WHAT ARE QUANTUM LOGICS AND WHAT OUGHT THEY TO BE?

D.J. Foulis and C.H. Randall

Department of Mathematics and Statistics
University of Massachusetts
Amherst, MA 01003, U.S.A.

We, and our students and colleagues at the University of Massachusetts, have erected the foundations of a general scientific language capable of elucidating, in the spirit of Leibniz, the physical theories that are of concern to us here. We feel that this has now been accomplished with a clarity and precision that has been wanting up to this point. As a consequence of this work, we feel qualified to submit our answer to the question in the title of this paper. It is both important and illuminating to understand the roots of any work of such a fundamental nature. Therefore, to begin with, we sketch the mundane pragmatic background of our own work and the history of the subject -- as we see it.

The aim of an empirical science, for us, is to order, explain, and predict the observable events associated with certain physical situations or experiments. As such, its mathematical foundations ought to be erected on a "general scientific language" capable of describing these physical occurrences with adequate precision. In particular, a flexible symbolic logic, or calculus of experimental propositions, is required to serve this purpose. In this regard, it should be appreciated at the outset that a symbolic logic is not a collection of physical laws; it is not even a language (an instrument of communication). A symbolic logic does not become a language until its symbols are assigned objective significance, and a language expresses physical laws only when it, in some sense, predicts the consequences of actual physical procedures. Thus, for the most part, we have not been concerned with physical laws, but with the pristine grammar of a language to describe physical experience. The direct approach, and the one we utilized, is to synthesize such a language from the physical procedures or operations employed in the empirical sciences. The result of these efforts we have called empirical logic.

The statistical theory that we have constructed on this foundation we have called operational statistics.

This ambitious program, however, was not the original intent. One of us (Randall), motivated by problems in nuclear reactor engineering, simply sought an adequate mathematical formalism to deal with mechanical mixtures of materials. The approach was to regard such media as stochastic products of some well-defined fabrication process. Stochastic models and methods were developed, after some effort, that did predict the behaviour of some of these materials with reasonable accuracy. However, there remained some difficulties. Manufacturers appeared to concentrate on producing useful materials for which no models were available. As a consequence, the stochastic models became more complex and the necessary mathematical manipulations became less tractable. It soon was evident that perennial model making, however clever, at best offered only a temporary and partial solution.

In addition to these obvious difficulties, there were a number of nagging doubts as to the adequacy of the formalism. In practice, it is only of interest to predict some of the material properties from data obtained from an effectively fixed class of scanning instruments. It appeared that, in some way, the stochastic models overdescribed the materials and were far more complicated than the situation demanded. Furthermore, in many instances, it was quite evident that the measurements significantly perturbed the scanned materials. Hence there seemed to be irreducible uncertainties, at least in the pragmatic sense, for which the formalism provided no adequate description.

In order to satisfy these doubts, it was innocently decided to investigate, in some detail, the formal mathematical structure of the problem. These investigations led to a jungle of fundamental problems whose subtleties had hardly been anticipated. The venerable question of the meaning of probability (subjective or objective -- epistemic or ontic) and the related metaphysical problems of induction demanded attention at the very outset of the study. In due time even the basic logic, that is the rules of inference, became a matter of concern.

In the process, and over a period of several years, it became necessary to abandon one cherished elementary concept after another. The continuum of real numbers was the first to go, followed by conventional probability theory and classical logic. Eventually nothing remained but the operational basis on which the present formalism is founded. The adoption of the operational approach, at this point in the development of an empirical language, does not imply radical empiricism, logical positivism, or even operationalism. The development of an operationally based language should not be presumed to be a simultaneous rejection of subjective methods. In particular, explicitly identifying the observables of an experimental science does not automatically deny the unifying power of idealized models. In fact, the essentially subjective logic has been erected on these

operational foundations.[18] As a matter of fact, the formalism, as it ultimately developed, is, as far as possible, independent of any epistemological point of view. For instance, it neither requires nor discourages the realist view of the world.

Even prior to the decision to study the basic character of the problem of mixtures of materials, the striking similarities to parts of quantum theory was evident. This should come as no surprise, for, as G. Mackey observed,[15] the formalism of quantum mechanics might become necessary when the precision of measurements becomes high with respect to the disturbance they cause. This is certainly the case in the materials problem, as indeed it is in many of the behavioural and social sciences, which have stubbornly resisted formal treatment.

As a consequence of these reflections, the aims of this study became more ambitious. It seemed as though a solution of the original problem would, in effect, permit one to deduce the formal structure of quantum mechanics from notions more primitive than customarily is the case. Although it appeared that a paper on quantum theory might be more scholarly than one on mixtures of materials, in time it became evident that the initial quest was surely the more ambitious one, in spite of its mundane motivations; for it ultimately demanded a sounder basis for empirical science. Since quantum mechanics purports to be part of empirical science, it ought to be subsumed by the more general results.

Our current formalism is, in effect, a blend of the earlier work on mixtures of materials and the Baer *-semigroup approach to orthomodular lattices (quantum logics) initiated by one of us (Foulis). Although semigroups do play an important role in our work,[7,8,19] we have, as yet, been unable to incorporate Baer *-semigroups in our general formalism without imposing ad hoc assumptions. In fact, the structures that we have been forced to consider in order to represent composite physical systems have "logics" that are not even orthomodular posets.

Although logic is the general science of inference, it began, and for the most part has been formally developed as a theory of deduction. This has been particularly true of its ideographic transcription, symbolic or mathematical logic. Leibniz is said to have been the first serious student of symbolic logic; almost two hundred years before Boole, he proposed a universal scientific language (characteristica universalis) and a calculus of reasoning (calculus ratiocinator) for its manipulation. Unfortunately, little came of his projected scientific reforms.

The history of modern symbolic logic properly begins with George Boole, who in 1847 published the mathematical foundations on which symbolic logic has since enjoyed continuous development. In brief, he proposed the algebra that now bears his name -- in lattice terminology, a distributive orthomodular lattice. It is noteworthy that

Boole's work was motivated by statistical considerations. It is even more noteworthy that he did not form the disjunction of propositions (events) unless they were disjoint (orthogonal).

