, by restricting to the (1,1)-component, we obtain U_{ψ_2, ψ_1} ; similarly, by considering the restriction of U_{ρ_2, ρ_1} to the (2,3)-component, we get $U_{\mathfrak{H}, \mathfrak{K}}^{\psi_2, \psi_1}$.

Now, let us turn to unicity. Let $\mathfrak{A}_i (= \mathfrak{A}_{\psi_i})$, for $i = 1, 2$, be the left Hilbert algebras associated with $\{\mathcal{N}, \psi_i\}$, and $(\mathfrak{A}_i)_0$ be the corresponding Tomita algebras. Set $\mathfrak{a}_i = \mathfrak{n}_{\psi_i} \cap \mathfrak{n}_{\psi_i}^* = \pi_\ell(\mathfrak{A}_i)$, and $(\mathfrak{a}_i)_0 = \pi_\ell((\mathfrak{A}_i)_0)$, $i = 1, 2$. For each $\xi \in \mathfrak{D}(\mathfrak{H}, \psi_1)$, $\eta \in \mathfrak{D}'(\mathfrak{K}, \psi_1)$ and $y_1, y_2 \in (\mathfrak{a}_1)_0$, if we take $a_1 = L_{\psi_1}(\xi)^*$ and $b_1 = R_{\psi_1}(\eta)$, then by

(4.10) we obtain

$$\begin{aligned}
U_{\mathfrak{H}, \mathfrak{K}}^{\psi_2, \psi_1}(\xi y_1 \boxtimes_{\psi_1} y_2 \eta) &= U_{\mathfrak{H}, \mathfrak{K}}^{\psi_2, \psi_1}(L_{\psi_1}(\xi) \eta'_{\psi_1}(y_1) \boxtimes_{\psi_1} R_{\psi_1}(\eta) \eta_{\psi_1}(y_2)) \\
&= U_{\mathfrak{H}, \mathfrak{K}}^{\psi_2, \psi_1}(L_{\psi_1}(\xi) \boxtimes_{\psi_1} R_{\psi_1}(\eta))(\eta'_{\psi_1}(y_1) \boxtimes_{\psi_1} \eta_{\psi_1}(y_2)) \\
&= U_{\mathfrak{H}, \mathfrak{K}}^{\psi_2, \psi_1}(a_1^* \boxtimes_{\psi_1} b_1)(U_{L^2(\mathcal{N}, \psi_1)}^{\psi_1})^*(\sigma_{-i/2}^{\psi_1}(y_1) \eta_{\psi_1}(y_2)) \\
&= (a_2^* \boxtimes_{\psi_2} b_2)(U_{L^2(\mathcal{N}, \psi_2)}^{\psi_2})^* U_{\psi_2, \psi_1}(\sigma_{-i/2}^{\psi_1}(y_1) \eta_{\psi_1}(y_2)).
\end{aligned}$$

This means that $U_{\mathfrak{H}, \mathfrak{K}}^{\psi_2, \psi_1}$ is uniquely determined on the vectors of the form

$$\{(\xi y_1) \boxtimes_{\psi_1} (y_2 \eta) : \xi \in \mathfrak{D}(\mathfrak{H}, \psi_1), \eta \in \mathfrak{D}'(\mathfrak{K}, \psi_1), y_1, y_2 \in (\mathfrak{a}_1)_0\},$$

which is dense in $\mathfrak{H} \boxtimes_{\psi_1} \mathfrak{K}$. Hence, $U_{\mathfrak{H}, \mathfrak{K}}^{\psi_2, \psi_1}$ is uniquely determined by the commutative diagram of (4.10).

The chain rule (4.11) follows from the uniqueness of $U_{\mathfrak{H}, \mathfrak{K}}^{\psi_2, \psi_1}$. ■

At this point it is necessary to make the following remark: *The bimodule isomorphism $U_{\mathfrak{H}, \mathfrak{K}}^{\psi_2, \psi_1}$ does not send $\xi \boxtimes_{\psi_1} \eta$ into $\xi \boxtimes_{\psi_2} \eta$ for $\xi \in \mathfrak{H}$ and $\eta \in \mathfrak{K}$.* One must always be careful not to make this mistake when performing calculations involving the relative tensor product.

