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FUNCTIONAL ANALYSIS

Finite Dimensional Orlicz Spaces

by

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Summary. The problem of description of the unit balls of the n-dimensional Orlicz and Musielak-Orlicz spaces in the space of all compact convex subset of R^n is studied. For $n = 2$ every compact symmetric body is the unit ball of some Orlicz space. This result cannot be extended to arbitrary $n \geqslant 3$. The unit ball of the n-dimensional Musielak-Orlicz space is stable.

It is well-known that to every compact centrally symmetric convex set with non-empty interior there corresponds a norm defined by the Minkowski functional. Consider a finite dimensional Orlicz space. More precisely, let $\varphi: [0, \infty) \to [0, \infty)$ be a convex function with $\varphi(0) = 0$. By l_n^φ we denote the space of sequences $(x_k) \in \mathbf{R}^n$ endowed with the Luxemburg norm

$$\|(x_k)\|_\varphi = \inf\{\alpha: \sum_{k=1}^{n} \varphi(|x_k/\alpha|) \leqslant 1.$$

We refer the reader to [3] for basic facts about Orlicz spaces. There is a natural question, whether each compact symmetric convex subset of $\mathbf{R}^n$ with non-empty interior can be a unit ball $B(l_n^\varphi)$ of some n-dimensional Orlicz space l_n^φ. In this note we discuss the above question. An answer is affirmative if $n = 2$ and negative if $n \geqslant 3$.

The condition $\|(x_k)\|_\varphi = \|(|x_k|)\|_\varphi$ geometrically means that $B(l_n^\varphi)$ is symmetric with respect to each hyperplane $\{x_k: x_{k_0} = 0\}$, $k_0 = 1, 2, \ldots, n$. The convex set $Q \subset \mathbf{R}^n$ such that $(x_k) \in Q$ if and only if $(|x_{\pi(i)}|) \in Q$ for all permutations π of $1, 2, \ldots, n$ will be called symmetric. Obviously the unit ball $B(l_n^\varphi)$ of l_n^φ is a symmetric convex subset of $\mathbf{R}^n$.

Because l_n^φ is an Orlicz space defined on the atomic measure space with mass of atoms equal to one, the domain of φ may be restricted to $[0, 1]$ if it is assumed that $\|e_i\|_\varphi = 1$.

THEOREM 1. *Every compact symmetric convex subset of* $\mathbf{R}^2$ *with non-empty interior is a unit ball of some Orlicz space* l_2^φ.

Proof. Let Q be a compact symmetric convex subset of $\mathbf{R}^2$ with non-empty interior. We denote by $\|\cdot\|$ the Minkowski functional of Q. We may and do assume that $\|e_i\| = 1$. We define a function $f\colon [0, 1] \to [0, 1]$ by

$$f(x) = \max\{z \colon \|(x, z)\| = 1\}.$$

Note that if $x < 1$ then there exists exactly one $z \geqslant 0$ with $\|(x, z)\| = 1$. The function f is concave and decreasing and $f(0) = 1$, $\|(f(x), x)\| = 1$. Let $x_0 > 0$ be such that $\|(x_0, x_0)\| = 1$. We have $0 \leqslant f(1) \leqslant f(x_0) = x_0 \leqslant 1$. If $x_0 < 1$ then $0 \geqslant f'_-(x_0) \geqslant -1 \geqslant f'_+(x_0)$, since f^{-1} exists in some neighbourhood of x_0 (and $f^{-1} = f$).

If $x_0 = 1$, then $l_2^\varphi = l_2^\infty$. In this case Q is a unit ball of an Orlicz space generated by a function

$$\varphi(t) = \begin{cases} 0 & \text{for} \quad 0 \leqslant t \leqslant 1 \\ +\infty & \text{for} \quad t > 1 \end{cases}$$

Now assume that $x_0 < 1$. Define

$$\varphi(t) = \begin{cases} (1 - f(t))/2(1 - x_0) & \text{if} \quad 0 \leqslant t \leqslant x_0 \\[2ex] \dfrac{1}{2} + (t - x_0)/2(1 - x_0) & \text{if} \quad x_0 < t. \end{cases}$$

The function φ is convex. Indeed, the restricted functions $\varphi|_{[0, x_0]}$ and $\varphi|_{(x_0, \infty)}$ are convex. We only need to show that

$$\varphi'_-(x_0) = \lim_{h \to 0_-} \frac{\varphi(x_0 + h) - \varphi(x_0)}{h} \leqslant \frac{1}{2(1 - x_0)} = \varphi'_+(x_0)$$

The end of the above equality holds since $0 \geqslant f'_-(x_0) \geqslant -1$.