W.S. Jevons, John Venn, and Charles Saunders Peirce are among those associated with the transition from Boole's original system to the modern form of Boolean algebra. In a number of further steps in the evolution of symbolic logic, the rigorous deductive methods of pure mathematics were brought together with Boole's system, ultimately culminating in the monumental Principia Mathematica of Whitehead and Russell. Algebraic logic is a natural product of this line of development. As Halmos observed, algebraic logic is more algebra than logic. In our view, the process leading to this final abstraction has thus tended to obscure the empirical content of logic!

It should not be supposed that this Boolean juggernaut went unchallenged. Over the years a number of interesting alternative logics have been advanced. The "intuitionist school", led by Brouwer, proposed the dual of a relatively pseudo-complemented lattice. Others, such as Post, Lukasiewicz, and Tarski, proposed modal logics -- that is, logics with propositions admitting more than two truth values. Kolmogorov suggested that conventional probability theory could be regarded as modal logic with a continuum of truth values. It is worth recalling, in this regard, that probability theory was originally introduced as a symbolic logic for plausible reasoning. When Laplace's Théorie Analytique was first published in 1812, it was widely regarded as the long awaited calculus of inductive reasoning fully developed.

Keynes, in agreement with Koopman, suggested much later that the modes of probability (viewed as an inductive logic) only form a partly ordered set. However, most of the modal logics that have been advanced have had linearly ordered modes. Birkhoff pointed out that the Brouwerian logic mentioned above and the so-called quantum logics can be valid only if they are modal logics (that is, only if they do admit propositions that can be neither true nor false). It is not surprising then, that Reichenbach explicitly proposed a three-valued (true-indeterminate-false) logic for quantum mechanics.

The impressive successes of classical mechanics (the prototype of modern empirical science) made its authority so complete that its epistemological foundations remained virtually unchallenged for almost two hundred years after Newton, Leibniz, and Descartes. Materialism, in fact, transformed this model of physical reality into reality itself. Its logical foundations, however, were finally subjected to careful scrutiny in 1883 in Mach's The Science of Mechanics. This work influenced Einstein and others to press these inquiries further. In due time, it was duly noted that the implied logic of classical mechanics was not quite the atomic Boolean algebra originally proposed. Nevertheless, it was a Boolean algebra, the universal separable measure algebra -- isomorphic to the quotient algebra of Borel subsets modulo sets of Lebesque measure zero of any n-dimensional Euclidean space. It thus appeared that the twin edifices of classical thought -- Boolean logic and classical mechanics -- were in essential, if not exact, agreement.

Einstein's relativity theory did not alter the essentials of this reassuring state of affairs; quantum mechanics, however, was another matter. Birkhoff and von Neumann demonstrated that the logic underlying quantum mechanics could not be a Boolean algebra and, in effect, they proved that it was a separable, atomic, orthomodular lattice. Experimental logic, in the original sense of Leibniz, owes its current renaissance to this seminal work and the general dissatisfaction with the logical foundations of quantum physics (or more precisely, the lack of them). In their original paper, Birkhoff and von Neumann also suggested tentatively that a quantum logic ought to be modular. However, most of the subsequently proposed logics have only been required to be orthomodular lattices or posets. As we have already observed, even these conditions may be overly stringent.

These logical discrepancies should come as no surprise for, as we have seen, the logics of empirical science have been consequences of, not foundations for, models of reality. Characteristically, deterministic or stochastic models of Nature are established and then, if at all, the implied logics are investigated. In brief, these posterior logics depend not on reality, the purported authority of empirical science, but on models of reality. This topsy-turvy practice that leaves the conventional wisdom so vulnerable to crucial physical tests (such as the Michelson-Morley experiment and the recent tests of the Bell inequality) has been noted by Mach, among others. Physicists (such as Tisza[22]) have sought a self-healing physics with which to escape the havoc caused by these periodic calamities. It would seem that a large step in this direction will have been taken when all physical laws can be founded on a common a priori logic.

It would be difficult to justify a preference for any of the known logics in the face of the noted discrepancies. In addition, there are many deficiencies implicit in these logics; for example, the lack of a suitable "tensor product". The former deficiency is readily appreciated by anyone who has ever attempted to formally discuss the consequences of performing temporally ordered observations on a physical system. Birkhoff, for one, has discussed this matter in connection with the problem of the interpretation of the infimum operation in quantum logic. In an excellent review of the axioms of quantum logic, MacLaren[16] pointed out that there are temporally ordered observables that can be operationally described, but for which there appear to be no corresponding Hermitian operators. The root of this difficulty is the absence of an adequate conditioning operation in the underlying logic. The conditionings that were proposed by Koopman and Copeland,[14,4] in classical Boolean settings were also in part motivated by such deficiencies. On the other hand, the lack of a suitable tensor product has been apparent to all those who have studied the measurement problem from an abstract point of view. Perhaps the most critical deficiency in the known logics is their inability to formally describe the necessary and sufficient conditions for an experiment (or physical situation) to take place.

Thus, we have developed the foundations of an adequate a priori logic independent of any particular model of nature or epistemic view; but nevertheless, one that may still reflect the collective experience of science. This has been accomplished by initially adopting a strict operational point of view. Such an approach has been proposed by many authors, but surely P.W. Bridgman is its most articulate modern spokesman. In brief, this point of view requires that all concepts be defined in terms of physically realizable operations. This procedure is in accord with Birkhoff's injunction, "Scientifically, quantum logic should draw its authority directly from experiments. This approach is not only scientific; it has the mathematical advantage of making the lattice theory of quantum logic autonomous."[1] In any case, introspection will reveal that there is no acceptable alternative, since we reject the aid of any specific physical model. Again note that this should not be construed to mean that we have rejected the concept of a physical model -- for indeed we have not! Here the important point is that the logic is established first, and then physical models are employed.