Before concluding this section, we wish to address (briefly) the following: is it possible to construct the “tensor product” $\mathfrak{H} \otimes_{\mathcal{N}} \mathfrak{K}$ directly from the right \mathcal{N} -module \mathfrak{H} and the left \mathcal{N} -module \mathfrak{K} , i.e., without recourse to a reference weight? It is, in fact, possible to do so if one abandons the notion of the tensor product $\xi \otimes_{\mathcal{N}} \eta$ of the vectors themselves. To see this, suppose we are given a von Neumann algebra \mathcal{N} , a right \mathcal{N} -module \mathfrak{H} , and a left \mathcal{N} -module \mathfrak{K} . We define $\tilde{\mathfrak{H}} \triangleq L^2(\mathcal{N}) \oplus \mathfrak{H} \oplus \overline{\mathfrak{K}}$, recognizing that $L^2(\mathcal{N})$ has meaning independent of any choice of fns weight on

\mathcal{N} . Then $\tilde{\mathfrak{H}}$ is a right \mathcal{N} -module in the obvious way, and we may view $L^2(\mathcal{N})$, \mathfrak{H} and $\bar{\mathfrak{K}}$ as closed subspaces, with e , f and g the projections down to these; note once again that e , f and g are all projections in $\mathcal{R} = \mathcal{L}(\tilde{\mathfrak{H}}_{\mathcal{N}})$. Then we have seen that we have $\mathfrak{H} = fL^2(\mathcal{R})e$ and $\bar{\mathfrak{K}} = gL^2(\mathcal{R})e$, which implies that $\mathfrak{K} = eL^2(\mathcal{R})g$. We may then *define* the “relative tensor product of \mathfrak{H} and \mathfrak{K} over \mathcal{N} ”, $\mathfrak{H} \otimes_{\mathcal{N}} \mathfrak{K}$, to be $fL^2(\mathcal{R})g$. It is clear that, when defined in such a way, $\mathfrak{H} \otimes_{\mathcal{N}} \mathfrak{K}$ has a natural $\mathcal{L}(\mathfrak{H}_{\mathcal{N}})$ - $\mathcal{L}(\mathfrak{K}_{\mathcal{N}})$ ° bimodule structure. In fact, it is a straightforward exercise to show that there exists a bimodule isomorphism $\mathfrak{H} \otimes_{\mathcal{N}} \mathfrak{K} \rightarrow \mathfrak{H} \boxtimes_{\psi} \mathfrak{K}$ for any $\psi \in \mathfrak{M}_0(\mathcal{N})$.

4.2 An Example and a Theorem Involving Relative Tensor Products

In order to make the ideas presented in the previous section more concrete, we begin this section with an example of the relative tensor product. While this example will deal exclusively with matrix algebras (and hence the spaces in question will be finite dimensional), all the essential notions regarding the relative tensor product will be evident. In particular, it is not the finite dimensionality which distinguishes this example; we will have more to say about this later.

Example. Let \mathcal{M} be $M_n(\mathbb{C})$, $\mathfrak{H} = \{M_n(\mathbb{C}), (\cdot | \cdot)_{\mathfrak{H}}\}$, and $\psi = \text{Tr}(H\cdot)$, $H \in \mathcal{M}_+$, non-singular. As any (faithful) positive linear functional on \mathcal{M} is of this form, this is, in fact, the general case. We take \mathfrak{H} to be the Hilbert space which arises, using ψ , via the GNS construction. In order to differentiate between elements of \mathcal{M} and those in \mathfrak{H} , we will denote the latter using the usual $\eta_{\psi}(\cdot)$ notation, e.g., $(\eta_{\psi}(X) | \eta_{\psi}(Y))_{\mathfrak{H}} \triangleq \text{Tr}(HY^*X)$.

\mathfrak{H} has an \mathcal{M} - \mathcal{M} bimodule structure, in which the left and right actions of \mathcal{M} on \mathfrak{H} are given by

$$(4.12) \quad A\eta_{\psi}(X) \triangleq \eta_{\psi}(AX), \quad \eta_{\psi}(X)B \triangleq \eta_{\psi}(XH^{\frac{1}{2}}BH^{-\frac{1}{2}}),$$

where $A, B \in \mathcal{M}$. It is important to realize that, while the left action of \mathcal{M} on \mathfrak{H} coincides with the usual matrix multiplication, the right action is “twisted” via conjugation by $H^{\frac{1}{2}}$. This definition of the right action is necessary in order to have

$$(\eta_\psi(X)B \mid \eta_\psi(Y))_{\mathfrak{H}} = (\eta_\psi(X) \mid \eta_\psi(Y)B^*)_{\mathfrak{H}}.$$