We claim that $B(l_2^\varphi) = Q$. Let $0 \leqslant x \leqslant y \leqslant 1$ be such that $\|(x, y)\| = 1$. To prove our claim it is sufficient to show that $\|(x, y)\|_\varphi = 1$. We have

$$\|(x, y)\|_\varphi = \inf\left\{\alpha \colon \varphi\left(\frac{x}{\alpha}\right) + \varphi\left(\frac{y}{\alpha}\right) \leqslant 1\right\} = \inf A$$

where $A = \{\alpha \colon x/\alpha \leqslant f(y/\alpha)\}$.

Obviously $1 \in A$. Suppose that some $\alpha_0 < 1$ belongs to A. Then $\|(y/\alpha_0, x/\alpha_0)\| \leqslant 1$, but this contradices with $\|(x, y)\| = 1$. Therefore $\|(x, y)\|_\varphi = \inf A = 1$. This completes the proof of Theorem.

REMARK 1. Instead of φ in the proof of Theorem 1 we can use the following Orlicz functions

$$\varphi_1(t) = \begin{cases} t/2x_0 & \text{if} \quad 0 \leqslant t \leqslant x_0 \\ 1 - f(t)/2x_0 & \text{if} \quad x_0 \leqslant t \leqslant 1 \\ +\infty & \text{if} \quad t > 1 \end{cases}$$

or

$$\varphi_2(t) = \begin{cases} h(t) & \text{if} \quad 0 \leqslant t \leqslant x_0 \\ 1 - h(f(x)) & \text{if} \quad x_0 < t \leqslant 1 \end{cases}$$

where we choose a function h in such a way that φ_2 is convex. Obviously $h(x_0)$ must be equal to 1/2.

Therefore $\mathbf{R}^2$ the same Orlicz space can be generated by two distinct Orlicz functions. For instance the Euclidean norm in R^2 is generated by φ, φ_1 where $f(t) = \sqrt{1-t^2}$ and by $\varphi_3 = t^2$ etc. Note that from the con-from the construction presented in the proof of Theorem 1 follows that the space l_2^1 is generated by exactly one Orlicz function (because $x_0 = 1/2$).

It should be pointed out that in the two dimensional case there exists strict convex Orlicz space generated by no strict convex Orlicz function (cf. [5], [6], [1], [2]).

Remark 2. There exists a compact symmetric convex subset of $\mathbf{R}^3$, which is a unit ball of no Orlicz space l_3^φ. For example let

$$Q = \operatorname{conv}\left\{\pm e_1, \pm e_2, \pm e_3, \left(\pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}\right)\right\}.$$

Indeed, suppose that there exists an Orlicz function such that $Q = B(l_3^\varphi)$. Since intersection Q with the plane $\{(x_1, x_2, 0) \in \mathbf{R}^3 : x_1, x_2 \in \mathbf{R}\}$ give l_2^1-ball. Thus $\varphi(0) = 0$, $\varphi(1/2) = 1/2$ and $\varphi(1) = 1$, so $\varphi(t) = t$ for $t \in [0, 1]$. Therefore $l_3^\varphi = l_3^1$, but $Q \neq B(l_3^1)$.

We will need the following fact.

Lemma. *Let H denote subset of the unit interval* $(0, 1)$ *such that*

(i) $1/2 \in H$,

(ii) $a \in H$ *implies* $(1-a) \in H$,

(iii) $a \in H$ *implies* $(1-a)/2 \in H$.

Then H is dense in $(0, 1)$.

Proof. Applaing (ii) and (iii) we obtain

(iv) $a \in H$ implies $a/2 \in H$.

Suppose that $k/2^n \in H$, $k = 1, 2, 3, \ldots, 2^n - 1$, $n \in \mathbf{N}$. It is sufficient to show that $l/2^{n+1} \in H$ for all $l = 1, \ldots, 2^{n+1} - 1$. If $l \in 2^n$, then $l/2^n \in H$ and by (iv) $l/2^{n+1} \in H$. If $2^n < l < 2^{n+1}$, then $(2^{n+1} - l)/2^{n+1} \in H$ and by (ii) $l/2^{n+1} \in H$.

Proposition. *The sections of the unit ball $B(l_3^\varphi)$ by the planes $\{(x_1, x_2, 0)\in \mathbf{R}^3 : x_1, x_2 \in \mathbf{R}\}$ and $\{(x_1, x_1, x_2)\in \mathbf{R}^3 \; x_1, x_2\in \mathbf{R}\}$ uniquely determines the Orlicz function φ.*

Proof. Let $\|\cdot\|_\varphi$ be the Luxemburg norm of l_3^φ. We can and do assume that $\|e_1\|_\varphi = 1$. Let $x_0 > 0$ be such that $\|(x_0, x_0, 0)\| = 1$. We define functions $f: [0, 1] \to [0, 1]$, $g: [0, x_0] \to [0, 1]$ by

$$f(x) = \max\{z: \|(x, z, 0)\|_\varphi = 1\}$$

$$g(x) = \max\{z: \|(x, x, z)\|_\varphi = 1\}.$$

It should be pointed out that the functions f and g can be defined in the case if only plane sections of $B(l_3^\varphi)$ presented in statement of Proposition are known.