The need for a fundamental operational logic is also implicit in much of the contemporary scientific literature. In Atomic Physics and Human Knowledge,[2] Neils Bohr restated his well-known views on the critical significance of the language with which experiments are described in atomic physics. He noted, for example, with regard to the unambiguous use of the concepts of classical physics, that, "The decisive point is to recognize that the description of the experimental arrangement and the recording of observations must be given in plain language, suitably refined by the usual physical terminology. This is a simple logical demand, since by the word 'experiment' we can only mean a procedure regarding which we are able to communicate to others what we have done and what we have learnt." In the very first sentence of his monograph on quantum mechanics, Heisenberg states, "The experiments of physics and their results can be described in the language of daily life."[12]

At a Colston Symposium, W. Kneale[13] remarked with regard to empirical science, "Since language is not merely a vehicle of communication of thought but also an instrument of learning itself, it is a mistake to be impatient about linguistic questions unless they are manifestly of the sort that can be solved by tossing a coin."

In the light of the above considerations and as a consequence of many years of investigation, deliberation, and debate, we give our answer to the question "What ought quantum logics to be?" in the form of the following list of desiderata:

I

Quantum logic should be founded on the notion of physical operations, experiments, procedures, measurements, or tests; or, at the very least, it should admit an explicit operational paradigm.

II

It should, insofar as possible, be independent of any epistemological or ontological prejudice. Its form should be neutral with respect to all world-views and explanations of physical events.

III

The elements of a quantum logic (events, propositions, questions, two-valued observables,...) must be testable; that is, for each such element, there must exist at least one physical operation every realization of which unequivocally determines one of two truth values (occur-nonoccur, true-false, yes-no, 0-1,...). It is not required that two realizations of a test operation produce the same truth value. If E is a test operation, we denote by T(E) the set of all elements of the quantum logic that are tested by E. Of course, we suppose that $T(E) \neq \emptyset$.

IV

If an element of a quantum logic admits two or more test operations, it is understood that no significance is attached to which of these tests is employed to obtain its truth value.

V

For each test operation E, the set T(E) should admit the Boolean notions of conjunction, disjunction, negation, and so forth. For instance, if $p,q \in T(E)$, there must exist an element $r \in T(E)$ that is effective as the conjunction of p and q in the classical sense: Whenever E is realized, the resulting truth value of r is 1 if and only if the resulting truth values of both p and q are 1. The remaining Boolean notions are understood in analogous ways.

VI

A quantum logic should be capable of formally describing compound operations and measurements, and their consequences. In particular, it should be possible to represent sequences of operations and operations on unions of physical systems.

If the above desiderata are satisfied, most of the standard notions of quantum logic can be introduced. For instance, a collection of elements is said to be jointly orthogonal if they admit a common test operation and, whenever such an operation is realized, at most one of these elements is assigned truth value 1. If, in addition, such a realization assigns truth value 1 to one and only one of these elements, we say that they form a maximal jointly orthogonal collection. In view of Desideratum IV, it is presumed that, if such a condition holds for any one common test operation, it holds for all common test operations. As usual, a state can be regarded as a function, mapping elements of a quantum logic to the closed unit interval, that sums to 1 over maximal jointly orthogonal collections.

The above notion of orthogonality allows us to introduce the following useful concepts: If $\{p,q\}$ is a maximal jointly orthogonal collection of elements of a quantum logic, we say that p and q are operational complements of each other, and we write $p\ \underline{oc}\ q$. If two elements p and q share a common operational complement, we say that they are operationally perspective and we write $p\ \underline{op}\ q$. If $p_i\ \underline{op}\ p_{i+1}$ for $i = 1,2,\ldots,n$, then we say that p_1 is weakly perspective to p_{n+1} and we write $p_1\ \underline{wp}\ p_{n+1}$. Finally, if $\{p,q\}$ is a jointly orthogonal set, we say that p and q are orthogonal and we write $p \perp q$. In this connection, it is important to realize that, in general, a pairwise orhtogonal collection need not be jointly orthogonal. A quantum logic in which every finite pairwise orthogonal set of elements is jointly orthogonal is said to be orthocoherent.

We can now turn our attention to the matter of deductive inference in quantum logic. We propose to consider this question axiomatically in terms of a binary relation $\leqslant$. If p and q are elements of a quantum logic, let us write $p \leqslant q$ to mean that, in some sense or the other, q can be deduced from p. The sense in which this is so will depend on the axioms imposed upon the binary relation $\leqslant$. Among the axioms that we have studied (but not necessarily adopted) are the following:

(1) $p \perp r,\ r\ \underline{oc}\ p \Longrightarrow p \leqslant q$.

(2) $p_1 \leqslant q_1,\ p_1\ \underline{op}\ p,\ q_1\ \underline{op}\ q \Longrightarrow p \leqslant q$.

(3) $q' \leqslant p',\ q'\ \underline{oc}\ q,\ p'\ \underline{oc}\ p \Longrightarrow p \leqslant q$.

(4) $p_j \leqslant q$ for all j, (p_j) a family of elements with a common test operation, p a disjunction of $(p_j) \Longrightarrow p \leqslant q$.

(5) $p \leqslant q_j$ for all j, (q_j) a family of elements with a common test operation, q a conjunction of $(q_j) \Longrightarrow p \leqslant q$.

(6) $p \leqslant r,\ r \leqslant q \Longrightarrow p \leqslant q$.

(7) $p \leqslant q \Longrightarrow$ there exists r, $r \perp p$, there exists s, s a disjunction of r and p such that $s \leqslant q$ and $q \leqslant s$.

(8) $p \leqslant q$ and $q \leqslant p \Longrightarrow p\ \underline{wp}\ q$.

Suppose that E is a test operation and that $p, q \in T(E)$. By Desideratum V, the usual Boolean notions are available in T(E). If q is deducible from p in the classical sense that p is disjoint from (that is, orthogonal to) the negation of q in T(E), then Axiom 1 guarantees that $p \leqslant q$.

For some purposes,[10] two elements p and q that are operationally perspective can be regarded as being logically equivalent. (Note that $\underline{wp}$ is the transitive closure of $\underline{op}$, and thus, from this point of view, $\underline{wp}$ is a logical equivalence.) Thus, Axiom 2 is simply the corresponding substitution rule.