We also note the following:

$$(4.13) \quad \begin{aligned} J_\psi \eta_\psi(X) &= \eta_\psi(H^{\frac{1}{2}}X^*H^{-\frac{1}{2}}), & \Delta_\psi^{it} \eta_\psi(X) &= \eta_\psi(H^{it}XH^{-it}) \\ &\Rightarrow \sigma_t^\psi(A) &= H^{it}AH^{-it}. \end{aligned}$$

Now, define φ by $\varphi \triangleq \text{Tr}(K \cdot)$, where K too is a positive, non-singular element of \mathcal{M} . How can we give a realization of $\mathfrak{H} \boxtimes_\varphi \mathfrak{H}$? More precisely, by combining the results of Theorem 4.7 with Proposition 4.5, we see that $\mathfrak{H} \boxtimes_\varphi \mathfrak{H} \simeq \mathfrak{H}$. (Note that \mathfrak{H} is really just $L^2(\mathcal{M}, \psi)$.) What we would like to do is to exhibit this \mathcal{M} - \mathcal{M} bimodule isomorphism explicitly.

We know that

$$\begin{aligned} &(\eta_\psi(X_1) \boxtimes_\varphi \eta_\psi(Y_1) \mid \eta_\psi(X_2) \boxtimes_\varphi \eta_\psi(Y_2))_{\mathfrak{H} \boxtimes_\varphi \mathfrak{H}} \\ &= (\eta_\psi(X_1) J_\psi R_\psi(Y_1)^* R_\psi(Y_2) J_\psi \mid \eta_\psi(X_2))_{\mathfrak{H}} \end{aligned}$$

from (4.3). Using (4.13), we can compute $J_\psi R_\psi(Y_1)^* R_\psi(Y_2) J_\psi$; we obtain

$$(4.14) \quad J_\psi R_\psi(Y_1)^* R_\psi(Y_2) J_\psi = K^{-\frac{1}{2}} Y_1 H Y_2^* K^{-\frac{1}{2}}.$$

Now, using (4.12), we know how $K^{-\frac{1}{2}} Y_1 H Y_2^* K^{-\frac{1}{2}} \in \mathcal{M}$ acts (from the right) on \mathfrak{H} . Hence, we can calculate

$$\begin{aligned}
& (\eta_\psi(X_1)K^{-\frac{1}{2}}Y_1HY_2^*K^{-\frac{1}{2}} \mid \eta_\psi(X_2))_{\mathfrak{H}} \\
&= (\eta_\psi(X_1H^{\frac{1}{2}}K^{-\frac{1}{2}}Y_1HY_2^*K^{-\frac{1}{2}}H^{-\frac{1}{2}}) \mid \eta_\psi(X_2))_{\mathfrak{H}} \\
&= \text{Tr}(HX_2^*X_1H^{\frac{1}{2}}K^{-\frac{1}{2}}Y_1HY_2^*K^{-\frac{1}{2}}H^{-\frac{1}{2}}) \\
&= \text{Tr}(HY_2^*K^{-\frac{1}{2}}H^{\frac{1}{2}}X_2^*X_1H^{\frac{1}{2}}K^{-\frac{1}{2}}Y_1) \\
&= \text{Tr}(H(X_2H^{\frac{1}{2}}K^{-\frac{1}{2}}Y_2)^*X_1H^{\frac{1}{2}}K^{-\frac{1}{2}}Y_1) \\
&= (\eta_\psi(X_1H^{\frac{1}{2}}K^{-\frac{1}{2}}Y_1) \mid \eta_\psi(X_2H^{\frac{1}{2}}K^{-\frac{1}{2}}Y_2))_{\mathfrak{H}}.
\end{aligned}$$

So we see that the \mathcal{M} - \mathcal{M} bimodule isomorphism is implemented by the map $\mathfrak{H} \boxtimes_{\varphi} \mathfrak{H} \rightarrow \mathfrak{H}$ given by

$$\eta_\psi(X) \boxtimes_{\varphi} \eta_\psi(Y) \mapsto \eta_\psi(XH^{\frac{1}{2}}K^{-\frac{1}{2}}Y)$$