Put $y_1 = \max\{x: f(x) = 1\}$, $y_2 = \max\{x: g(x) = 1\}$. Because f and g are concave and decreasing, the restricted functions $f_1 = f|_{[y_1, 1]}$ and $g_1 = g|_{[y_2, 1]}$ are strictly decreasing. Therefore f_1^{-1} and g_1^{-1} exist.

Since φ is increasing, convex and $\varphi([0, 1]) \subset [0, 1]$ it is sufficient to find a set B such that $H = \{\varphi(x): x\in B\}$ is a dense subset of $(0, 1)$. Note that if $\|(x_1, x_2, x_3)\| = 1$ and $0 \leqslant x_i < 1$ $i = 1, 2, 3$, then $\varphi(x_1)+\varphi(x_2)+\varphi(x_3) = 1$. Thus if $x, f(x), g(x)\in(0, 1)$, then $\varphi(x)+\varphi(f(x)) = 1$ and $\varphi(x)+\varphi(x)+\varphi(g(x)) = 1$. Therefore if the value $b = \varphi(y)$ is known, then we can determine $\varphi(f_1^{-1}(y)) = 1-\varphi(y)$, and analogously $\varphi(g_1^{-1}(y)) = [1-\varphi(y)]/2$. Let B be a set such that

(a) $x_0 \in H$

(b) $x\in B$ implies $f_1^{-1}(x)\in B$

(c) $x\in B$ implies $g_1^{-1}(x)\in B$.

Then $\varphi(x_0) = 1/2 \in H$ and $b\in H$ implies $(1-b)\in H$ (by (*b*)) and $(1-b)/2$ H (by (*c*)). Invoking the Lemma we conclude H is dense in $(0, 1)$. Therefore φ is uniquely determined by the functions f and g.

Remark 3. Above Proposition can be written for arbitrary l_n^φ, $n \geqslant 3$ and l^φ.

Problem. Characterize all $B(l_n^\varphi)$ in the space of compact symmetric convex subsets of $\mathbf{R}^n$ $(n \geqslant 3)$.

The case of Musielak-Orlicz spaces. Consider more general class of spaces: Musielak-Orlicz spaces. In the 2-dimensional case the unit ball of the Musielak-Orlicz space generated by φ_1, φ_2 is a set

$$B = \{(x, y)\in \mathbf{R}^2 : \varphi_1(|x|)+\varphi_2(|y|) \leqslant 1$$

where φ_i are convex functions with $\varphi_i(0) = 0$, $i = 1, 2$. Obviously the unit ball of each Musielak-Orlicz space is centrally symmetric convex body. It is also symmetric with respect to x and y — axes when we consider the plane.

THEOREM 2. *Every compact convex set $B \subset \mathbf{R}^2$ with int $B \neq \emptyset$ such that $(x, y) \in B$ implies $(\pm x, \pm y) \in B$ is a unit ball of some 2-dimensional Musielak--Orlicz space.*

Proof. Let B be a subset of $\mathbf{R}^2$ satisfying the assumption of the Theorem. We denote by $\|\cdot\|$ the norm corresponding to B. Let $a, b > 0$ be such that $\|(a, 0)\| = \|(0, b)\| = 1$. Put

$$\varphi_1(t) = \frac{t}{a}$$

$$\varphi_2(t) = \begin{cases} 1 - \max\{z : \|(az, t)\| = 1\} & \text{if} \quad 0 \leqslant t \leqslant b \\ +\infty & \text{if} \quad t \geqslant b. \end{cases}$$

It is not hard to see that φ_i are convex functions with $\varphi_i(0) = 0$, $i = 1, 2$, and the unit ball of the Musielak-Orlicz space generated by φ_1, φ_2 coincides with B.

REMARK 4. There exists a compact symmetric convex subset of $\mathbf{R}^3$ which is the unit ball of no Musielak-Orlicz space. For example the set Q from Remark 2. Indeed, suppose, to get a contradiction, that Q is the unit ball of a 3-dimensional Musielak-Orlicz space generated by convex functions $\varphi_1, \varphi_2, \varphi_3$ i.e.

$$Q = \{(x, y, z) \in \mathbf{R}^3 \ \ \varphi_1(|x|) + \varphi_2(|y|) + \varphi_3(|z|) \leqslant 1\}.$$