Evidently, Axioms 3 through 6 are simply shreds of classical reasoning that may or may not be desirable. Axiom 7 is also in the classical tradition; it amounts to the celebrated orthomodular law.

If Axioms 1 and 6 are imposed, then $\leqslant$ is indeed a quasi-order of the elements of the quantum logic. In this case, denote by $\equiv$ the corresponding equivalence relation ($p \equiv q \Longleftrightarrow p \leqslant q$ and $q \leqslant p$). It follows from Axiom 1 that $p\ \underline{wp}\ q \Longrightarrow p \equiv q$. Given Axiom 8, the converse holds and the two equivalence relations coincide.

To many authors, (perhaps most), a quantum logic is an orthomodular poset. In such a poset the orthogonality relation is available from the start ($p \perp q \Longleftrightarrow p \leqslant q'$), orthocoherence holds, two elements are operational complements if an only if they are orthocomplements, and $\underline{wp}$ (as well as $\underline{op}$) are the relation of equality. Furthermore, there is one and only one binary relation that satisfies Axioms 1 through 8; namely, the original partial order relation.

A simple and realizable example due to Ron Wright[24] permits us to illustrate the ideas introduced above. Thus, let us suppose we have a supply of balls on each of which letters a,b,c,x,y, or z are printed in color as follows:

Ball Type	Red	Green	Blue
I	a	a	x
II	b	z	b
III	y	c	c
IV	y	z	x

For instance, on a ball of type I is printed a red a, a green a and a blue x. Balls of these types are placed in an urn. We consider three test operations -- called the red, the green, and the blue test. To conduct the red test, for instance, we put on a pair of red spectacles, select a ball from the urn, and read whatever letter we see on it. (It is supposed that we see only the letter printed in red.) The blue and the green tests are defined analogously.

Since the red test admits only three possible outcomes -- namely a, y, or b -- it is natural to introduce the sample space R = {a,y,b} corresponding to this test. Therefore, it is reasonable to let T(red test) = ℘(R), the set of all subsets of R, and to interpret the elements of T(red test) as events in the conventional sense. Similarly, T(green test) = ℘(G), G = {a,z,c} and T(blue test) = ℘(B), B = {b,x,c} The nineteen events in

T(red test) ∪ T(green test) ∪ T(blue test)

constitute the elements of a (quantum) logic for this physical arrangement. Given such an element, for instance D ∈ T(red test), we assign a truth value of occur or nonoccur (1 or 0 if you prefer) to D by executing the red test and ascertaining whether the outcome belongs or does not belong to D, respectively.

The Boolean notions required in Desideratum V are, of course, supplied by the ordinary set-theoretic operations and relations. For instance, if A,B ∈ T(red test), then A ∪ B ∈ T(red test) and A ∪ B is effective as the disjunction of A and B, while the relative complements R \ A and R \ B are effective as their respective negations. Clearly, a collection of events is jointly orthogonal if and only if they are pairwise disjoint and their union is contained in R, G, or B. Such a collection is maximally jointly orthogonal if and only if, in addition, the union is R, G, or B. Here a state, regarded as a probability in the frequency sense, can be prepared by mixing the four types of balls in a certain proportion. However, there exist states that cannot be prepared in this manner -- for instance, the state that assigns ½ to {a}, {b}, and {c}.

Two events are operational complements if and only if they form a partition of R, G, or B into two disjoint sets. Here, we do have distinct events that are operationally perspective, for instance, {a,y} op {x,c} via the common operational complement {b}. Furthermore, in this example, op is transitive and consequently op coincides with wp.

As was mentioned earlier, operationally perspective events may be regarded as being logically equivalent in some sense. In the present case, this is rather obvious; for instance, the operationally perspective events {a,y} and {x,c} occur under precisely the same circumstances -- namely that the selected ball is not of type II. It should come as no surprise that one might wish to regard the equivalence class, consisting of these two events, as representing the proposition the selected ball is not of type II.

For this particular physical situation, there does not exist a relation ≤ on the logic of nineteen events that satisfies all of Axioms 1 through 8. Indeed, suppose that such a relation exists.

(i) {b} ≤ {b,y} by Axiom 1.

(ii) {b,y} op {z,c} via {a}.

(iii) {b} ≤ {z,c} by (i), (ii) and Axiom 2.

(iv) {b} ≤ {b,x} by Axiom 1.

(v) {b,x} op {z,a} via {c}.

(vi) {b} ≤ {z,a} by (iv), (v) and Axiom 2.

(vii) {z,c} and {z,a} have a common test (the green test).

(viii) {b} ≤ {z,c} ∩ {z,a} = {z} by (iii), (vi), (vii) and Axiom 5.

(ix) There exists an event C such that C ⊥ {b} and {b} ∪ C ≤ {z} ≤ {b} ∪ C by (viii) and Axiom 7.

(x) {b} ∪ C wp {z} by (ix) and Axiom 8.

In the example at hand we have already observed that wp coincides with op. From (x), we conclude that {b} ∪ C op {z}. But inspection reveals that no event other than {z} is operationally perspective to {z}, and yields a contradiction.

In spite of the fact that no relation on the (quantum) logic under discussion can satisfy Axioms 1 through 8, certain implication relations do recommend themselves. Perhaps the most natural of these is defined by A ≤ B if and only of there exist events A_1 and B_1 such that A_1 wp A, B_1 wp B and $A_1 \subseteq B_1$. This relation satisfies all of Axioms 1 through 8 with the exceptions of Axioms 4 and 5. Furthermore, ≤ respects states in the sense that A ≤ B ⟹ α(A) ≤ α(B) for all states α.

An alternative implication relation ⥽ that satisfies all of Axioms 1 through 8, except for Axiom 7 (the orthomodular law) can be defined as follows: For each event A, let A^1 denote the set of all ball types such that, if a ball of that type is selected, then A must occur when tested. (This is essentially the same as the set of all dispersion free states α for which α(A) = 1.) Then define A ⥽ B if and only if $A^1 \subseteq B^1$. It is easy to see that A ≤ B ⟹ A ⥽ B: The converse does not hold; in fact, {b} ⥽ {z}, but {b} ≰ {z}. Note that ⥽ does not preserve the states; in fact, if α is that state for which α({a}) = α({b}) = α({c}) = ½, then α({z}) = 0, {b} ⥽ {z}, but ½ = α({b}) ≰ 0 = α({z}).