□

Let's examine our example further. Suppose we were interested in formulating a theory of "bimodule tensor products," and proceeded naively: then we would expect the elements of \mathcal{M} to merely "move through" the \boxtimes_{φ} , i.e., we anticipate

$$(4.15) \quad \eta_\psi(X)A \boxtimes_{\varphi} \eta_\psi(Y) = \eta_\psi(X) \boxtimes_{\varphi} A\eta_\psi(Y).$$

Using calculations found in the example, we have

$$\begin{aligned}
\eta_\psi(X)A \boxtimes_{\varphi} \eta_\psi(Y) &= \eta_\psi(XH^{\frac{1}{2}}AH^{-\frac{1}{2}}) \boxtimes_{\varphi} \eta_\psi(Y) \\
&\leftrightarrow \eta_\psi(XH^{\frac{1}{2}}AH^{-\frac{1}{2}}H^{\frac{1}{2}}K^{-\frac{1}{2}}Y) = \eta_\psi(XH^{\frac{1}{2}}AK^{-\frac{1}{2}}Y),
\end{aligned}$$

while

$$\eta_\psi(X) \boxtimes_{\varphi} A\eta_\psi(Y) \leftrightarrow \eta_\psi(XH^{\frac{1}{2}}K^{-\frac{1}{2}}AY).$$

So, if we want (4.15), we must have $XH^{\frac{1}{2}}AK^{-\frac{1}{2}}Y = XH^{\frac{1}{2}}K^{-\frac{1}{2}}AY$, or equivalently $AK^{-\frac{1}{2}} = K^{-\frac{1}{2}}A$. This in turn yields $K^{\frac{1}{2}}AK^{-\frac{1}{2}} = A$, $\forall A \in \mathcal{M}$, which says that $\sigma_{-i/2}^\varphi(A) = A$. Hence we see that the modular automorphism group comprises only the identity automorphism, which says $\varphi = \text{Tr}$, so $K = I$, the identity matrix in $\mathcal{M} = M_n(\mathbb{C})$.

Of course, we could have obtained the above directly from (4.6), which told us what happens to elements when they move through \mathfrak{K}_φ . However, it was our intention to illustrate the theory derived in the previous section directly.

The example presented above is not entirely unmotivated. We now present a theorem which demonstrates that the relative tensor product is really the most natural product construction possible in the category of von Neumann bimodules. As the theorem will show, attempts to formulate a theory motivated solely by algebraic construction can succeed only under restrictive circumstances.

Theorem 4.8 *Let \mathcal{M} be a σ -finite von Neumann algebra. Take ψ a faithful state on \mathcal{M} , and let \mathfrak{H}_ψ denote, as usual, $L^2(\mathcal{M}, \psi)$. Suppose there exists a \mathbb{C} -linear map $I: \mathfrak{H}_\psi \times \mathfrak{H}_\psi \rightarrow \mathfrak{K}$, where \mathfrak{K} (like \mathfrak{H}_ψ) is a faithful \mathcal{M} - \mathcal{M} bimodule. We also assume I is continuous in each variable separately, and satisfies the following conditions:*

(i)

$$aI(\xi, \eta) = I(a\xi, \eta),$$

$$I(\xi b, \eta) = I(\xi, b\eta),$$

$$I(\xi, \eta c) = I(\xi, \eta)c,$$

where a, b and c are in \mathcal{M} , $\xi, \eta \in \mathfrak{H}_\psi$.

(ii) $\text{Span}_{\mathbb{C}}\{I(\xi, \eta) : \xi, \eta \in \mathfrak{H}_\psi\}$ is dense in \mathfrak{K} .

(iii) I is non-degenerate, i.e., for any $0 \neq \xi \in \mathfrak{H}_\psi$, there exists $\eta \in \mathfrak{H}_\psi$ such that

$$I(\xi, \eta) \neq 0.$$

Then \mathcal{M} is an atomic von Neumann algebra (hence semi-finite), and $\mathfrak{K} \simeq \mathfrak{H}_\psi \boxtimes_{\tau} \mathfrak{H}_\psi$, where τ is a faithful, normal and semi-finite trace on \mathcal{M} . (Note that τ may be a tracial weight; \mathcal{M} may possess no tracial state.)

Proof. We will prove the above assertion in stages. We begin with an observation, viz., that the usual appeal to Uniform Boundedness allows us to conclude that $\exists 0 < C < +\infty$ such that

$$(4.16) \quad \|I(\xi, \eta)\|_{\mathfrak{K}} \leq C\|\xi\|\|\eta\| \quad \forall \xi, \eta \in \mathfrak{H}_\psi;$$

hence, I is actually jointly continuous.