Because $(1, 0, 0), (0, 1, 0), (0, 0, 1) \in B$ we have $\varphi_i(1) \leqslant 1$. Because $\left(\frac{1}{2}, \frac{1}{2}, 0\right)$, $\left(\frac{1}{2}, 0, \frac{1}{2}\right), \left(0, \frac{1}{2}, \frac{1}{2}\right)$ belong to the unit sphere we obtain

$$\varphi_1\left(\frac{1}{2}\right) + \varphi_2\left(\frac{1}{2}\right) = 1$$

$$\varphi_1\left(\frac{1}{2}\right) + \varphi_3\left(\frac{1}{2}\right) = 1$$

$$\varphi_2\left(\frac{1}{2}\right) + \varphi_3\left(\frac{1}{2}\right) = 1$$

After solving the above three equations we obtain $\varphi_i\left(\frac{1}{2}\right) = \frac{1}{2}$ $i = 1, 2, 3$. Thus we have $\varphi_i(t) = t$ for $t \in [0, 1]$, since φ_i are convex. This contradicts with

$$\varphi_1\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) \in Q$$

and

$$\varphi_1(1/2)+\varphi_2(1/2)+\varphi_3(1/2) = 3/2 > 1.$$

The problem of description of all the unit balls of Musielak-Orlicz spaces in the space of compact convex subset of R^n ($n \geqslant 3$) remains open.

The affine structure of the unit ball. For a point x a convex compact set B we define a face generated by x in B as follows

$$F_x = \{y \in B: \text{there exist } z \in B \quad \text{and} \quad \alpha \in (0, 1] \quad \text{such that} \quad x = \alpha y + (1-\alpha) z\}.$$

Note that $x \in \operatorname{ext} B$ if and only if $\dim F_x = 0$.

Remark 5. Let $\varphi_i: \mathbf{R} \to [0, \infty)$ be convex functions s.t. $\varphi_i(t) = 0$ iff $t = 0$. Let U_i be maximal open subsets of $\mathbf{R}$ such that φ_i is linear on each connected component of U_i, $i = 1, 2, \ldots, n$. Denote by $l_n^{(\varphi_i)}$ the n dimensional Musielak-Orlicz space generated by φ_i. Let $x \in l_n^{(\varphi_i)}$ with $\|x\| = 1$. We have $\dim F_x = \dim \operatorname{lin} Y$, where $Y = \{y: x \pm y \in B(l_n^{(\varphi_i)})$. By convexity of φ_i for $z \in Y$

$$1 = \sum_{i=1}^{n} \varphi_i(x_i) = \sum_{i=1}^{n} \frac{1}{2} [\varphi_i(x_i - z_i) + \varphi_i(x_i + z_i)] = 1$$

so

$$\varphi_i(x_i) = \frac{1}{2} [\varphi_i(x_j - z_i) + \varphi_i(x_i + z_i)] \quad \text{for} \quad i = 1, 2, \ldots, n.$$

Therefore if $x_i \notin U_i$ then $z_i = 0$, i.e. $z \in Y$ and $i \in J(x) = \{i: x_i \in U_i\}$ implies $z_i = 0$. Hence

$$\dim F_x = \begin{cases} 0 & \text{if} \quad k = 0 \quad \text{and} \quad \|x\| = 1 \\ k-1 & \text{if} \quad k \geqslant 1 \quad \text{and} \quad \|x\| = 1 \\ n & \text{if} \quad \|x\| < 1 \end{cases}$$

where $k = \operatorname{card} J(x)$.

The m-skeleton of a convex set B is the set of all $x \in B$ such that $\dim F_x \leqslant m$. We recall that a convex compact set B in an Euclidean space is said to be stable if all m-skeletions of B are closed (see [4]).

Theorem 3. *The unit ball of $l_n^{(\varphi i)}$ is stable.*

Proof. Fix $m \leqslant n$. Let a sequence $\{x^k\}_{k=1}^n$ with $\dim F_{x^k} \leqslant n$ converges to x^0. We need to show that $\dim F_{x^0} \leqslant m$. Suppose, to get a contradiction, that $\dim F_{x^0} = m' > m$. Then $\operatorname{card} J(x^0) = m'+1$ and there exists K such that x_i^k and x_i^0 belong to the same component of U_i for all $k \geqslant K$ and all $i \in J(x^0)$ (since U_i are open). By Remark 5 it follows that $\dim F_{x^m} \geqslant m' > m$ for all $k \geqslant K$. This contradiction ends the proof.

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Р. Гжонслевич, **Конечномерные пространства Орлича**

В работе рассматривается проблема изображения шара n-мерного пространства Орлича и Муселяка–Орлича в пространстве всех компактных выпуклых множеств в R^n. Для $n = 2$ симметрическое компактное выпуклое множество является шаром определенного пространства Орлича. Дается пример на то, что не существует такого изображения для $n \geqslant 3$. Доказывается, что шар n-мерного пространства Муселяка–Орлича является стабильным.