Since both ≤ and ⥽ satisfy Axioms 2 and 8, we can factor out the equivalence relation wp and thus induce partial orders, also denoted by ≤ and ⥽, respectively, on the equivalence classes (propositions). The quotient structure, in fact, becomes an orthocomplemented poset under either of these partial orders. However, it is not an orthomodular poset in either case. On the one hand, for $\leqslant$, the supremum of orthogonal elements need not exist. What does exist is the equivalence class corresponding to the disjunction of the underlying orthogonal events and, with respect to such "disjunctions" (rather than suprema), the orthomodular law does hold for $\leqslant$.[10] On the other hand, for $\dashv$, the supremum of orthogonal elements does exist; in fact, it is the "disjunction" just mentioned. However, the orthomodular law fails for $\dashv$.

As this example plainly shows, a quantum logic (whether or not an appropriate logical equivalence has been factored out) may well admit more than one significant implication relation. In general, such order relations are a consequence of adopting a particular type of model or explanation of the behaviour of a physical arrangement. A choice of a particular implication relation can therefore amount to a rejection of a class of possible models. Such a choice, as part of the general language of empirical logic, is contrary to Desideratum II, and for us is unacceptable.

Incidentally, the practice of identifying elements of a quantum logic that are imagined to be equivalent in some sense can be quite useful, but it must be applied with considerable caution. For example, as R. Cooke and J. Hilgevoord have clearly shown in their paper in these transactions,[3] the common practice of identifying elements of a quantum logic that cannot be separated by states can lead to absurdities. Furthermore, in his paper on spin experiments,[23] Ron Wright made it absolutely clear that logically equivalent events can condition states differently and thus must not be identified unless they are "final events" (that is, events that will not be used for conditioning).

The details of our own empirical logic have been published elsewhere,[5,6,9,10,17,19,21] so we shall only provide a sketch here. As a matter of fact, the urn example discussed above, in itself, provides a compact illustration of our approach.

We always begin with a collection of well-defined, physically realizable, reproducible test procedures (in our urn example, the red, green, and blue tests). With each such test procedure, we associate a sample space (in our example R, G, and B) the elements of which are called outcomes. Since each sample space corresponds to some physical operation, we refer to these sample spaces as operations. Since the collection of sample spaces corresponds to a catalogue or manual of experimental procedures, we refer to the set of all operations as a (quasi) manual. (The quasimanual in the urn example is {R, G, B}.)

Thus, a quasimanual $\mathfrak{A}$ is a nonempty collection of (possibly overlapping) nonempty sets -- called $\mathfrak{A}$-operations. By an $\mathfrak{A}$-outcome, we mean any element $e \in E$, for any $E \in \mathfrak{A}$. By an $\mathfrak{A}$-event, we mean any subset $A \subseteq E$, for any $E \in \mathfrak{A}$. We denote by $\mathcal{E}(\mathfrak{A})$ the set of all $\mathfrak{A}$-events and, of course, $\bigcup\mathfrak{A}$ denotes the set of all $\mathfrak{A}$-outcomes. Thus, the basic ideas with which empirical logic begins are very simple indeed -- only one step removed from the sample spaces of classical probability theory. The operations fit together in a quasimanual much in the same way Euclidean patches fit together in a manifold. Precision is obtained by working within this well-defined mathematical framework.

If $\mathfrak{A}$ is a quasimanual, an $E \in \mathfrak{A}$ is said to be a test operation for an event $A \in \mathcal{E}(\mathfrak{A})$ if $A \subseteq E$. Thus (just as in the urn example) $T(E) = \mathcal{P}(E)$ and the Boolean notions required in Desideratum V are supplied by the ordinary set-theoretic operations and relations. In particular, a collection of $\mathfrak{A}$-events is jointly orthogonal if and only if they are all contained in some $\mathfrak{A}$-operation E and they are pairwise disjoint. Therefore, two $\mathfrak{A}$-events A and B are operational complements if and only if they are disjoint and their union is an $\mathfrak{A}$-operation.

An event $A \in \mathcal{E}(\mathfrak{A})$ is said to occur (or to nonoccur) if any test operation for A is executed and an $e \in A$ (respectively, $e \notin A$) is secured as the outcome. This is but one of the many modalities that can be defined for events. For example, we can also say that an event $A \in \mathcal{E}(\mathfrak{A})$ is confirmed (or refuted) whenever an event $B \in \mathcal{E}(\mathfrak{A})$ occurs (respectively, nonoccurs) and B <u>wp</u> A. Similarly, for any implication relation $\dashv$ on $\mathcal{E}(\mathfrak{A})$, we can say that an event A is $\dashv$-confirmed (or $\dashv$-refuted) if an event B occurs (respectively, nonoccurs) and $B \dashv A$ (respectively, $A \dashv B$).

An $\mathfrak{A}$-weight is a function $\omega : \bigcup\mathfrak{A} \longrightarrow [0,1]$ such that, for every $E \in \mathfrak{A}$,

$$\sum_{e \in E} \omega(e) = 1,$$

the sum being understood in the unordered sense. The set of all $\mathfrak{A}$-weights is denoted by $\Omega(\mathfrak{A})$. For $\omega \in \Omega(\mathfrak{A})$ and $A \in \mathcal{E}(\mathfrak{A})$, define

$$\omega(A) = \sum_{e \in A} \omega(e).$$

The resulting map $\omega : \mathcal{E}(\mathfrak{A}) \longrightarrow [0,1]$ is called a regular $\mathfrak{A}$-state. Notice that, if $\omega \in \Omega(\mathfrak{A})$, and if $A,B \in \mathcal{E}(\mathfrak{A})$ with $A \perp B$, then $\omega(A \cup B) = \omega(A) + \omega(B)$. A regular $\mathfrak{A}$-state ω is to be regarded as a logically possible complete stochastic model, in the objective sense, for the empirical situation described by the quasimanual $\mathfrak{A}$. Notice that, if $A,B \in \mathcal{E}(\mathfrak{A})$ with A <u>wp</u> B, then $\omega(A) = \omega(B)$ for every $\omega \in \Omega(\mathfrak{A})$.