Now, define $\xi_\psi \triangleq \eta_\psi(\mathbf{1}_{\mathcal{M}})$; then we know that ξ_ψ is both cyclic and separating

for \mathfrak{H}_ψ – so, $\{x\xi_\psi : x \in \mathcal{M}\}$ and $\{\xi_\psi y : y \in \mathcal{M}\}$ are both dense in \mathfrak{H} . Now, we have

$$\begin{aligned} \mathfrak{K} &= [\{I(\xi, \eta) : \xi, \eta \in \mathfrak{H}_\psi\}] = [\{I(\xi_\psi x, y\xi_\psi) : x, y \in \mathcal{M}\}] \\ &= [\{I(\xi_\psi, xy\xi_\psi) : x, y \in \mathcal{M}\}] = [\{I(\xi_\psi, a\xi_\psi) : a \in \mathcal{M}\}] \\ &= \overline{\{I(\xi_\psi, \xi_\psi)b : b \in \mathcal{M}\}}. \end{aligned}$$

If we define $\eta_0 \in \mathfrak{K}$ as $\eta_0 \triangleq I(\xi_\psi, \xi_\psi)$, then the preceding calculation shows that η_0 is separating for \mathcal{M} in \mathfrak{K} (since it is cyclic for $\mathcal{L}_\mathcal{M}(\mathfrak{K})$). A similar argument shows that $\overline{\{a\eta_0 : a \in \mathcal{M}\}} = \mathfrak{K}$; hence η_0 is both cyclic and separating for \mathcal{M} in \mathfrak{K} .

Certainly, we lose nothing if we assume that $\|\eta_0\|_{\mathfrak{K}} = 1$. Then, defining $\varphi \triangleq (\cdot \eta_0 | \eta_0)_{\mathfrak{K}}$, we see that $\varphi \in \mathfrak{S}_*(\mathcal{M})$, and $\mathfrak{K} \cong \mathfrak{H}_\varphi$. (Here, $\mathfrak{S}_*(\mathcal{M})$ represents the set of normal states of \mathcal{M} .) Now, we can compute

$$\begin{aligned} (4.17) \quad \varphi(x^*x) &= (x^*x\eta_0 | \eta_0)_{\mathfrak{K}} = \|x\eta_0\|_{\mathfrak{K}}^2 \\ &= \|xI(\xi_\psi, \xi_\psi)\|_{\mathfrak{K}}^2 = \|I(x\xi_\psi, \xi_\psi)\|_{\mathfrak{K}}^2 \\ &\leq C^2\|x\xi_\psi\|_{\mathfrak{H}_\psi}^2 = C^2\psi(x^*x), \quad \forall x \in \mathcal{M}. \end{aligned}$$

Note that we have used (4.16). Hence, we see that $\varphi \leq C^2\psi$. From the theory of the cocycle derivative (see [9]), this allows us to infer the following:

- (i) The map $t \mapsto (D\varphi : D\psi)_t = u_t$ extends to a map $(z \mapsto u_z) \in \mathcal{A}_\mathcal{M}(\mathbf{D}_{1/2})$.
- (ii) $\varphi(x^*x) = \psi(u_{-i/2}^* x^* x u_{-i/2})$, $\forall x \in \mathcal{M}$.

Here,

$$\mathcal{A}_\mathcal{M}(\mathbf{D}_{1/2}) \triangleq \left\{ f : \mathbf{D}_{1/2} \rightarrow \mathcal{M} : f \text{ is analytic on the interior of } \mathbf{D}_{1/2}, \right. \\ \left. \text{and continuous and bounded on all } \mathbf{D}_{1/2} \right\},$$

where

$$\mathbf{D}_r \triangleq \begin{cases} \{z \in \mathbf{C} : -r \leq \Im(z) \leq 0\}, & \text{if } r \geq 0 \\ \{z \in \mathbf{C} : 0 \leq \Im(z) \leq -r\}, & \text{otherwise.} \end{cases}$$

We now note that $\varphi \leq C^2\psi$ tells us that $\mathcal{M}_a^\psi \subset \mathcal{M}_a^\varphi$, i.e., the σ -weakly dense *-subalgebra of ψ -analytic elements of \mathcal{M} is actually contained in the set of φ -analytic elements. We can therefore compute as follows: $\forall a \in \mathcal{M}_a^\psi$, we have

$$\begin{aligned} a\xi_\psi &= \xi_\psi \sigma_{i/2}^\psi(a), & \text{while} \\ a\eta_0 &= \eta_0 \sigma_{i/2}^\varphi(a). \end{aligned}$$

However,

$$\begin{aligned} a\eta_0 &= a\mathbf{I}(\xi_\psi, \xi_\psi) = \mathbf{I}(a\xi_\psi, \xi_\psi) = \mathbf{I}(\xi_\psi \sigma_{i/2}^\psi(a), \xi_\psi) \\ &= \mathbf{I}(\xi_\psi, \sigma_{i/2}^\psi(a)\xi_\psi) = \mathbf{I}(\xi_\psi, \xi_\psi \sigma_i^\psi(a)) = \mathbf{I}(\xi_\psi, \xi_\psi) \sigma_i^\psi(a) = \eta_0 \sigma_i^\psi(a). \end{aligned}$$

Hence we are forced to conclude

$$(4.18) \quad \sigma_i^\psi(a) = \sigma_{i/2}^\varphi(a), \quad \forall a \in \mathcal{M}_a^\psi.$$

Now, it is a fundamental property of the cocycle derivative $(D\varphi : D\psi)_t = u_t$ that

$$\sigma_t^\varphi(x) = u_t \sigma_t^\psi(x) u_t^*, \quad \forall x \in \mathcal{M},$$

or, equivalently,

$$(4.19) \quad \sigma_t^\varphi(x) u_t = u_t \sigma_t^\psi(x).$$

Therefore we obtain, $\forall a \in \mathcal{M}_a^\psi$,

$$\sigma_{-i/2}^\varphi(a) u_{-i/2} = u_{-i/2} \sigma_{-i/2}^\psi(a),$$

since the product of the analytic maps is again analytic, and these agree on all of \mathbf{R} , by virtue of (4.19). This gives us

$$\sigma_{i/2}^\varphi(a^*)^* u_{-i/2} = u_{-i/2} \sigma_{i/2}^\psi(a^*)^* \implies u_{-i/2}^* \sigma_{i/2}^\varphi(a^*) = \sigma_{i/2}^\psi(a^*) u_{-i/2}^*.$$

Now, using (4.18), we obtain

$$u_{-i/2}^* \sigma_i^\psi(a^*) = \sigma_{i/2}^\psi(a^*) u_{-i/2}^*;$$

setting $b = \sigma_{i/2}^\psi(a^*)$, we then have

$$(4.20) \quad u_{-i/2}^* \sigma_{i/2}^\psi(b) = b u_{-i/2}^*, \quad \forall b \in \mathcal{M}_a^\psi.$$

Furthermore, we see from (4.20)

$$u_{-i/2} b^* = \sigma_{i/2}^\psi(b)^* u_{-i/2} \implies u_{-i/2} b^* = \sigma_{-i/2}^\psi(b^*) u_{-i/2};$$

hence we also obtain

$$(4.20') \quad \sigma_{-i/2}^\psi(b) u_{-i/2} = u_{-i/2} b, \quad \forall b \in \mathcal{M}_a^\psi.$$

By combining (4.20) and (4.20'), we observe

$$u_{-i/2} u_{-i/2}^* b = u_{-i/2} \sigma_{-i/2}^\psi(b) u_{-i/2}^* = \sigma_{-i}^\psi(b) u_{-i/2} u_{-i/2}^*.$$

So, we define $h \triangleq u_{-i/2} u_{-i/2}^*$, and rewrite the above:

$$(4.21) \quad hb = \sigma_{-i}^\psi(b) h, \quad \forall b \in \mathcal{M}_a^\psi.$$

Now, we make use of a result from [14], which states that any positive, non-singular h which satisfies (4.21) on the set of analytic elements for the one parameter

automorphism group σ^ψ must in fact be an *analytic generator* for σ^ψ , i.e., we must have

$$(4.22) \quad \sigma_t^\psi(x) = h^{it} x h^{-it}, \quad \forall x \in \mathcal{M}.$$

In particular, this means that σ_t^ψ is inner, and hence \mathcal{M} is semi-finite.