Let $\mathfrak{A}$ and β be quasimanuals. By an interpretation morphism $\mathfrak{A} \xrightarrow{\phi} \beta$, we mean a map $\phi : \bigcup\mathfrak{A} \longrightarrow \mathcal{E}(\beta)$ with the following two properties:

(i) For $E \in \mathfrak{A}$, $\bigcup_{e \in E} \phi(e) \in \beta$.

(ii) If x and y are $\mathcal{A}$-outcomes and $\{x\} \perp \{y\}$, then $\phi(x) \cap \phi(y) = \emptyset$

For such a ϕ, if $A \subseteq \cup\mathcal{A}$, define $\phi(A) = \bigcup_{e \in A} \phi(e)$. We call ϕ positive in case $\phi(e) \neq \emptyset$ for all $e \in \cup\mathcal{A}$. Notice that $\omega \in \Omega(\beta)$ implies that $\omega \circ \phi \in \Omega(\mathcal{A})$, so we can define the conjugate map

$$\phi^\dagger : \Omega(\beta) \longrightarrow \Omega(\mathcal{A})$$

by the equation $\phi^\dagger(\omega) = \omega \circ \phi$ for all $\omega \in \Omega(\beta)$.

We use the term 'interpretation' because we have in mind the situation in which $\mathcal{A}$ is a phenomenological or laboratory quasimanual explained or interpreted by the theoretical or model quasimanual β. In practice, of course, the model quasimanual β will have considerably more mathematical structure than the laboratory quasimanual $\mathcal{A}$ -- for instance, β may be the orthocoherent quasimanual of all orthonormal bases of a Hilbert space. In particular, notice that the conjugate map $\phi^\dagger$ interprets a theoretical regular state ω as a laboratory frequency model $\phi^\dagger(\omega)$. Naturally, $\phi^\dagger$ need not be one-to-one or onto -- there may be many theoretical models that predict the same frequency for laboratory events and there may be regular $\mathcal{A}$-states that are not consistent with the interpretation ϕ.

A quasimanual in which the equivalence relation wp respects orthogonality (in the sense that A wp B, $C \perp A \Longrightarrow C \perp B$) is called a manual. A quasimanual $\mathcal{A}$ is called a premanual if there exists a manual β and a positive interpretation morphism

$$\mathcal{A} \xrightarrow{\phi} \beta.$$

If $\mathcal{A}$ is a premanual, there exists a unique smallest manual $\langle\mathcal{A}\rangle$ for which $\cup\langle\mathcal{A}\rangle = \cup\mathcal{A}$. Moreover, $\Omega(\mathcal{A}) = \Omega(\langle\mathcal{A}\rangle)$. We call $\langle\mathcal{A}\rangle$ the manual generated by $\mathcal{A}$. A quasimanual with a unital set of regular states is a premanual. (A set of regular states is said to be unital if every nonempty event is assigned the value 1 by some state in the set.)

In those instances in which it is desirable to factor out the logical equivalence relation wp it is essential to be able to lift the orthogonality relation to the quotient structure and this is precisely the condition imposed on manuals. Now, let $\mathcal{A}$ be a manual. It is easy to show that op is a transitive relation on $\mathcal{E}(\mathcal{A})$; hence, op coincides with wp. If $A \in \mathcal{E}(\mathcal{A})$, define the proposition p(A) affiliated with A by

$$p(A) = \{B \in \mathcal{E}(\mathcal{A}) \mid B \text{ wp } A\}$$

and let

$$\Pi(\mathcal{A}) = \{p(A) \mid A \in \mathcal{E}(\mathcal{A})\}$$

denote the collection of all such propositions. As noted, we can lift the orthogonality relation $\perp$ to $\Pi(\mathcal{A})$ by defining

$$p(A) \perp p(B) \Longleftrightarrow A \perp B$$

for $A, B \in \mathcal{E}(\mathcal{A})$. Furthermore, if $p(A) \perp p(B)$, we define

$$p(A) \oplus p(B) = p(A \cup B).$$

(It is easy to show that the partially defined binary operation $\oplus$ is, in fact, well-defined for a manual $\mathcal{A}$.) Recall that, if the $\mathcal{A}$-events A and B are orthogonal, then $A \cup B$ is the Boolean disjunction of A and B in T(E), where E is any common test operation for A and B. Thus, $p(A) \oplus p(B)$ is the result of lifting this disjunction to the quotient structure $\Pi(\mathcal{A})$. A negation map $p(A) \longrightarrow p(A)'$ is defined on $\Pi(\mathcal{A})$ by $p(A)' = p(B)$, where B is any $\mathcal{A}$-event such that B oc A. Of course, the special propositions 0 and 1 in $\Pi(\mathcal{A})$ are defined by $0 = p(\emptyset)$ and $1 = 0'$. It can be shown that $\Pi(\mathcal{A})$ is an associative ortho-algebra in the sense of Hardegree and Frazer.[11]

Recall that a quasimanual $\mathcal{A}$ is said to be orthocoherent if every finite collection of pairwise orthogonal events is jointly orthogonal. Thus, a manual $\mathcal{A}$ is orthocoherent if and only if $\Pi(\mathcal{A})$ is an orthocoherent associative ortho-algebra in the sense that, if $p(A)$, $p(B)$, and $p(C) \in \Pi(\mathcal{A})$ with $p(A) \perp p(B)$, $p(A) \perp p(C)$, and $p(B) \perp p(C)$, then $p(A) \perp p(B) \oplus p(C)$. If $\Pi(\mathcal{A})$ is orthocoherent, then there exists a unique partial order relation $\leqslant$ on $\Pi(\mathcal{A})$ such that $(\Pi(\mathcal{A}), \leqslant, \perp, ', 0, 1)$ is an orthomodular poset for which the orthogonal join is the orthogonal sum $\oplus$. In fact this order relation is the unique relation satisfying Axioms 1 through 8 introduced earlier. Explicitly, we have

$$p(A) \leqslant p(B) \Longleftrightarrow \text{there exists } p(C) \text{ such that } p(A) \perp p(C) \text{ and } p(A) \oplus p(C) = p(B),$$

for $p(A), p(B), p(C) \in \Pi(\mathcal{A})$.

As observed earlier, the model manual β for an interpretation morphism $\mathcal{A} \xrightarrow{\phi} \beta$ normally will have a richer structure than the phenomenological quasimanual $\mathcal{A}$ -- in particular, it will generally be an orthocoherent manual for which $\Pi(\beta)$ carries the distinguished order relation $\leqslant$ discussed above. Given such an interpretation morphism ϕ, it is natural to define a corresponding quasi-order $\leqslant_\phi$ on $\mathcal{E}(\mathcal{A})$ by

$$A \leqslant_\phi B \Longleftrightarrow p(\phi(A)) \leqslant p(\phi(B))$$

for $A, B \in \mathcal{E}(\mathcal{A})$.

Certain classes of interpretation morphisms are of special interest. For instance, a Boolean manual is an orthocoherent manual β for which $\Pi(\beta)$ is a Boolean algebra, and a Boolean interpretation of a quasimanual $\mathcal{A}$ is an interpretation morphism $\mathcal{A} \xrightarrow{\phi} \beta$, where β is a Boolean manual. Similarly, a Hilbert manual is an orthocoherent manual $\mathcal{H}$ for which $\Pi(\mathcal{H})$ is isomorphic to the lattice of projection operators on a Hilbert space, and a Hilbert interpretation of $\mathcal{A}$ is an interpretation morphism $\mathfrak{A} \xrightarrow{\phi} \mathcal{H}$, where $\mathcal{H}$ is a Hilbert manual. If, as often is the case, we wish to consider such a special class of interpretations of a phenomenological quasimanual $\mathfrak{A}$, we can study the corresponding class of implication relations $\leq_\phi$ on $\mathcal{E}(\mathfrak{A})$. In particular, we can form the intersection of all such relations $\leq_\phi$ and obtain a single quasi-order on $\mathcal{E}(\mathfrak{A})$ corresponding to the class of models under consideration. In this way, we can consider the implications common to all models of a certain type before committing ourselves to a particular model.

Thus, by means of interpretations, we can pull back relations, as well as states, from models. Therefore, as promised, we introduce idealistic models in empirical logic.

The subjective logics also mentioned at the beginning of this paper are constructed as follows: For a quasimanual $\mathfrak{A}$ we regard subsets of $\Omega(\mathfrak{A})$ as hypotheses making assertions concerning the frequencies with which events occur and propositions are confirmed. We denote by $\mathfrak{J}(\mathfrak{A})$ the sigma field of subsets of $\Omega(\mathfrak{A})$ generated by sets of the form $\{\omega \in \Omega(\mathfrak{A}) \mid p \leq \omega(A) \leq q\}$, where p and q are rational numbers and $A \in \mathcal{E}(\mathfrak{A})$. The elements of $\mathfrak{J}(\mathfrak{A})$ are interpreted as inductively accessible statistical hypotheses concerning the physical operations described in $\mathfrak{A}$.

It is clear that $\mathfrak{J}(\mathfrak{A})$ is a classical Boolean algebra. Hence it is natural to introduce the convex set $\mathfrak{M}(\mathfrak{A})$ of all countably additive probability measures on $\mathfrak{J}(\mathfrak{A})$. The elements of $\mathfrak{M}(\mathfrak{A})$ are regarded as subjective probabilities, or credibilities, that amount to consistent models of belief concerning the statistics associated with the physical procedures represented in $\mathfrak{A}$. It has been shown that a generalization of Bayesian inference works well in this mathematical framework and permits us to modify beliefs in the face of experimental data.[18] In fact, most of the methods of modern statistics (for example, maximum likelihood, confidence intervals, and hypothesis testing) can be employed in this formalism.

In many cases, the elements of a manual represent physical operations that a single observer may execute. In other instances, the operations of a manual represent the measurements that are permitted on a given type of physical system We represent the combined activities of two observers and the union of two physical systems by means of suitable "products". If $\mathfrak{A}$ and β are manuals, the products in question are defined as follows:

$$\overrightarrow{\mathfrak{A}\beta} = \left\{ \bigcup_{e \in E} \{e\} \times F_e \mid E \in \mathfrak{A},\ F_e \in \beta \right\},$$

$$\overleftarrow{\mathfrak{A}\beta} = \left\{ \bigcup_{f \in F} E_f \times \{f\} \mid F \in \beta,\ E_f \in \mathfrak{A} \right\},$$

$$\mathfrak{A} \times \beta = \overrightarrow{\mathfrak{A}\beta} \cap \overleftarrow{\mathfrak{A}\beta}, \quad \text{and}$$

$$\mathfrak{A} \otimes \beta = \langle \overrightarrow{\mathfrak{A}\beta} \cup \overleftarrow{\mathfrak{A}\beta} \rangle.$$

It can be shown that the forward and backward operational products, $\overrightarrow{\mathfrak{A}\beta}$ and $\overleftarrow{\mathfrak{A}\beta}$, respectively, are manuals and, as a consequence, so is the cartesian product $\mathfrak{A} \times \beta$. If $\mathfrak{A}$ and β admit unital sets of regular states, then the tensor product $\mathfrak{A} \otimes \beta$ exists and, of course, it is a manual. The cartesian product represents situations that admit bilateral influence or interference between the two factor manuals, while the operational products represent situations admitting only unilateral influence or interference. The tensor product represents situations that admit no influence or interference, but that do admit correlation.[21] Of course the union of two quantum mechanical systems would normally be represented by the tensor product. Evidently, it is possible to construct iterated products, thus satisfying Desideratum VI.