Thus, we are lead to define $\tau \triangleq \psi_{h^{-1}}$, where

$$\psi_{h^{-1}}(x) \triangleq \lim_{\varepsilon \searrow 0} \psi(h^{-1}(1 + \varepsilon h)x), \quad x \in \mathcal{M}_+;$$

note that this makes sense, since, once again, $h \in \mathcal{M}_\psi$. Then, this gives $(D\tau : D\psi)_t = h^{-it}$, and $\sigma_t^\tau = \text{id}$; so τ is a trace on \mathcal{M} . (However, notice that τ may be a tracial weight.) Now, let k be such that $(D\varphi : D\tau)_t = k^{it}$; we remark that, due to the fact that φ is a state, k is a non-singular, positive element in \mathcal{M} . From the chain rule for cocycle derivatives,

$$(D\varphi : D\tau)_t = (D\varphi : D\psi)_t (D\psi : D\tau)_t,$$

we may conclude that $k^{it} = u_t h^{it}$. Using (4.18), and the fact that $h \in \mathcal{M}_\psi \subset \mathcal{M}_\alpha^\psi$,

we have

$$k^{\frac{1}{2}} \sigma_i^\psi(h) h = k^{\frac{1}{2}} \sigma_{i/2}^\phi(h) h,$$

or

$$(4.23) \quad k^{\frac{1}{2}} h = h k^{\frac{1}{2}} \implies kh = hk,$$

i.e., h and k commute.

So, we may write $u_t = k^{it}h^{-it} = (kh^{-1})^{it}$, which yields $u_{-i/2} = (kh^{-1})^{\frac{1}{2}}$, i.e., $u_{-i/2}$ is a positive element in \mathcal{M} . (Strictly speaking, $u_{-i/2} = (kh^{-1})^{\frac{1}{2}}$ is valid only on $h\mathfrak{K}$, the range of h ; however, this set is dense in \mathfrak{K} , and since we already know that $u_{-i/2}$ is a bounded operator, its positivity follows by continuity.) But recall that, by definition, $h^{\frac{1}{2}} = |u_{-i/2}^*|$; this means that we must have $h^{\frac{1}{2}} = u_{-i/2}^* = u_{-i/2}$, and, from the above calculations we may conclude

$$(4.24) \quad k = h^2.$$

We will now demonstrate that \mathfrak{K} is actually isomorphic, as an \mathcal{M} - \mathcal{M} bimodule, to $\mathfrak{H}_\psi \boxtimes_\tau \mathfrak{H}_\psi$. We define the map V via

$$V: \mathbb{I}(x\xi_\psi, y\xi_\psi) \mapsto x\xi_\psi \boxtimes_\tau y\xi_\psi, \quad \forall x, y \in \mathcal{M},$$

noting that, given the following:

$$\begin{aligned} \|xy\xi_\psi\|_{\mathfrak{H}_\psi}^2 &= \psi(y^*x^*xy) = \tau(hy^*x^*xy) \\ &= \tau(xyhy^*x^*) \leq \|yhy^*\| \tau(xx^*) = \|yhy^*\| \tau(x^*x) \end{aligned}$$

we have $\eta_\psi(\mathfrak{n}_\psi) \subset \mathcal{D}'(\mathfrak{H}_\psi, \tau)$. (In fact, such a fact characterizes τ as a trace.) This makes V well-defined.

This map is an isometry: first, we compute

$$\begin{aligned} \|\mathbb{I}(x\xi_\psi, y\xi_\psi)\|_{\mathfrak{K}}^2 &= \|\mathbb{I}(x\sigma_{-i/2}^\psi(y)\xi_\psi, \xi_\psi)\|_{\mathfrak{K}}^2 = \|\mathbb{I}(x\sigma_{-i/2}^\psi(y)\eta_0)\|_{\mathfrak{K}}^2 \\ &= \|\mathbb{I}(xh^{\frac{1}{2}}yh^{-\frac{1}{2}}\eta_0)\|_{\mathfrak{K}}^2 = \varphi(h^{-\frac{1}{2}}y^*h^{\frac{1}{2}}x^*xh^{\frac{1}{2}}yh^{-\frac{1}{2}}) \\ &= \tau(kh^{-\frac{1}{2}}y^*h^{\frac{1}{2}}x^*xh^{\frac{1}{2}}yh^{-\frac{1}{2}}) = \tau(h^2h^{-\frac{1}{2}}y^*h^{\frac{1}{2}}x^*xh^{\frac{1}{2}}yh^{-\frac{1}{2}}) \\ &= \tau(hy^*h^{\frac{1}{2}}x^*xh^{\frac{1}{2}}y) = \psi((xh^{\frac{1}{2}}y)^*xh^{\frac{1}{2}}y), \quad x \in \mathcal{M}, y \in \mathcal{D}(\sigma_{-i/2}^\psi). \end{aligned}$$