In general, $\Pi(\mathfrak{A} \times \beta)$ need not be an orthomodular poset, even when $\Pi(\mathfrak{A})$ and $\Pi(\beta)$ are orthomodular posets. Althought it might be supposed that this is a defect in our version of the tensor product, it can be shown that there is no tensor product in the category of orthomodular posets that satisfies even the most fundamental requirements.[10,20] In fact, then, the apparent "defect" is really an essential property of tensor products.

REFERENCES

1. G. Birkhoff, Lattices in applied mathematics, in: "Proceedings of Symposia in Pure Mathematics, Vol. II Lattice Theory", American Mathematical Society, Providence, R.I. (1961), pp. 154-184.
2. N. Bohr, "Essays on Atomic Physics and Human Knowledge" (1958-1962), Interscience Publishers, New York (1963).
3. R.M. Cooke and J. Hilgevoord, A new approach to quantum logic, (these transactions).
4. A.H. Copeland, Sr., Probabilities, observations, and predictions, in "Third Berkeley Symposium on Mathematical Statistics and Probability", Vol. II, University of California Press, Berkeley, Ca. (1956), pp. 41-48.
5. D.J. Foulis and C.H. Randall, Operational statistics I. Basic concepts, J. Math. Phys. 13 (1972), pp. 1667-1675.
6. D.J. Foulis and C.H. Randall, Empirical logic in quantum mechanics, Synthèse 29 (1974), pp. 81-111.
7. D.J. Foulis and C.H. Randall, The empirical logic approach to the physical sciences, in "Foundations of Quantum Mechanics and Ordered Linear Spaces", A. Härtkamper and H. Neumann eds, Springer Verlag, Berlin (1974), pp. 81-111.
8. D.J. Foulis and C.H. Randall, The stability of pure weights under conditioning, Glasgow Math. J. 15 (1974), pp. 5-12.
9. D.J. Foulis and C.H. Randall, Manuals, morphisms, and quantum mechanics", in "Mathematical Foundations of Quantum Theory", A.R. Marlow ed., Academic Press, New York (1978), pp. 105-126.

10. D.J. Foulis and C.H. Randall, Empirical logic and tensor products, in "Proceedings of the Colloquium on the Interpretations and Foundations of Quantum Theories", (Fachbereich Physik der Philipps Universität, Marburg, Germany, May 1979), H. Neumann ed., (To appear).

11. G. Hardegree and P.J. Frazer, Charting the labyrinth of quantum logics; a progress report, (These Transactions).

12. W. Heisenberg, "The Physical Principles of Quantum Theory", University of Chicago Press, Chicago (1930).

13. W. Kneale, What can we see? in "Observation and Interpretation in the Philosophy of Physics", S. Körner ed., Dover Publishers, New York (1957), pp. 151-159.

14. B.O. Koopman, The axioms and algebra of intuitive probability, Ann. of Math. 41 (1940), pp. 269-292.

15. G. Mackey, Quantum mechanics and Hilbert space, American Math. Monthly Suppl. 64 (1957), pp. 45-57.

16. M.D. MacLaren, "Notes on Axioms for Quantum Mechanics", ANL 7065, Argonne National Laboratory, Argonne, Ill. (1965).

17. C.H. Randall and D.J. Foulis, Operational statistics II. Manuals of operations and their logics, J. Math. Phys. 14 (1973), pp. 1472-1480.

18. C.H. Randall and D.J. Foulis, A mathematical setting for inductive reasoning, in "Foundations of Probability Theory, Statistical Inference, and Statistical Theories of Science, III", C.A. Hooker ed., D. Reidel Publishing Co., Dordrecht (1976), pp. 169-205.

19. C.H. Randall and D.J. Foulis, The operational approach to quantum mechanics, in "Physical Theory as Logico-Operational Structure", C.A. Hooker ed., D. Reidel Publishing Company, Dordrecht (1978), pp. 167-201.

20. C.H. Randall and D.J. Foulis, Tensor products of quantum logics do not exist, Notices of the American Math. Soc. 26:6 (1979), A-557.

21. C.H. Randall and D.J. Foulis, Operational statistics and tensor products, in "Proceedings of the Colloquium on the Interpretations and Foundations of Quantum Theories", (Fachbereich Physik der Philipps Universität, Marburg, Germany, May 1979), H. Neumann ed., (To appear).

22. L. Tisza, The conceptual structure of physics, Rev. Mod. Phys. 35 (1963), pp. 151-185.

23. R. Wright, Spin manuals: empirical logic talks quantum mechanics, in "Mathematical Foundations of Quantum Theory", A.R. Marlow ed., Academic Press, New York (1978), pp. 177-254.

24. R. Wright, The state of the pentagon: a nonclassical example, in "Mathematical Foundations of Quantum Theory", A.R. Marlow ed., Academic Press, New York (1978), pp. 255-274.

CHARTING THE LABYRINTH OF QUANTUM LOGICS:

A PROGRESS REPORT

Gary M. Hardegree

Department of Philosophy
University of Massachusetts

and

Patricia J. Frazer†

Department of Mathematics and Statistics
University of Massachusetts

1. INTRODUCTION

Quantum logic traces to the investigations of Birkhoff and von Neumann[1]* who suggested that the (closed) subspaces of a (separable) Hilbert space may be interpreted as representing the propositions pertaining to a physical system. Subsequent research has attempted to clarify this suggestion, and a large assortment of mathematical systems have been investigated under the general rubric of quantum logic (QL).

All quantum logical systems take the family of subspaces of a Hilbert space as their point of departure, but beyond this there are important differences concerning the appropriate parent category. Birkhoff and von Neumann emphasized the lattice structure, and their heirs have proposed the class of orthomodular lattices as the appropriate parent category. Other researchers have concentrated on different structural features of quantum propositions, and have accordingly proposed different parent categories. The most prominent

†Research supported by Zonta International, Amelia Earhart Fellowship.

*The 1936 investigation stems from von Neumann's earlier work (1932)[13] Chapter 3, Section 5. Strauss made a similar suggestion in 1937/8.[11]