Now, we calculate

$$\begin{aligned}
\|x\xi_\psi \boxtimes_\tau y\xi_\psi\|_{\mathfrak{H}_\psi \boxtimes_\tau \mathfrak{H}_\psi}^2 &= (x\xi_\psi J_\tau R_\tau(y\xi_\psi)^* R_\tau(y\xi_\psi) J_\tau \mid x\xi_\psi)_{\mathfrak{H}_\psi} = (x\xi_\psi y h y^* \mid x\xi_\psi)_{\mathfrak{H}_\psi} \\
&= (x h^{\frac{1}{2}} y h y^* h^{-\frac{1}{2}} \xi_\psi \mid x\xi_\psi)_{\mathfrak{H}_\psi} = \psi(x^* x h^{\frac{1}{2}} y h y^* h^{-\frac{1}{2}}) \\
&= \tau(h x^* x h^{\frac{1}{2}} y h y^* h^{-\frac{1}{2}}) = \tau(h y^* h^{\frac{1}{2}} x^* x h^{\frac{1}{2}} y) \\
&= \psi((x h^{\frac{1}{2}} y)^* x h^{\frac{1}{2}} y), \quad x \in \mathcal{M}, y\xi_\psi \in \mathcal{D}'(\mathfrak{H}_\psi, \tau).
\end{aligned}$$

Hence, V is an isometry, and can be extended to a map from \mathfrak{K} onto $\mathfrak{H}_\psi \boxtimes_\tau \mathfrak{H}_\psi$.

It is also immediate that \mathfrak{K} and $\mathfrak{H}_\psi \boxtimes_\tau \mathfrak{H}_\psi$ have the same \mathcal{M} - \mathcal{M} bimodule structure.

Thus, V is the desired \mathcal{M} - \mathcal{M} bimodule isomorphism.

Finally, we wish to demonstrate that \mathcal{M} must be an atomic von Neumann algebra. To see this, we simply note that the map $x\xi_\psi \mapsto \eta_\tau(xh^{\frac{1}{2}})$ implements the standard isometry between \mathfrak{H}_ψ and $\mathfrak{H}_\tau = L^2(\mathcal{M}, \tau)$. Because of the preceding argument, we may then view I as a map from $\mathfrak{H}_\tau \times \mathfrak{H}_\tau \rightarrow \mathfrak{H}_\tau \boxtimes_\tau \mathfrak{H}_\tau$, via

$$I : (\eta_\tau(xh^{\frac{1}{2}}), \eta_\tau(yh^{\frac{1}{2}})) \mapsto \eta_\tau(xh^{\frac{1}{2}}) \boxtimes_\tau \eta_\tau(yh^{\frac{1}{2}}).$$

However, $\mathfrak{H}_\tau \boxtimes_\tau \mathfrak{H}_\tau$ is isometrically isomorphic to \mathfrak{H}_τ as an \mathcal{M} - \mathcal{M} bimodule under the map

$$(4.25) \quad \eta_\tau(xh^{\frac{1}{2}}) \boxtimes_\tau \eta_\tau(yh^{\frac{1}{2}}) \mapsto \eta_\tau(xh^{\frac{1}{2}} y h^{\frac{1}{2}}).$$

If we restrict ourselves to $\eta_\tau(\mathfrak{n}_\tau^* \cap \mathfrak{n}_\tau)$, i.e., the left Hilbert algebra \mathfrak{A}_τ , then (4.25) is just telling us that the usual multiplication operation “lifts” to the relative tensor product $\mathfrak{H}_\tau \boxtimes_\tau \mathfrak{H}_\tau$. However, when we combine this with our previous results regarding the map I , specifically (4.16), we must conclude that (left or right)

multiplication by *any* element of \mathfrak{H}_τ acts as a *bounded* operator on \mathfrak{H}_τ . This can only be the case when \mathcal{M} is atomic. ■

We now see that the example with which we began this Section is actually quite general: we have discovered that if we wish to define a “naive” tensor product of L^2 -von Neumann modules we may do so only under very restrictive conditions, viz., when the von Neumann algebra is essentially a “matrix algebra.”

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